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Asymptotic Properties of Heavy-Tailed Distributions

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JŪRATĖ KARASEVIČIENĖ,
Vilnius.

Contents

The main notations

$\mathbb{N}$	set of positive integers
$\mathbb{N}_0$	set of non-negative integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of non-negative real numbers
$\mathcal{H}$	class of heavy-tailed distributions
$\mathcal{S}$	class of subexponential distributions
$\mathcal{OS}$	class of generalized subexponential distributions
$\mathcal{L}$	class of long-tailed distributions
$\mathcal{OL}$	class of $\mathcal{O}$-exponential distributions
$\mathcal{L}(\gamma)$	class of exponential distributions
$\mathcal{D}$	class of dominatedly varying distributions
$F_\xi(x) = \mathbb{P}(\xi \leqslant x)$	distribution function of the random variable ξ
$\overline{F}_\xi = 1 - F_\xi$	tail of the distribution function F_ξ
$\lfloor x \rfloor$	integer part of the real number x
$\langle x \rangle$	fractional part of the real number x
$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A \end{cases}$	indicator function of the set A
d.f.	distribution function
d.f.s	distribution functions
r.v.	random variable
r.v.s	random variables
i.i.d.	independent and identically distributed
supp(X) = {x ∈ $\mathbb{R}$: $\mathbb{P}(X = x) > 0\}$	support of the discrete random variable X

For positive functions f and g we have the following notations:

$f(x) = o(g(x))$	denotes that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$
$f(x) = O(g(x))$	denotes that $\limsup\limits_{x\to\infty} \frac{f(x)}{g(x)} < \infty$
$f(x) \sim cg(x), c > 0$	denotes that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = c$
$f(x) \asymp g(x)$	denotes that $0 < \liminf\limits_{x\to\infty} \frac{f(x)}{g(x)} \leqslant \limsup\limits_{x\to\infty} \frac{f(x)}{g(x)} < \infty$

1 Introduction

1.1 Research problem, topicality and novelty

The research objects of the thesis are the randomly stopped sum S_η, the randomly stopped minimum $\xi_{(\eta)}$, maximum $\xi^{(\eta)}$ and the randomly stopped minimum of sums $S_{(\eta)}$ and maximum of sums $S^{(\eta)}$. These randomly stopped structures are defined in Chapter 2 by the following equalitiess:

$$
\begin{aligned}
S_\eta &= \xi_1 + \cdots + \xi_\eta, \quad S_0 = 0,\\
\xi^{(\eta)} &= \max\{0, \xi_1, \ldots, \xi_\eta\},\\
S^{(\eta)} &= \max\{S_0, S_1, \ldots, S_\eta\},\\
\xi_{(\eta)} &= \begin{cases} 0 & \text{if } \eta = 0,\\ \min\{\xi_1, \ldots, \xi_\eta\} & \text{if } \eta \geqslant 1, \end{cases}\\
S_{(\eta)} &= \begin{cases} 0 & \text{if } \eta = 0,\\ \min\{S_1, \ldots, S_\eta\} & \text{if } \eta \geqslant 1, \end{cases}
\end{aligned}
$$

where $\{\xi_1, \xi_2, \ldots\}$ is a sequence of random variables and η is a counting random variable.

In this thesis, we consider conditions under which distribution functions of randomly stopped S_η, $\xi_{(\eta)}$, $\xi^{(\eta)}$, $S_{(\eta)}$, $S^{(\eta)}$ belong to the class of generalized subexponential distributions and to the class of heavy-tailed distributions. The results presented in this thesis complement the closure properties of randomly stopped sums considered in monographs [6, 30, 50] and in the references therein. In this thesis the primary random variables are supposed to be independent and real-valued, but not necessarily identically distributed. The counting random variable describing the stopping moment of random structures is supposed to be nonnegative, integer-valued and not degenerate at zero. In addition, it is supposed that counting random variable and the sequence of the primary random variables are independent. The motivation for investigating randomly stopped structures primarily arises from insurance and finance, where questions related to extreme or rare events are traditionally considered, see e.g., [3, 22, 62]. In particular, exponential, Pareto, gamma, lognormal and loggamma distributions are extremely popular in actuarial mathematics. Mathematical aspects of risk theory related to the calculation of ruin probabilities are addressed in numerous works; see [3, 7, 33, 34, 62, 63] and references therein. From the mathematical point of view, the success of any insurance business depends on the asymptotic behavior of the distribution of S_η, $S^{(\eta)}$ and $S_{(\eta)}$. If the distribution of the individual claim size Z is light-tailed, i.e.,

$$\mathbb{E}e^{\gamma Z} < \infty$$

for some $\gamma > 0$, then the ruin probability of the corresponding risk process is also relatively small for large values of the initial surplus and decreases with an exponential rate; see, e.g., [3, 34, 62, 63]. If the individual claim size is heavy-tailed, i.e.,

$$\mathbb{E}e^{\gamma Z} = \infty$$

for all $\gamma > 0$, then the ruin probability of the corresponding risk process decreases much more slowly as the initial surplus increases; see, e.g., [63]. Therefore, it is worth finding out at the beginning of the investigation whether the distribution of individual claim sizes is light-tailed or heavy-tailed. One of the most significant research directions in risk theory is the investigation of the ruin probability when the distribution of claim sizes is heavy-tailed. We will consider the class of generalized subexponential distribution $\mathcal{OS}$. From the description of this class it follows that some distributions in this class have heavy tails, while others have light tails. However, even in the event that the random variable generating the claim flow has the $\mathcal{OS}$-class or similar regularity, it is possible to provide some considerable information about the ruin probability of the model [9, 43, 65, 66, 72, 76]. Results regarding the asymptotic behavior of the ruin probability typically differ across classes.

The closure properties in probability theory have a long history, back from the middle of the previous century. They appear as substantial supports in reliability theory, queuing theory, branching processes, risk theory, stochastic control, asset pricing and others fields. Bingham, Goldie and Teugels [6], Seneta [64] and Resnick [61] were among the first researchers to study closure problems. In the mentioned monographs, these authors fully explored the properties of slowly varying, regularly varying, and O-regularly varying functions, which are closely related to the closure properties of distribution functions. It is worth mentioning that the majority of the initial results related to the closure problems were obtained for the d.f.s of identically distributed r.v.s. A detailed analysis of closure problems for two random variables is given in the book by Leipus et al. [50]. The main novelty of this thesis is that not only identically distributed r.v.s and not only two, but also a random number of primary r.v.s are considered.

1.2 Aim and tasks

The main aim of the thesis is to find conditions for the independent random variables $\{\xi_1, \xi_2, \dots\}$ and the counting random variable η under which the distribution functions of S_η, $\xi_{(\eta)}$, $\xi^{(\eta)}$, $S_{(\eta)}$ and $S^{(\eta)}$ belong to some regularity classes of distribution functions.

To achieve the aim, the following tasks are raised:

- To establish conditions under which the randomly stopped sum S_η belongs to the class of generalized subexponential distributions.
- To find conditions under which the randomly stopped $\xi_{(\eta)}$, $\xi^{(\eta)}$, $S_{(\eta)}$ and $S^{(\eta)}$ belong to the class of generalized subexponential distributions.
- To determine the conditions on the primary random variables, under which the randomly stopped structures are either heavy-tailed or light-tailed.

1.3 Methodology of the investigation

Belonging to the classes of heavy-tailed distributions is usually associated with the tail behavior of the distribution function. Therefore, to estimate tail probabilities for sums of random variables and the minimum, maximum, minimum of sums, maximum of sums, we use standard methods of probability theory in this thesis. To investigate the tails of randomly stopped sums, randomly stopped maximums and minimums, randomly stopped manimum and maximums of sums, the set of all possible values of the counting random variable is usually divided into a few subsets, where the tails are studied separately using different methods. The tails of sums of random variables are evaluated using classical methods when the values of the counting random variables are held constant.

1.4 Defended propositions

- Distribution function of randomly stopped sums belongs to the class of generalized subexponential distributions if the first distribution belongs to this class and other members do not interfere, and the counting random variable has finite support.

- Distribution of randomly stopped sum is generalized subexponential, if the counting random variable is ultralight-tailed and tails of distributions of primary r.v.s are asymptotically equivalent to the same generalized subexponential distribution.

- If distributions of independent r.v.s along with an independent counting r.v. belong to the class of generalized subexponential distributions, then their minimum and minimum of randomly stopped sums of independent r.v.s. belong to the same class.

- In order to satisfy the condition for maximum, it is required that first distribution is generalized subexponential, other tails of distributions of primary r.v.'s are asymptotically equivalent to the first generalized subexponential distribution and a counting random variable has finite expectation.

- Conditions under which the distribution function of the randomly stopped maximum of sums is generalized subexponential are as follows: the distribution function of the first r.v. belongs to the class of generalized subexponential distributions, other tails of distributions of primary r.v.'s are asymptotically equivalent to the first generalized subexponential distribution and counting random variable has finite exponential expectation.

- For light-tailed primary r.v.'s there exist randomly stopped structures that belong to the class of heavy-tailed distributions.

1.5 Publications

- Karasevičienė, J., Šiaulys, J. (2023). Randomly stopped sums with generalized subexponential distribution. *Axioms*, 12: 641.

- Karasevičienė, J., Šiaulys, J. (2024). Randomly stopped minimum, maximum, minimum of sums and maximum of sums with generalized subexponential distribution. *Axioms*, 13: 85.

- Leipus, R., Šiaulys, J., Danilenko, S., Karasevičienė , J. (2024). Randomly stopped sums, minima and maxima for heavy- tailed and light-tailed distribution *Axioms*, 13: 335.

1.6 Conferences

- Atsitiktinai sustabdytos apibendrintų subeksponentinių skirstinių sumos. *64th conference of Lithuanian Mathematical Society*, June 21-22, 2023, Vilnius.

- Randomly stopped sums with generalized subexponential distribution, properties of min, max and min, max of sums. *The international scientific conference dedicated to the 160th anniversary of Prof. Dr Hermann Minkowski*, June 20-22, 2024, Kaunas.

- Randomly stopped sums with generalized subexponential distribution, properties of min, max and min, max of sums. *11th Tartu Conference on Multivariate Statistics*, June 25-28, 2024, Tartu.

1.7 Structure of the thesis

In Chapters $1-2$, the necessary notations is introduced and an overview of the known results is given. In subsection 2.1, we formulate the main notions and recall the definitions of the classes $\mathcal{H}$, $\mathcal{S}$, $\mathcal{OS}$, $\mathcal{L}$, $\mathcal{OL}$, $\mathcal{D}$ and $\mathcal{L}(\gamma)$, $\gamma > 0$, along with some other related classes. In addition, we describe some interrelationships among the classes of heavy-tailed distributios. In subsection 2.3, we give a number of typical examples of d.f.'s from all classes under consideration. In subsection 2.4, we formulate a few known results for several classes of distributions.

In Chapters $3-5$, we present our main results.

In Chapter 3, we determine conditions under which the distribution function of the randomly stopped sum $S_\eta = \xi_1 + \xi_2 + \ldots + \xi_\eta$ belongs to the class of generalized subexponential distributions, where: $\{\xi_1, \xi_2 \ldots\}$ is a sequence of independent possibly differently distributed random variables with distribution functions $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ and η is a counting random variable independent of the sequence $\{\xi_1, \xi_2 \ldots\}$. Theorems and proofs are based on the publication [40].

In Chapter 4, we determine conditions under which the distribution functions of randomly stopped minimum, maximum, minimum of sums and maximum of sums belong to the class of generalized subexponential distributions. As in Chapter 3 the primary random variables are supposed to be independent and real-valued, but not necessarily identically distributed. The counting random variable describing the stopping moment of random structures is supposed to be nonnegative, integer-valued and not degenerate at zero. In addition, it is supposed that a counting random variable and the sequence of the primary random variables are independent. In subsection 4.2 it is demonstrated how randomly stopped structures can be applied to the construction of new generalized subexponential distributions. Theorems and proofs are based on the publication [41].

Chapter 5 investigates the randomly stopped sums, minima and maxima of heavy- and light-tailed random variables. The conditions on the primary random variables, which are independent but generally not identically distributed, and a counting random variable are given in order that the randomly stopped sum, random minimum and maximum is heavy- tailed or light-tailed. The results generalize several existing findings in the previous literature. The examples illustrating the results are provided. Theorems and proofs are based on the publication [51].

Finally, the conclusions are formulated in Chapter 6.

Chapters 3 - 5 present a collection of auxiliary lemmas, and the proofs of the main results.

2 Preliminaries

2.1 Randomly stopped structures

In this subsection we define all randomly stopped structures which we consider in Chapters 3, 4, 5. Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting random variable, that is, a nonnegative, nondegenerate at 0, and integer-valued r.v. In addition, we suppose that the r.v. η and the sequence $\{\xi_1, \xi_2, \ldots\}$ are independent.

Let $S_0 := 0$, $S_n := \xi_1 + \ldots + \xi_n$ for $n \in \mathbb{N}$, and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the *randomly stopped sum* of the r.v.s $\xi_1, \xi_2, \ldots$.

By F_{S_η} we denote the d.f. of S_η, and by $\overline{F}$ we denote the tail function (t.f.) of a d.f. F, that is, $\overline{F}(x) = 1 - F(x)$ for $x \in \mathbb{R}$. It is obvious that the following equalities hold for positive x:

$$F_{S_\eta}(x) = \mathbb{P}(\eta = 0) + \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n \leqslant x)$$

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n > x).$$

For instance, if

$$\mathbb{P}(\eta = k) = \frac{1}{2^k}, k \in \mathbb{N}$$

and

$$\overline{F}_{\xi_k}(x) = \mathbb{1}_{(-\infty,0)}(x) + \frac{1}{(1+x)^3}\mathbb{1}_{[0,\infty)}(x), \quad k \in 1,2,3...$$

then

$$\begin{aligned}
\overline{F}_{S_\eta}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n > x) \\
&= \frac{1}{2}\frac{1}{(1+x)^3} + \sum_{n=2}^{\infty} \frac{1}{2^n}\overline{F_{\xi_1} * F_{\xi_2} .... * F_{\xi_n}} \\
&= \frac{1}{2}\frac{1}{(1+x)^3} + \sum_{n=2}^{\infty} \frac{1}{2^n}\overline{F^{*n}_{\xi_1}}(x), \quad x > 0,
\end{aligned}$$

where $F_1 * F_2 = \int\limits_0^x F_1(x-y)\mathrm{d}F_2(y)$.

In Chapter 3, we consider a sequence $\{\xi_1, \xi_2, \ldots\}$ of independent and possibly nonidentically distributed r.v.s. We suppose that some of the d.f.s of these r.v.s belong to the specific class of distributions, and we find conditions under which the d.f. F_{S_η} remains in that class.

In our thesis the main results are obtained for the class of generalized subexponential distributions $\mathcal{OS}$ and the class of heavy-tailed distributions $\mathcal{H}$. We present a discussion of these considered classes and related classes in subsection 2.3.

Other randomly stopped structures which we consider in this thesis are randomly stopped minimum, randomly stopped maximum, randomly stopped minimum of sums and randomly stopped maximum of sums. Suppose, as before, that $\{\xi_1, \xi_2, \ldots\}$ is a sequence of r.v.s and η is a counting r.v. independent of $\{\xi_1, \ldots, \xi_n\}$.

Let $\xi_{(0)} = 0$, $\xi_{(n)} = \min\{\xi_1, \ldots, \xi_n\}$ for $n \in \mathbb{N}$, and let

$$\xi_{(\eta)} = \begin{cases} 0 & \text{if } \eta = 0, \\ \min\{\xi_1, \ldots, \xi_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

be the *randomly stopped minimum* of r.v.s $\{\xi_1, \xi_2, \ldots\}$.

Let $\xi^{(0)} = 0$, $\xi^{(n)} = \max\{\xi_1, \ldots, \xi_n\}$ for $n \in \mathbb{N}$, and let

$$\xi^{(\eta)} = \begin{cases} 0 & \text{if } \eta = 0, \\ \max\{\xi_1, \ldots, \xi_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

be the *randomly stopped maximum* of r.v.s $\{\xi_1, \xi_2, \ldots\}$.

Let, as before, S_η be the *randomly stopped sum* of r.v.s $\{\xi_1, \xi_2, \ldots\}$, then

$$S_{(\eta)} = \begin{cases} 0 & \text{if } \eta = 0, \\ \min\{S_1, \ldots, S_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

be the *minimum of randomly stopped sums* of r.v.s $\{\xi_1, \xi_2, \dots\}$, and

$$S^{(\eta)} = \begin{cases} 0 & \text{if } \eta = 0, \\ \max\{S_1, \dots, S_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

be the *maximum of randomly stopped sums* of r.v.s $\{\xi_1, \xi_2, \dots\}$.

We observe that the following equalities hold for positive x:

$$\overline{F}_{\xi_{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\xi_{(n)} > x)\mathbb{P}(\eta = n),$$

$$\overline{F}_{\xi^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\xi^{(n)} > x)\mathbb{P}(\eta = n),$$

$$\overline{F}_{S_{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(S_{(n)} > x)\mathbb{P}(\eta = n),$$

$$\overline{F}_{S^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(S^{(n)} > x)\mathbb{P}(\eta = n),$$

where $\overline{F}_\xi = 1 - F_\xi = \mathbb{P}(\xi > x)$ denotes the tail function (t.f.) of r.v. ξ. For instance, let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent r.v.s such that the first member ξ_1 has the Pareto distribution:

$$F_{\xi_1}(x) = (1 - \frac{1}{(1+x)^3})\,\mathbb{1}_{[0,\infty)}(x)$$

and other elements of the sequence are identically exponentially distributed:

$$F_{\xi_k}(x) = \big(1 - \mathrm{e}^{-x}\big)\,\mathbb{1}_{[0,\infty)}(x) \quad k \in \{2, 3, \dots\}$$

In the case of the discrete uniform counting r.v. η with parameter $N = 3$, we have:

$$\begin{aligned}
\overline{F}_{\xi^{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\xi^{(n)} > x)\mathbb{P}(\eta = n) = \sum_{n=1}^{3} \mathbb{P}\Big(\bigcup_{k=1}^{n} \{\xi_k > x\}\Big)\mathbb{P}(\eta = n) \\
&= \frac{1}{3} \sum_{n=1}^{3} \mathbb{P}\Big(\bigcup_{k=1}^{n} \{\xi_k > x\}\Big) \\
&= \frac{1}{3}\Big(\mathbb{P}(\xi_1 > x) + \mathbb{P}\big((\xi_1 > x) \cup (\xi_2 > x)\big) \\
&+ \mathbb{P}\big((\xi_1 > x) \cup (\xi_2 > x) \cup (\xi_3 > x)\big)\Big) \\
&= \frac{1}{3}\Big(\mathbb{P}(\xi_1 > x) + \mathbb{P}(\xi_1 > x) + \mathbb{P}(\xi_2 > x) - \mathbb{P}(\xi_1 > x)\mathbb{P}(\xi_2 > x)
\end{aligned}$$

$$
\begin{aligned}
&+ \mathbb{P}(\xi_1 > x) + \mathbb{P}(\xi_2 > x) + \mathbb{P}(\xi_3 > x) \\
&- \mathbb{P}(\xi_1 > x)\mathbb{P}(\xi_2 > x) - \mathbb{P}(\xi_1 > x)\mathbb{P}(\xi_3 > x) - \mathbb{P}(\xi_2 > x)\mathbb{P}(\xi_3 > x) \\
&+ \mathbb{P}(\xi_1 > x)\mathbb{P}(\xi_2 > x)\mathbb{P}(\xi_3 > x)\Big) \\
&= \mathbb{I}_{(-\infty,0)}(x) + \frac{1}{3}\Big(\frac{1}{(1+x)^3} + \frac{1}{(1+x)^3} + e^{-x} - \frac{e^{-x}}{(1+x)^3} \\
&+ \frac{1}{(1+x)^3} + e^{-x} + e^{-x} - \frac{2e^{-x}}{(1+x)^3} - e^{-2x} + \frac{e^{-2x}}{(1+x)^3}\Big)\mathbb{I}_{[0,\infty)}(x) \\
&= \mathbb{I}_{(-\infty,0)}(x) + \Big(\frac{1}{(1+x)^3} \\
&+ \Big(\mathrm{e}^{-x} - \frac{\mathrm{e}^{-2x}}{3}\Big)\Big(1 - \frac{1}{(1+x)^3}\Big)\Big)\mathbb{I}_{[0,\infty)}(x).
\end{aligned}
$$

$$
\begin{aligned}
\overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\min\{\xi_1, \ldots, \xi_n\} > x)\mathbb{P}(\eta = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \prod_{k=1}^{n} \overline{F}_{\xi_k}(x) \\
&= \mathbb{I}_{(-\infty,0)}(x) + \frac{1}{3}(\frac{1}{(1+x)^3} + \frac{\mathrm{e}^{-\mathrm{x}}}{(1+x)^3} \\
&+ \frac{\mathrm{e}^{-2\mathrm{x}}}{(1+x)^3})\mathbb{I}_{[0,\infty)}(x) \\
&= \mathbb{I}_{(-\infty,0)}(x) + \frac{1}{3(1+x)^3}(1 + \mathrm{e}^{-x} + \mathrm{e}^{-2x})\mathbb{I}_{[0,\infty)}(x).
\end{aligned}
$$

$$
\begin{aligned}
\overline{F}_{S_{(\eta)}}(x) &= \sum_{n=1}^{3} \overline{F}_{S_{(n)}}(x)\mathbb{P}(\eta = n) \\
&= \frac{1}{3}\overline{F}_{S_{(1)}} + \frac{1}{3}\overline{F}_{S_{(2)}} + \frac{1}{3}\overline{F}_{S_{(3)}} \\
&= \mathbb{P}(S_1 > x) \\
&= \overline{F}_{S_1}(x), \;\; x \geqslant 0.
\end{aligned}
$$

$$
\begin{aligned}
\overline{F}_{S^{(\eta)}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{S^{(n)}}(x)\ \mathbb{P}(\eta = n) = \frac{1}{3}\sum_{n=1}^{3} \overline{F}_{S^{(n)}}(x) \\
&= \frac{1}{3}(\mathbb{P}(S_1 > x) + \mathbb{P}(\max(S_1, S_2) > x) + \mathbb{P}(\max(S_1, S_2, S_3) > x)) \\
&= \frac{1}{3}(\mathbb{P}(S_1 > x) + \mathbb{P}(S_2 > x) + \mathbb{P}(S_3 > x)) \\
&= \frac{1}{3}\sum_{n=1}^{3} \overline{\coprod_{k=1}^{n} F_{\xi_k}(x)} \\
&= \frac{1}{3}(\mathbb{P}(\xi_1 > x) + \mathbb{P}(\xi_1 + \xi_2 > x) + \mathbb{P}(\xi_1 + \xi_2 + \xi_3 > x), \;\; x \geqslant 0.
\end{aligned}
$$

2.2 Regularity classes of distribution functions

In this subsection we describe $\mathcal{OS}, \mathcal{H}$ and related classes of d.f.s.

DEFINITION 2.2.1. *R.v. ξ is said to be heavy-tailed ($F_\xi \in \mathcal{H}$) if for any $\lambda > 0$*

$$\mathbb{E}\, \mathrm{e}^{\lambda\xi} = \int_{-\infty}^{\infty} \mathrm{e}^{\lambda x} \mathrm{d}F_\xi(x) = \infty.$$

The class of heavy-tailed random variables $\mathcal{H}$ has a very rich structure. The most important subclasses of $\mathcal{H}$ are defined below.

Subexponential and long-tailed distributions were first introduced and studied by Chistyakov [10] in the context of the branching process. In particular, he proved that the subexponential distribution class is contained in the class of long-tailed distributions and in the class of heavy-tailed distributions. Later subexponential distributions have found broad application in probability theory, renewal theory and the theory of infinitely divisible distributions (see, e.g., [24, 25, 30, 58, 71]). The class of subexponential distributions was studied also by Athreya and Ney [4], Chover et al. [11, 12], Embrechts and Goldie [23], Embrechts and Omey [26], Cline [14] and Cline and Samorodnitsky [15], among others.

DEFINITION 2.2.2. *A d.f. F_ξ of a non-negative r.v. ξ is said to be subexponential, denoted $F_\xi \in \mathcal{S}$, if*

$$\overline{F_\xi * F_\xi}(x) \sim 2\overline{F}_\xi(x).$$

Here and subsequently, $*$ denotes the convolution of d.f.'s.

- *A d.f. F_ξ of a real-valued r.v. ξ is called subexponential $F_\xi \in \mathcal{S}$ if:*

$$F_\xi^+(x) = F_\xi(x)\mathbb{I}_{[0,\infty)}(x)$$

belongs to the class $\mathcal{S}$.

The class of subexponential distributions was introduced by Chistyakov [10] and later studied by Athreya and Ney [4], Chover et al. [11, 12], Embrechts and Goldie [23], Embrechts and Omey [26], Cline [14] and Cline and Samorodnitsky [15], among others.

DEFINITION 2.2.3. *A d.f. F_ξ of a real-valued r.v. is said to be generalized subexponential, denoted $F_\xi \in \mathcal{OS}$, if*

$$\limsup_{x\to\infty} \frac{\overline{F_\xi * F_\xi}(x)}{\overline{F}_\xi(x)} < \infty,$$

Definition 2.2.4. *A d.f. F is said to be long-tailed ($F \in \mathcal{L}$) if for any fixed $y > 0$ we have*

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1.$$

Shimura and Watanabe [65] introduced the class $\mathcal{OL}$, which is the direct generalization of the class $\mathcal{L}$. In that paper, Shimura and Watanabe established the main properties of class $\mathcal{L}$ and investigated some of its subclasses.

Definition 2.2.5. *A d.f. F is said to be $\mathcal{O}$-exponential ($F \in \mathcal{OL}$) if for any fixed $y \in \mathbb{R}$ we have*

$$0 < \liminf_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} < \infty.$$

The last definition implies that $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ if $F \in \mathcal{OL}$.
It is obvious that $F \in \mathcal{OL}$ if and only if

$$\limsup_{x\to\infty} \frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty, \tag{2.2.1}$$

or, equivalently,

$$\sup_{x\geqslant 0} \frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty.$$

The last condition shows that the class $\mathcal{OL}$ is quite wide. Now we describe the most popular subclasses of $\mathcal{OL}$ because we will present some results on the random convolution of distributions from these subclasses later.

Definition 2.2.6. *A d.f. F is said to belong to the class of exponential distributions ($F \in \mathcal{L}(\gamma)$) with some $\gamma > 0$ if for any fixed $y > 0$, we have*

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-y\gamma}.$$

Definition 2.2.7. *A d.f. F is said to be dominatingly varying ($F \in \mathcal{D}$) if for any fixed $y \in (0,1)$, we have*

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

The class of dominatedly varying d.f.s $\mathcal{D}$ was introduced by Feller [28] and later considered in [6, 8, 45, 64, 69, 70, 75], among others.
The class $\mathcal{L}(\gamma)$ with $\gamma > 0$ was introduced by Chover et al. [11, 12]. Later the various properties of long-tailed and exponential-like-tailed

d.f.s were considered in [8, 22, 31, 32, 42, 59, 73]. Now we summarize the interrelationships among the most important classes of heavy-tailed distributions introduced above. Most of these interrelationships are well known.

$$\mathcal{L}\cap\mathcal{D}\subset\mathcal{S}\subset\mathcal{L}\subset\mathcal{H},\quad \mathcal{D}\subset\mathcal{H}.$$

Namely implication

$$\mathcal{L}\cap\mathcal{D}\subset\mathcal{S}$$

is proven in Proposition 1.4.4 in [25]. The inclusion

$$\mathcal{S}\subset\mathcal{L}$$

is derived in Lemma 1.3.5 (a) at [25]. While the inclusion

$$\mathcal{L}\subset\mathcal{H}$$

follows from the representation formula (2.2.2). Namely, acording to such formula (see Theorem 1.3.1 in [6]) for $F\in\mathcal{L}$, we have that

$$\overline{F}(\log x)=a(x)\exp\left\{\int_1^x\frac{\varepsilon(u)}{u}\mathrm{d}u\right\},x>0, \tag{2.2.2}$$

where

$$a(x)\underset{x\to\infty}{\to}a$$

and

$$\varepsilon(x)\underset{x\to\infty}{\to}0.$$

This implies that

$$\overline{F}(x)=a^*(x)\exp\left\{\int_0^x\varepsilon^*(u)\mathrm{d}u\right\},x>0,$$

with measurable functions a^* and ε^* such that

$$a^*(x)\underset{x\to\infty}{\to}a^*$$

and

$$\varepsilon^*(x)\underset{x\to\infty}{\to}0.$$

Hence, for an arbitrary $\varrho>0$,

$$\begin{aligned}\limsup_{x\to\infty}\overline{F}(x)\mathrm{e}^{\varrho x} &= \limsup_{x\to\infty}a^*(x)\exp\left\{\int_0^x(\varrho+\varepsilon^*(u))du\right\}\\ &= \infty.\end{aligned}$$

And $F \in \mathcal{H}$ according to Definition 2.2.1.
Figure 1 shows the interrelationships among the classes of heavy-tailed distributions $\mathcal{D}$, $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{H}$.

The above definitions imply directly the following interrelationships which can be seen in Figure 2

$$\mathcal{L} \subset \mathcal{OL}, \quad \mathcal{D} \subset \mathcal{OL} \quad \text{and} \quad \bigcup_{\gamma>0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

$$\mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}, \quad \mathcal{D} \subset \mathcal{H}$$

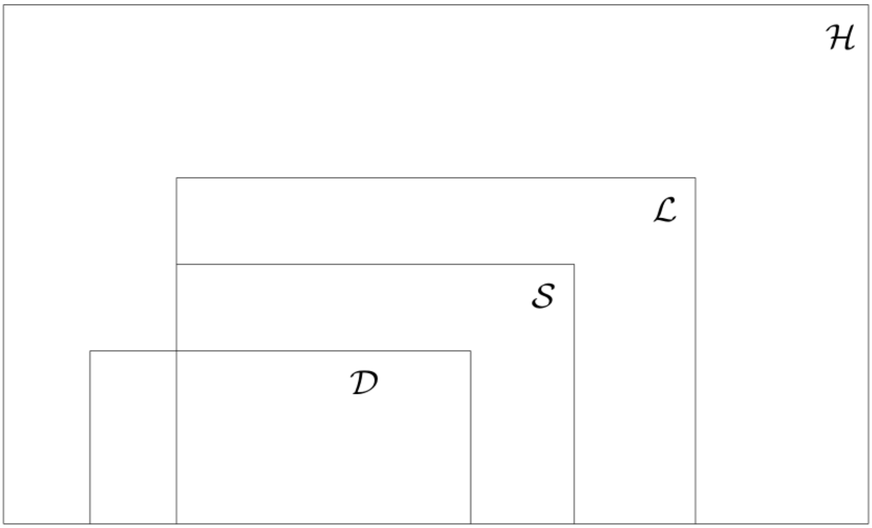

Figure 1: *Interrelationships between d.f.s regularity classes*

$$\mathcal{L} \subset \mathcal{OL}, \quad \mathcal{D} \subset \mathcal{OL}, \quad \bigcup_{\gamma>0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

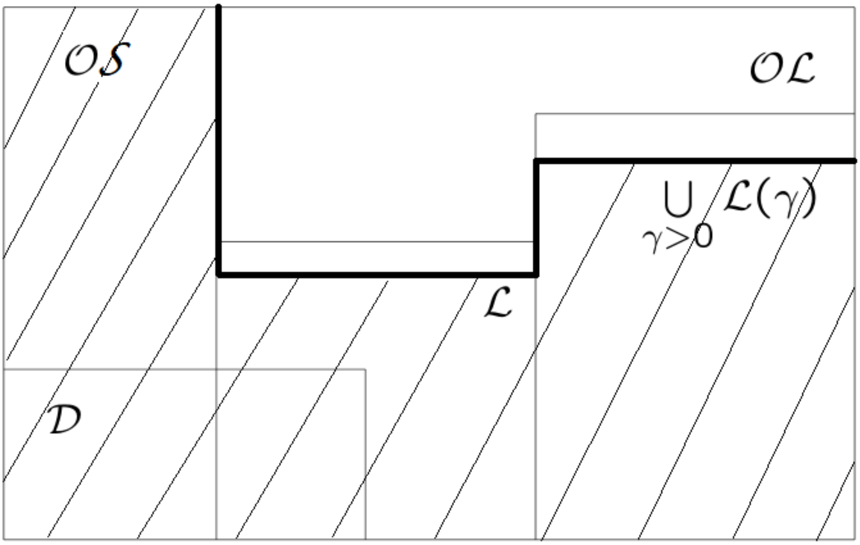

Figure 2: *Interrelationships between d.f.s regularity classes*

2.3 Typical representatives of regularity classes

In this subsection, we give a number of d.f.'s which belong to the classes defined in subsection 2.2.

EXAMPLE 2.3.1. *The Pareto distribution with d.f.*

$$F(x) = 1 - \left(1 + \frac{x}{b}\right)^{-a}, \ x \geqslant 0,$$

where $b > 0$ is the scale parameter and $a > 0$ is the shape parameter.

With $a = 1$, $b = 5$, we have Figure 3 with the tail function $\overline{F}(x)$.

For the Pareto distribution we show, that $F \in \mathcal{L} \cap \mathcal{D}$. It is clear that $\overline{F}(x) \sim (x/b)^{-a}$ as $x \to \infty$. For this reason, the Pareto distribution is sometimes referred to as the power-law distribution. The Pareto distribution has finite moments of order $k < a$, whereas all moments of order $k \geqslant a$ are infinite. We have the following:

• $F \in \mathcal{L}$ for fixed $y \in \mathbb{R}$ because

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \lim_{x\to\infty} \left(\frac{x+y}{x}\right)^{-a}$$
$$= \lim_{x\to\infty} \left(1 + \frac{y}{x}\right)^{-a} = 1.$$

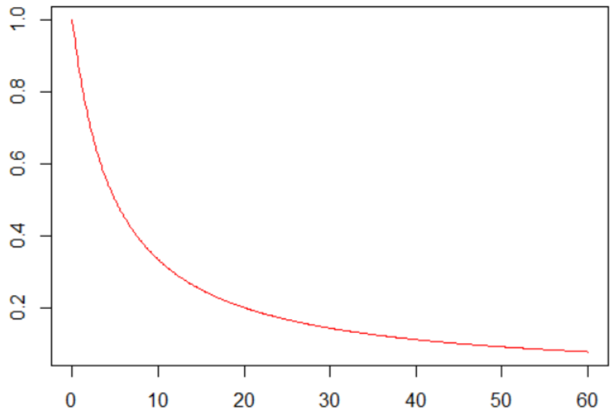

Figure 3: *Graph of the function* $\overline{F}(x) = (1 + x/5)^{-1}$

- $F \in \mathcal{D}$ for fixed $y \in (0, 1)$ because

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \limsup_{x\to\infty} \left(\frac{xy}{x}\right)^{-a}$$
$$= \limsup_{x\to\infty} \left(\frac{1}{y}\right)^{a} < \infty.$$

Example 2.3.2. *The Weibull distribution with d.f.*

$$F(x) = (1 - \exp\{-(x)^{\alpha}\})\mathbb{1}_{[0,\infty)}(x)$$

is subexponential with $\alpha \in (0, 1)$.

With $\alpha = 0.5$, we have Figure 4 of the t.f. $\overline{F}(x)$.
Since $F(x) = 0$ for $x < 0$, it follows that

$$\liminf_{x\to\infty} \frac{\overline{F^{*2}(x)}}{\overline{F}(x)} \geq 2.$$

On the other hand, for positive x,

$$F^{*2}(x) = \int_{[0,x]} F(x-y)\mathrm{d}F(y)$$
$$= \int_0^x F(x-y)\mathrm{d}F(y).$$

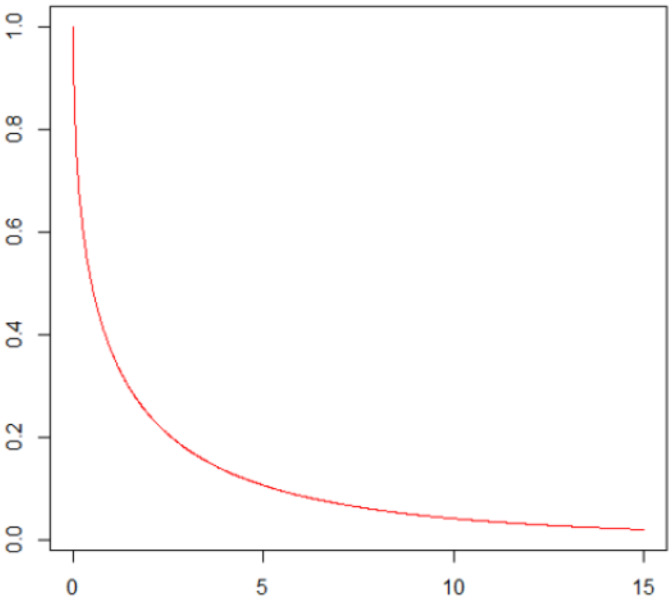

Figure 4: *Graph of the function* $\overline{F}(x) = e^{-(x)^{0.5}}$

Therefore,

$$\overline{F^{*2}}(x) = \int_0^\infty \mathrm{d}F(y) - \int_0^x F(x-y)\mathrm{d}F(y)$$
$$= \overline{F}(x) + \int_0^x \overline{F}(x-y)\mathrm{d}F(y).$$

Let the notation $f(x) = O_\alpha(g(x))$ means, that

$$\limsup_{x\to\infty} \frac{f(x)}{g(x)} = C_\alpha < \infty$$

for some constant C_α depending on α.

For any function $\phi(x)$ with a property $0 < \phi(x) < x$,

$$\begin{aligned}
\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} =&1 + \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \mathrm{d}F(y) \\
&= 1 + \alpha \int_0^{\phi(x)} \exp\{x^\alpha - (x-y)^\alpha - y^\alpha) y^{\alpha-1} \mathrm{d}y \\
&+ \alpha \int_{\phi(x)}^{x-\phi(x)} \exp\{x^\alpha - (x-y)^\alpha - y^\alpha\} y^{\alpha-1} \mathrm{d}y \\
&+ \alpha \int_{x-\phi(x)}^{x} \exp\{x^\alpha - (x-y)^\alpha - y^\alpha\} y^{\alpha-1} \mathrm{d}y \qquad (2.3.1) \\
&= 1 + \alpha \int_0^{\phi(x)} \exp\{x^\alpha - (x-y)^\alpha - y^\alpha\} y^{\alpha-1} \mathrm{d}y \\
&+ \alpha \int_{\phi(x)}^{x-\phi(x)} \exp\{x^\alpha - (x-y)^\alpha - y^\alpha\} y^{\alpha-1} \mathrm{d}y \\
&+ \alpha \int_0^{\phi(x)} \exp\{x^\alpha - (x-y)^\alpha - y^\alpha\} (x-y)^{\alpha-1} \mathrm{d}y.
\end{aligned}$$

By choosing $\phi(x) = \log^{\frac{2}{\alpha}} x$, we get

$$\begin{aligned}
&\alpha \int_0^{\phi(x)} \exp\{x^\alpha - (x-y)^\alpha - y^\alpha\} y^{\alpha-1} \mathrm{d}y \\
&= \int_0^{\phi(x)} \exp\left\{x^\alpha \left(\alpha \frac{y}{x} + O_\alpha\left(\frac{y^2}{x^2}\right)\right)\right\} \mathrm{d}(-\exp(-y^\alpha)) \\
&= \left(1 + O_\alpha\left(\frac{\log^{\frac{2}{\alpha}} x}{x^{1-\alpha}}\right)\right)\left(1 - \exp\{-\log^2 x\}\right).
\end{aligned}$$

In a similar manner,

$$\alpha \int\limits_0^{\phi(x)} \exp\{x^\alpha - (x-y)^\alpha - y^\alpha\}(x-y)^{\alpha-1} \mathrm{d}y$$

$$= \int\limits_0^{\phi(x)} \exp\left\{x^\alpha \left(\alpha \frac{y}{x} + O_\alpha\left(\frac{y^2}{x^2}\right)\right)\right\}\left(\frac{y}{x-y}\right)^{1-\alpha} \mathrm{d}(-\exp(-y^\alpha))$$

$$= \left(1 + O_\alpha\left(\frac{\log^{\frac{2}{\alpha}} x}{x^{1-\alpha}}\right)\right) \int\limits_0^{\phi(x)} \left(\frac{y}{x-y}\right)^{1-\alpha} \mathrm{d}(-\exp(-y^\alpha))$$

$$= \left(1 + O_\alpha\left(\frac{\log^{\frac{2}{\alpha}} x}{x^{1-\alpha}}\right)\right) O_\alpha\left(\frac{\log^{\frac{2}{\alpha}} x}{x^{1-\alpha}}\right)^{1-\alpha} \left(1 - \exp\{-\log^2 x\}\right).$$

Finally, we can evaluate the remaining integral as follows:

$$I(x) = \alpha \int\limits_{\phi(x)}^{x-\phi(x)} \exp\{x^\alpha - (x-y)^\alpha - y^\alpha\} y^{\alpha-1} \mathrm{d}y$$

$$\leq \max_{\phi(x) \leq y \leq x-\phi(x)} \exp\{-y^\alpha - (x-y)^\alpha\} \exp\{x^\alpha\} \alpha \int\limits_{\phi(x)}^{x-\phi(x)} y^{\alpha-1} \mathrm{d}y.$$

When $y \in [\phi(x), x-\phi(x)]$, the function acquires minima at the endpoints of the interval. Hence,

$$I(x) \leq x^\alpha \exp\{x^\alpha - \phi^\alpha(x) - (x-\phi(x))^\alpha\}$$
$$= x^\alpha \exp\left\{x^\alpha\left(1 - \left(1 - \frac{\phi(x)}{x}\right)^\alpha\right) - \log^2 x\right\}$$
$$= \frac{x^\alpha}{\exp\{\log^2 x\}}\left(1 + O_\alpha\left(\frac{\log^{\frac{2}{\alpha}} x}{x^{1-\alpha}}\right)\right).$$

After inserting the obtained integral estimates into the inequality (2.3.1), we have:

$$\limsup_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \leq 1$$
$$+ \limsup_{x\to\infty} \left(1 + O_\alpha\left(\frac{\log^{\frac{2}{\alpha}} x}{x^{1-\alpha}}\right)\right)\left(1 + O_\alpha\left(\frac{\log^{\frac{2}{\alpha}} x}{x^{1-\alpha}}\right)^{1-\alpha}\right)\left(1 - \exp\{-\log^2 x\}\right)$$
$$+ \limsup_{x\to\infty} \frac{x^\alpha}{\exp\{\log^2 x\}}\left(1 + O_\alpha\left(\frac{\log^{\frac{2}{\alpha}} x}{x^{1-\alpha}}\right)\right) = 2.$$

From the established boundary relations it follows that

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2.$$

EXAMPLE 2.3.3. *The distribution from the paper by Cline and Samorodnitsky [15] with t.f.*

$$\overline{F}(x) = \exp\big\{ - \lfloor \log(1+x) \rfloor \\ - \min\{(1+x)(\log(1+x) - \lfloor \log(1+x) \rfloor), 1\}\big\}, \quad x \geqslant 0,$$

where $\lfloor z \rfloor$ *denotes the integer part of* z *and* $\langle z \rangle$ *denotes the fractional part of* z*.*

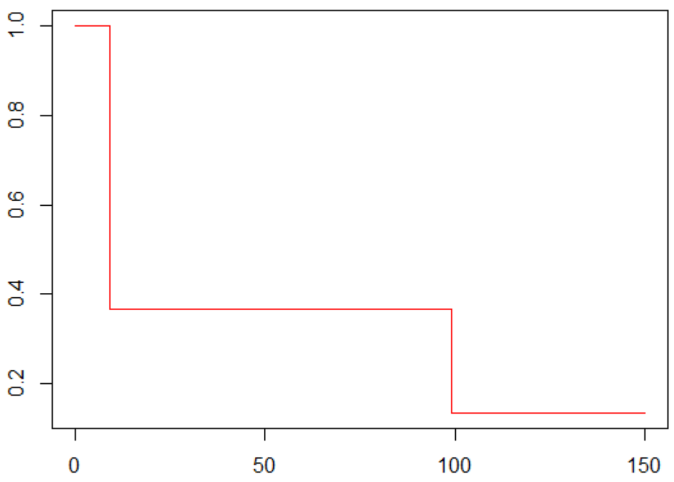

Figure 5: *Graph of the tail function*

The grapf of the tail function from example 2.3.3 is presented in Figure 5 In this case, we have $F \in \mathcal{D}$ but $F \notin \mathcal{L}$. By using the formula of the t.f. we derive that

$$\overline{F}(x) = \exp\{ - \lfloor \log(1+x) \rfloor - \min\{(1+x)\langle \log(1+x) \rangle, 1\}\} \\ \leq \exp\{ - \lfloor \log(1+x) \rfloor\},$$

and

$$\overline{F}(x) \geq \exp\{ - \lfloor \log(1+x) \rfloor - 1\},$$

for any $x \geq 0$.
Therefore, for each fixed $y \in (0,1)$,

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} &\leq \limsup_{x\to\infty} e^{-\lfloor \log(1+xy)\rfloor + \lfloor \log(1+x)\rfloor + 1} \\
&= e \limsup_{x\to\infty} e^{\log(1+x) - \langle \log(1+x)\rangle - \log(1+xy) + \langle \log(1+xy)\rangle} \\
&= e \limsup_{x\to\infty} \frac{1+x}{1+xy} e\langle \log(1+xy)\rangle - \langle \log(1+x)\rangle \\
&= \frac{e}{y} \limsup_{x\to\infty} e^{\langle \log(1+xy)\rangle - \langle \log(1+x)\rangle} \\
&\leq \frac{e^2}{y} < \infty.
\end{aligned}$$

This estimate implies that $F \in \mathcal{D}$.
By choosing the sequence $x_n = e^n - 1$, $n \in \mathbb{N}$ we get that

$$\overline{F}(x_n) = \exp\{ - \lfloor \log(e^n)\rfloor - \min\{(e^n)\langle \log(e^n)\rangle, 1\}\}$$
$$= e^{-n}$$

and

$$\begin{aligned}
&\overline{F}(x_n+1) = \exp\{ - \lfloor \log(e^n+1)\rfloor \\
&- \min\{(e^n+1)(\log(e^n+1) - \lfloor \log(e^n+1)\rfloor), 1\}\} \\
&= \exp\{ - n - \min\left\{(e^n+1)\log\frac{e^n+1}{e^n}, 1\right\}\} \\
&= e^{-n-1}
\end{aligned}$$

for large n.
The derived relations show that

$$\limsup_{x\to\infty} \frac{\overline{F}(x_n+1)}{\overline{F}(x_n)} = \frac{1}{e}$$

implying that $F \notin \mathcal{L}$.

Example 2.3.4. *By Proposition 2.6 from [1], an absolutely continuous d.f. F belongs to the class $\mathcal{L}(\gamma)$ if and only if*

$$\overline{F}(x) = \exp\left\{ - \int\limits_{-\infty}^{x} (a(u)+b(u))\mathrm{d}u \right\}$$

for $x \in \mathbb{R}$, where measurable functions a and b satisfy the following conditions:

(i) $a(u)+b(u) \geqslant 0, u \in \mathbb{R}$;

(ii) $\lim_{u\to\infty} a(u) = \gamma$;

(iii) $\lim_{x\to\infty} \int_{-\infty}^{x} a(u)\mathrm{d}u = \infty$;

(iv) $\lim_{x\to\infty} \int_{-\infty}^{x} b(u)\mathrm{d}u$ *exists.*

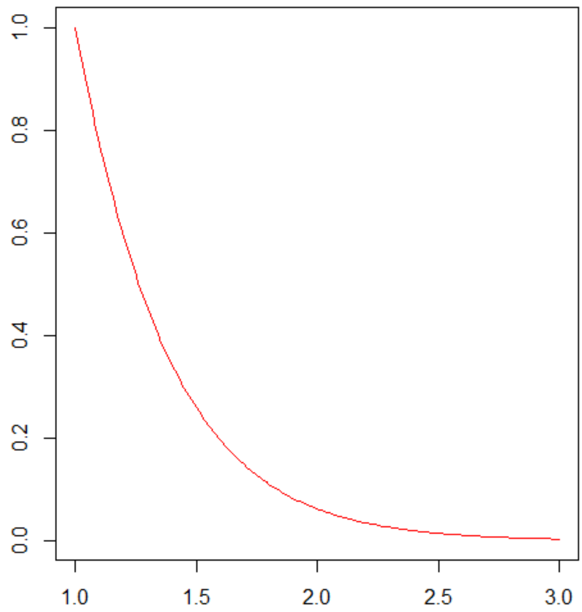

Figure 6: *Graph of the function* $\overline{F}(x) = \exp\left\{-\left(3x + \frac{1}{x} + \frac{1}{2}\arctan x^2 - 4 - \frac{\pi}{8}\right)\right\}$

If we choose

$$a(u) = \left(3 - \frac{1}{u^2}\right) \mathbb{I}_{[1,\infty)}(u) \tag{2.3.2}$$

and

$$b(u) = \left(\frac{u}{1+u^4}\right) \mathbb{I}_{[1,\infty)}(u), \tag{2.3.3}$$

then we get the d.f. F with the tail

$$\overline{F}(x) = \exp\left\{-\left(3x + \frac{1}{x} + \frac{1}{2}\arctan x^2 - 4 - \frac{\pi}{8}\right)\right\}, x \geqslant 1,$$

which belongs to the class $\mathcal{L}(3)$ because

$$\begin{aligned}
\lim_{u\to\infty} a(u) &= \lim_{u\to\infty}\left(3-\frac{1}{u^2}\right)=3,\\
\int\limits_{-\infty}^{x} a(u)du &= \int\limits_{-\infty}^{x}\left(3-\frac{1}{u^2}\right)\mathrm{d}u = 3x+\frac{1}{x}-4 \underset{x\to\infty}{\longrightarrow} \infty,\\
\lim_{x\to\infty}\int\limits_{-\infty}^{x} b(u)\mathrm{d}u &= \lim_{x\to\infty}\int\limits_{-\infty}^{x}\left(\frac{u}{1+u^4}\right)\mathrm{d}u\\
&= \frac{1}{2}\lim_{x\to\infty}(\arctan x^2-\arctan 1)=\frac{\pi}{8}.
\end{aligned}$$

Example 2.3.5. *By Proposition 2.6 from [1], an absolutely continuous d.f. F belongs to the class $\mathcal{OL}$ if its tail $\overline{F}$ has the representation*

$$\overline{F}(x)=\exp\left\{-\int\limits_{-\infty}^{x}(a(u)+b(u))\mathrm{d}u\right\} \tag{2.3.4}$$

for $x\in\mathbb{R}$, where some measurable functions a and b satisfy the following conditions:

(i) $a(u)+b(u)\geqslant 0, u\in\mathbb{R}$;

(ii) $\limsup\limits_{u\to\infty}|a(u)|<\infty$;

(iii) $\liminf\limits_{x\to\infty}\int\limits_{-\infty}^{x} a(u)\mathrm{d}u=\infty$;

(iv) $\limsup\limits_{x\to\infty}\left|\int\limits_{-\infty}^{x} b(u)\mathrm{d}u\right|<\infty$.

If we choose

$$a(u)=(3+\cos u)\,\mathbb{I}_{[1,\infty)}(u) \tag{2.3.5}$$

and

$$b(u)=\left(\sin u\cos^2 u\right)\mathbb{I}_{[1,\infty)}(u), \tag{2.3.6}$$

then we get the d.f. F with the tail

$$\overline{F}(x)=\exp\left\{-\left(3x+\sin x-\frac{\cos^3 x}{3}-3+\frac{\cos^3 1}{3}-\sin 1\right)\right\}, x\geqslant 1,$$

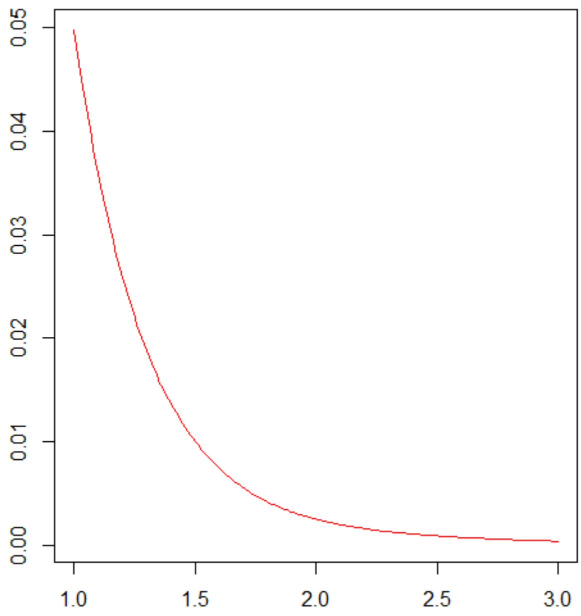

Figure 7: *Graph of the function*
$\overline{F}(x) = \exp\left\{-\left(3x + \sin x - \frac{\cos^3 x}{3} - 3 + \frac{\cos^3 1}{3} - \sin 1\right)\right\}$

which belongs to the class $\mathcal{OL}$ because

$$\limsup_{u\to\infty} |a(u)| = \limsup_{u\to\infty} |3 + \cos u| = 4,$$

$$\liminf_{x\to\infty} \int_{-\infty}^{x} a(u)du = \liminf_{x\to\infty} \int_{1}^{x} (3 + \cos u)\mathrm{d}u = \infty,$$

$$\limsup_{x\to\infty} \left|\int_{-\infty}^{x} b(u)du\right| = \limsup_{x\to\infty} \left|\int_{1}^{x} \cos^2 ud\cos \mathrm{u}\right| \leqslant \frac{1}{3}.$$

In Figure 6, the tail function is presented according selected functions 2.3.2 and 2.3.3.
In Figure 7, the tail function is presented according selected functions 2.3.5 and 2.3.6.

Example 2.3.6. *Let the r.v. ξ have the generalized geometric distribution with parameters $p \in (0;1), 0 < \alpha \leq 1$:*

$$\mathbb{P}(\xi = k) = \frac{\alpha p^k(1-p)}{(1-(1-\alpha)p^{k+1})(1-(1-\alpha)p^k)}, \quad k = 0, 1, 2, \ldots$$

For this r.v. tail of d.f. has the following form:

$$\overline{F}_\xi(x) = \sum_{k>x} \mathbb{P}(\xi = k) = \alpha p(1-p) \sum_{k=\lfloor x \rfloor + 1}^{\infty} \frac{p^k}{(1-(1-\alpha)p^{k+1})(1-(1-\alpha)p^k)}$$

This distribution belongs to the class $\mathcal{OL}$ based on the results presented in [21]

EXAMPLE 2.3.7. *Let the r.v. ξ be distributed according to the Peter and Paul law with parameters $\frac{1}{3}$ and $\frac{1}{7}$:*

$$\overline{F}_\xi(x) = 3 \sum_{\substack{3^l > x \\ l \geqslant 1}} \frac{1}{7^l}, \quad x \geqslant 1.$$

Since $\overline{F}_\xi$ is a piecewise constant function, ξ is a discrete random variable.

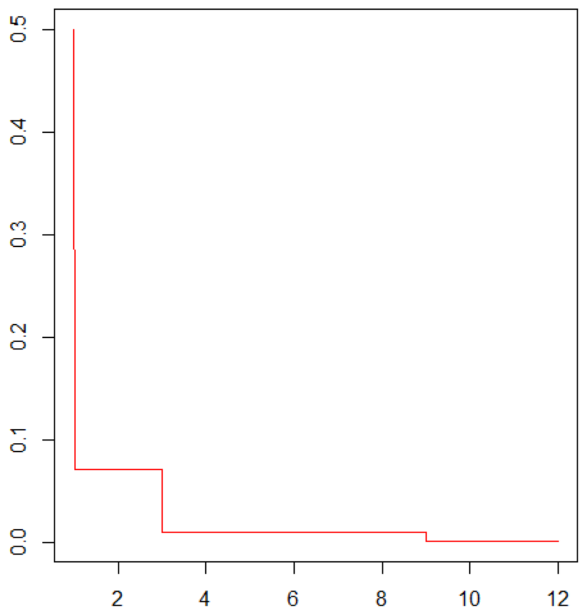

Figure 8: *Graph of the tail* $\overline{F}_\xi(x) = \frac{7^{-\lfloor \frac{\log x}{\log 3} \rfloor}}{2}$

.

In Figure 8, the tail function of d.f. Peter and Paul law is presented with parameters $\frac{1}{3}$ and $\frac{1}{7}$. For all $x \geqslant 1$, we have:

$$\begin{aligned}
\overline{F}_\xi(x) &= 3\sum_{3^l>x}\frac{1}{7^l} = 3\sum_{l>\frac{\log x}{\log 3}}\frac{1}{7^l} = 3\sum_{l=\lfloor\frac{\log x}{\log 3}\rfloor+1}^{\infty}\frac{1}{7^l} \\
&= 3\left(\frac{1}{7^{\lfloor\frac{\log x}{\log 3}\rfloor+1}}+\frac{1}{7^{\lfloor\frac{\log x}{\log 3}\rfloor+2}}+\dots\right) \\
&= 7^{-\lfloor\frac{\log x}{\log 3}\rfloor}3\left(\frac{1}{7}+\frac{1}{7^2}+\dots\right) \\
&= \frac{7^{-\lfloor\frac{\log x}{\log 3}\rfloor}}{2}.
\end{aligned}$$

Therefore, this distribution belongs to the class $\mathcal{D}$ because

$$\begin{aligned}
\limsup_{x\to\infty}\frac{\overline{F}_\xi(yx)}{\overline{F}_\xi(x)} &= \limsup_{x\to\infty}\frac{7^{\lfloor\frac{\log x}{\log 3}\rfloor}}{7^{\lfloor\frac{\log yx}{\log 3}\rfloor}} \\
&= \limsup_{x\to\infty}\frac{7^{\frac{\log x}{\log 3}-\langle\frac{\log x}{\log 3}\rangle}}{7^{\frac{\log yx}{\log 3}-\langle\frac{\log yx}{\log 3}\rangle}} \\
&\leqslant 7^{-\frac{\log y}{\log 3}}7 = 7^{1+\frac{1}{\log 3}\log\frac{1}{y}} < \infty
\end{aligned}$$

for any fixed $y \in (0,1)$.

Example 2.3.8. *Let ξ_1 be r.v. having the t.f.*

$$\overline{F}_\xi(x) = \mathbb{1}_{(-\infty,0)}(x) + \frac{\mathrm{e}^{-x}}{(1+x)^3}\left(1+\frac{\sin x}{d}\right)\mathbb{1}_{[0,\infty)}(x),$$

where $d > 2$. According to the results of [16], the d.f. F_ξ belongs to class $\mathcal{OS}$

In the particular case, if $d = 3$, then

$$\overline{F}_\xi(x) = \frac{\mathrm{e}^{-x}}{(1+x)^3}\left(1+\frac{\sin x}{3}\right)$$

for $x \geq 0$.
The graph of this function is presented in Figure 9:

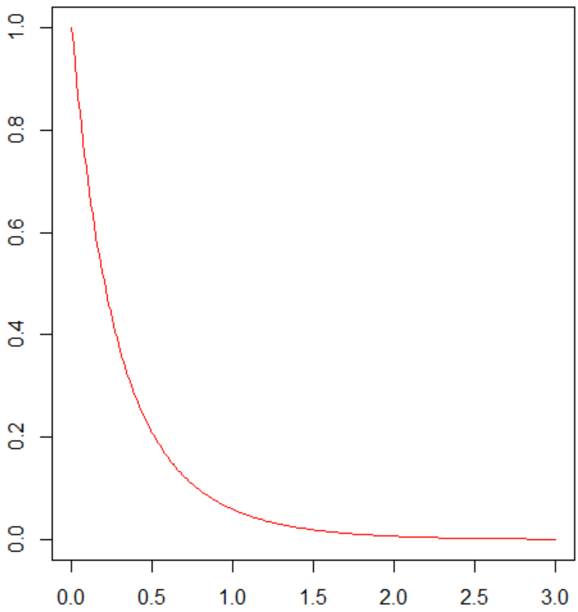

Figure 9: *Graph of the function*
$\overline{F}_\xi(x) = \mathbb{1}_{(-\infty,0)}(x) + \frac{e^{-x}}{(1+x)^3}\left(1 + \frac{\sin x}{3}\right)\mathbb{1}_{[0,\infty)}(x)$

EXAMPLE 2.3.9. *The distribution with d.f.*

$$F(x) = (1 - \exp\{-(\gamma x)\})\mathbb{1}_{[0,\infty)}(x)$$

and the tail function

$$\overline{F}(x) = \exp\{-(\gamma x)\}\,\mathbb{1}_{[0,\infty)}(x) + \mathbb{1}_{(-\infty,0)}(x)$$

belongs to $\mathcal{L}(\gamma)$*, with* $\gamma > 0$ *but does not belong to* $\mathcal{OS}$ *.*

Namely, for fixed $y > 0$,

$$\begin{aligned}&\lim_{x\to\infty}\frac{\overline{F}(x+y)}{\overline{F}(x)}\\ &= \lim_{x\to\infty}\frac{\exp\{-\gamma(x+y)\}}{\exp\{-\gamma x\}}\\ &= e^{-\gamma y}.\end{aligned}$$

Hence $F \in \mathcal{L}(\gamma)$.
On the other hand, according to the convolution formula:

$$\begin{aligned}F * F(x) &= (1 - \exp\{-(\gamma x)\})(1+\gamma x)\mathbb{1}_{[0,\infty)}(x)\\ \overline{F * F}(x) &= \exp\{-(\gamma x)\}\,(1+\gamma x)\mathbb{1}_{[0,\infty)}(x) + \mathbb{1}_{(-\infty,0)}(x).\end{aligned}$$

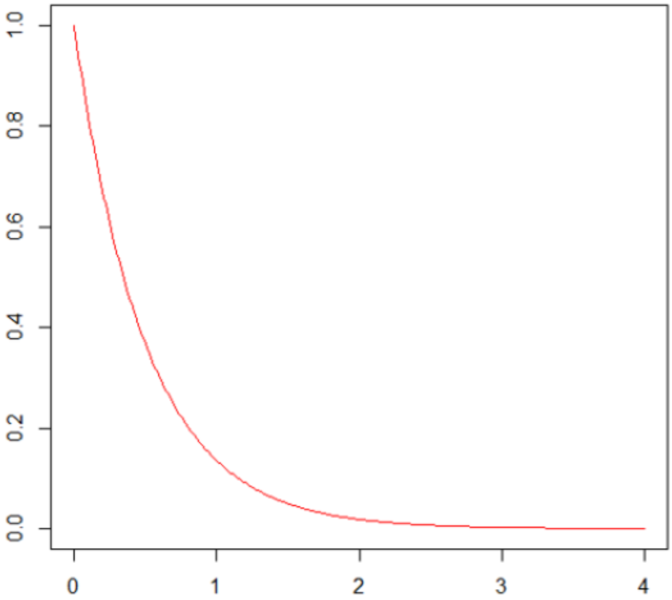

Figure 10: *Graph of the function* $\overline{F}(x) = e^{-2x}$

Therefore,

$$\limsup_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{\exp\{-(\gamma x)\}(1+\gamma x)}{\exp\{-(\gamma x)\}} = \infty$$

implying that $F \notin \mathcal{OS}$.
The graph of the tail function with $\gamma = 2$ is presented in Figure 10.

2.4 Known results for several distributions classes

In this subsection, we will discuss the previously known results regarding the classes under consideration.

For the class $\mathcal{S}$ the following result is obtained in Theorem 3.37 of [31], (see also [4, 25, 2, 20]).

Theorem 2.4.1. *Let* $\{\xi_1, \xi_2, \dots\}$ *be a sequence of independent real-valued r.v.s with a common distribution* $F_\xi \in \mathcal{S}$, *and let* η *be an independent counting r.v. with an expectation* $\mathbb{E}\eta$, *such that* $\mathbb{E}(1+\varepsilon)^\eta < \infty$ *for some* $\varepsilon > 0$. *Then,*

$$\overline{F}_{S_\eta}(x) \sim \mathbb{E}\eta \overline{F}_\eta(x),$$

and $F_{S_\eta} \in \mathcal{S}$.

• For any distribution F, define its Laplace-Stieltjes transform as

$$\widehat{F}(\lambda) := \int_{-\infty}^{\infty} \mathrm{e}^{\lambda x} \mathrm{d}F(x), \quad \lambda \in \mathbb{R}.$$

• A d.f. F_ξ of a real-valued r.v. is said to belong to class $F_\xi \in \mathcal{OS}(\gamma)$, $\gamma \geq 0$, if

$$\widehat{F}(\gamma) < \infty, \qquad F \in \mathcal{L}(\gamma),$$

and there exists finite limit

$$\limsup_{x\to\infty} \frac{\overline{F^*F}(x)}{\overline{F}_(x)} = 2c < \infty, c - constant.$$

Theorem 2.4.2. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent real-valued r.v.s with a common distribution $F_\xi \in \mathcal{S}(\gamma)$, $\gamma > 0$, and let η be an independent counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If*

$$\sum_{n=0}^{\infty} \mathbb{P}(\eta = n) \max\left\{\left(\widehat{F}_\xi(\gamma) + \varepsilon\right)^n, 1\right\} < \infty$$

for some $\varepsilon > 0$, then $F_{S_\eta} \in \mathcal{S}(\gamma)$.

We note that in Theorems 2.4.1 and 2.4.2 r.v.s in the sequences $\{\xi_1, \xi_2, \ldots\}$ are identically distributed. However, there are related regularity classes for which similar results can be obtained in cases where r.v.s in $\{\xi_1, \xi_2, \ldots\}$ are not necessarily identically distributed. Here we discuss two such classes.

The following assertion regarding $F_{S_\eta} \in \mathcal{D}$ is presented in Theorem 4 of [49].

Theorem 2.4.3. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s with a common d.f. $F_\xi \in \mathcal{D}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then $F_{S_\eta} \in \mathcal{D}$ if $\mathbb{E}\eta^{p+1} < \infty$ for some*

$$p > J^+_{F_\xi} := -\lim_{y\to\infty} \frac{1}{\log y} \log \liminf_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)}.$$

In the inhomogeneous case, when sumands are not necessary identically distributed, the following statement is obtained in Theorem 2.1 of [18].

Theorem 2.4.4. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent nonnegative r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then $F_{S_\eta} \in \mathcal{D}$ if the following three conditions are satisfied:*

(i) $F_{\xi_\varkappa} \in \mathcal{D}$ *for some* $\varkappa \in \mathrm{supp}(\eta) := \{n \in \mathbb{N}_0 : \mathbb{P}(\eta = n) > 0\}$,

(ii) $\limsup\limits_{x\to\infty} \sup\limits_{n>\varkappa} \frac{1}{n\overline{F}_{\xi_\varkappa}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$,

(iii) $\mathbb{E}\eta^{p+1} < \infty$ *for some* $p > J^+_{F_{\xi_\varkappa}}$.

Examples of conditions for the function F_{S_η} to belong to the class $\mathcal{L}(\gamma)$ are provided in the theorems below. Theorem 2.4.5 proved in [49] present conditions for the homogeneous case for the class $\mathcal{L} = \mathcal{L}(0)$, while Theorem 2.4.6 proved in [17] gives conditions for the inhomogeneous case for the class $\mathcal{L}(\gamma)$ with $\gamma \geqslant 0$.

Theorem 2.4.5. *Suppose that $\{\xi_1, \xi_2, \ldots\}$ are independent nonnegative r.v.s with a common distribution $F_\xi \in \mathcal{L}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If*

$$\overline{F}_\eta(\delta x) = o\big(\sqrt{x}\,\overline{F}_\xi(x)\big)$$

for any $\delta \in (0,1)$, then $F_{S_\eta} \in \mathcal{L}$.

Theorem 2.4.6. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent r.v.s such that for some $\gamma \geqslant 0$*

$$\sup_{k \geqslant 1} \left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y} \right| \underset{x\to\infty}{\to} 0$$

for each fixed $y > 0$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If

$$\frac{\mathbb{P}(\eta = k+1)}{\mathbb{P}(\eta = k)} \underset{k\to\infty}{\to} 0,$$

then $F_{S_\eta} \in \mathcal{L}(\gamma)$.

In the context of the randomly stopped sums, the class $\mathcal{OS}$ was considered by Shimura and Watanabe [65]. In Proposition 3.1 of that paper the following assertion is presented.

Theorem 2.4.7. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of nonnegative independent r.v.s with a common d.f. F_ξ, and let η be a counting r.v. such that*

$$\mathbb{P}(\eta > 1) > 0, \quad \sup\left\{ x \geqslant 1 : \sum_{k=0}^{\infty} \mathbb{P}(\eta = k) x^k < \infty \right\} = \infty.$$

Then $F_\xi \in \mathcal{OS}$ if and only if $\overline{F}_{S_\eta}(x) \underset{x\to\infty}{\asymp} \overline{F}_\xi(x)$.

From the information presented, it is evident that that our main theorems 3.1.1 and 3.1.2 are in fact inhomogeneous versions of the formulated theorem 2.4.7. In [65], the class of distributions $\mathcal{OS}$ is considered together with other distribution regularity classes. In that paper, several closedness properties of the the class $\mathcal{OS}$ were proved. For example, it is shown that class $\mathcal{OS}$ is not closed under convolution roots. This means that there exists r.v. ξ such that the n-fold convolution $F_\xi^{*n} \in \mathcal{OS}$ for all $n \geqslant 2$, but $F_\xi \notin \mathcal{OS}$. In [5], the simple conditions are provided under which the d.f. of the special form

$$F_\xi(x) = 1 - \exp\Big\{ - \int\limits_0^x q(u)\mathrm{d}u \Big\}$$

belongs to the class $\mathcal{OS}$, where q is some integrable hazard rate function. For distributions of the class $\mathcal{OS}$, the closure under tail-equivalence and the closure under convolution are established in [75]. The detailed proofs of these closures for nonnegative r.v.s are presented in [42] and for real-valued r.v.s in [76]. The closure under convolution means that in the case of independent r.v.s ξ_1, ξ_2 conditions $F_{\xi_1} \in \mathcal{OS}$, $F_{\xi_2} \in \mathcal{OS}$ imply that $F_{\xi_1} * F_{\xi_2} = F_{\xi_1+\xi_2} \in \mathcal{OS}$. The closure under tail-equivalence means that conditions $F_{\xi_1} \in \mathcal{OS}$, $\overline{F}_{\xi_1}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_2}(x)$ imply $F_{\xi_2} \in \mathcal{OS}$.

A counterexample demonstrating that $F_{\xi_1}, F_{\xi_2} \in \mathcal{OS}$ for independent r.v.s ξ_1, ξ_2 does not imply that $F_{\xi_1 \vee \xi_2} \in \mathcal{OS}$ can be found in [52]. Moreover, in that paper, the closure under minimum is established, which means that $F_{\xi_1}, F_{\xi_2} \in \mathcal{OS}$ for independent r.v.s ξ_1, ξ_2 implies $F_{\xi_1 \wedge \xi_2} \in \mathcal{OS}$. The authors of the articles [46, 55] consider when the distribution of the product of two independent random variables ξ, θ belongs to the class $\mathcal{OS}$. For instance, in [55], it is proved that d.f. $F_{\xi\theta}$ is a generalized subexponential if $F_\xi \in \mathcal{OS}$ and θ is independent of ξ, nonnegative and not degenerated at zero.

3 Randomly stopped sums with a generalized subexponential distribution

3.1 Main results

In this subsection, we present the main results regarding randomly stopped sums which belong to the class of generalized subexponential distributions. Our first assertion describes the situation when the primary r.v.s. belong to the class $\mathcal{OS}$ and the counting r.v. has a finite support.

Theorem 3.1.1. *[40] Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent r.v.s, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If η is bounded, $F_{\xi_1} \in \mathcal{OS}$, and for other indices $k \neq 1$ either $F_{\xi_k} \in \mathcal{OS}$ or $\overline{F}_{\xi_k}(x) = O\big(\overline{F}_{\xi_1}(x)\big)$, then d.f. of the randomly stopped sum F_{S_η} belongs to the class $\mathcal{OS}$.*

Our second result describes situation when the counting r.v. can have unbounded support. In such a case primary r.v.s. satisfy the additional requirements and the counting r.v. must have a very light tail.

Theorem 3.1.2. *[40] Let $\{\eta, \xi_1, \xi_2, ...\}$ be independent random variables, where counting r.v. η be such that $\mathbb{E}e^{\lambda\eta} < \infty$ for all $\lambda > 0$. Then $F_{S_\eta} \in \mathcal{OS}$, if $F_{\xi_1} \in \mathcal{OS}$ and one of the conditions below is satisfied:*

$$\text{(i)}\ \ \mathbb{P}(\eta = 1) > 0 \ \ \text{and} \ \ \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty;$$

$$\text{(ii)}\ 0 < \liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty.$$

3.2 Illustration of the results

In this subsection, we present two examples that demonstrate how Theorems 3.1.1 and 3.1.2 can be used to construct distributions belonging to the class $\mathcal{OS}$. It is practically impossible to write the analytical expression of d.f F_{S_η} in the general case, but according to Theorems 3.1.1 and 3.1.2, we can establish whether the constructed distributions are generalized subexponential.

Example 1. Let ξ_1 be r.v. having the t.f.

$$\overline{F}_{\xi_1}(x) = \mathbb{1}_{(-\infty,0)}(x) + \frac{e^{-x}}{(1+x)^3}\left(1 + \frac{\sin x}{a}\right)\mathbb{1}_{[0,\infty)}(x),$$

where $a > 2$. According to the results of [16], the d.f. F_{ξ_1} belongs to the class $\mathcal{OS}$. Therefore, Theorem 3.1.1 states that the d.f. F_{S_η} belongs to $\mathcal{OS}$ for each sequence of independent r.v.s $\{\xi_1, \xi_2, \ldots\}$ such that

$$\overline{F}_{\xi_k}(x) = O\left(\frac{e^{-x}}{(1+x)^3}\right),\ \ k \in \{2, 3, \ldots\},$$

and for each bounded counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$.

In particular, the d.f. with the tail

$$\overline{F}_{S_\eta}(x) = \mathbb{1}_{(-\infty,0)}(x) + \frac{1}{3}\Big(\overline{F}_{\xi_1}(x) + \overline{F_{\xi_1} * F_{\xi_2}}(x) + \overline{F_{\xi_1} * F_{\xi_2} * F_{\xi_3}}(x)\Big)\mathbb{1}_{[0,\infty)}(x)$$

belongs to the class $\mathcal{OS}$ with

$$\overline{F}_{\xi_1}(x) = \mathbb{1}_{(-\infty,0)}(x) + \frac{e^{-x}}{(1+x)^3}\left(1 + \frac{\sin x}{3}\right)\mathbb{1}_{[0,\infty)}(x),$$

$$\overline{F}_{\xi_2}(x) = \mathbb{1}_{(-\infty,0)}(x) + \frac{e^{-x}}{(1+x)^3}\left(1 + \frac{\sin x}{4}\right)\mathbb{1}_{[0,\infty)}(x),$$

$$\overline{F}_{\xi_3}(x) = \mathbb{1}_{(-\infty,0)}(x) + \frac{e^{-x}}{(1+x)^3}\mathbb{1}_{[0,\infty)}(x).$$

Example 2. Let $\{\eta, \xi_1, \xi_2, \ldots\}$ be independent r.v.s, where the counting r.v. η is distributed according to the Poisson law with the parameter $\mu > 0$, and

$$\overline{F}_{\xi_k}(x) = \begin{cases} \mathbb{I}_{(-\infty,1)}(x) + \mathrm{e}^{1-x}x^{-2}\,\mathbb{I}_{[1,\infty)}(x) & \text{if} \quad k \in \{1,3,5,\ldots\}, \\ \mathbb{I}_{(-\infty,2)}(x) + 4\,\mathrm{e}^{2-x}x^{-2}\,\mathbb{I}_{[2,\infty)}(x) & \text{if} \quad k \in \{2,4,6,\ldots\}. \end{cases}$$

According to the results of [13], d.f. F_{ξ_1} belongs to the class $\mathcal{OS}$. In addition,

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} = 4\,\mathrm{e}.$$

Hence, the d.f. F_{S_η} with the t.f.

$$\overline{F}_{S_\eta}(x) = \mathbb{I}_{(-\infty,1)}(x) + \mathrm{e}^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^n}{n!}\, \overline{F_{\xi_1} * F_{\xi_2} * \ldots F_{\xi_n}}(x)\,\mathbb{I}_{[1,\infty)}(x)$$

is generalized subexponential due to Theorem 3.1.2.

3.3 Auxiliary results for Theorems 3.1.1-3.1.2

In this subsection, we will present and prove some auxiliary lemmas that will be applied to the derivations of the main Theorems 3.1.1 and 3.1.2. The first lemma will be used many times.

Lemma 3.3.1. *Let X and Y be two real valued r.v.s with corresponding d.f.s F_X and F_Y. The following statements hold:*

(i) $F_X \in \mathcal{OS}$ *if and only if* $\sup\limits_{x\in\mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty$.

(ii) *If $F_X \in \mathcal{OS}$ and $\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$, then $F_Y \in \mathcal{OS}$.*

(iii) *If $F_X \in \mathcal{OS}$ and $F_Y \in \mathcal{OS}$, then $F_X * F_Y \in \mathcal{OS}$.*

(iv) *If $F_X \in \mathcal{OS}$, then $F_X \in \mathcal{OL}$ i.e.* $\limsup\limits_{x\to\infty} \frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} < \infty$.

(v) *If $F_X \in \mathcal{OS}$ and $\overline{F}_Y(x) = O\big(\overline{F}_X(x)\big)$, then $F_X * F_Y \in \mathcal{OS}$ and $\overline{F_X * F_Y}(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$.*

Proof. A large part of the properties of the class $\mathcal{OS}$ listed in Lemma 3.3.1 can be found, for instance, in [42, 65, 75, 76]. However, for the sake of exposition completeness, we present the full proof of the formulated lemma.

Part(i). If $F_X \in \mathcal{OS}$, then

$$\limsup_{x\to\infty} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty \tag{3.3.1}$$

according to the definition. This estimate implies that $\overline{F}_X(x) > 0$ for each $x \in \mathbb{R}$. In addition, the inequality (3.3.1) indicates that

$$\frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \leqslant M$$

if $x \geqslant x_M$ for some M and x_M.

If $x < x_M$, then, obviously, $\overline{F}_X(x) \geqslant \overline{F}_X(x_M)$ and $\overline{F_X * F_X}(x) \leqslant 1$. Therefore, for each $x \in \mathbb{R}$ we get that

$$\frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \leqslant \max\left\{M, \frac{1}{\overline{F}_X(x_M)}\right\} < \infty$$

because $\overline{F}_X(x_M) > 0$. The last estimate finishes the proof of the Part (i) because the condition

$$\sup_{x \in \mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty$$

implies (3.3.1) obviously.

Part(ii). The condition $\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$ implies

$$\liminf_{x\to\infty} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} > 0 \quad \text{and} \quad \limsup_{x\to\infty} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} < \infty. \tag{3.3.2}$$

It follows from this that

$$\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant M, \ x \geqslant x_M,$$

for some M and x_M. If $x < x_M$, then

$$\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant \frac{1}{\overline{F}_X(x_M)} < \infty$$

because $F_X \in \mathcal{OS}$ implying that $\overline{F}_X(x) > 0$ for each $x \in \mathbb{R}$. According to the derived estimates,

$$\sup_{x \in \mathbb{R}} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leq \max\left\{M, \frac{1}{\overline{F}_X(x_M)}\right\} = C < \infty.$$

Therefore, for each $x \in \mathbb{R}$,

$$
\begin{aligned}
\overline{F_Y * F_Y}(x) &= \int\limits_{-\infty}^{\infty} \frac{\overline{F}_Y(x-y)}{\overline{F}_X(x-y)} \overline{F}_X(x-y) \mathrm{d}F_Y(y) \\
&\leqslant C \int\limits_{-\infty}^{\infty} \overline{F}_X(x-y) \mathrm{d}F_Y(y) \\
&= C \int\limits_{-\infty}^{\infty} \overline{F}_Y(x-y) \mathrm{d}F_X(y) \\
&= C \int\limits_{-\infty}^{\infty} \frac{\overline{F}_Y(x-y)}{\overline{F}_X(x-y)} \overline{F}_X(x-y) \mathrm{d}F_X(y) \\
&\leqslant C^2 \int\limits_{-\infty}^{\infty} \overline{F}_X(x-y) \mathrm{d}F_X(y) = C^2\, \overline{F_X * F_X}(x).
\end{aligned}
$$

This estimate implies that:

$$
\begin{aligned}
\limsup_{x\to\infty} \frac{\overline{F_Y * F_Y}(x)}{\overline{F}_Y(x)} &\leqslant C^2 \limsup_{x\to\infty} \frac{\overline{F_X * F_X}(x)}{\overline{F}_Y(x)} \\
&\leqslant C^2 \limsup_{x\to\infty} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \frac{1}{\liminf\limits_{x\to\infty} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)}} < \infty
\end{aligned}
$$

due to the assumption $F_X \in \mathcal{OS}$ and the first inequality in (3.3.2). The last estimate indicates that the d.f. F_Y belongs to the class $\mathcal{OS}$. Part (ii) of the lemma has been proved.

Part(iii). According to part (i) we have that

$$
\sup_{x\in\mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} = C_1 < \infty
$$

and

$$
\sup_{x\in\mathbb{R}} \frac{\overline{F_Y * F_Y}(x)}{\overline{F}_Y(x)} = C_2 < \infty
$$

Let X_1, X_2, Y_1, Y_2 be independent r.v.s. Suppose that X_1, X_2 are distributed according to the d.f. F_X, and Y_1, Y_2 are distributed according to the d.f. F_Y. For each $x \in \mathbb{R}$ we get

$$\begin{aligned}
\overline{(F_X * F_Y)^{*2}}(x) &= \overline{(F_X * F_Y) * (F_X * F_Y)}(x) \\
&= \mathbb{P}(X_1 + Y_1 + X_2 + Y_2 > x) \\
&= \mathbb{P}(X_1 + X_2 + Y_1 + Y_2) > x) \\
&= \int\limits_{-\infty}^{\infty} \mathbb{P}(X_1 + X_2 > x - y) \mathrm{d}\mathbb{P}(Y_1 + Y_2 \leqslant y) \\
&= \int\limits_{-\infty}^{\infty} \frac{\overline{F_X * F_X}(x-y)}{\overline{F}_X(x-y)} \overline{F}_X(x-y) \mathrm{d}\mathbb{P}(Y_1 + Y_2 \leqslant y) \\
&\leqslant C_1 \int\limits_{-\infty}^{\infty} \overline{F}_X(x-y) \mathrm{d}\mathbb{P}(Y_1 + Y_2 \leqslant y) \\
&= C_1 \mathbb{P}(X_1 + Y_1 + Y_2 > x) \\
&= C_1 \int\limits_{-\infty}^{\infty} \frac{\overline{F_Y * F_Y}(x-y)}{\overline{F}_Y(x-y)} \overline{F}_Y(x-y) \mathrm{d}\mathbb{P}(X_1 \leqslant y) \\
&\leqslant C_1 C_2 \int\limits_{-\infty}^{\infty} \overline{F}_Y(x-y) \mathrm{d}F_X(y) = C_1 C_2 \overline{F_X * F_Y}(x).
\end{aligned}$$

Hence,

$$\sup_{x \in \mathbb{R}} \frac{\overline{(F_X * F_Y)^{*2}}(x)}{\overline{F_X * F_Y}(x)} \leqslant C_1 C_2$$

implying that $F_X * F_Y \in \mathcal{OS}$ by Part (i). Part (iii) of the lemma has been proved.

Part(iv). From Part (i), we have

$$\sup_{x \in \mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F_X}(x)} = C_3 < \infty.$$

In addition, for $x > 2$, we obtain:

$$\begin{aligned}
\overline{F_X * F_X}(x) &= \int\limits_{-\infty}^{\infty} \overline{F}_X(x-t) \mathrm{d}F_X(t) \geqslant \int\limits_{(1,x]} \overline{F}_X(x-t) \mathrm{d}F_X(t) \\
&\geqslant \overline{F}_X(x-1)(F_X(x) - F_X(1))
\end{aligned}$$

By condition $F_X \in \mathcal{OS}$, we have that $\overline{F}_X(x) > 0$ for each $x \in \mathbb{R}$. Hence, when x is large enough we have $F_X(x) - F_X(1) > 0$, and, therefore,

$$\frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} \leqslant \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \frac{1}{F_X(x) - F_X(1)}.$$

Implying that

$$\limsup_{x\to\infty} \frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} \leqslant \frac{C_3}{\overline{F}_X(1)} < \infty,$$

and Part (iv) of the lemma has been proved.

Part(v). Since $\overline{F}_Y(x) = O\big(\overline{F}_X(x)\big)$, we have

$$\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant M,\ x \geq x_M,$$

with certain constants M and x_M. If $x < x_M$, then

$$\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant \frac{1}{\overline{F}_X(x_M)} < \infty$$

because $F_X \in \mathcal{OS}$ implies $\overline{F}_X(x_M) > 0$. From both inequalities above it follows that

$$\sup_{x\in\mathbb{R}} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant \max\left\{M, \frac{1}{\overline{F}_X(x_M)}\right\} = C_4$$

Consequently, for $x \in \mathbb{R}$ we get

$$\begin{aligned}
\overline{F_X * F_Y}(x) &= \int_{-\infty}^{\infty} \overline{F}_Y(x-y)\mathrm{d}F_X(y) \\
&\leqslant C_4 \int_{-\infty}^{\infty} \overline{F}_X(x-y)\mathrm{d}F_X(y) \\
&= C_4\overline{F_X * F_X}(x) \leqslant C_5\overline{F_X}(x) \qquad (3.3.3)
\end{aligned}$$

with some positive constant C_5, where the last step in the above derivation follows from Part (i) of the lemma.

On the other hand, there exists a real $b \in \mathbb{R}$ for which

$$\overline{F}_Y(b) = 1 - F_Y(b) \geqslant \frac{1}{2}$$

For this b, we get

$$\begin{aligned}
\overline{F_X * F_Y}(x) &\geqslant \int_{(b,\infty)} \overline{F}_X(x-y)\mathrm{d}F_Y(y) \\
&\geqslant \overline{F}_X(x-b) \int_{(b,\infty)} \mathrm{d}F_Y(y) \\
&= \overline{F}_X(x-b)\overline{F}_Y(b) \geqslant \frac{1}{2}\overline{F}_X(x)\frac{\overline{F}_X(x-b)}{\overline{F}_X(x)}
\end{aligned}$$

Hence,

$$\liminf_{x\to\infty} \frac{\overline{F_X * F_Y}(x)}{\overline{F}_X(x)} \geqslant \frac{1}{2} \liminf_{x\to\infty} \frac{\overline{F}_X(x-b)}{\overline{F}_X(x)}. \tag{3.3.4}$$

In Part (iv) of the lemma we proved that $F_X \in \mathcal{OL}$. It is obvious that

$$F_X \in \mathcal{OL} \Leftrightarrow \overline{F}_X(x-b) \underset{x\to\infty}{\asymp} \overline{F}_X(x) \text{ for each } b \in \mathbb{R}.$$

Therefore, the estimate (3.3.4) implies that

$$\liminf_{x\to\infty} \frac{\overline{F_X * F_Y}(x)}{\overline{F}_X(x)} > 0. \tag{3.3.5}$$

From (3.3.3) and (3.3.5) inequalities it follows that $\overline{F_X * F_Y}(x) \underset{x\to\infty}{\asymp} \overline{F_X}(x)$. Moreover by Part (ii) of the lemma $F_X * F_Y \in \mathcal{OS}$. This finishes the proof of the last part of the lemma. □

Lemma 3.3.2. *Let $\{X_1, X_2 ...\}$ be a sequence of independent r.v.s, for which $F_{X_1} \in \mathcal{OS}$, and for other indices $k \geqslant 2$ either $F_{X_k} \in \mathcal{OS}$ or $\overline{F}_{X_k}(x) = O(\overline{F}_{X_1}(x))$. Then $F_{S_n} \in \mathcal{OS}$ for all $n \in \mathbb{N}$.*

Proof. If $n = 1$, then the statement is obvious because $S_1 = X_1$. If $n = 2$, then two options are possible: $F_{X_2} \in \mathcal{OS}$ or $\overline{F}_{X_2} = O(\overline{F}_{X_1}(x))$. In the first case $F_{S_2} = F_{X_1} * F_{X_2} \in \mathcal{OS}$ according to Part (iii) of Lemma 3.3.1. In the second case $F_{S_2} \in \mathcal{OS}$ by the Part (v) of the same lemma. Let now $n > 2$, denote

$$\mathcal{K} = \{k \in \{2, ..., n\} : \overline{F}_{X_k}(x) = O(\overline{F}_{X_1}(x))\}.$$

Initially let us assume that the set $\mathcal{K}$ is empty. In such a case, $F_{X_k} \in \mathcal{OS}$ for all indices $k \in \mathcal{K}^c = \{1, 2, 3, \ldots, n\}$. By Part (iii) of Lemma 3.3.1 we get that $F_{S_n} \in \mathcal{OS}$.

Let now the index set $\mathcal{K} = \{k_1, k_2, ...k_r\} \subset \{2, ...n\}$ be non-empty. Since

$$\overline{F}_{X_{k_1}}(x) = O(\overline{F}_{X_1}(x)),$$

Part (v) of Lemma 3.3.1 implies that

$$F_{X_1} * F_{X_{k_1}} \in \mathcal{OS}, \tag{3.3.6}$$

and

$$\overline{F_{X_1} * F_{X_{k_1}}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{X_1}(x). \tag{3.3.7}$$

According the the relation (3.3.7),

$$\limsup_{x\to\infty} \frac{\overline{F}_{X_{k_2}}(x)}{\overline{F_{X_1} * F_{X_{k_1}}}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_{X_{k_2}}(x)}{\overline{F}_{X_1}(x)} \frac{1}{\liminf\limits_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_{k_1}}}(x)}{\overline{F}_{X_1}(x)}} < \infty$$

because $\overline{F}_{X_{k_2}}(x) = O(\overline{F}_{X_1}(x))$ by definition of the set $\mathcal{K}$. This means that

$$\overline{F}_{X_{k_2}}(x) = O(\overline{F_{X_1} * F_{X_{k_1}}}(x)).$$

Hence, according to (3.3.6) and part (v) of Lemma 3.3.1, we get

$$F_{X_1} * F_{X_{k_1}} * F_{X_{k_2}} = (F_{X_1} * F_{X_{k_1}}) * F_{X_{k_2}} \in \mathcal{OS},$$

and

$$\overline{F_{X_1} * F_{X_{k_1}} * F_{X_{k_2}}}(x) \underset{x\to\infty}{\asymp} \overline{F_{X_1} * F_{X_{k_1}}}(x).$$

By continuing the process, we obtain

$$F_{\mathcal{K}} := F_{X_1} * \prod_{j=1}^{r}{}^{*} F_{X_{k_j}} = F_{X_1} * F_{X_{k_1}} * F_{X_{k_2}} * ... * F_{X_{k_r}} \in \mathcal{OS},$$

and

$$\overline{F_{X_1} * F_{X_{k_1}} * F_{X_{k_2}} * ... * F_{X_{k_r}}}(x) \underset{x\to\infty}{\asymp} \overline{F_{X_1} * F_{X_{k_1}} * F_{X_{k_2}} * ... * F_{X_{k_{r-1}}}}(x).$$

For the remaining indices $k \in \mathcal{K}^c = \{2, 3, ..., n\} \setminus \{k_1, k_2, ..., k_r\}$ d.f. $F_{X_k} \in \mathcal{OS}$. By Part (iii) of Lemma 3.3.1 we get

$$F_{\mathcal{K}^c} := \prod_{k\in\mathcal{K}^c}{}^{*} F_{X_k} \in \mathcal{OS}.$$

By using Part (iii) of Lemma 3.3.1 again we derive that

$$F_{S_n} = F_{\mathcal{K}} * F_{\mathcal{K}^c} \in \mathcal{OS}.$$

This finishes the proof of Lemma 3.3.2. □

Lemma 3.3.3. *Let $X_1, X_2, ...$ be a sequence of independent random variables, for which $F_{X_1} \in \mathcal{OS}$ and*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} < \infty. \tag{3.3.8}$$

Then there exists a constant $\widehat{C}$ for which

$$\overline{F}_{S_n}(x) \leqslant \widehat{C}^{\,n-1}\overline{F}_{X_1}(x) \tag{3.3.9}$$

for all $x \in \mathbb{R}$ and for all $n \geqslant 2$.

Proof. The condition (3.3.8) implies that

$$\sup_{k\geqslant 1} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} \leqslant C_6$$

for all $x \geqslant A$ with some positive constants C_6 and A. If $x < A$, then

$$\sup_{k\geqslant 1} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} \leqslant \frac{1}{\overline{F}_{X_1}(x)} \leqslant \frac{1}{\overline{F}_{X_1}(A)} < \infty.$$

Therefore, for each $x \in \mathbb{R}$

$$\sup_{k\geqslant 1} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} \leqslant \max\left\{C_6, \frac{1}{\overline{F}_{X_1}(A)}\right\} := C_7. \tag{3.3.10}$$

In addition, the Part (i) of Lemma 3.3.1 gives that

$$\overline{F_{X_1} * F_{X_1}}(x) \leqslant C_8 \overline{F}_{X_1}(x) \tag{3.3.11}$$

for all $x \in \mathbb{R}$ with some positive constant C_8.

We will prove the inequality (3.3.9) with the constant $\widehat{C} = C_7\, C_8$.
If $n = 1$, the inequality (3.3.9) holds evidently because
$\overline{F}_{S_1}(x) = \overline{F}_{X_1}(x)$.
If $n = 2$, then by (3.3.10) and (3.3.11) for $x \in \mathbb{R}$ we have

$$\begin{aligned}\overline{F}_{S_2}(x) &= \int_{-\infty}^{\infty} \overline{F}_{X_2}(x-y)\mathrm{d}F_{X_1}(y) \leqslant C_7 \int_{-\infty}^{\infty} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_1}(y) \\ &= C_7\overline{F_{X_1} * F_{X_1}}(x) \leqslant \widehat{C}\, \overline{F}_{X_1}(x).\end{aligned}$$

Suppose now that the inequality (3.3.9) holds for $n = m \geqslant 2$, i.e.

$$\frac{\overline{F}_{S_m}(x)}{\overline{F}_{X_1}(x)} \leqslant \widehat{C}^{\,m-1},\ x \in \mathbb{R}.$$

After choosing $n = m + 1$, from this assumption and from (3.3.10), (3.3.11) we get

$$\begin{aligned}\overline{F}_{S_{m+1}}(x) &= \int_{-\infty}^{\infty} \overline{F}_{S_m}(x-y)\mathrm{d}F_{X_{m+1}}(y) \\ &\leqslant \widehat{C}^{m-1} \int_{-\infty}^{\infty} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_{m+1}}(y) \\ &= \widehat{C}^{m-1} \int_{-\infty}^{\infty} \overline{F}_{X_{m+1}}(x-y)\mathrm{d}F_{X_1}(y) \\ &\leqslant \widehat{C}^{\,m-1} C_7 \int_{-\infty}^{\infty} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_1}(y) \\ &= \widehat{C}^{\,m-1} C_7\, \overline{F_{X_1} * F_{X_1}}(x) \\ &\leqslant \widehat{C}^{m-1} C_7 C_8\, \overline{F}_{X_1}(x), \\ &= \widehat{C}^{m}\, \overline{F}_{X_1}(x),\ x \in \mathbb{R}.\end{aligned}$$

According to the induction principle, the inequality (3.3.9) holds for all $n \in \mathbb{N}$. Lemma 3.3.3 has been proved. $\square$

3.4 Proofs of Theorems 3.1.1-3.1.2

In this subsection, we present proofs of the main results of this Chapter.

Proof of Theorem 3.1.1.
Suppose that $0 \le \eta \le L$ with $\mathbb{P}(\eta = L) > 0$ for some $L \in \mathbb{N}$. In such a case, we have

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{L} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x),\ x > 0.$$

By Lemma 3.3.2 we have that $F_{S_L} \in \mathcal{OS}$ implying that $\overline{F}_{S_L}(x) > 0$ for all $x \in \mathbb{R}$. Hence, for each $x > 0$

$$\frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{S_L}(x)} \geqslant \frac{\mathbb{P}(\eta = L)\overline{F}_{S_L}(x)}{\overline{F}_{S_L}(x)} = \mathbb{P}(\eta = L) > 0. \tag{3.4.1}$$

On the other hand,

$$\overline{F}_{S_\eta}(x) = \sum_{k=0}^{L-1} \mathbb{P}(\eta = L-k)\mathbb{P}(S_{L-k} > x) \tag{3.4.2}$$

For any random variable ξ_k, $k \in \{1, 2, \ldots, L\}$, there exists a negative number $-a_k$, for which $\mathbb{P}(\xi_k \geqslant -a_k) \geqslant 1/2$. We have

$$\begin{aligned}\mathbb{P}(S_{L-1} > x) &= \mathbb{P}(S_{L-1} - a_L > x - a_L, \xi_L \geqslant -a_L) + \mathbb{P}(S_{L-1} > x, \xi_L < -a_L) \\ &\leqslant \mathbb{P}(S_L > x - a_L) + \mathbb{P}(S_{L-1} > x)\mathbb{P}(\xi_L < -a_L)\end{aligned}$$

From this we derive that

$$\mathbb{P}(S_{L-1} > x) \leqslant 2\mathbb{P}(S_L > x - a_L)$$

for each $x \in \mathbb{R}$. Similarly,

$$\mathbb{P}(S_{L-2} > x) \leqslant 2\mathbb{P}(S_{L-1} > x - a_{L-1}) \leqslant 4\mathbb{P}(S_L > x - a_{L-1} - a_L)$$

also for each real number x. By continuing the process we obtain

$$\mathbb{P}(S_{L-k} > x) \leqslant 2^k\mathbb{P}\left(S_L > x - \sum_{j=0}^{k-1} a_{L-j}\right)$$

for all $x \in \mathbb{R}$ and for all $k = 1, 2, ..., L-1$. After inserting the obtained estimates into inequality (3.4.2), we get that

$$\begin{aligned}\overline{F}_{S_\eta}(x) &\leqslant \sum_{k=0}^{L-1} \mathbb{P}(\eta = L-k)\, 2^k\, \mathbb{P}(S_L > x - \sum_{j=0}^{k-1} a_{L-j}) \\ &\leqslant \mathbb{P}(S_L > x - a) \sum_{k=0}^{L-1} 2^k\, \mathbb{P}(\eta = L-k) \\ &= C^* \overline{F}_{S_L}(x-a),\end{aligned}$$

where

$$C^* = \sum_{k=0}^{L-1} 2^k\, \mathbb{P}(\eta = L-k), \quad \text{and} \quad a = \sum_{j=2}^{L} a_j.$$

Consequently, for all $x > 0$

$$\frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{S_L}(x)} \leqslant \frac{C^* \overline{F}_{S_L}(x-a)}{\overline{F}_{S_L}(x)}$$

By Lemma 3.3.2 and Part (iv) of Lemma 3.3.1 it follows that $F_{S_L} \in \mathcal{OS} \subset \mathcal{OL}$. Therefore

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{S_L}(x)} < \infty. \tag{3.4.3}$$

By (3.4.1) and (3.4.3) we have,that:

$$\overline{F}_{S_\eta}(x) \asymp \overline{F}_{S_L}(x)$$

Therefore $F_{S_\eta} \in \mathcal{OS}$ together with F_{S_L} by the part (ii) of Lemma 3.3.1. Theorem 3.1.1 has been proved. □

Proof of Theorem 3.1.2. Part(i). Whereas

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \overline{F}_{S_n}(x), \ x > 0,$$

by Lemma 3.3.3 we obtain that:

$$\frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \frac{\sum\limits_{n=1}^{\infty} \widehat{C}^{\,n-1} \mathbb{P}(\eta = n) \overline{F}_{\xi_1}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \mathbb{E}\mathrm{e}^{\widehat{C}\eta} < \infty, \tag{3.4.4}$$

where $\widehat{C}$ is some positive constant.
On the other hand,

$$\overline{F}_{S_\eta}(x) \geqslant \mathbb{P}(\eta = 1) \overline{F}_{\xi_1}(x).$$

Hence under conditions of Part (i), we have that $\overline{F}_{S_\eta}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$. Therefore $F_{S_\eta} \in OS$ according to part (ii) of Lemma 3.3.1. Part (i) of Theorem 3.1.2 has been proved.

Part(ii). If $\mathbb{P}(\eta = 1) > 0$, then the assertion of this part follows from the proved part (i). Let now $\mathbb{P}(\eta = 1) = 0$. Since $\mathbb{E}e^{\lambda\eta} < \infty$ for each $\lambda > 0$, the inequality (3.4.4) implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{\xi_1}(x)} < \infty. \tag{3.4.5}$$

In addition, conditions of Part (ii) of the theorem give that

$$\inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \Delta$$

for all $x \geqslant x_\Delta$ and some positive Δ. If $x < x_\Delta$, then

$$\inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} = \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x_\Delta)}{\overline{F}_{\xi_1}(x_\Delta)} \overline{F}_{\xi_1}(x_\Delta) \geqslant \Delta\, \overline{F}_{\xi_1}(x_\Delta) := \widetilde{C} > 0$$

due to the assumption $F_{\xi_1} \in \mathcal{OS}$. The derived inequalities imply that

$$\overline{F}_{\xi_k}(x) \geqslant \widetilde{C}\, \overline{F}_{\xi_1}(x)$$

for some positive constant $\widetilde{C}$, and for all $x \in \mathbb{R}$, $k \in \{1, 2, \ldots\}$.

Using the last estimate we get

$$\begin{aligned}\overline{F}_{S_2}(x) &= \int_{-\infty}^{\infty} \frac{\overline{F}_{\xi_2}(x-y)}{\overline{F}_{\xi_1}(x-y)} \overline{F}_{\xi_1}(x-y)\, \mathrm{d}F_{\xi_1}(y) \\ &\geqslant \widetilde{C}\ \overline{F_{\xi_1} * F_{\xi_1}}(x) \geqslant \widetilde{C}\ \overline{F}_{\xi_1}(0)\overline{F}_{\xi_1}(x),\ x \in \mathbb{R}.\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{F}_{S_3}(x) &= \int_{-\infty}^{\infty} \frac{\overline{F}_{S_2}(x-y)}{\overline{F}_{\xi_1}(x-y)} \overline{F}_{\xi_1}(x-y)\, \mathrm{d}F_{\xi_1}(y) \\ &\geqslant \widetilde{C}\ \overline{F}_{\xi_1}(0)\overline{F_{\xi_1} * F_{\xi_1}}(x) \geqslant \widetilde{C}\ \left(\overline{F}_{\xi_1}(0)\right)^2 \overline{F}_{\xi_1}(x),\ x \in \mathbb{R}.\end{aligned}$$

By continuing the process, we obtain that

$$\overline{F}_{S_n}(x) \geqslant \widetilde{C}\, \left(\overline{F}_{\xi_1}(0)\right)^{n-1} \overline{F}_{\xi_1}(x)$$

for all $x \in \mathbb{R}$ and $n \in \{2, 3, \ldots\}$.

Therefore,

$$\liminf_{x\to\infty}\frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \liminf_{x\to\infty}\frac{\mathbb{P}(\eta=\widetilde{L})\overline{F}_{S_{\widetilde{L}}}(x)}{\overline{F}_{\xi_1}(x)}$$
$$\geqslant \mathbb{P}(\eta=\widetilde{L})\,\widetilde{C}\left(\overline{F}_{\xi_1}(0)\right)^{\widetilde{L}-1}>0, \qquad (3.4.6)$$

where $\widetilde{L}=\min\{n\geqslant 2:\mathbb{P}(\eta=n)>0\}$.

The derived inequalities (3.4.5) and (3.4.6) imply $\overline{F}_{S_\eta}(x)\underset{x\to\infty}{\asymp}\overline{F}_{\xi_1}(x)$. By part (ii) of Lemma 3.3.1 we get $F_{S_\eta}\in\mathcal{OS}$. Theorem 3.1.2 has been proved.

4 Randomly stopped minimum, maximum, minimum of sums and maximum of sums with generalized subexponential distributions

In this chapter we continue to consider the closure properties of the class $\mathcal{OS}$. We suppose that some elements of primary r.v.s. belong to the class $\mathcal{OS}$ and we find conditions under which the randomly stopped structures described in Chapter 2 remain in $\mathcal{OS}$.

4.1 Main results

Our first result is on the randomly stopped minimum and the randomly stopped minimum of sums.

Theorem 4.1.1. *Let $\{\xi_1,\xi_2,\ldots\}$ be a sequence of independent real-valued r.v.s, and let η be a counting r.v. independent of $\{\xi_1,\xi_2,\ldots\}$. If $F_{\xi_k}\in\mathcal{OS}$ for each k, then $F_{\xi_{(\eta)}}$ and $F_{S_{(\eta)}}$ belong to the class $\mathcal{OS}$, and it holds the following asymptotic relations:*

$$\overline{F}_{\xi_{(\eta)}}(x)\underset{x\to\infty}{\asymp}\overline{F}_{\xi_{(\varkappa)}}(x)=\prod_{k=1}^{\varkappa}\overline{F}_{\xi_k}(x), \qquad (4.1.1)$$
$$\overline{F}_{S_{(\eta)}}(x)\underset{x\to\infty}{\asymp}\overline{F}_{S_{(\varkappa)}}(x), \qquad (4.1.2)$$

where $\varkappa=\min\{k\geqslant 1:\mathbb{P}(\eta=k)>0\}$.

The second theorem states the conditions for the maximum of randomly stopped, possibly differently distributed random variables to belong to the class $\mathcal{OS}$.

Theorem 4.1.2. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s such that $F_{\xi_1} \in \mathcal{OS}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ with the finite expectation $\mathbb{E}\eta$. If*

$$0 < \liminf_{x\to\infty} \inf_{n\geqslant 1} \frac{1}{n}\sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n}\sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty, \quad (4.1.3)$$

then $F_{\xi^{(\eta)}} \in \mathcal{OS}$ and $\overline{F}_{\xi^{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$.

The third and the last theorem describes the conditions under which the d.f. of the randomly stopped maximum of sums is generalized subexponential.

Theorem 4.1.3. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s such that $F_{\xi_1} \in \mathcal{OS}$ and the condition*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty \quad (4.1.4)$$

is satisfied. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ such that $\mathbb{E}\mathrm{e}^{\lambda\eta} < \infty$ for all $\lambda > 0$. Then $F_{S^{(\eta)}} \in \mathcal{OS}$ and $\overline{F}_{S^{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$.

4.2 Illustration of the results

Example 4.2.1. *Let $\{\xi_1, \xi_2, \ldots,\}$ be a sequence of independent and identically distributed r.v.s such that*

$$\overline{F}_{\xi_k}(x) = \mathbb{I}_{(-\infty,0)}(x) + \frac{\mathrm{e}^{-x}}{(1+x)^3}\left(1 + \frac{\sin x}{3}\right)\mathbb{I}_{[0,\infty)}(x),\ k \in \mathbb{N}.$$

According to the results of [16], the d.f. F_{ξ_1} belongs to the "edge" of the class $\mathcal{OS}$. In addition, requirements (4.1.3) *and* (4.1.4) *are certainly satisfied. Therefore, Theorems 4.1.1–4.1.3 can be applied for sequence of independent and identically distributed r.v.s $\{\xi_1, \xi_2, \ldots,\}$.*

Theorem 4.1.1 gives that d.f.s $F_{\xi_{(\eta)}}$ and $F_{S_{(\eta)}}$ belong to $\mathcal{OS}$ for each counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$. In particular, if

$$\mathbb{P}(\eta = n) = (1-p)p^{n-2},\ n \in \{2, 3, \ldots\},\ p \in (0,1), \quad (4.2.1)$$

then d.f.s $F_{\xi_{(\eta)}}$ and $F_{S_{(\eta)}}$ with tails

$$\begin{aligned}
\overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbb{P}(\eta = n) = \sum_{n=2}^{\infty} (\overline{F}_{\xi_1}(x))^n (1-p)p^{n-2} \\
&= (1-p)(\overline{F}_{\xi_1}(x))^2 \sum_{k=0}^{\infty} (\overline{F}_{\xi_1}(x))^k p^k \\
&= \frac{(1-p)\big(\overline{F}_{\xi_1}(x)\big)^2}{1-p\overline{F}_{\xi_1}(x)}, \ x \geqslant 0, \\
\overline{F}_{S_{(\eta)}}(x) &= \sum_{n=2}^{\infty} \overline{F}_{S_{(n)}}(x)\mathbb{P}(\eta = n) \qquad\qquad (4.2.2) \\
&= \sum_{n=2}^{\infty} \mathbb{P}\big(\min\{S_1, S_2, \dots S_n\} > x\big)\mathbb{P}(\eta = n) \\
&= \mathbb{P}(S_1 > x)\mathbb{P}(\eta \geq 2) = \overline{F}_{\xi_1}(x), \ x \geqslant 0,
\end{aligned}$$

belong to the class of generalized subexponential distributions.

Theorem 4.1.2 implies that d.f. $F_{\xi^{(\eta)}}$ belongs to the class $\mathcal{OS}$ for each counting r.v. η independent of $\{\xi_1, \xi_2, \dots\}$ such that $\mathbb{E}\eta < \infty$. In a special case, when the counting random variable is defined by the equality (4.2.1), according to Theorem 4.1.2, we obtain that the distribution function with the tail

$$\begin{aligned}
\overline{F}_{\xi^{(\eta)}}(x) &= \sum_{n=2}^{\infty} \overline{F}_{\xi^{(n)}}(x)\mathbb{P}(\eta = n) \\
&= \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n \ \mathbb{P}\big(\max\{\xi_1, \xi_2, \dots \xi_n\} > x\big) \\
&= \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n \ \Big(1 - \mathbb{P}\big(\max\{\xi_1, \xi_2, \dots \xi_n\} \leq x\big)\Big) \\
&= \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n \ \Big(1 - \mathbb{P}\Big(\bigcap_{k=1}^{n} \{\xi_k \leq x\}\Big)\Big) \\
&= \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n \ \Big(1 - \prod_{k=1}^{n} \mathbb{P}(\xi_k \leq x)\Big) \\
&= \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n \ \Big(1 - (F_{\xi_1}(x))^n\Big) \\
&= \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n - \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n (F_{\xi_1}(x))^n \\
&= 1 - \frac{1-p}{p^2} \ \frac{(pF_{\xi_1}(x))^2}{1-pF_{\xi_1}(x)} = 1 - (1-p) \ \frac{F_{\xi_1}^2(x)}{1-pF_{\xi_1}(x)}
\end{aligned}$$

belongs to the class $\mathcal{OS}$.

Finally, by Theorem 4.1.3, we obtain that d.f. $F_{S(\eta)}$ is generalized subexponential if a counting r.v. η is independent of $\{\xi_1, \xi_2, \ldots\}$ and $\mathbb{E}\,\mathrm{e}^{\lambda\eta} < \infty$ for each $\lambda > 0$. In particular, if

$$\mathbb{P}(\eta = n) = \frac{1}{C_9}\,\mathrm{e}^{-n^2},\ n \in \mathbb{N},\ C_9 = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2},$$

then d.f. $F_{S(\eta)}$ with the tail

$$\begin{aligned}
\overline{F}_{S(\eta)}(x) &= \sum_{n=1}^{\infty} \overline{F}_{S(n)}(x)\ \mathbb{P}(\eta = n) \\
&= \frac{1}{C_9} \sum_{n=1}^{\infty} \mathrm{e}^{-n^2}\ \mathbb{P}(\max\{S_1, S_2, \ldots S_n\} > x) \\
&= \frac{1}{C_9} \sum_{n=1}^{\infty} \mathrm{e}^{-n^2}\ \mathbb{P}(S_n > x) \\
&= \frac{1}{C_9} \sum_{n=1}^{\infty} \mathrm{e}^{-n^2}\ \mathbb{P}(\xi_1 + \xi_2 + ... + \xi_n > x) \\
&= \frac{1}{C_9} \sum_{n=1}^{\infty} \mathrm{e}^{-n^2}\ \overline{\mathop{\large\ast}_{k=1}^{n} F_{\xi_k}(x)},\ x \geqslant 0,
\end{aligned}$$

belongs to the class $\mathcal{OS}$.

EXAMPLE 4.2.2. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s such that*

$$\overline{F}_{\xi_1}(x) = \mathbb{I}_{(-\infty,1)}(x) + \frac{\mathrm{e}^{1-x}}{x^2}\,\mathbb{I}_{[1,\infty)}(x),$$
$$\overline{F}_{\xi_k}(x) = \mathbb{I}_{(-\infty,1)}(x) + \Big(\mathrm{e} - \frac{1}{k-1}\Big)\frac{\mathrm{e}^{-x}}{x^2}\,\mathbb{I}_{[1,\infty)}(x),\ k \in \{2, 3, \ldots\}.$$

According to the results of [13], [74] d.f.s F_{ξ_k} belongs to the class $\mathcal{OS}$ for all $k \in \mathbb{N}$. In addition,

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} = 1, \tag{4.2.3}$$

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} = 1, \tag{4.2.4}$$

and

$$\liminf_{x\to\infty} \inf_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} = 1 - \frac{1}{2\mathrm{e}}. \tag{4.2.5}$$

Hence, the sequence of r.v.s $\{\xi_1, \xi_2, \ldots,\}$ also satisfies the conditions of Theorems 3.1.1–4.1.1.

Namely,

$$\begin{aligned}
&\limsup_{x\to\infty}\sup_{k\geqslant 2}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)}\\
&=\limsup_{x\to\infty}\sup_{k\geqslant 2}\frac{(\mathrm{e}-\frac{1}{k-1})\mathrm{e}^{-x}x^{-2}}{\mathrm{e}^{1-x}x^2}\\
&=\limsup_{x\to\infty}\sup_{k\geqslant 2}(1-\frac{1}{\mathrm{e}(k-1)})=1.
\end{aligned}$$

Hence (4.2.3) and (4.2.4) conditions hold.
Let's check condition (4.2.5), since for $x \geq 1$

$$\begin{aligned}
\overline{F}_1(x) &= \mathrm{e}^{1-x}x^{-2}\\
\overline{F}_k(x) &= (\mathrm{e}-\frac{1}{k-1})^{-x}x^{-2}, k=2,3..,
\end{aligned}$$

we get that:

(a) $n=1:$

$$\frac{1}{1}\frac{\overline{F}_1(x)}{\overline{F}_1(x)}=1,$$

(b) $n=2:$

$$\begin{aligned}
&\frac{1}{2}(\frac{\overline{F}_1(x)}{\overline{F}_1(x)}+\frac{\overline{F}_2(x)}{\overline{F}_1(x)})\\
&=\frac{1}{2}(1+\frac{\mathrm{e}-1}{\mathrm{e}})=1-\frac{1}{2\mathrm{e}},
\end{aligned}$$

(c) $n=3:$

$$\begin{aligned}
&\frac{1}{3}(\frac{\overline{F}_1(x)}{\overline{F}_1(x)}+\frac{\overline{F}_2(x)}{\overline{F}_1(x)}+\frac{\overline{F}_3(x)}{\overline{F}_1(x)})\\
&=\frac{1}{3}(1+\frac{\mathrm{e}-1}{\mathrm{e}}+\frac{\mathrm{e}-\frac{1}{2}}{\mathrm{e}})\\
&=\frac{1}{3}(3-\frac{3}{2\mathrm{e}})=1-\frac{1}{2\mathrm{e}}.
\end{aligned}$$

By continuing the process we derive condition (4.2.5).
Theorem 4.1.1 gives that d.f.s $F_{\xi_{(\eta)}}$ and $F_{S_{(\eta)}}$ belong to $\mathcal{OS}$ for each counting r.v. η independent of $\{\xi_1,\xi_2,\ldots\}$. In a particular case, when r.v. η is distributed according to the geometric law (4.2.1), distribution

functions $F_{\xi_{(\eta)}}$ and $F_{S_{(\eta)}}$ belong to the class $\mathcal{OS}$.
In our case, we have that

$$\overline{F}_{\xi_{(\eta)}}(x) = 1\!\!1_{(-\infty,1)}(x) + \frac{(1-p)\,\mathrm{e}}{p^2}\sum_{n=2}^{\infty}\Big(\frac{\mathrm{e}^{-x}p}{x^2}\Big)^n\prod_{k=2}^{n}\Big(\mathrm{e}-\frac{1}{k-1}\Big)1\!\!1_{[1,\infty)}(x),$$

because for $x \geq 1$

$$\begin{aligned}\sum_{n=2}^{\infty}\overline{F}_{\xi_{(n)}}(x)\mathbb{P}(\eta=n) &= \overline{F}_{\xi_1}(x)\overline{F}_{\xi_2}(x)\mathbb{P}(\eta=2)\\ &+\overline{F}_{\xi_1}(x)\overline{F}_{\xi_2}(x)\overline{F}_{\xi_3}(x)\mathbb{P}(\eta=3)\\ &+\ldots+\overline{F}_{\xi_1}(x)\overline{F}_{\xi_2}(x)\overline{F}_{\xi_3}(x)\ldots\overline{F}_{\xi_n}(x)\mathbb{P}(\eta=n)\\ &= \frac{e^{1-x}}{x^2}(\mathrm{e}-\frac{1}{2-1})\frac{\mathrm{e}^{-x}}{x^2}(1-p)\\ &+\frac{\mathrm{e}^{1-x}}{x^2}(\mathrm{e}-\frac{1}{2-1})\frac{\mathrm{e}^{-x}}{x^2}(\mathrm{e}-\frac{1}{3-1})\frac{\mathrm{e}^{-x}}{x^2}(1-p)p+\ldots\\ &+\frac{\mathrm{e}^{1-x}}{x^2}(\mathrm{e}-\frac{1}{2-1})\frac{\mathrm{e}^{-x}}{x^2}\ldots\\ &(\mathrm{e}-\frac{1}{n-1})\frac{\mathrm{e}^{-x}}{x^2}(1-p)p^{n-2}\\ &= \frac{(1-p)\,\mathrm{e}}{p^2}\sum_{n=2}^{\infty}\Big(\frac{\mathrm{e}^{-x}p}{x^2}\Big)^n\prod_{k=2}^{n}\Big(\mathrm{e}-\frac{1}{k-1}\Big).\end{aligned}$$

In addition, $\overline{F}_{S_{(\eta)}}(x) = \overline{F}_{\xi_1}(x)$ based on (4.2.2).
Similarly, Theorem 4.1.2 implies that d.f. $F_{\xi^{(\eta)}}$ belong to the class $\mathcal{OS}$ for each counting r.v. η independent of $\{\xi_1,\xi_2,\ldots\}$ such that $\mathbb{E}\eta<\infty$. If the counting random variable is defined by equality (4.2.1), then by Theorem 4.1.2, we obtain that the distribution function with the tail

$$\begin{aligned}\overline{F}_{\xi^{(\eta)}}(x) &= 1\!\!1_{(-\infty,1)}(x)\\ &+\frac{1-p}{p^2}\sum_{n=2}^{\infty}p^n\sum_{k=1}^{n}\overline{F}_{\xi_k}(x)\prod_{j=1}^{k-1}F_{\xi_j}(x)\,1\!\!1_{[1,\infty)}(x),\end{aligned}$$

belongs to the class $\mathcal{OS}$, because for $x \geq 1$

$$\begin{aligned}
\overline{F}_{\xi^{(\eta)}}(x) &= \sum_{n=2}^{\infty} \overline{F}_{\xi^{(n)}}(x)\mathbb{P}(\eta = n) \\
&= \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n \ \mathbb{P}\big(\max\{\xi_1, \xi_2, \ldots \xi_n\} > x\big) \\
&= \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n \ \Big(1 - \mathbb{P}\big(\max\{\xi_1, \xi_2, \ldots \xi_n\} \leq x\big)\Big) \\
&= \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n - \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n \prod_{k=1}^{n} \mathbb{P}(\xi_k \leq x) \\
&= 1 - \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n \prod_{k=1}^{n} F_{\xi_k}(x).
\end{aligned}$$

Theorem 4.1.3 states that d.f. $F_{S^{(\eta)}}$ belongs to the class $\mathcal{OS}$ if the counting r.v. η is independent of $\{\xi_1, \xi_2, \ldots\}$ and $\mathbb{E}\, \mathrm{e}^{\lambda\eta} < \infty$ for each $\lambda > 0$. In particular, if η is distributed according to the Poisson law

$$\mathbb{P}(\eta = n) = \frac{\mu^n}{n!}\, \mathrm{e}^{-\mu}, \ n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \ \mu > 0,$$

then d.f. $F_{S^{(\eta)}}$ with the tail

$$\overline{F}_{S^{(\eta)}}(x) = \mathrm{e}^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \overline{\prod_{k=1}^{n}\!{}^{*} F_{\xi_k}}(x), \ x \geqslant 1,$$

because

$$\begin{aligned}
\overline{F}_{S^{(\eta)}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{S^{(n)}}(x) \ \mathbb{P}(\eta = n) \\
&= \mathrm{e}^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \mathbb{P}\big(\max\{S_1, S_2, \ldots S_n\} > x\big) \\
&= \mathrm{e}^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \mathbb{P}(S_n > x) \\
&= \mathrm{e}^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \mathbb{P}(\xi_1 + \xi_2 + ... + \xi_n > x) \\
&= \mathrm{e}^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \overline{\prod_{k=1}^{n}\!{}^{*} F_{\xi_k}}(x), \ x \geqslant 0,
\end{aligned}$$

belongs to the class $\mathcal{OS}$.

4.3 Auxiliary results for Theorems 4.1.1-4.1.3

In this subsection, we present Lemmas 4.3.1 and 4.3.3 that will be applied to the derivations of the main Theorems 4.1.1–4.1.3.

The first new lemma which we use in the proofs is a refined version of Lemma 3.3.2 from subsection 3.3. As a result, we present this lemma together with a modified proof.

Lemma 4.3.1. *Let* $\{X_1, X_2 \ldots\}$ *be a sequence of independent real-valued r.v.s, for which* $F_{X_1} \in \mathcal{OS}$*, and for other indices* $k \geqslant 2$*, either* $F_{X_k} \in \mathcal{OS}$ *or* $\overline{F}_{X_k}(x) = O(\overline{F}_{X_1}(x))$*. Then,*

$$\overline{F}_{S_n}(x) \underset{x\to\infty}{\asymp} \overline{\mathop{\Huge\ast}_{k\in\mathcal{A}_n} F_{X_k}}(x) \tag{4.3.1}$$

where $\mathcal{A}_n := \{k \in \{1, 2, \ldots, n\} : F_{X_k} \in \mathcal{OS}\}$*, and hence,* $F_{S_n} \in \mathcal{OS}$ *for all* $n \in \mathbb{N}$*.*

Proof. If $n = 1$, then the statement is obvious because $S_1 = X_1$. If $n = 2$, then two options are possible: $F_{X_2} \in \mathcal{OS}$ or $\overline{F}_{X_2}(x) = O(\overline{F}_{X_1}(x))$. In the first case, $F_{S_2} = F_{X_1} * F_{X_2}$ In the second case, $\overline{F}_{S_2}(x) \underset{x\to\infty}{\asymp} \overline{F}_{X_1}(x)$ by Part (v) of the Lemma 3.3.1. The asymptotic relation of the lemma holds for both cases.

Let us suppose that the asymptotic relation (4.3.1) is valid for some $n = N$, i.e.,

$$\overline{F}_{S_N}(x) \underset{x\to\infty}{\asymp} \overline{\mathop{\Huge\ast}_{k\in\mathcal{A}_N} F_{X_k}}(x) = \overline{F_{X_1} * F_{X_{k_1}} * \ldots F_{X_{k_r}}}(x) \tag{4.3.2}$$

where $\mathcal{A}_N = \{1, k_1, \ldots, k_r\} = \{k \in \{1, 2, \ldots, N\} : F_{X_k} \in \mathcal{OS}\}$.

The above relation and parts (ii), (iii) of Lemma 3.3.1 imply that $F_{S_N} \in \mathcal{OS}$. The tail of any d.f. from class $\mathcal{OS}$ remains positive throughout the set of real numbers. Hence, the relation (4.3.2) implies that

$$0 < \inf_{x\in\mathbb{R}} \frac{\overline{F}_{S_N}(x)}{\overline{\mathop{\Huge\ast}_{k\in\mathcal{A}_N} F_{X_k}}(x)} \leqslant \sup_{x\in\mathbb{R}} \frac{\overline{F}_{S_N}(x)}{\overline{\mathop{\Huge\ast}_{k\in\mathcal{A}_N} F_{X_k}}(x)} < \infty. \tag{4.3.3}$$

For $n = N + 1$, we have two possibilities:
either $F_{X_{N+1}} \in \mathcal{OS}$, or $\overline{F}_{X_{N+1}}(x) = O\big(\overline{F}_{X_1}(x)\big)$. If $F_{X_{N+1}} \in \mathcal{OS}$, then according to (4.3.3), it holds that

$$\begin{aligned}
\overline{F}_{S_{N+1}}(x) &= \int_{-\infty}^{\infty} \overline{F}_{S_N}(x-y)\mathrm{d}F_{X_{N+1}}(y) \\
&\leqslant c_{1N} \int_{-\infty}^{\infty} \overline{\mathop{\Huge\ast}_{k\in\mathcal{A}_N} F_{X_k}}(x-y)\mathrm{d}F_{X_{N+1}}(y) \\
&= c_{1N} \overline{\mathop{\Huge\ast}_{k\in\mathcal{A}_{N+1}} F_{X_k}}(x), x \in \mathbb{R}
\end{aligned}$$

for some positive quantity c_{1N} not depending on x, where $\mathcal{A}_{N+1} = \mathcal{A}_N \cup \{N+1\}$. Similarly,

$$\overline{F}_{S_{N+1}}(x) \geqslant c_{2N} \overline{\coprod_{k\in\mathcal{A}_{N+1}}^{*} F_{X_k}}(x), x \in \mathbb{R}$$

for some positive quantity c_{2N}.

The above two estimates imply the asymptotic relation (4.3.1) in the case $n = N + 1$.

If $\overline{F}_{X_{N+1}} = O(\overline{F}_{X_1}(x))$, then according to (4.3.3), for positive x, we have

$$\frac{\overline{F}_{X_{N+1}}(x)}{\overline{F}_{S_N}(x)} \leqslant \frac{1}{c_{2N}} \frac{\overline{F}_{X_{N+1}}(x)}{\overline{\coprod_{k\in\mathcal{A}_N}^{*} F_{X_k}}(x)} \leqslant \frac{1}{c_{2N} \prod\limits_{k\in\mathcal{A}_N\setminus\{1\}} \overline{F}_{X_k}(0)} \frac{\overline{F}_{X_{N+1}}(x)}{\overline{F}_{X_1}(x)}$$

because

$$\begin{aligned}\overline{F}_{X_1}(x) \prod_{k\in\mathcal{A}_N\setminus\{1\}} \overline{F}_{X_k}(0) &= \mathbb{P}(X_1 > x, X_{k_1} > 0, \ldots, X_{k_r} > 0) \\ &\leqslant \mathbb{P}(X_1 + X_{k_1} + \ldots + X_{k_r} > x),\end{aligned}$$

where $\{k_1, \ldots, k_r\} = \mathcal{A}_N \setminus \{1\}$. Consequently, $\overline{F}_{X_{N+1}}(x) = O(\overline{F}_{S_N}(x))$, and according to the Part (v) of Lemma 3.3.1, we obtain that

$$\overline{F}_{S_{N+1}}(x) = \overline{F_{S_N} * F_{X_{N+1}}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{S_N}(x) \underset{x\to\infty}{\asymp} \overline{\coprod_{k\in\mathcal{A}_{N+1}}^{*} F_{X_k}}(x)$$

with $\mathcal{A}_{N+1} = \mathcal{A}_N$ in the case.

We derived the asymptotic relation (4.3.1) for $n = N+1$ by supposing that this relation holds for $n = N$. Due to the induction principle, the asymptotic relation (4.3.1) is valid for all $n \in \mathbb{N}$. The assertion $F_{S_n} \in \mathcal{OS}$ follows from (4.3.1) after using part (ii) and (iii) of Lemma 3.3.1. This finishes the proof of the lemma.

□

For the maximum of sums $S^{(n)} = \max\{S_1, S_2, \ldots, S_n\}$ the following statement holds.

Lemma 4.3.2. *Let $\{X_1, X_2, \ldots\}$ be a sequence of independent real-valued r.v.s, for which $F_{X_1} \in \mathcal{OS}$ and*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{X_1}(x)} < \infty$$

there exists a constant c_4 such that

$$\overline{F}_{S^{(n)}}(x) \leqslant c_4^n \overline{F}_{X_1}(x)$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proof. If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$\begin{aligned}\overline{F}_{S^{(n)}}(x) &= \mathbb{P}\big(\max\{S_1, S_2, \dots S_n\} > x\big)\\ &= \mathbb{P}\Big(\bigcup_{k=1}^{n}\{S_k > x\}\Big)\\ &\leqslant \sum_{k=1}^{n}\mathbb{P}\{S_k > x\}.\end{aligned}$$

Let us suppose $c_4 = \max\{c_3, 2\}$, with $c_3 = \widehat{C}$ from (3.3.9). According to the estimate (3.3.9) from Lemma 3.3.3, we have that

$$\begin{aligned}\overline{F}_{S^{(n)}}(x) &\leqslant \sum_{k=1}^{n} c_3^{k-1}\overline{F}_{X_1}(x) \leqslant \overline{F}_{X_1}(x)\frac{c_4^n - 1}{c_4 - 1}\\ &\leqslant c_4^n\overline{F}_{X_1}(x).\end{aligned}$$

The estimate of the lemma has been proved. □

The next lemma is on the minimum $X_{(n)} = \min\{X_1, \dots, X_n\}$ of a collection of independent r.v.s $\{X_1, \dots, X_n\}$. The similar assertion is presented in Lemma 3.1 of [52]. Unfortunately, the proof presented there is suitable for nonnegative absolutely continuous r.v.s. Hence, we present the detailed proof of the lemma below.

Lemma 4.3.3. *Let $\{X_1, X_2, \dots X_n\}$, $n \in \mathbb{N}$ be a collection of independent real-valued r.v.s with d.f.s $\{F_{X_1}, F_{X_2}, \dots F_{X_n}\}$. If $F_{X_k} \in \mathcal{OS}$ for all $k \in \{1, 2, \dots n\}$, then d.f. $F_{X_{(n)}}$ of $X_{(n)} = \min\{X_1, X_2, \dots, X_n\}$ belongs to the class $\mathcal{OS}$ as well.*

Proof. If $n = 1$, then the assertion of the lemma is trivial. If $n \geqslant 2$, then

$$\min\{X_1, X_2, \dots, X_n\} = \min\big\{\min\{X_1, X_2, \dots, X_{n-1}\}, X_n\big\}.$$

It follows from this equality that it is sufficient to prove the statement of the lemma for the case $n = 2$, i.e., it is sufficient for us to prove such a statement:

$$X, Y \ \text{ independent r.v.s } \ F_X \in \mathcal{OS}, F_Y \in \mathcal{OS} \ \Rightarrow \ F_{X\wedge Y} \in \mathcal{OS}. \tag{4.3.4}$$

- At first, let us suppose that X and Y are absolutely continuous r.v.s. In such a case,

$$\begin{aligned}\overline{F}_{\min(X,Y)}(x) &= \overline{F}_{X\wedge Y}(x) = \overline{F}_X(x)\overline{F}_Y(x)\\ &= \int_x^{\infty} f_X(y)\mathrm{d}y \int_x^{\infty} f_Y(y)\mathrm{d}y\end{aligned}$$

with density functions f_X and f_Y. For $x \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{\overline{F^{*2}_{X\wedge Y}}(x)}{\overline{F}_{X\wedge Y}(x)} &= \frac{1}{\overline{F}_{X\wedge Y}(x)} \int\limits_{-\infty}^{\infty} \overline{F}_{X\wedge Y}(x-y)\mathrm{d}F_{X\wedge Y}(y) \\
&= -\frac{1}{\overline{F}_X(x)\overline{F}_Y(x)} \int\limits_{-\infty}^{\infty} \overline{F}_X(x-y)\overline{F}_Y(x-y)\mathrm{d}\overline{F}_X(y)\overline{F}_Y(y) \\
&= \int\limits_{-\infty}^{\infty} \frac{\overline{F}_X(x-y)\overline{F}_Y(x-y)\overline{F}_Y(y)f_X(y)\mathrm{d}y}{\overline{F}_X(x)\overline{F}_Y(x)} \\
&+ \int\limits_{-\infty}^{\infty} \frac{\overline{F}_X(x-y)\overline{F}_Y(x-y)\overline{F}_X(y)f_Y(y)\mathrm{d}y}{\overline{F}_X(x)\overline{F}_Y(x)}. \qquad (4.3.5)
\end{aligned}$$

If Y_1 and Y_2 are independent copies of Y, then

$$\overline{F}_Y(x-y)\overline{F}_Y(y) = \mathbb{P}(Y_1 > x-y)\mathbb{P}(Y_2 > y) \leqslant \mathbb{P}(Y_1+Y_2 > x) = \overline{F^{*2}_Y}(x).$$

for all $x, y \in \mathbb{R}$. Hence, the condition $F_Y \in \mathcal{OS}$ implies that

$$\sup_{x,y\in\mathbb{R}} \frac{\overline{F}_Y(x-y)\overline{F}_Y(y)}{\overline{F}_Y(x)} \leqslant \sup_{x\in\mathbb{R}} \frac{\overline{F^{*2}_Y}(x)}{\overline{F}_Y(x)} \leqslant c_5$$

for some constant c_5 according to Lemma 3.3.1(i). Similarly for r.v. X, we obtain that

$$\sup_{x,y\in\mathbb{R}} \frac{\overline{F}_X(x-y)\overline{F}_X(y)}{\overline{F}_X(x)} \leqslant \sup_{x\in\mathbb{R}} \frac{\overline{F^{*2}_X}(x)}{\overline{F}_X(x)} \leqslant c_6$$

for some another constant c_6. Hence, according to the decomposition (4.3.5),

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \frac{\overline{F^{*2}_{X\wedge Y}}(x)}{\overline{F}_{X\wedge Y}(x)} &\leqslant \max\{c_5, c_6\} \sup_{x\in\mathbb{R}} \Bigg(\int\limits_{-\infty}^{\infty} \frac{\overline{F}_X(x-y)f_X(y)\mathrm{d}y}{\overline{F}_X(x)} \\
&+ \int\limits_{-\infty}^{\infty} \frac{\overline{F}_Y(x-y)f_Y(y)\mathrm{d}y}{\overline{F}_Y(x)} \Bigg) \\
&= \max\{c_5, c_6\} \sup_{x\in\mathbb{R}} \left(\frac{\overline{F^{*2}_X}(x)}{\overline{F}_X(x)} + \frac{\overline{F^{*2}_Y}(x)}{\overline{F}_Y(x)} \right) \\
&\leqslant \max\{c_5, c_6\}(c_5 + c_6),
\end{aligned}$$

which implies that $F_{X\wedge Y} \in \mathcal{OS}$ due to Lemma 3.3.1(i) again.

- Now, let us suppose that r.v.s X and Y are not necessarily absolutely continuous.

  At first, let us consider r.v. X. Since $\mathcal{OS} \subset \mathcal{OL}$ (see Lemma 3.3.1(iv)), we have that $F_X \in \mathcal{OL}$. If function F_X belongs to $\mathcal{OL}$, then the function $F_X(\log x)$, $x > 0$, is nonincreasing O-regularly varying, according to Bingham [6], because

$$\limsup_{x\to\infty} \frac{\overline{F}_X(\log xy)}{\overline{F}_X(\log x)} = \limsup_{x\to\infty} \frac{\overline{F}_X(\log x + \log y)}{\overline{F}_X(\log x)} < \infty.$$

  for an arbitrary $y > 0$.

  From the representation Theorem - see Theorem 2.2.7 in [6], or Theorem A.1 together with Definition A.4 and Remark on page 100 of [64] - we have that

$$\overline{F}_X(\log x) = \exp\left\{a(x) - \int\limits_A^x \frac{b(t)}{t}\mathrm{d}t\right\}$$

  for all $x \geqslant A$. Here, A is a positive number, a and $b \geqslant 0$ are bounded integrable functions. Therefore, for all $x \geqslant \log A$

$$\overline{F}_X(x) = \exp\left\{a^*(x) - \int\limits_{\log A}^x b^*(u)\mathrm{d}u\right\} \tag{4.3.6}$$

  with bounded and integrable functions a^* and $b^* \geqslant 0$. Since b^* is a positive-bounded and integrable function,

$$G(x) = \left(1 - \exp\left\{-\int\limits_{\log A}^x b^*(u)\mathrm{d}u\right\}\right)\mathbb{1}_{[\log A,\infty)}(x)$$

  is an absolutely continuous d.f. from the class $\mathcal{OL}$ with a tail function

$$\overline{G}(x) = \mathbb{I}_{(-\infty,\log A)}(x) + \exp\left\{-\int\limits_{\log A}^x b^*(u)\mathrm{d}u\right\}\mathbb{1}_{[\log A,\infty)}(x)$$

  In addition, the boundedness of the function a^* in (4.3.6) implies that

$$\overline{F}_X(x) \underset{x\to\infty}{\asymp} \overline{G}(x)$$

  In a similar way, we derive that

$$\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{H}(x)$$

for some absolutely continuous d.f. $H \in \mathcal{OL}$. According to Lemma 3.3.1(ii) and the first part of the proof d.f. $1-\overline{GH}$ belongs to the class $\mathcal{OS}$, and from Lemma 3.3.1(ii), again, we obtain $F_{X\wedge Y} \in \mathcal{OS}$ because

$$\overline{F_{X\wedge Y}}(x) = \overline{F_X}(x)\overline{F_Y}(x) \underset{x\to\infty}{\asymp} \overline{G}(x)\overline{H}(x).$$

It follows from both parts of the proof that the relation (4.3.4) holds. At the same time, the assertion of the lemma has been proved. □

The last technical lemma is an asymetric assertion for the assertion of Lemma 4.3.2. The lemma below can be used for the examination of the randomly stopped maximum of sums. For the proof of the lemma below, we use the revised episodes of the proof of Theorem 3.1.2.

Lemma 4.3.4. *Let $\{X_1, X_2, \ldots\}$ be a sequence of independent real-valued r.v.s such that d.f. $F_{X_1} \in \mathcal{OS}$ and*

$$\liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} > 0,$$

Then,

$$\inf_{x\in\mathbb{R}} \frac{\overline{F}_{S^{(n)}}(x)}{\overline{F}_{X_1}(x)} \geqslant \inf_{x\in\mathbb{R}} \frac{\overline{F}_{S_n}(x)}{\overline{F}_{X_1}(x)} \geqslant c_7^{n-1} \tag{4.3.7}$$

for all $n \geqslant 1$ and some $c_7 > 0$, where $S_n = X_1 + \ldots + X_n$ and $S^{(n)} = \max\{S_1, S_2, \ldots S_n\}$.

Proof. The conditions of the lemma give that

$$\inf_{k\geqslant 1} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} \geqslant \Delta \tag{4.3.8}$$

for all $x \geqslant x_\Delta$ and some positive Δ. If $x < x_\Delta$, then

$$\begin{aligned}\inf_{k\geqslant 1} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} &\geqslant \inf_{k\geqslant 1} \overline{F}_{X_k}(x_\Delta) = \inf_{k\geqslant 1} \frac{\overline{F}_{X_k}(x_\Delta)}{\overline{F}_{X_1}(x_\Delta)}\, \overline{F}_{X_1}(x_\Delta) \\ &\geqslant \Delta\, \overline{F}_{X_1}(x_\Delta) := c_8 > 0\end{aligned}$$

due to the assumption $F_{X_1} \in \mathcal{OS}$. The derived inequalities imply that

$$\overline{F}_{X_k}(x) \geqslant c_8\, \overline{F}_{X_1}(x)$$

for all $x \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. Using the last estimate, we obtain

$$\begin{aligned}\overline{F}_{S_2}(x) &= \int_{-\infty}^{\infty} \frac{\overline{F}_{X_2}(x-y)}{\overline{F}_{X_1}(x-y)}\, \overline{F}_{X_1}(x-y)\, \mathrm{d}F_{X_1}(y) \\ &\geqslant c_8\, \overline{F_{X_1} * F_{X_1}}(x) \\ &\geqslant c_8\, \overline{F}_{X_1}(0)\overline{F}_{X_1}(x),\ x \in \mathbb{R}.\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{F}_{S_3}(x) &= \int\limits_{-\infty}^{\infty} \frac{\overline{F}_{S_2}(x-y)}{\overline{F}_{X_1}(x-y)}\, \overline{F}_{X_1}(x-y)\, \mathrm{d}F_{X_3}(y) \\
&\geqslant c_8 \overline{F}_{X_1}(0) \int\limits_{-\infty}^{\infty} \overline{F}_{X_1}(x-y)\, \mathrm{d}F_{X_3}(y) \\
&= c_8\, \overline{F}_{X_1}(0) \int\limits_{-\infty}^{\infty} \overline{F}_{X_3}(x-y)\, \mathrm{d}F_{X_1}(y) \\
&\geqslant c_8^2\, \overline{F}_{X_1}(0) \int\limits_{-\infty}^{\infty} \overline{F}_{X_1}(x-y)\, \mathrm{d}F_{X_1}(y) \\
&= c_8^2\, \overline{F}_{X_1}(0) \overline{F_{X_1} * F_{X_1}}(x) \\
&\geqslant c_8^2\, \left(\overline{F}_{X_1}(0)\right)^2 \overline{F}_{X_1}(x), \quad x \in \mathbb{R}.
\end{aligned}$$

By continuing the process, we obtain

$$\overline{F}_{S_n}(x) \geqslant \left(c_8 \overline{F}_{X_1}(0)\right)^{n-1} \overline{F}_{X_1}(x)$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence, the second estimate in (4.3.7) holds with $c_7 = c_8\, \overline{F}_{X_1}(0)$. The first inequality in (4.3.7) is obvious because

$$\overline{F}_{S_n}(x) = \mathbb{P}(S_n > x) \leqslant \mathbb{P}(\max(S_1, S_2, ..S_n) > x) = \overline{F}_{S^{(n)}}(x)$$

for an arbitrary real number x. Lemma 4.3.4 has been proved. □

4.4 Proofs of Theorems 4.1.1-4.1.3

In this subsection, we prove all main results.

4.4.1 Proof of Theorem 4.1.1

Proof. At first, let us consider the first part of the theorem. Due to Lemma 3.3.1(ii), it is enough to prove the asymptotic relations (4.1.1) because $F_{\xi_{(\varkappa)}} \in \mathcal{OS}$ by Lemma 4.3.3. By the definition of the randomly stopped minimum for positive x, we have

$$
\begin{aligned}
\overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbb{P}(\eta = n) \\
&= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa) + \sum_{n=\varkappa+1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbb{P}(\eta = n) \\
&= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa) + \overline{F}_{\xi_{(\varkappa)}}(x) \sum_{n=\varkappa+1}^{\infty} \mathbb{P}(\eta = n) \prod_{k=\varkappa+1}^{n} \overline{F}_{\xi_k}(x) \\
&= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)\Big(1 + \frac{1}{\mathbb{P}(\eta = \varkappa)} \sum_{n=\varkappa+1}^{\infty} \mathbb{P}(\eta = n) \prod_{k=\varkappa+1}^{n} \overline{F}_{\xi_k}(x)\Big) \\
&\leqslant \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)\Big(1 + \overline{F}_{\xi_{\varkappa+1}}(x)\frac{\mathbb{P}(\eta \geqslant \varkappa + 1)}{\mathbb{P}(\eta = \varkappa)}\Big).
\end{aligned}
$$

On the other hand, for all $x > 0$,

$$
\overline{F}_{\xi_{(\eta)}}(x) \geqslant \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa).
$$

On this basis, we can assert that

$$
0 < \liminf_{x\to\infty} \frac{\overline{F}_{\xi_{(\eta)}}(x)}{\overline{F}_{\xi_{(\varkappa)}}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_{\xi_{(\eta)}}(x)}{\overline{F}_{\xi_{(\varkappa)}}(x)} < \infty.
$$

Hence, the relation (4.1.1) holds, and by Lemma 4.3.3, it follows that $\overline{F}_{\xi_{(\eta)}} \in \mathcal{OS}$.

Let us consider the second part of the theorem. By Lemma 4.3.1, we have that $F_{S_k} \in \mathcal{OS}$ for each $k \in \mathbb{N}$, and by Lemma 4.3.3, we have $F_{S_{(\varkappa)}} \in \mathcal{OS}$. Hence, it suffices to prove the asymptotic relation (4.1.2) in order to obtain $F_{S_{(\eta)}} \in \mathcal{OS}$. Similar to the first part of the proof, we obtain that

$$
\overline{F}_{S_{(\eta)}}(x) \leqslant \overline{F}_{S_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)\Big(1 + \overline{F}_{S_{\varkappa+1}}(x)\frac{\mathbb{P}(\eta \geqslant \varkappa + 1)}{\mathbb{P}(\eta = \varkappa)}\Big),
$$

and

$$
\overline{F}_{S_{(\eta)}}(x) \geqslant \overline{F}_{S_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)
$$

for all positive x. Therefore,

$$
0 < \liminf_{x\to\infty} \frac{\overline{F}_{S_{(\eta)}}(x)}{\overline{F}_{S_{(\varkappa)}}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_{S_{(\eta)}}(x)}{\overline{F}_{S_{(\varkappa)}}(x)} < \infty,
$$

and the desired relation (4.1.2) follows. This finishes the proof of the theorem. □

4.4.2 Proof of Theorem 4.1.2

Proof. For $x > 0$, we have

$$\begin{aligned}
\overline{F}_{\xi^{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\xi^{(n)} > x)\mathbb{P}(\eta = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\Big(\bigcup_{k=1}^{n} \{\xi_k > x\} \Big) \mathbb{P}(\eta = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\bigg(\bigcup_{k=1}^{n} \Big(\xi_k > x, \bigcap_{j=1}^{k-1} \{\xi_j \le x\} \Big) \bigg) \mathbb{P}(\eta = n).
\end{aligned}$$

Therefore,

$$\frac{\overline{F}_{\xi^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \prod_{j=1}^{k-1} F_{\xi_j}(x). \tag{4.4.1}$$

The condition (4.1.3) implies that

$$\sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant c_9 n$$

for all $n \in \mathbb{N}$, some c_9 and sufficiently large x, say $x > x_1$. Hence,

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} \leqslant c_9 \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)n = c_9 \mathbb{E}\eta. \tag{4.4.2}$$

In a similar way from (4.1.3), we derive that there exists $c_{10} > 0$, such that

$$\sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \geqslant c_{10}\, n$$

for all n and sufficiently large x, say $x > x_2$. The decomposition (4.4.1) implies that for each $N \geqslant 1$,

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\overline{F}_{\xi^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} &\geqslant \liminf_{x\to\infty} \sum_{n=1}^{N} \mathbb{P}(\eta = n) \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \prod_{j=1}^{k-1} F_{\xi_j}(x) \\
&= \sum_{n=1}^{N} \mathbb{P}(\eta = n) \liminf_{x\to\infty} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \prod_{j=1}^{k-1} F_{\xi_j}(x) \\
&\geqslant \sum_{n=1}^{N} \mathbb{P}(\eta = n) \liminf_{x\to\infty} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \liminf_{x\to\infty} \prod_{j=1}^{n-1} F_{\xi_j}(x) \\
&\geqslant c_{10} \sum_{n=1}^{N} n\mathbb{P}(\eta = n).
\end{aligned}$$

By passing N to infinity, we obtain

$$\liminf_{x\to\infty} \frac{\overline{F}_{\xi(\eta)}(x)}{\overline{F}_{\xi_1}(x)} \geqslant c_{10}\mathbb{E}\eta. \tag{4.4.3}$$

Since $0 < \mathbb{E}\eta < \infty$, the derived relations (4.4.2) and (4.4.3) imply that $\overline{F}_{\xi(\eta)}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$, and hence, $F_{\xi(\eta)} \in \mathcal{OS}$ due to Lemma 3.3.1(ii). Theorem 4.1.2 has been proved. □

4.4.3 Proof of Theorem 4.1.3

Proof. For $x > 0$, we have

$$\overline{F}_{S^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(S^{(n)} > x)\mathbb{P}(\eta = n).$$

Therefore, for such x,

$$\begin{aligned}
\frac{\overline{F}_{S^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} &= \sum_{n=1}^{\infty} \frac{\overline{F}_{S^{(n)}}(x)}{\overline{F}_{\xi_1}(x)}\mathbb{P}(\eta = n) \\
&\leqslant \sum_{n=1}^{\infty} \sup_{x>0} \frac{\overline{F}_{S^{(n)}}(x)}{\overline{F}_{\xi_1}(x)} \mathbb{P}(\eta = n) \\
&\leqslant \sum_{n=1}^{\infty} (c_{11})^n \mathbb{P}(\eta = n) \\
&\leqslant \mathbb{E}(c_{11})^\eta < \infty
\end{aligned}$$

with $c_{11} \geqslant 2$ due to the conditions of the theorem and Lemma 4.3.2. Consequently,

$$\limsup_{x\to\infty} \frac{\overline{F}_{S^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} < \infty. \tag{4.4.4}$$

Now, we check if

$$\liminf_{x\to\infty} \frac{\overline{F}_{S^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} > 0. \tag{4.4.5}$$

For positive x, it holds that

$$\begin{aligned}
\frac{\overline{F}_{S^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} &= \sum_{n=1}^{\infty} \frac{\overline{F}_{S^{(n)}}(x)}{\overline{F}_{\xi_1}(x)}\mathbb{P}(\eta = n) \\
&= \sum_{n=1}^{\infty} \frac{\mathbb{P}(\max\{\xi_1, \xi_2, \dots \xi_n\} > x)}{\mathbb{P}(\xi_1 > x)}\mathbb{P}(\eta = n) \\
&\geq \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \\
&= \mathbb{P}(\eta \geq 1) > 0.
\end{aligned}$$

The derived estimates (4.4.4) and (4.4.5) imply that $\overline{F}_{S^{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$. Hence, $F_{S^{(\eta)}} \in \mathcal{OS}$ due to Lemma 3.3.1(ii). Theorem 4.1.3 has been proved. □

5 Randomly stopped sums, minima and maxima for heavy-tailed and light-tailed distributions

In this chapter, we continue to explore the same structures, randomly stopped sums, randomly stopped minimum, randomly stopped maximum, randomly stopped maximum of sums and randomly stopped minimum of sums. In the previous chapters, we focused on the class of generalized subexponential distributions, and in this chapter we change the main object of consideration to the class of heavy tailed distributions $\mathcal{H}$.

5.1 Main results

In this subsection we formulate the main results of the chapter. We begin with the randomly stopped sums. We notice that the d.f. F_{S_η} can become heavy-tailed because of the heavy tail of some element in $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ or because of the heavy tail of the counting random variable η.

Theorem 5.1.1. *Let $\xi_1, \xi_2, \ldots$ be independent real-valued r.v.s and let η be a counting r.v. independent of the sequence $\{\xi_1, \xi_2, \ldots\}$. The distribution F_{S_η} is heavy-tailed if at least one of the following conditions is satisfied:*

(i) $\inf\limits_{k\geq 1} \mathbb{E}\, \mathrm{e}^{\lambda\xi_k} > 1$ *for any* $\lambda > 0$, *and* $F_\eta \in \mathcal{H}$;

(ii) $\inf\limits_{k\geq 1} \mathbb{P}(\xi_k \geq a) = 1$ *for some* $a > 0$, *and* $F_\eta \in \mathcal{H}$;

(iii) $F_{\xi_\varkappa} \in \mathcal{H}$ *for some* $\varkappa \geq 1$, *and* $\overline{F}_\eta(x) > 0$ *for all* $x \in \mathbb{R}$;

(iv) $F_{\xi_\varkappa} \in \mathcal{H}$ *for some* $1 \leq \varkappa \leq \max \operatorname{supp}(\eta)$ *and* $\operatorname{supp}(\eta)$ *is finite.*

The distribution F_{S_η} is light-tailed if at least one of the following conditions is satisfied:

(v) $F_{\xi_1} \in \mathcal{H}^c$, $F_\eta \in \mathcal{H}^c$, $\overline{F}_{\xi_1}(x) > 0$ *for all* $x \in \mathbb{R}$ *and*

$$\limsup_{x\to\infty} \sup_{k\geq 1} \frac{\overline{F_{\xi_k}}(x)}{\overline{F}_{\xi_1}(x)} < \infty; \tag{5.1.1}$$

(vi) $\sup\limits_{k\geq 1} \mathbb{E}\, \mathrm{e}^{\lambda\xi_k} < \infty$ *for some* $\lambda > 0$, *and* $F_\eta \in \mathcal{H}^c$.

Our next statement is about the randomly stopped maximum of r.v.s. We observe that some conditions under which the distribution of the randomly stopped maximum $F_{\xi^{(\eta)}}$ becomes heavy-tailed are the same as in Proposition 5.1.1. Unfortunately, we were unable to construct a heavy-tailed distribution $F_{\xi^{(\eta)}}$ from the light-tailed primary r.v.s $\{\xi_1, \xi_2, \ldots\}$.

Theorem 5.1.2. *Let $\xi_1, \xi_2, \ldots$ be independent real-valued r.v.s and let η be a counting r.v. independent of the sequence $\{\xi_1, \xi_2, \ldots\}$.*

(i) *If $F_{\xi_\varkappa} \in \mathcal{H}$ for some $\varkappa \geq 1$ and $\overline{F}_\eta(x) > 0$ for all $x \in \mathbb{R}$, then $F_{\xi^{(\eta)}} \in \mathcal{H}$;*

(ii) *If $F_{\xi_\varkappa} \in \mathcal{H}$ for some $\varkappa \leq \max\{\mathrm{supp}(\eta)\} < \infty$, then $F_{\xi^{(\eta)}} \in \mathcal{H}$;*

(iii) *The distribution $F_{\xi^{(\eta)}}$ belongs to the class $\mathcal{H}^c$ if $F_{\xi_1} \in \mathcal{H}^c$, $\overline{F}_{\xi_1}(x) > 0$ for all $x \in \mathbb{R}$, $\mathbb{E}\eta < \infty$ and*

$$\limsup_{x\to\infty} \sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty. \tag{5.1.2}$$

The statement below is on the distribution of the randomly stopped minimum of r.v.s. From the formulation below, we observe that the tail of the d.f. $F_{\xi_{(\eta)}}$ has significantly less chance of becoming heavy compared to the d.f.s F_{S_η} and $F_{\xi^{(\eta)}}$.

Theorem 5.1.3. *Let $\xi_1, \xi_2, \ldots$ be independent real-valued r.v.s and let η be a counting r.v. independent of the sequence $\{\xi_1, \xi_2, \ldots\}$.*

(i) *If $F_{\xi_1} \in \mathcal{H}$ and*

$$\liminf_{x\to\infty} \min_{1\leq k\leq\varkappa} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} > 0$$

for $\varkappa = \min\{\mathrm{supp}(\eta) \setminus \{0\}\}$, then $F_{\xi_{(\eta)}} \in \mathcal{H}$ and

$$\overline{F}_{\xi_{(\eta)}}(x) \underset{x\to\infty}{\sim} \mathbb{P}(\eta = \varkappa)\, \overline{F}_{\xi_{(\varkappa)}}(x);$$

(ii) *If $F_{\xi_k} \in \mathcal{H}^c$ for $1 \leq k \leq \varkappa = \min\{\mathrm{supp}(\eta)\setminus\{0\}\}$, then $F_{\xi_{(\eta)}} \in \mathcal{H}^c$.*

The next two statements pertain to the heaviness of the randomly stopped minimum of sums and randomly stopped maximum of sums. It can be seen from the presented formulations that some of the conditions were already present in the previous statements. However, for the sake of clarity, we present the complete statements regarding the heaviness of $F_{S_{(\eta)}}$ and $F_{S^{(\eta)}}$.

Theorem 5.1.4. *Let $\xi_1, \xi_2, \ldots$ be independent real-valued r.v.s and let η be a counting r.v. independent of the sequence $\{\xi_1, \xi_2, \ldots\}$.*

(i) *If* $F_{\xi_1} \in \mathcal{H}$ *and* $\min\limits_{1 \le k \le \varkappa} \mathbb{P}(\xi_k \ge 0) > 0$ *for* $\varkappa = \min\{\operatorname{supp}(\eta) \setminus \{0\}\}$, *then* $F_{S_{(\eta)}} \in \mathcal{H}$ *and*

$$\overline{F}_{S_{(\eta)}}(x) \underset{x \to \infty}{\asymp} \overline{F}_{\xi_1}(x). \tag{5.1.3}$$

(ii) *If* $F_{\xi_1} \in \mathcal{H}^c$, *then* $F_{S_{(\eta)}} \in \mathcal{H}^c$ *for any r.v.* η.

Theorem 5.1.5. *Let* $\{\xi_1, \xi_2, \ldots\}$ *and* η *be r.v.'s. as described in Theorems 5.1.1-5.1.4. Then* $F_{S(\eta)} \in \mathcal{H}$ *if at least one of the following conditions is satisfied:*

(i) $\inf\limits_{k \ge 1} \mathbb{E}\, \mathrm{e}^{\lambda \xi_k} > 1$ *for all* $\lambda > 0$ *and* $F_\eta \in \mathcal{H}$*;*

(ii) $\inf\limits_{k \ge 1} \mathbb{P}(\xi_k \ge a) = 1$ *for some* $a > 0$ *and* $F_\eta \in \mathcal{H}$*;*

(iii) $F_{\xi_1} \in \mathcal{H}$*;*

(iv) $F_{\xi_\varkappa} \in \mathcal{H}$ *for some* $\varkappa \ge 1$ *in the case of infinite* $\operatorname{supp}(\eta)$ *or for some* $1 \le \varkappa \le \max \operatorname{supp}(\eta)$ *in the case of finite* $\operatorname{supp}(\eta)$*;*

(v) *The distribution* $F_{S(\eta)}$ *is light- tailed if* $\sup\limits_{k \ge 1} \mathbb{E}\, \mathrm{e}^{\lambda \xi_k} < \infty$ *for some* $\lambda > 0$ *and* $F_\eta \in \mathcal{H}^c$*.*

In the i.i.d. case, Theorem 5.1.1 immediately leads to following corollaries. It is worth noting that the first two corollaries can be found in the monograph [31] as Problems 2.12 and 2.13.

Corollary 5.1.1. *Let* $\xi_1, \xi_2, \ldots$ *be i.i.d. real-valued r.v.s with a common distribution* F_{ξ_1}*, and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$*. If* $F_{\xi_1} \in \mathcal{H}^c$ *and* $F_\eta \in \mathcal{H}^c$ *then* $F_{S_\eta} \in \mathcal{H}^c$*.*

Corollary 5.1.2. *Let* $\xi_1, \xi_2, \ldots$ *be i.i.d. nonnegative not degenerate at zero r.v.s, and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$*. If* $F_\eta \in \mathcal{H}$ *then* $F_{S_\eta} \in \mathcal{H}$*.*

Corollary 5.1.3. *Let* $\xi_1, \xi_2, \ldots$ *be i.i.d. real-valued r.v.s with common distribution* F_{ξ_1}*, and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$*. If* $F_{\xi_1} \in \mathcal{H}$ *then* $F_{S_\eta} \in \mathcal{H}$*.*

Similar corollaries can be formulated for randomly stopped minima and maxima.

5.2 Examples

In this subsection, we present two examples that demonstrate how heavy-tailed distributions can be constructed using the randomly stopped structures discussed above.

Example 5.2.1. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent r.v.s such that the first member ξ_1 has the Pareto distribution*

$$F_{\xi_1}(x) = (1 - \frac{1}{(1+x)^3})\mathbb{1}_{[0,\infty)}(x)$$

and other elements of the sequence are identically exponentially distributed:

$$F_{\xi_k}(x) = \left(1 - \mathrm{e}^{-x}\right)\mathbb{1}_{[0,\infty)}(x) \quad k \in \{2, 3, \ldots\}$$

According to Theorem 5.1.1 (Parts (iii) and (iv)) and Theorem 5.1.5 (iii) distributions F_{S_η} and $F_{S(\eta)}$ are heavy-tailed for any counting r.v. independent of the sequence $\{\xi_1, \xi_2, \ldots\}$. This is due to the fact that the first primary distributions has a significantly heavier tail than the other elements of the infinite primary sequence. For instance, in the case of a discrete uniform counting r.v. with the parameter $N \geq 2$:

$\mathbb{P}(\eta = n) = \frac{1}{N}$, $n = 1, 2, ...N$.

In this case we have:

- If $k = 1$ and $x > 0$, then

$$\overline{F}_{S_1}(x) = \overline{F}_{\xi_1}(x) = \frac{1}{(1+x)^3},$$

- If $k = 2$ and $x > 0$, then

$$\begin{aligned}
\overline{F}_{S_2}(x) &= \mathbb{P}(\xi_1 + \xi_2 > x, \xi_2 > x) + \mathbb{P}(\xi_1 + \xi_2 > x, \xi_2 \leq x) \\
&= \overline{F}_{\xi_2}(x) + \int_0^x \overline{F}_{\xi_1}(x-y)\mathrm{d}F_{\xi_2}(y) \\
&= \overline{F}_{\xi_2}(x) - \int_0^x \overline{F}_{\xi_1}(x-y)\mathrm{d}\overline{F}_{\xi_2}(y) \\
&= \overline{F}_{\xi_2}(x) - \overline{F}_{\xi_1}(x-y)\overline{F}_{\xi_2}(y)\ \Big|_0^x \\
&+ \int_0^x \overline{F}_{\xi_2}(y)\mathrm{d}\overline{F}_{\xi_1}(x-y) \\
&= \overline{F}_{\xi_2}(x) - \overline{F}_{\xi_2}(x) + \overline{F}_{\xi_1}(x) + \int_0^x \overline{F}_{\xi_2}(y)\mathrm{d}\overline{F}_{\xi_1}(x-y) \\
&= \frac{1}{(1+x)^3} + \int_0^x \mathrm{e}^{-y}\ \mathrm{d}\frac{1}{(1+(x-y))^3}.
\end{aligned}$$

- If $k > 2$ and $x > 0$, then similarly

$$\begin{aligned}\overline{F}_{S_k}(x) &= \mathbb{P}(\xi_1 + \xi_2 + ... + \xi_k > x) = \mathbb{P}(\xi_1 + \xi_2 + ... + \xi_k > x, \xi_1 > x)\\ &+ \mathbb{P}(\xi_1 + \xi_2 + ... + \xi_k > x, \xi_1 \leq x)\\ &= \overline{F}_{\xi_1}(x) + \int\limits_0^x \overline{F}_{\xi_2+...+\xi_k}(y) \mathrm{d}F_{\xi_1}(x-y)\\ &= \frac{1}{(1+x)^3} + \sum_{j=1}^{k-1} \int\limits_0^x \frac{\mathrm{e}^{-y} y^j}{(j-1)!} \, \mathrm{d}\frac{1}{(1+(x-y))^3}.\end{aligned}$$

Therefore, in the case of uniform discrete r.v. with parameter $N \geq 2$ we have that distribution with the tail:

$$\begin{aligned}\overline{F}_{S^{(\eta)}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{S^{(n)}}(x) \mathbb{P}(\eta = n) = \sum_{n=1}^{\infty} \mathbb{P}\Big(\bigcup_{k=1}^{n} \{S_k > x\} \Big) \mathbb{P}(\eta = n)\\ &= \frac{1}{N} \sum_{k=1}^{n} \overline{F}_{S_k} = \frac{1}{N} \Bigg(\sum_{k=1}^{N} \Bigg(\frac{1}{(1+x)^3}\\ &+ \sum_{j=1}^{k-1} \int\limits_0^x \frac{\mathrm{e}^{-y} y^j}{(j-1)!} \, \mathrm{d}\frac{1}{(1+(x-y))^3} \Bigg) \Bigg)\end{aligned}$$

$$\begin{aligned}&= \frac{1}{(1+x)^3} + \frac{1}{N} \sum_{k=1}^{N} \sum_{j=1}^{k-1} \frac{1}{(j-1)!} \int\limits_0^x \mathrm{e}^{-y} y^j \, \mathrm{d}\frac{1}{(1+(x-y))^3}\\ &= \frac{1}{(1+x)^3} + \frac{1}{N} \sum_{k=1}^{N-1} \frac{N-k}{(k-1)!} \int\limits_0^x \mathrm{e}^{-y} y^k \, \mathrm{d}\frac{1}{(1+(x-y))^3}.\end{aligned}$$

belong to the class $\mathcal{H}$. Theorem 5.1.2 (ii) implies that the distribution $F_{\xi^{(\eta)}}$ belongs to the class $\mathcal{H}$ for any counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$. Meanwhile Theorem 5.1.3 (i) and Theorem 5.1.4 (i) imply that $F_{\xi_{(\eta)}}$ and $F_{S_{(\eta)}}$ are heavy-tailed for a counting r.v. under condition $1 \in \operatorname{supp}(\eta)$.

In the case of the discrete uniform counting r.v. η with parameter $N = 3$, we have that $F_{S_{(\eta)}} = F_{\xi_1}$ and distributions with the following tails are also heavy-tailed:

$$
\begin{aligned}
\overline{F}_{\xi^{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\xi^{(n)} > x)\mathbb{P}(\eta = n) \\
&= \mathbb{I}_{(-\infty,0)}(x) + \Big(\frac{1}{(1+x)^3} \\
&+ \Big(\mathrm{e}^{-x} - \frac{\mathrm{e}^{-2x}}{3}\Big)\Big(1 - \frac{1}{(1+x)^3}\Big)\Big)\mathbb{I}_{[0,\infty)}(x). \\
\overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\min\{\xi_1, \ldots, \xi_n\} > x)\mathbb{P}(\eta = n) \\
&= \mathbb{I}_{(-\infty,0)}(x) + \frac{1}{3(1+x)^3}(1 + \mathrm{e}^{-x} + \mathrm{e}^{-2x})\mathbb{I}_{[0,\infty)}(x).
\end{aligned}
$$

Complete calculations are performed in subsection 2.1.

EXAMPLE 5.2.2. *Let* $\{\xi_1, \xi_2, \ldots\}$ *be a sequence of independent r.v.s uniformly distributed on the interval* $[0, 1]$*, i.e.*

$$
F_{\xi_k}(x) = x\mathbb{I}_{[0,1)}(x) + \mathbb{I}_{[1,\infty)}(x)
$$

for each $k \in \mathbb{N}$*.*

Obviously,

$$
\mathbb{E}\,\mathrm{e}^{\lambda\xi_k} = \int_0^1 \mathrm{e}^{\lambda \mathrm{x}}\mathrm{dx} = \frac{\mathrm{e}^{\lambda} - 1}{\lambda} > 1
$$

for any $\lambda > 0$ and all $k \in \mathbb{N}$. Therefore, by Theorem 5.1.1 (i) and Theorem 5.1.5 (i) we get that distributions F_{S_η} and $F_{S^{(\eta)}}$ are heavy-tailed for an arbitrary heavy tailed counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$. Suppose that a counting r.v. η is distributed according to the zeta distribution with parameter 2:

$$
\mathbb{P}(\eta = n) = \frac{1}{n^2}\frac{1}{\zeta(2)}, \quad n \in \mathbb{N},
$$

where

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \; s \in \mathbb{C},
$$

denotes the Riemann zeta function. Such η is heavy-tailed. Theorems 5.1.1 (i) and 5.1.5 (i) imply that the distribution

$$
F_{S_\eta}(x) = F_{S^{(\eta)}}(x) = \frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{1}{n^2} F_{\xi_1}^{*n}(x)\mathbb{I}_{[0,n]}(x)
$$

belongs to the class $\mathcal{H}$, where

$$F_{\xi_1}^{*n}(x) = \frac{1}{n!}\sum_{k=0}^{\lfloor x \rfloor}(-1)^k \binom{n}{k}(x-k)^n$$

is the well-known Irwin-Hall distribution with the parameter n, see [35, 36] or subsection 26.9 in [37]. Meanwhile Theorem 5.1.3 (ii) and 5.1.4 (ii) imply that distributions with tails

$$\overline{F}_{S^{(\eta)}}(x) = \overline{F_{\xi_1}}(x),$$

$$\overline{F}_{\xi_{(\eta)}}(x) = \mathbb{I}_{(-\infty,0)}(x) + \frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{1}{n^2}(1-x)^n \mathbb{I}_{[0,1)}(x)$$

are light-tailed despite the fact that the counting r.v. η distributed according to the zeta distribution is heavy-tailed.

Example 5.2.3. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent r.v.s distributed according to the Burr type XII law, i.e.*

$$F_{\xi_k}(x) = \left(1 - \Big(\frac{1}{1+\sqrt{kx}}\Big)^{3/2}\right)\mathbb{I}_{[0,\infty)}(x),\ k = 1, 2, \ldots,$$

and let the counting r.v. η be independent of $\{\xi_1, \xi_2, \ldots\}$ and distributed according to the shifted Poisson law, i.e.

$$\mathbb{P}(\eta = k) = \frac{1}{\mathrm{e}(k-3)!},\ k = 3, 4, \ldots. \tag{5.2.1}$$

Since $F_{\xi_1} \in \mathcal{H}$ and

$$\liminf_{x\to\infty}\min_{1\leq k\leq 3}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} = \min_{1\leq k\leq 3}\Big(\frac{1}{\sqrt{k}}\Big)^{3/2} = 3^{-3/4} > 0,$$

we get from Theorem 5.1.3(i) that $F_{\xi_{(\eta)}} \in \mathcal{H}$ and

$$\overline{F}_{\xi_{(\eta)}}(x) \underset{x\to\infty}{\sim} \frac{1}{\mathrm{e}}\overline{F}_{\xi_{(3)}}(x) \tag{5.2.2}$$

with

$$\begin{aligned}
\overline{F}_{\xi_{(3)}}(x) &= \mathbb{P}\big(\min\{\xi_1, \xi_2, \xi_3\} > x\big) = \prod_{k=1}^{3}\overline{F}_{\xi_k}(x) \\
&= \left(\frac{1}{(1+\sqrt{x})(1+\sqrt{2x})(1+\sqrt{3x})}\right)^{3/2}, \\
\overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty}\mathbb{P}\big(\min\{\xi_1, \ldots, \xi_n\} > x\big)\mathbb{P}(\eta = n) \\
&= \sum_{n=1}^{\infty}\mathbb{P}(\eta = n)\prod_{k=1}^{n}\overline{F}_{\xi_k}(x) \\
&= \frac{1}{\mathrm{e}}\sum_{n=3}^{\infty}\frac{1}{(n-3)!}\prod_{k=1}^{n}\Big(\frac{1}{1+\sqrt{kx}}\Big)^{3/2}.
\end{aligned}$$

A graphical representation of the asymptotic (5.2.2) is shown in Figure 11.

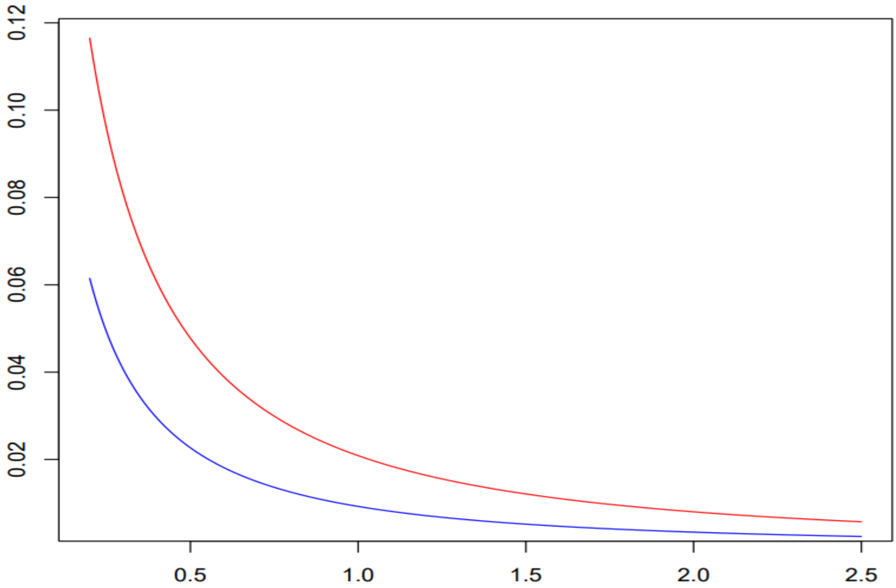

Figure 11: *Comparison of tails* $\overline{F}_{\xi_{(\eta)}}$ *(blue line) and* $\overline{F}_{\xi_{(3)}}$ *(red line) from example* 5.2.3

We note that Theorem 5.1.3(i) can also be applied to other Burr type XII distributions whose distribution functions have the form

$$F(x) = \left(1 - \left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{-\gamma}\right) \mathbb{I}_{[0,\infty)}(x),$$

where α, β, γ are positive parameters, see [67] for instance.

EXAMPLE 5.2.4. *Let* $\{\xi_1, \xi_2, \ldots\}$ *be a sequence of independent r.v.s such that* F_{ξ_1} *is distributed according to the Weibull law with the scale parameter* 1 *and the shape parameter* $1/2$*, i.e.*

$$\overline{F}_{\xi_1}(x) = \mathbb{I}_{(-\infty,0)}(x) + \mathrm{e}^{-\sqrt{x}}\, \mathbb{I}_{[0,\infty)}(x).$$

Since $F_{\xi_1} \in \mathcal{H}$, due to the Theorem 5.1.4(i) we get that d.f. of the randomly stopped minimum of sums $F_{S_{(\eta)}}$ is heavy-tailed and

$$\overline{F}_{S_{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F_{\xi_1}}(x)$$

if $\min\limits_{2 \le k \le \varkappa} \mathbb{P}(\xi_k \ge 0) > 0$ for $\varkappa = \min\{\mathrm{supp}(\eta) \setminus \{0\}\}$.

For example, if

$$\mathbb{P}(\xi_k = -1) = \mathbb{P}(\xi_k = 1) = \frac{1}{2},\ k \in \{2, 3, \ldots\},$$

and η is distributed according to the shifted Poisson law (5.2.1), then $F_{S_{(\eta)}} \in \mathcal{H}$ and

$$\overline{F}_{S_{(\eta)}}(x) \underset{x\to\infty}{\asymp} \mathrm{e}^{-\sqrt{x}}.$$

A graphical representation of the last relation is shown in Figure 12 having in mind that

$$\frac{1}{4\mathrm{e}}\,\mathrm{e}^{-\sqrt{x}} \le \overline{F}_{S_{(\eta)}}(x) \le \mathrm{e}^{-\sqrt{x}},\ x \ge 0,$$

and

$$\overline{F}_{S_{(\eta)}}(x) = \frac{1}{\mathrm{e}} \sum_{n=3}^{\infty} \mathbb{P}\Big(\bigcap_{k=1}^{n} \{S_k > x\}\Big) \frac{1}{(n-3)!} = \frac{1}{\mathrm{e}} \sum_{n=3}^{\infty} \frac{\Delta_n(x)}{(n-3)!},$$

where

$$\begin{aligned}\Delta_{2m}(x) &= \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m-1}{k} \Big(\mathrm{e}^{-\sqrt{x+2(m-k)-1}} \\ &\quad + \mathrm{e}^{-\sqrt{x+2(m-k)-2}}\Big),\ m \in \{2, 3, \ldots\};\end{aligned} \tag{5.2.3}$$

$$\begin{aligned}\Delta_{2m+1}(x) &= \frac{1}{2^{2m}} \sum_{k=0}^{m-1} \binom{2m}{k} \Big(\mathrm{e}^{-\sqrt{x+2(m-k)}} + \mathrm{e}^{-\sqrt{x+2(m-k)-1}}\Big) \\ &\quad + \frac{1}{2^{2m}} \binom{2m}{m} \mathrm{e}^{-\sqrt{x}},\ m \in \{1, 2, \ldots\}.\end{aligned} \tag{5.2.4}$$

The last formula follows from the consideration below. Namely,

$$\begin{aligned}
\Delta_3(x) &= \frac{1}{4}(\mathrm{e}^{-\sqrt{\mathrm{x}+2}} + \mathrm{e}^{-\sqrt{\mathrm{x}+1}}) + \frac{2}{4}\mathrm{e}^{-\sqrt{\mathrm{x}}},\\
\Delta_4(x) &= \frac{1}{8}(\mathrm{e}^{-\sqrt{\mathrm{x}+3}} + \mathrm{e}^{-\sqrt{\mathrm{x}+2}} + 3\mathrm{e}^{-\sqrt{\mathrm{x}+1}} + 3\mathrm{e}^{-\sqrt{\mathrm{x}}}),\\
\Delta_5(x) &= \frac{1}{16}(\mathrm{e}^{-\sqrt{\mathrm{x}+4}} + \mathrm{e}^{-\sqrt{\mathrm{x}+3}} + 4\mathrm{e}^{-\sqrt{\mathrm{x}+2}} + 4\mathrm{e}^{-\sqrt{\mathrm{x}+1}}) + \frac{6}{16}\mathrm{e}^{-\sqrt{\mathrm{x}}},\\
\Delta_6(x) &= \frac{1}{32}(\mathrm{e}^{-\sqrt{\mathrm{x}+5}} + \mathrm{e}^{-\sqrt{\mathrm{x}+4}} + 5\mathrm{e}^{-\sqrt{\mathrm{x}+3}} + 5\mathrm{e}^{-\sqrt{\mathrm{x}+2}} + 10\mathrm{e}^{-\sqrt{\mathrm{x}+1}} + 10\mathrm{e}^{-\sqrt{\mathrm{x}}}),
\end{aligned}$$

and in the general case we get (5.2.3) and (5.2.4).

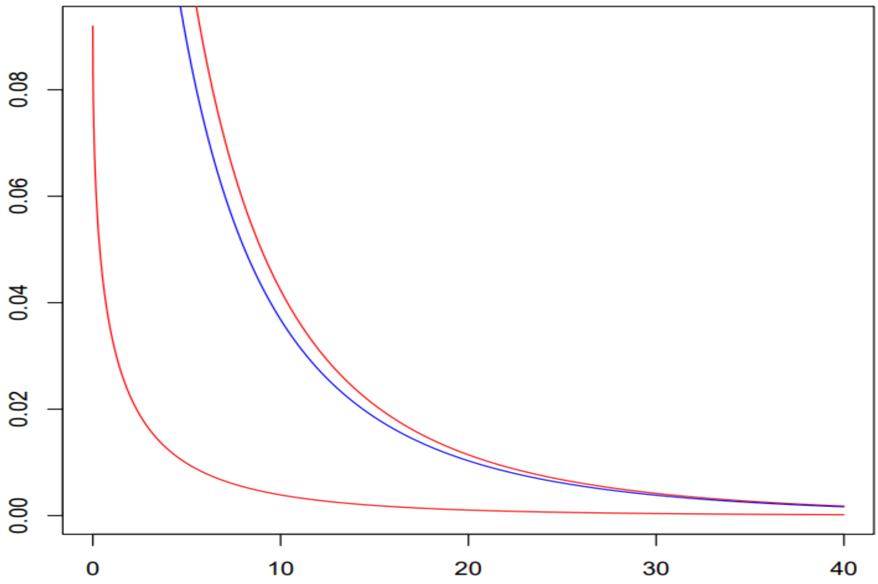

Figure 12: *Tail of d.f.* $\overline{F}_{S_{(\eta)}}$ *(blue line) and its bounds (red lines) from example* 5.2.4

5.3 Auxiliary results for Theorems 5.1.1-5.1.5

We formulate two lemmas that were used in the proofs of several main theorems. Although the results of the lemmas are well-known and can be found, e.g., in [31], [50], [57], we provide the proofs for the sake of convenience. the first lemma provides equivalent conditions for the distribution F to be classified as heavy or light tailed.

Lemma 5.3.1. *Suppose that F is the d.f. of a real-valued r.v. The following statements are equivalent:*

(i) *F is heavy-tailed*, i.e. $\int\limits_{-\infty}^{\infty} \mathrm{e}^{\lambda x} \mathrm{d}F_\xi(x) = \infty$ *for any* $\lambda > 0$,

(ii) $\limsup\limits_{x\to\infty} \mathrm{e}^{\lambda x}\overline{F}(x) = \infty$ *for any* $\lambda > 0$,

(iii) $\limsup\limits_{x\to\infty} x^{-1} \log \overline{F}(x) = 0$.

Similarly, the following statements are equivalent:

(i') *F is light-tailed*, i.e. $\int\limits_{-\infty}^{\infty} \mathrm{e}^{\lambda x} \mathrm{d}F_\xi(x) < \infty$ *for some* $\lambda > 0$,

(ii') $\limsup\limits_{x\to\infty} \mathrm{e}^{\lambda x}\,\overline{F}(x) < \infty$ *for some* $\lambda > 0$,

(iii') $\limsup\limits_{x\to\infty} x^{-1} \log \overline{F}(x) < 0$.

Proof. We prove only the first part of the lemma.

(i) $\Rightarrow$ (iii). Suppose that $\widehat{F}(\lambda) = \infty$ for any $\lambda > 0$. Let, on the contrary,

$$\limsup_{x\to\infty} \frac{\log \overline{F}(x)}{x} < 0.$$

Then there exist constants $c > 0$ and $x_c > 0$ such that $x^{-1} \log \overline{F}(x) \le -c$ for $x \ge x_c$ or, equivalently,

$$\overline{F}(x) \le \mathrm{e}^{-cx}, \quad x \ge x_c. \tag{5.3.1}$$

For any $\delta \in (0, c)$, using (5.3.1) and the alternative expectation formula (see [53], for instance), we obtain

$$\begin{aligned}
\int_{[0,\infty)} \mathrm{e}^{\delta u} \mathrm{d}F(u) &= 1 + \delta \int_0^\infty \mathrm{e}^{\delta u} \overline{F}(u) \mathrm{d}u \\
&= 1 + \left(\int_1^{\mathrm{e}^{\delta x_c}} + \int_{\mathrm{e}^{\delta x_c}}^\infty \right) \overline{F}\left(\delta^{-1} \log u\right) \mathrm{d}u \\
&\le \mathrm{e}^{\delta x_c} + \int_{\mathrm{e}^{\delta x_c}}^\infty \mathrm{e}^{-c\delta^{-1} \log u} \mathrm{d}u \\
&= \mathrm{e}^{\delta x_c} + \int_{\mathrm{e}^{\delta x_c}}^\infty u^{-c\delta^{-1}} \mathrm{d}u.
\end{aligned}$$

Since $c\delta^{-1} > 1$, the last integral is finite, hence

$$\widehat{F}(\delta) \le F(0) + \int_{[0,\infty)} \mathrm{e}^{\delta u} \mathrm{d}F(u) \ < \ \infty,$$

leading to a contradiction.

(iii) $\Rightarrow$ (ii). From the condition

$$\limsup_{x\to\infty} x^{-1} \log \overline{F}(x) = 0$$

we deduce that there exists an infinitely increasing sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} x_n^{-1} \log \overline{F}(x_n) = 0.$$

For any given $\lambda > 0$, this implies that there exists $n_\lambda \ge 1$ such that

$$x_n^{-1} \log \overline{F}(x_n) \ge -\lambda/2$$

for all $n \ge n_\lambda$. Equivalently,

$$\mathrm{e}^{\lambda x_n} \overline{F}(x_n) \ge \mathrm{e}^{\lambda x_n/2}, \quad n \ge n_\lambda.$$

Hence, $\mathrm{e}^{\lambda x_n} \overline{F}(x_n)$ tends to infinity as $n \to \infty$, and thus

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x} \overline{F}(x) \ge \lim_{n\to\infty} \mathrm{e}^{\lambda x_n} \overline{F}(x_n) = \infty.$$

Since this holds for any $\lambda > 0$, we have (ii).
(ii) $\Rightarrow$ (i). Let

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\overline{F}(x) = \infty$$

for any $\lambda > 0$. For any $x \in \mathbb{R}$ we can write

$$\int\limits_{-\infty}^{\infty} \mathrm{e}^{\lambda u}\mathrm{d}F(u) \geq \int\limits_{(x,\infty)} \mathrm{e}^{\lambda u}\mathrm{d}F(u) \geq \mathrm{e}^{\lambda x}\overline{F}(x).$$

Thus,

$$\widehat{F}(\lambda) \geq \limsup_{x\to\infty} \mathrm{e}^{\lambda x}\overline{F}(x) = \infty \quad \text{for any} \quad \lambda > 0.$$

Lemma 5.3.1 has been proved. □

The next lemma implies that $\mathcal{H}$ and $\mathcal{H}^c$ are closed with respect to weak tail equivalence.

Lemma 5.3.2. *Let F and G be two distributions of real-valued r.v.s.*
(i) *If $F \in \mathcal{H}$ and*

$$\liminf_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} > 0, \tag{5.3.2}$$

then $G \in \mathcal{H}$.
(ii) *If $F \in \mathcal{H}^c$, and $\overline{G}(x) \leq \tilde{c}\,\overline{F}(x)$ for some $\tilde{c} > 0$ and large x ($x > x_{\tilde{c}}$), then $G \in \mathcal{H}^c$.*

Proof. Consider Part (i). By condition (5.3.2) we obtain that

$$\overline{G}(x) \geq \hat{c}\,\overline{F}(x)$$

for some $\hat{c}$ and sufficiently large x ($x > x_{\hat{c}}$). Therefore,

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\,\overline{G}(x) \geq \hat{c}\,\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\,\overline{F}(x) = \infty$$

for any positive λ implying $G \in \mathcal{H}$ by definition.
The proof of Part (ii) can be constructed in a similar way by using Lemma 5.3.1 (ii') showing that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\,\overline{G}(x) < \infty$$

for some $\lambda > 0$. Lemma 5.3.2 has been proved. □

5.4 Proofs of Theorems 5.1.1-5.1.5

In this subsection, we present the proofs of all the main propositions. The proof of each proposition is assigned to a separate subsubsection.

5.4.1 Proof of Theorem 5.1.1

Proof of Part (i). For any $\lambda > 0$ and an arbitrary $K \geq 1$, we have

$$\begin{aligned}
\mathbb{E}e^{\lambda S_\eta} &= \mathbb{E}\left(e^{\lambda S_\eta}\sum_{n=0}^{\infty}\mathbb{I}_{\{\eta=n\}}\right) = \mathbb{E}\left(\sum_{n=0}^{\infty}e^{\lambda S_n}\mathbb{I}_{\{\eta=n\}}\right) \\
&\geq \mathbb{E}\left(\sum_{n=0}^{K}e^{\lambda S_n}\mathbb{I}_{\{\eta=n\}}\right) \\
&= \sum_{n=0}^{K}\mathbb{E}\,e^{\lambda S_n}\mathbb{P}(\eta=n) \qquad (5.4.1)
\end{aligned}$$

From the condition

$$\inf_{k\geq 1}\mathbb{E}\,e^{\lambda\xi_k} > 1$$

we derive that the estimate

$$\mathbb{E}\,e^{\lambda\xi_k} \geq \Delta$$

holds for some $\Delta = \Delta(\lambda) > 1$ and all $k \geq 1$. Therefore, for all $n \in \{1,\ldots,K\}$ we obtain

$$\mathbb{E}\,e^{\lambda S_n} = \prod_{k=1}^{n}\mathbb{E}\,e^{\lambda\xi_k} \geq \Delta^n. \qquad (5.4.2)$$

This together with (5.4.1) imply that

$$\mathbb{E}\,e^{\lambda S_\eta} \geq \sum_{n=0}^{K}\Delta^n\mathbb{P}(\eta=n).$$

Since $F_\eta \in \mathcal{H}$, we have

$$\sum_{n=0}^{K}\Delta^n\mathbb{P}(\eta=n) = \mathbb{E}\,e^{\eta\log\Delta}\mathbb{I}_{\{\eta\leq K\}} \underset{K\to\infty}{\to} \infty.$$

Hence, $\mathbb{E}\,e^{\lambda S_\eta} = \infty$ implying $F_{S_\eta} \in \mathcal{H}$ by definition. Part (i) of the theorem has been proved. □

Proof of Part (ii). Let us fix an arbitrary $\lambda > 0$. Due to the conditions of part (ii), for such λ we have

$$\begin{aligned}
\inf_{k\geq 1}\mathbb{E}\ e^{\lambda\xi_k} &= \inf_{k\geq 1}\mathbb{E}\ \left(e^{\lambda\xi_k}\mathbb{I}_{\{\xi_k\geq a\}} + e^{\lambda\xi_k}\mathbb{I}_{\{\xi_k<a\}}\right) \\
&\geq \inf_{k\geq 1}\mathbb{E}\ e^{\lambda\xi_k}\mathbb{I}_{\{\xi_k\geq a\}} \\
&\geq \inf_{k\geq 1}e^{\lambda a}\,\mathbb{P}(\xi_k\geq a) \\
&= e^{\lambda a} > 1.
\end{aligned}$$

Hence the assertion of Part (ii) follows from Part (i) of the theorem. □

Proof of Part (iii). The requirement $\overline{F_\eta}(x) > 0$ for all $x \in \mathbb{R}$ implies that a counting r.v. η has an unbounded support. Thus we can find $K \geq \varkappa$ such that $\mathbb{P}(\eta = K) > 0$. Let λ be any positive number and $M \geq 1$. Then

$$\begin{aligned}\mathbb{E}\,\mathrm{e}^{\lambda S_K} &\geq \mathbb{E}\exp\left\{\lambda\sum_{k=1}^{K}\xi_k \mathbb{I}_{\{\xi_k\leq M\}}\right\}\\ &= \mathbb{E}\,\mathrm{e}^{\lambda\xi_\varkappa \mathbf{I}_{\{\xi_\varkappa\leq M\}}}\prod_{\substack{k=1\\k\neq\varkappa}}^{K}\mathbb{E}\,\mathrm{e}^{\lambda\xi_k \mathbf{I}_{\{\xi_k\leq M\}}} \underset{M\to\infty}{\to}\infty\end{aligned}$$

because $F_\varkappa \in \mathcal{H}$ and $\mathbb{E}\mathrm{e}^{\lambda\xi_k} > 0$ for each $k \in \{1,\ldots,K\}$. Therefore, $F_{S_K} \in \mathcal{H}$. By representation (5.4.1) we obtain that

$$\mathbb{E}\,\mathrm{e}^{\lambda S_\eta} \geq \mathbb{P}(\eta = K)\mathbb{E}\,\mathrm{e}^{\lambda S_K}$$

which implies that $F_{S_\eta} \in \mathcal{H}$. This completes the proof of Part (iii) of the theorem. □

Proof of Part (iv). Let K be such that $\mathbb{P}(\eta = K) > 0$ and $\varkappa \leq K$. Clearly, the conditions of Part (iv) imply the existence of such K. To finish the proof of this part, it is sufficient to repeat the arguments of Part (iii). □

Proof of Part (v). Suppose that $0 < \delta \leq \lambda$, and $\lambda > 0$ is such that $\mathbb{E}\,\mathrm{e}^{\lambda\xi_1^+} < \infty$ with $\xi_1^+ := \xi_1\mathbb{I}_{\{\xi_1\geq 0\}}$.
By the standard representation (5.4.1) we have

$$\begin{aligned}\mathbb{E}\,\mathrm{e}^{\delta S_\eta} &= \sum_{n=0}^{\infty}\mathbb{E}\,\mathrm{e}^{\delta S_n}\mathbb{P}(\eta = n)\\ &\leq \sum_{n=0}^{\infty}\mathbb{E}\,\mathrm{e}^{\delta S_n^{(+)}}\mathbb{P}(\eta = n),\end{aligned} \tag{5.4.3}$$

where $S_0^{(+)} = 0$ and

$$S_n^{(+)} = \sum_{k=1}^{n}\xi_k^+ = \sum_{k=1}^{n}\xi_k\mathbb{I}_{\{\xi_k\geq 0\}},\ n \in \{1,2,\ldots\}.$$

Condition (5.1.1) and the requirement

$$\overline{F}_{\xi_1}(x) > 0, x \in \mathbb{R}$$

imply that

$$\overline{F_{\xi_k}}(x) \leq c_1\overline{F_{\xi_1}}(x) \tag{5.4.4}$$

for some $c_1 > 0$, all $k \geq 1$ and all $x \in \mathbb{R}$. Therefore, by the alternative expectation formula (see, for instance, [19]), we derive from (5.4.4) that

$$\begin{aligned}
\mathbb{E}\,\mathrm{e}^{\delta\xi_k^+} &= 1 + \delta \int_0^\infty \mathrm{e}^{\delta u}\, \overline{F_{\xi_k^+}}(u)\,\mathrm{d}u \\
&\leq 1 + \delta c_1 \int_0^\infty \mathrm{e}^{\lambda u}\, \overline{F_{\xi_1}}(u)\,\mathrm{d}u \\
&\leq 1 + \frac{\delta}{\lambda} c_1 \Big(\mathbb{E}\,\mathrm{e}^{\lambda X_1^+} - 1\Big) := c_2(\delta)
\end{aligned}$$

for any $k \geq 1$, where $1 < c_2(\delta) < \infty$ for $0 < \delta \leq \lambda$, and

$$\lim_{\delta \downarrow 0} c_2(\delta) = 1.$$

Since $\xi_1^+, \xi_2^+, \ldots$ are independent r.v.s, we obtain

$$\mathbb{E}\,\mathrm{e}^{\delta S_n^{(+)}} = \prod_{k=1}^n \mathbb{E}\,\mathrm{e}^{\delta\xi_k^+} \leq \big(c_2(\delta)\big)^n.$$

Hence, by inequality (5.4.3) and the condition $F_\eta \in \mathcal{H}^c$ we derive that

$$\mathbb{E}\,\mathrm{e}^{\delta S_\eta} \leq \sum_{n=0}^\infty \big(c_2(\delta)\big)^n\, \mathbb{P}(\eta = n) = \mathbb{E}\,\mathrm{e}^{\eta \log c_2(\delta)} < \infty$$

if $\delta \in (0, \lambda]$ is chosen to be sufficiently small. This implies that $F_{S_\eta} \in \mathcal{H}^c$. □

Proof of Part (vi). The statement of this part can be proved in manner similar to that of Part (v). Namely, the conditions of part (vi) imply that

$$\sup_{k \geq 1} \mathrm{E}\,\mathrm{e}^{\lambda\xi_k^+} = c_\lambda$$

for some constants $\lambda > 0$ and $c_\lambda \geq 1$. Therefore, using the alternative expectation formula, we derive

$$\begin{aligned}
\mathrm{E}\,\mathrm{e}^{\delta\xi_k^+} &= 1 + \delta \int_{[0,\infty)} \mathrm{e}^{\delta u} \overline{F}_{\xi_k}(u)\,\mathrm{d}u \\
&\leq 1 + \frac{\delta}{\lambda}\Big(\lambda \int_{[0,\infty)} \mathrm{e}^{\lambda u} \overline{F}_{\xi_k}(u)\,\mathrm{d}u\Big) \\
&= 1 + \frac{\delta}{\lambda}(c_\lambda - 1)
\end{aligned}$$

for all $\delta \in (0, \lambda)$ and $k \geq 1$. The last estimation and inequality (5.4.3) imply that

$$\mathrm{E}\,\mathrm{e}^{\delta S_\eta} \leq \sum_{n=0}^\infty \prod_{k=1}^n \mathrm{E}\,\mathrm{e}^{\delta\xi_k^+}\, \mathbb{P}(\eta = n) \leq \mathrm{E}\,\mathrm{e}^{\eta \log\left(1 + \frac{\delta}{\lambda}(c_\lambda - 1)\right)}.$$

If $\delta \in (0, \lambda]$ is sufficiently small, then the last expectation is finite because of $F_\eta \in \mathcal{H}^c$. Hence $F_{S_\eta} \in \mathcal{H}^c$ as well. Part (vi) of the theorem has been proved. □

5.4.2 Proof of Theorem 5.1.2

Proof of Part (i). By the standard representation we have

$$\overline{F}_{\xi^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\xi^{(n)} > x)\mathbb{P}(\eta = n)$$
$$\geq \mathbb{P}(\xi^{(K)} > x)\mathbb{P}(\eta = K) \tag{5.4.5}$$

for $x > 0$ and any K such that $\mathbb{P}(\eta = K) > 0$, $K \geq \varkappa$. Due to the conditions of Part (i) there exists a sequence of numbers K with the above property. Obviously,

$$\mathbb{P}(\xi^{(K)} > x) = \mathbb{P}(\max\{0, \xi_1, \ldots, \xi_K\} > x)$$
$$\geq \mathbb{P}(\xi_\varkappa > x). \tag{5.4.6}$$

Consequently, for an arbitrary $\lambda > 0$, we get from (5.4.5) and (5.4.6)

$$\limsup_{x\to\infty} e^{\lambda x}\, \overline{F}_{\xi^{(\eta)}}(x) \geq \mathbb{P}(\eta = K) \limsup_{x\to\infty} e^{\lambda x}\, \overline{F}_{\xi_\varkappa}(x).$$

The assertion of Part (i) follows now from Lemma 5.3.1. □

Proof of part (ii). The proof of this part is similar to the proof of part (i), because the conditions of part (ii) imply that there exists at least one K such that $K \geq \varkappa$ and $\mathbb{P}(\eta = K) > 0$. □

Proof of part (iii). The standard representation implies that

$$\overline{F}_{\xi^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\xi^{(n)} > x)\mathbb{P}(\eta = n)$$
$$= \sum_{n=1}^{\infty} \mathbb{P}\Big(\bigcup_{k=1}^{n} \{\xi_k > x\}\Big)\mathbb{P}(\eta = n)$$
$$\leq \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \sum_{k=1}^{n} \overline{F}_{\xi_k}(x) \tag{5.4.7}$$

for positive x.

Due to Lemma 5.3.1, there is $\lambda > 0$ such that

$$\limsup_{x\to\infty} e^{\lambda x}\, \overline{F}_{\xi_1}(x) < \infty. \tag{5.4.8}$$

It follows from the estimate (5.4.7) that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{\xi(\eta)}}(x) \le \limsup_{x\to\infty} \mathrm{e}^{\lambda x} \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \sum_{k=1}^{n} \overline{F}_{\xi_k}(x).$$

Condition (5.1.2) of part (iii) implies that

$$\sum_{k=1}^{n} \overline{F}_{\xi_k}(x) \le c_4 n \overline{F_{\xi_1}}(x) \tag{5.4.9}$$

for all $n \ge 1$, for some $c_4 > 0$ and for sufficiently large $x\,(x \ge x_1)$. Therefore, by (5.4.8) and (5.4.9) we get that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{\xi(\eta)}}(x) \le c_4 \mathbb{E}\eta \limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F}_{\xi_1}(x) < \infty.$$

The assertion of part (iii) follows now by Lemma 5.3.1. □

5.4.3 Proof of Theorem 5.1.3

Proof of Part (i). By the standard representation we have

$$\begin{aligned}
\overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\min\{\xi_1, \ldots, \xi_n\} > x)\mathbb{P}(\eta = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \prod_{k=1}^{n} \overline{F}_{\xi_k}(x) \\
&= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa) + \sum_{n=\varkappa+1}^{\infty} \mathbb{P}(\eta = n)\overline{F_{\xi_{(\varkappa)}}}(x) \prod_{k=\varkappa+1}^{n} \overline{F}_{\xi_k}(x) \\
&\le \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)\left(1 + \overline{F}_{\xi_{\varkappa+1}}(x)\frac{\mathbb{P}(\eta > \varkappa)}{\mathbb{P}(\eta = \varkappa)}\right),
\end{aligned} \tag{5.4.10}$$

and

$$\overline{F}_{\xi_{(\eta)}}(x) \ge \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)$$

for each positive x. In addition, the conditions of Part (i) imply that $\overline{F}_{\xi_{(\varkappa)}}(x) > 0$ for all positive x. Therefore

$$\overline{F}_{\xi_{(\eta)}}(x) \underset{x\to\infty}{\sim} \mathbb{P}(\eta = \varkappa)\overline{F}_{\xi_{(\varkappa)}}(x).$$

From this, by using Lemma 5.3.2, we conclude that $F_{\xi_{(\eta)}} \in \mathcal{H}$ if $F_{\xi_{(\varkappa)}} \in \mathcal{H}$. Hence, to prove the assertion of part (i) it is enough to prove that $F_{\xi_{(\varkappa)}} \in \mathcal{H}$ for $1 \le \varkappa \le \min\{\mathrm{supp}(\eta) \setminus \{0\}\}$.

Given the condition $F_{\xi_1} \in \mathcal{H}$ and applying Lemma 5.3.1 we have

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F}_{\xi_1}(x) = \infty \tag{5.4.11}$$

for an arbitrary $\lambda > 0$. The requirement

$$\liminf_{x\to\infty} \min_{1\leqslant k\leqslant \varkappa} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} > 0$$

implies that

$$\overline{F}_{\xi_k}(x) \geq c_5 \overline{F}_{\xi_1}(x)$$

for some positive c_5, sufficiently large x ($x \geq x_2$) and for all $1 \leq k \leq \varkappa$. Therefore, for any positive λ and large x ($x \geq x_2$) we obtain

$$\begin{aligned}
\mathrm{e}^{\lambda x}\, \overline{F}_{\xi_{(\varkappa)}}(x) &= \mathrm{e}^{\lambda x} \prod_{k=1}^{\varkappa} \overline{F}_{\xi_k}(x) \\
&\geq c_5^{\varkappa}\, \mathrm{e}^{\lambda x}\ (\overline{F}_{\xi_1}(x))^{\varkappa} \\
&= \left(c_5\, \mathrm{e}^{\lambda x/\varkappa}\ \overline{F}_{\xi_1}(x)\right)^{\varkappa}.
\end{aligned}$$

By relation (5.4.11) we derive that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F}_{\xi_{(\varkappa)}}(x) = \infty$$

implying that $F_{\xi_{(\varkappa)}} \in \mathcal{H}$. Part (i) of the proposition has been proved. □

Proof of Part (ii). According to inequality (5.4.10) and Lemma 5.3.2, $F_{\xi_{(\eta)}} \in \mathcal{H}^c$ if $F_{\xi_{(\varkappa)}} \in \mathcal{H}^c$. Since $\varkappa$ is finite, the conditions $F_{\xi_k} \in \mathcal{H}^c$, $k \in \{1, 2, \ldots, \varkappa\}$ and Lemma 5.3.1 imply that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F}_{\xi_k}(x) < \infty \tag{5.4.12}$$

for some $\lambda > 0$ and each $k \in \{1, 2, \ldots, \varkappa\}$. For this λ and an arbitrary positive x, we have

$$\mathrm{e}^{\lambda x}\, \overline{F}_{\xi_{(\varkappa)}}(x) = \prod_{k=1}^{\varkappa} \Big(\mathrm{e}^{\lambda x/\varkappa} \overline{F}_{\xi_k}(x)\Big).$$

Since $\lambda/\varkappa \leq \lambda$, due to (5.4.12),

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x/\varkappa}\, \overline{F}_{\xi_k}(x) < \infty$$

for each $k \in \{1, 2, \ldots, \varkappa\}$. Therefore,

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F}_{\xi_{(\varkappa)}}(x) < \infty$$

implying that $F_{\xi_{(\varkappa)}} \in \mathcal{H}^c$ by Lemma 5.3.1. Hence $F_{\xi_{(\eta)}} \in \mathcal{H}^c$ as well, and Part (ii) of the proposition has been proved. □

5.4.4 Proof of Theorem 5.1.4

Proof of Part (i). If $\varkappa = 1$, then for $x > 0$ we have

$$\begin{aligned}\overline{F}_{S_{(\eta)}}(x) &= \sum_{n \in \operatorname{supp}(\eta)\setminus\{0\}} \overline{F}_{S_{(n)}}(x)\,\mathbb{P}(\eta = n)\\ &\geq \overline{F}_{S_{(1)}}(x)\,\mathbb{P}(\eta = 1)\\ &= \overline{F}_{\xi_1}(x)\,\mathbb{P}(\eta = 1),\end{aligned}$$

and

$$\begin{aligned}\overline{F}_{S_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{S_{(n)}}(x)\,\mathbb{P}(\eta = n)\\ &= \sum_{n=1}^{\infty} \mathbb{P}\left(\min\{S_1, \ldots, S_n\} > x\right)\,\mathbb{P}(\eta = n)\\ &= \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcap_{k=1}^{n}\{S_k > x\}\right)\mathbb{P}(\eta = n)\\ &\leq \sum_{n=1}^{\infty} \mathbb{P}(S_1 > x)\,\mathbb{P}(\eta = n)\\ &= \overline{F}_{\xi_1}(x)\,\mathbb{P}(\eta \geq 1).\end{aligned}$$

The derived estimates imply the asymptotic relation (5.1.3) in the case $\varkappa = 1$.
Let us now suppose that $\varkappa > 1$. Due to the conditions of Part (i)

$$\mathbb{P}(\xi_k \geq 0) \geq c_6$$

for some $c_6 > 0$ and all $1 \leq k \leq \varkappa$. Hence by the standard decomposition we get that for positive x

$$\begin{aligned}\overline{F}_{S_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{S_{(n)}}(x)\,\mathbb{P}(\eta = n)\\ &\geq \overline{F}_{S_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)\\ &= \mathbb{P}\left(\min\{S_1, \ldots, S_\varkappa\} > x\right)\,\mathbb{P}(\eta = \varkappa)\\ &= \mathbb{P}\Big(\bigcap_{k=1}^{\varkappa}\{\xi_1 + \ldots + \xi_k > x\}\Big)\mathbb{P}(\eta = \varkappa)\\ &\geq \mathbb{P}\left(\xi_1 > x, \xi_2 \geq 0, \ldots, \xi_\varkappa \geq 0\right)\,\mathbb{P}(\eta = \varkappa)\\ &= \mathbb{P}(\xi_1 > x)\prod_{k=2}^{\varkappa}\mathbb{P}(\xi_k \geq 0)\mathbb{P}(\eta = \varkappa)\\ &\geq c_6^{\varkappa-1}\,\mathbb{P}(\eta = \varkappa)\,\overline{F}_{\xi_1}(x). \qquad (5.4.13)\end{aligned}$$

On the other hand, similar to the case when $\varkappa = 1$, we have

$$\begin{aligned}
\overline{F}_{S_{(\eta)}}(x) &= \sum_{n\,\in\,\mathrm{supp}(\eta)\setminus\{0\}} \mathbb{P}\Big(\bigcap_{k=1}^{n}\{S_k > x\}\Big)\mathbb{P}(\eta = n)\\
&\le \sum_{n\,\in\,\mathrm{supp}(\eta)\setminus\{0\}} \mathbb{P}(S_1 > x)\mathbb{P}(\eta = n)\\
&= \overline{F}_{\xi_1}(x)\mathbb{P}(\eta \ge \varkappa). \qquad (5.4.14)
\end{aligned}$$

Estimates (5.4.13) and (5.4.14) imply that the asymptotic relation (5.1.3) holds for any possible $\varkappa$. In addition, we observe that, by Lemma 5.3.2, the distribution $F_{S_{(\eta)}}$ belongs to $\mathcal{H}$ together with F_{ξ_1}. Part (i) of the theorem has been proved. □

Proof of Part (ii). The statement of this part follows immediately from the estimate (5.4.14) and Lemma 5.3.1 because

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\,\overline{F}_{S_{(\eta)}}(x) \le \mathbb{P}(\eta \ge 1) \limsup_{x\to\infty} \mathrm{e}^{\lambda x}\,\overline{F}_{\xi_1}(x)$$

for any $\lambda > 0$. □

5.4.5 Proof of Theorem 5.1.5

Proof of Part (i). The proof of this part is similar to the proof of Part (i) of Theorem 5.1.1. Namely, for $\lambda > 0$ and $K \ge 2$ by using (5.4.2), we get that

$$\begin{aligned}
\mathbb{E}\,\mathrm{e}^{\lambda S^{(\eta)}} &\ge \mathbb{E}\Big(\mathrm{e}^{\lambda S^{(\eta)}}\mathbb{I}_{\{\eta\le K\}}\Big)\\
&= \sum_{n=0}^{K} \mathbb{E}\,\mathrm{e}^{\lambda S^{(n)}}\mathbb{P}(\eta = n)\\
&\ge \sum_{n=0}^{K} \mathbb{E}\,\mathrm{e}^{\lambda S_n}\mathbb{P}(\eta = n)\\
&\ge \sum_{n=0}^{K} \Delta^n\mathbb{P}(\eta = n)\\
&= \mathbb{E}\left(\mathrm{e}^{\eta\log\Delta}\mathbb{I}_{\{\eta\le K\}}\right)
\end{aligned}$$

with $\Delta = \Delta(\lambda) = \inf_{k\ge1} \mathbb{E}\mathrm{e}^{\lambda\xi_k} > 1$. The condition $F_\eta \in \mathcal{H}$ implies that

$$\lim_{K\to\infty} \mathbb{E}\left(\mathrm{e}^{\eta\log\Delta}\mathbb{I}_{\{\eta\le K\}}\right) = \infty.$$

Therefore, $\mathbb{E}\mathrm{e}^{\lambda S^{(\eta)}} = \infty$ for an arbitrary $\lambda > 0$, i.e. $F_{S^{(\eta)}} \in \mathcal{H}$. Part (i) of the theorem has been proved. □

Proof of part (ii). The assertion of this part is obvious because condition $\inf_{k\geq 1} \mathbb{P}(\xi_k \geq a) = 1$ with $a > 0$ implies that $\inf_{k\geq 1} \mathbb{E}e^{\lambda \xi_k} > 1$ for any $\lambda > 0$. The details of this implication are presented in the proof of Theorem 5.1.1(ii). □

Proof of Part (iii). For positive x we have

$$
\begin{aligned}
\overline{F}_{S(\eta)}(x) &= \sum_{n=1}^{\infty} \overline{F_{S(n)}}(x)\mathbb{P}(\eta = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\Big(\bigcup_{k=1}^{n} \{S_k > x\}\Big)\mathbb{P}(\eta = n) \\
&\geq \sum_{n=1}^{\infty} \mathbb{P}(S_1 > x)\,\mathbb{P}(\eta = n) \\
&= \overline{F}_{\xi_1}(x)\,\mathbb{P}(\eta \geq 1). \qquad (5.4.15)
\end{aligned}
$$

The assertion of Part (iii) follows now from Lemma 5.3.1 because by (5.4.15)

$$
\limsup_{x\to\infty} e^{\lambda x}\,\overline{F}_{S(\eta)}(x) \geq \mathbb{P}(\eta \geq 1) \limsup_{x\to\infty} e^{\lambda x}\,\overline{F}_{\xi_1}(x)
$$

for an arbitrary positive λ. □

Proof of Part (iv). The conditions of this part and and Theorem 5.1.1 (parts (iii) and (iv)) imply that $F_{S_\eta} \in \mathcal{H}$. In addition, for positive x

$$
\begin{aligned}
\overline{F}_{S(\eta)}(x) &= \sum_{n=1}^{\infty} \mathbb{P}\big(\max\{S_1, S_2, \ldots, S_n\} > x\big)\mathbb{P}(\eta = n) \\
&\geq \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n) \\
&= \overline{F}_{S_\eta}(x).
\end{aligned}
$$

Hence $F_{S(\eta)} \in \mathcal{H}$ according to the Lemma 5.3.2. Part (iv) of the theorem has been proved. □

Proof of Part (v). Let $\lambda > 0$ be a positive number specified by the condition of part (v), i.e.

$$
\sup_{k\geq 1} \mathrm{E}\, e^{\lambda \xi_k} = \hat{c}_\lambda
$$

with some positive constant $\hat{c}_\lambda$. For this λ we have

$$
\begin{aligned}
\sup_{k\geq 1} \mathrm{E}\, e^{\lambda \xi_k^+} &= \sup_{k\geq 1} \mathbb{E}\Big(e^{\lambda \xi_k^+}\,\mathbb{1}_{\{\xi_k \geq 0\}} + e^{\lambda \xi_k^+}\,\mathbb{1}_{\{\xi_k < 0\}}\Big) \\
&= \sup_{k\geq 1} \mathbb{E}\Big(e^{\lambda \xi_k}\,\mathbb{1}_{\{\xi_k \geq 0\}} + \mathbb{1}_{\{\xi_k < 0\}}\Big) \\
&\leq \hat{c}_\lambda + 1,
\end{aligned}
$$

where $\xi_k^+ = \xi_k \mathbb{I}_{\{\xi_k \geq 0\}}$ for $k \in \{1, 2, \ldots, \}$. Due to Theorem 5.1.1(vi) d.f. $F_{S_\eta^{(+)}}$ belongs to the class $\mathcal{H}^c$ with r.v. $S_\eta^{(+)} = \xi_1^+ + \ldots + \xi_\eta^+$.

According to the standard representation, for positive x, we have

$$\begin{aligned}
\overline{F}_{S^{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\max\{S_1, S_2, \ldots, S_n\} > x)\mathbb{P}(\eta = n) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}(\max\{S_1^{(+)}, S_2^{(+)}, \ldots, S_n^{(+)}\} > x)\mathbb{P}(\eta = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(S_n^{(+)} > x)\mathbb{P}(\eta = n) \\
&= \overline{F}_{S_\eta^{(+)}}(x).
\end{aligned}$$

By applying Lemma 5.3.2 we get that d.f. $F_{S^{(\eta)}}$ is light-tailed due to the light tail of d.f. $F_{S_\eta^{(+)}}$. Part (v) of the theorem has been proved. □

6 Conclusions

In this thesis, we studied distribution functions obtained by randomly stopping minimum, maximum, minimum of sums and maximum of sums of random variables. Primary random variables are considered to be real-valued, independent and possibly differently distributed. The random variable defining the stopping moment is integer-valued, nonnegative and not degenerate at zero. We have found conditions when the distribution functions of these randomly stopped structures belong to the class of generalized subexponential distributions. The belonging of the distributions of randomly stopped structures to the class of generalized subexponential distributions can be determined either by primary random variables or by counting random variables. In this thesis, we have considered the case when a set of primary random variables has a decisive value. Our main results are formulated in Theorems 4.1.1–4.1.3. The primary random variables considered in all theorems can be differently distributed. However, additional conditions of all theorems are satisfied in the case where the primary random variables are identically distributed.

In the future, it would be interesting to study the case when some randomly stopped structure belongs to the class of generalized subexponential distributions due to the specific properties of the counting random variable. In such cases, primary random variables would likely need to have significantly lighter tails compared to the tail counting of the random variable.

On the other hand, it would be interesting to explore the closure properties of randomly stopped structures that are not related to the sum or maximum, but related to the product of random variables, as was performed in paper [60], for instance. In the class of generalized subexponential distributions, this investigation would be easier compared to that of other classes due to the results obtained in article [55].

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7 Santrauka (Summary in Lithuanian)

7.1 Įžanga

7.1.1 Mokslinė problema, jos aktualumas ir naujumas

Disertacijoje nagrinėjama atsitiktinai sustabdyta suma $S_\eta = \xi_1 + \cdots + \xi_\eta$, $S_0 = 0$, atsitiktinai sustabdytas minimumas

$$\xi_{(\eta)} = \begin{cases} 0 & \text{kai } \eta = 0, \\ \min\{\xi_1, \ldots, \xi_\eta\} & \text{kai } \eta \geqslant 1, \end{cases}$$

atsitiktinai sustabdytas maksimumas $\xi(\eta) = \max\{0, \xi_1, \ldots, \xi_\eta\}$ ir atsitiktinai sustabdytų sumų minimumas

$$S_{(\eta)} = \begin{cases} 0 & \text{kai } \eta = 0, \\ \min\{S_1, \ldots, S_\eta\} & \text{kai } \eta \geqslant 1, \end{cases}$$

atsitiktinai sustabdytų sumų maksimumas $S_{(\eta)} = \max\{S_0, S_1, \ldots, S_\eta\}$, kai $\{\xi_1, \xi_2, \ldots\}$ yra nepriklausomi atsitiktiniai dydžiai, o η skaičiuojantis atsitiktinis dydis, nepriklausomas nuo atsitiktinių dydžių $\{\xi_1, \xi_2, \ldots\}$. Sakoma, kad η yra skaičiuojantis atsitiktinis dydis, jei η yra neneigiamas, sveikareikšmis ir neišsigimęs taške 0. Disertacijoje tiriamos sąlygos, kurioms esant atsitiktinių dydžių S_η, $\xi_{(\eta)}$, $\xi^{(\eta)}$, $S_{(\eta)}$, $S^{(\eta)}$ pasiskirstymo funkcijos F_{S_η}, $F_{\xi_{(\eta)}}$,$F_{\xi^{(\eta)}}$, $F_{S_{(\eta)}}$, $F_{S^{(\eta)}}$ priklauso apibendrintai subeksponentinių skirstinių klasei. Taip pat analizuojamos sąlygos, kada atsitiktinių dydžių minimumas, maksimumas, bei atsitiktinių dydžių sumų maksimumas ir minimumas priklausys sunkiauodegių skirstinių klasei. Šio tyrimo praktinė reikšmė yra susijusi su draudimine ir finansine veikla, kur tradiciškai nagrinėjamos problemos, atsirandančios dėl nenumatytų ekstremalių įvykių. Matematiniu požiūriu, bet kokio draudimo verslo sėkmė priklauso nuo asimptotinio atsitiktinių dydžių S_η, $S_{(\eta)}$, $S^{(\eta)}$ pasiskirstymo funkcijų elgesio. Jei atsitiktinių dydžių pasiskirstymo funkcijos uodegos yra lengvos, tai rizikos atstatymo modelio bankroto tikimybė yra maža, kai pradinis kapitalas yra santykinai didelis. Įpaprastai tokiomis sąlygomis ši tikimybė gęsta eksponentiniu greičiu pradiniam kapitalui neaprėžtai augant(žr. pvz., [3, 29, 33, 34, 56, 62, 63]). Jei dydžių S_η, $S_{(\eta)}$ ar $S^{(\eta)}$ skirstiniai priklauso sunkiauodegių klasei, tada rizikos atstatymo modelio bankroto tikimybė didėjant pradiniam kapitalui gęsta laipsniškai (žr. pvz., [27, 38, 39, 44, 47, 48, 54, 63, 68]). Taigi tyrimo pradžioje būtina išsiaiškinti, kaip "elgiasi" atsitiktinių dydžių pasiskirstymo funkcijų uodegos.

Viena iš svarbiausių rizikos teorijos ar draudimo matematikos tyrimų kryp-čių yra bankroto tikimybės nagrinėjimas, kai ieškinių dydžių pasiskirstymo funkcijos turi sunkias uodegas. Šiuo atveju bankrotas įprastai įvyksta dėl vieno didelio ieškinio. Mūsų tyrimo ojektas - apibendrinti subeksponentiniai skirstiniai. Todėl aiškinsimės ar suminių ieškinių S_η, $S_{(\eta)}$, $S^{(\eta)}$ paskirstymai priklauso disertacijoje tyrinėjamiems apibendrintiems subeksponentinams skirstiniams.

Antra vertus, darbe gauti rezultatai susiję su klasikine uždarumo problema. Bingham, Goldie ir Teugels (žr. [6]) vieni iš pirmųjų tyrinėjo šią sritį. Visi klasikiniai rezultatai, susiję su uždarumo problema, nagrinėja vienodai pasiskirsčiusius atsitiktinius dydžius $\{\xi_1, \xi_2, \ldots\}$. Tuo tarpu šioje disertacijoje yra nagrinėjami ne tik vienodai pasiskirstę atsitiktiniai dydžiai. Parodoma, kada atsitiktinių dydžių S_η, $\xi_{(\eta)}$, $S_{(\eta)}$ pasiskirstymo funkcijos pasilieka apibendrintų subeksponentinių (3,4 skyriai) arba sunkiauodegių skirstinių klasėje (5 skyrius). Yra nagrinėjami ne du atsitiktiniai dydžiai, baigtinis jų skaičius, o atsitiktinai stabdytas skaičius. Disertacijoje
pateikiami rezultatai atsitiktinai parinktam duomenų skaičiui. Visi disertacijoje pateikti rezultatai yra nauji ir originalūs. Disertacijos rezultatai publikuoti 3 moksliniuose straipsniuose.

7.1.2 Tikslas ir uždaviniai

Pagrindinis disertacijos tikslas - rasti sąlygas nepriklausomiems atsitiktiniams dydžiams $\{\xi_1, \xi_2, \ldots\}$ ir skaičiuojančiam atsitiktiniam dydžiui η, kurioms esant S_η, $\xi_{(\eta)}$, $\xi^{(\eta)}$, $S_{(\eta)}$ ir $S_{(\eta)}$ priklauso apibendrintų subeksponentinių skirstinių klasei. Taip pat siekiama išspręsti uždavinį, kada sunkiauodegių skirstinių klasė yra uždara atsitiktinai sustabdytos sumos, minimumo, maksimumo, sumų minimumo ir maksimumo atžvilgiu. Tikslas pasiekiamas įrodžius sekančius teiginius:

- Atsitiktinai sustabdytos sumos pasiskirstymo funkcija priklauso apibendrintų subeksponentinių skirstinių klasei tuo atveju, jei pirmasis skirs-tinys priklauso šiai klasei, kiti jam "netrukdo", o skaičiuojantis atsitiktinis dydis turi baigtinę atramą.

- Atsitiktinai sustabdytos sumos skirstinys yra apibendrintai subeksponentinis, kai skaičiuojantis atsitiktinis dydis yra labai lengvas, o atsitiktinių dydžių uodegos yra asimptotiškai suderintos su pirmuoju, apibendrintai subeksponentiniu skirstiniu.

- Jei nepriklausomų atsitiktinių dydžių skirstiniai su nepriklausomu nuo jų skaičiuojančiu atsitiktiniu dydžiu priklauso apibendrintų subeksponentinių skirstinių klasei, tai atsitiktinai sustabdytas minimumas ir atsitiktinai sustabdytos sumos minimumas taip pat priklausys šiai klasei.

- Kad būtų patenkintos maksimumo ir sumos maksimumo priklausymo apibendrintų subeksponentinių skirstinių klasei sąlygos, reikalaujama, kad pirmas skirstinys būtų apibendrintai subeksponentinis, skaičiuojantis atsitiktinis dydis turėtų baigtinį vidurkį, o sumos maksimumui - baigtinį eksponentinį vidurkį.

- Randamos sąlygos analogiškai užduočiai sunkiauodegių skirstinių klasėje. Taip pat pateikiamas atsakymas, ar įmanoma iš keleto lengvauodegių skirstinių gauti sunkų pagal atsitiktinai sustabdytas struktūras.

7.1.3 Tyrimo metodika

Kuriai iš skirstinių klasių priklauso tam tikras skirstinys yra susiję su pasiskirstymo funkcijos uodegos elgesiu. Vertinant atsitiktinai sustabdytos sumos, atsitiktinai sustabdytų sumų maksimumo ir minimumo uodegų pasiskirstymą šioje disertacijoje naudojami įprasti tikimybių teorijos ir matematinės analizės metodai. Visų įrodymų pradinis žingsnis atliekamas pagal pilno vidurkio formulę. Po to gautos sumos nariai analizuojami skirtingais metodais. Tokioje analizėje svarbiausia išskirti iš sumos pagrindinius narius.

7.2 Pagrindiniai rezultatai

7.2.1 Apibrėžimai

Disertacijoje nagrinėjame apibendrintų subeksponentinių skirstinių klasę, taip pat sunkiauodegių skirstinių klasę todėl pateiksime jų ir su jomis susijusių klasių apibrėžimus: $\mathcal{H}$, $\mathcal{OS}$, $\mathcal{OL}$, $\mathcal{S}$, $\mathcal{L}$ ir $\mathcal{L}(\gamma)$. Čia, ir toliau: $\overline{F}(x) = 1 - F(x)$, $x \in \mathbb{R}$ - pasiskirstymo funkcijos F uodega. Svarbiausius sunkiauodegių skirstinių klasių apibrėžimus pateiksime žemiau. Pradėsime nuo subeksponentinių ir ilgauodegių skirstinių klasių, kurias pirmasis aprašė Chistyakov [10]. Pavyzdžiui, jis įrodė, kad subeksponentinių skirstinių klasė yra ilgauodegių skirstinių klasės dalis.

- Sakoma, kad pasiskirstymo funkcija F turi sunkią uodegą ($F \in \mathcal{H}$) jeigu bet kuriam $\delta > 0$

$$\limsup_{x\to\infty} \overline{F}(x)\mathrm{e}^{\delta x} = \infty.$$

- Sakoma, kad neneigiamo atsitiktinio dydžio pasiskirstymo funkcija F yra subeksponentinė ($F \in \mathcal{S}$) jeigu

$$\lim_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2.$$

Čia ir toliau $*$ reiškia pasiskirstymo funkcijos sąsūką.

- Realias reikšmes įgyjančio a.d. pasiskirstymo funkcija F yra subeksponentinė, jeigu

$$\lim_{x\to\infty} \frac{\overline{F^+ * F^+}(x)}{\overline{F}(x)} = 2.$$

su $F^+(x) = F(x)\mathbb{1}_{[0,\infty)}(x)$

- Sakoma, kad realaus kintamojo pasiskirstymo funkcija F priklauso api-bendrintų subeksponentinių skirstinių klasei ($F \in \mathcal{OS}$), jeigu

$$\limsup_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} < \infty.$$

- Sakoma, kad pasiskirstymo funkcija F priklauso ilgauodegių skirstinių klasei $\mathcal{L}$ ($F \in \mathcal{L}$) jeigu bet kuriam teigiamam $y > 0$:

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1.$$

Shimura ir Watanabe [65] aprašė klasę $\mathcal{OL}$, kuri yra platesnė už klasę $\mathcal{L}$, bet kažkiek panaši į ją savo struktūra.

- Sakoma, kad pasiskirstymo funkcija F yra $\mathcal{O}$- subeksponentinė ($F \in \mathcal{OL}$), jeigu bet kuriam fiksuotam $y \in \mathbb{R}$, turime:

$$0 < \liminf_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} < \infty.$$

Faktiškai, pasiskirstymo funkcija F priklauso klasei $\mathcal{OL}$ tada ir tik tada, kai

$$\sup_{x\geqslant 0} \frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty.$$

Paskutinė sąlyga rodo, kad klasė $\mathcal{OL}$ yra gana plati. Dabar mes apibūdinsime populiariausias $\mathcal{OL}$ poklases.

- Sakoma, kad pasiskirstymo funkcija F priklauso eksponentinių skirstinių klasei $\mathcal{L}(\gamma)$, $\gamma > 0$, jeigu bet kuriam fiksuotam $y > 0$

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-y\gamma}.$$

Kai $\gamma > 0$, klasė $\mathcal{L}(\gamma)$ buvo pirmą kartą nagrinėjama Embrechts ir Goldie [22].

Kai $\gamma = 0$, akivaizdu, kad $\mathcal{L}(\gamma) = \mathcal{L}$.

Kita žinoma sunkiauodegių skirstinių klasė - dominuojamai kintančių skirstinių klasė $\mathcal{D}$.

- Sakoma, kad pasisikirstymo funkcija F priklauso dominuojamai kintančių skirstinių klasei $\mathcal{D}$, jeigu bet kuriam fiksuotam $y \in (0, 1)$

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

Dominuojančiai kintančių skirstinių klasė $\mathcal{D}$ minima Feller [28] ir vėliau tyrinėjama [6, 8, 45, 64, 69, 70, 75], ir kitur. Ilgauodegių skirstinių klasę $\mathcal{L} = \mathcal{L}(0)$ pirmą kartą aprašė Chistyakov [10] šakojimosi proceso kontekste. Chover ir kiti [11, 12] pirmą kartą aprašė klasę $\mathcal{L}(\gamma)$ su $\gamma > 0$. Vėliau įvairios pasiskirstymo funkcijų su ilgomis uodegomis ir su artimomis eksponentinėms uodegomis savybės buvo tyrinėjamos ir aprašomos šiuose
veikaluose [8, 22, 31, 32, 42, 59, 73] .

Apibendrinsime anksčiau pateiktų pagrindinių sunkiauodegių skirstinių kla-sių tarpusavio ryšius. Dauguma šių tarpusavio sąryšių yra gerai žinomi.

$$\mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}, \quad \mathcal{D} \subset \mathcal{H}.$$

Sąryšis

$$\mathcal{L} \cap \mathcal{D} \subset \mathcal{S}$$

yra įrodytas 1.4.4. teiginyje [25]. Priklausomybė

$$\mathcal{S} \subset \mathcal{L}$$

įrodyta lemoje 1.3.5 (a) [25]. O sąryšis

$$\mathcal{L} \subset \mathcal{H}$$

išplaukia iš reprezentacinės formulės.

Brėžinys 1 demonstruoja sąryšius tarp sunkiauodegių skirstinių klasių $\mathcal{D}$, $\mathcal{S}$, $\mathcal{L}$ ir $\mathcal{H}$.

Aukščiau pateikti apibrėžimai tiesiogiai reiškia šiuos tarpusavio ryšius, kuriuos galima pažiūrėti brėžinyje 2.

$$\mathcal{L} \subset \mathcal{OL}, \quad \mathcal{D} \subset \mathcal{OL} \quad \text{ir} \quad \bigcup_{\gamma>0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

7.2.2 Atsitiktinai sustabdytos apibendrintų subeksponentinių skirstinių sumos

Šiame skyriuje pristatysime pagrindinius rezultatus apie atsitiktinai sustabdytų sumų priklausymą apibendrintų subeksponentinių skirstinių klasei. Pirmas mūsų tvirtinimas aprašo situaciją, kai pradiniai atsitiktiniai dydžiai priklauso klasei $\mathcal{OS}$, o skaičiuojantis atsitiktinis dydis turi baigtinę atramą.

TEOREMA 7.2.1. *[40] Sakykime $\{\xi_1, \xi_2, \ldots\}$ yra nepriklausomų atsitiktinių dydžių seka, atsitiktinis dydis η skaičiuojantis, nepriklausomas nuo $\{\xi_1, \xi_2, \ldots\}$. Jei η yra aprėžtas, $F_{\xi_1} \in \mathcal{OS}$, o kitiems indeksams $k \neq 1$ arba $F_{\xi_k} \in \mathcal{OS}$ arba $\overline{F}_{\xi_k}(x) = O\big(\overline{F}_{\xi_1}(x)\big)$, tada atsitiktinai sustabdytos sumos pasiskirstymo funkcija F_{S_η} priklauso klasei $\mathcal{OS}$.*

TEOREMA 7.2.2. *[40] Sakykime $\{\eta, \xi_1, \xi_2, ...\}$ yra nepriklausomi atsitiktiniai dydžiai, o skaičiuojantis atsitiktinis dydis η turi baigtinį bet kurios eilės $\lambda > 0$ eksponentinį momentą $\mathbb{E}e^{\lambda\eta} < \infty$. Tada $F_{S_\eta} \in \mathcal{OS}$, jeigu $F_{\xi_1} \in \mathcal{OS}$ ir patenkinta kuri nors viena iš žemiau esančių sąlygų:*

$$\text{(i)} \quad \mathbb{P}(\eta = 1) > 0 \quad \text{ir} \quad \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty;$$

$$\text{(ii)} \quad 0 < \liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty.$$

7.2.3 Atsitiktinai sustabdyti apibendrintų subeksponentinių skirstinių minimumas, maksimumas, sumų minimumas, sumų ma-ksimumas

Šiame skyriuje mes pratęsime klasės $\mathcal{OS}$ uždarumo savybių tyrimą. Tardami, jog kai kurie pradiniai atsitiktiniai dydžiai priklauso klasei $\mathcal{OS}$, ieškosime, kokiomis sąlygomis toje pačioje klasėje išliks struktūros, aprašytos 7.1.1.

TEOREMA 7.2.3. *Sakykime $\{\xi_1, \xi_2, \ldots\}$ - seka nepriklausomų atsitiktinių dydžių, įgyjančių realias reikšmes, o η skaičiuojantis atsitiktinis dydis, nepriklausomas nuo $\{\xi_1, \xi_2, \ldots\}$. Jei $F_{\xi_k} \in \mathcal{OS}$ kiekvienam k, tai $F_{\xi_{(\eta)}}$ ir $F_{S_{(\eta)}}$ priklauso klasei $\mathcal{OS}$, ir galioja sekantys asimptotiniai ryšiai:*

$$\overline{F}_{\xi_{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_{(\varkappa)}}(x) = \prod_{k=1}^{\varkappa} \overline{F}_{\xi_k}(x), \tag{7.2.1}$$

$$\overline{F}_{S_{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{S_{(\varkappa)}}(x), \tag{7.2.2}$$

kur $\varkappa = \min\{k \geqslant 1 : \mathbb{P}(\eta = k) > 0\}$.

TEOREMA 7.2.4. *Sakykime $\{\xi_1, \xi_2, \ldots\}$- nepriklausomų atsitiktinių dydžių, įgyjančių realias reikšmes seka, tokia, kad $F_{\xi_1} \in \mathcal{OS}$. Tegul η skaičiuojantis atsitiktinis dydis, nepriklausomas nuo $\{\xi_1, \xi_2, \ldots\}$ ir turintis baigtinį vidurkį $\mathbb{E}\eta$. Jeigu*

$$0 < \liminf_{x\to\infty} \inf_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty, \tag{7.2.3}$$

tada $F_{\xi^{(\eta)}} \in \mathcal{OS}$ ir $\overline{F}_{\xi^{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$.

TEOREMA 7.2.5. *Sakykime $\{\xi_1, \xi_2, \ldots\}$- nepriklausomų atsitiktinių dydžių, įgyjančių realias reikšmes seka, tokia, kad $F_{\xi_1} \in \mathcal{OS}$ ir tenkinama sąlyga:*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty. \tag{7.2.4}$$

Tegul η skaičiuojantis atsitiktinis dydis, nepriklausomas nuo $\{\xi_1, \xi_2, \ldots\}$ ir turintis eksponentinį vidurkį $\mathbb{E}e^{\lambda\eta} < \infty$ visiems $\lambda > 0$. Tada $F_{S^{(\eta)}} \in \mathcal{OS}$ ir $\overline{F}_{S^{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$.

7.2.4 Atsitiktinai sustabdyti sunkiauodegių ir lengvauodegių skirstinių minimumas, maksimumas, sumų minimumas, sumų ma-ksimumas

Šiame skyriuje mes pratęsime tų pačių struktūrų: atsitiktinai sustabdytų sumų, atsitiktinai sustabdytų minimumo ir maksimumo, atsitiktinai sustabdytų sumų minimumo ir maksimumo tyrinėjimą. Ankstesniuose disertacijos skyriuose mes tyrinėjome apibendrintų subeksponentinių skirstinių klasę, šiame skyriuje dėmesys bus koncentruotas į sunkiauodegių skirstinių klasę $\mathcal{H}$.

TEOREMA 7.2.6. *Sakykime $\{\xi_1, \xi_2, \ldots\}$ nepriklausomi, įgyjantys realias reikš-mes atsitiktiniai dydžiai, o η skaičiuojantis atsitiktinis dydis, nepriklausomas nuo sekos $\{\xi_1, \xi_2, \ldots\}$. Skirstinys F_{S_η} bus sunkiauodegis, jei tenkinama bent viena iš sąlygų:*

(i) $\inf\limits_{k\geq 1} \mathbb{E}\, \mathrm{e}^{\lambda \xi_k} > 1$ *bet kuriam $\lambda > 0$, ir $F_\eta \in \mathcal{H}$;*

(ii) $\inf\limits_{k\geq 1} \mathbb{P}(\xi_k \geq a) = 1$ *kažkokiam $a > 0$, ir $F_\eta \in \mathcal{H}$;*

(iii) $F_{\xi_\varkappa} \in \mathcal{H}$ *kuriam nors $\varkappa \geq 1$, ir $\overline{F}_\eta(x) > 0$ visiems $x \in \mathbb{R}$;*

(iv) $F_{\xi_\varkappa} \in \mathcal{H}$ *kuriam nors $1 \leq \varkappa \leq \max \operatorname{supp}(\eta) < \infty$.*

Skirstinys F_{S_η} turės lengvą uodegą, jei bus patenkinta bent viena iš sąlygų:

(v) $F_{\xi_1} \in \mathcal{H}^c$, $F_\eta \in \mathcal{H}^c$, $\overline{F}_{\xi_1}(x) > 0$ *visiems $x \in \mathbb{R}$ ir*

$$\limsup_{x\to\infty} \sup_{k\geq 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty; \tag{7.2.5}$$

(vi) $\sup\limits_{k\geq 1} \mathbb{E}\, \mathrm{e}^{\lambda \xi_k} < \infty$ *kuriam nors $\lambda > 0$, ir $F_\eta \in \mathcal{H}^c$.*

TEOREMA 7.2.7. *Sakykime $\{\xi_1, \xi_2, \ldots\}$ nepriklausomų atsitiktinių dydžių, įgyjančių realias reikšmes seka , o η skaičiuojantis atsitiktinis dydis, nepriklausomas nuo sekos $\{\xi_1, \xi_2, \ldots\}$.*

(i) *Jei $F_{\xi_\varkappa} \in \mathcal{H}$ kuriam nors $\varkappa \geq 1$ ir $\overline{F}_\eta(x) > 0$ visiems $x \in \mathbb{R}$, tada $F_{\xi^{(\eta)}} \in \mathcal{H}$;*

(ii) *Jei $F_{\xi_\varkappa} \in \mathcal{H}$ kuriam nors $\varkappa \leq \max\{\operatorname{supp}(\eta)\} < \infty$, tai $F_{\xi^{(\eta)}} \in \mathcal{H}$;*

(iii) *Skirstinys $F_{\xi^{(\eta)}}$ priklauso klasei $\mathcal{H}^c$ jei $F_{\xi_1} \in \mathcal{H}^c$, $\overline{F}_{\xi_1}(x) > 0$ visiems $x \in \mathbb{R}$, $\mathbb{E}\eta < \infty$ ir*

$$\limsup_{x\to\infty} \sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty. \tag{7.2.6}$$

TEOREMA 7.2.8. *Sakykime* $\{\xi_1, \xi_2, \ldots\}$ *nepriklausomų atsitiktinių dydžių, įgyjančių realias reikšmes seka, o* η *skaičiuojantis atsitiktinis dydis, nepriklausomas nuo sekos* $\{\xi_1, \xi_2, \ldots\}$.

(i) *Jei* $F_{\xi_1} \in \mathcal{H}$ *ir*

$$\liminf_{x\to\infty} \min_{1\leq k\leq \varkappa} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} > 0,$$

su $\varkappa = \min\{\operatorname{supp}(\eta) \setminus \{0\}\}$, *tai* $F_{\xi_{(\eta)}} \in \mathcal{H}$ *ir*

$$F_{\xi_{(\eta)}}(x) \underset{x\to\infty}{\sim} \mathbb{P}(\eta = \varkappa)\, \overline{F}_{\xi_{(\varkappa)}}(x);$$

(ii) *Jei* $F_{\xi_k} \in \mathcal{H}^c$, *su visais* $1 \leq k \leq \varkappa = \min\{\operatorname{supp}(\eta) \setminus \{0\}\}$, *tai* $F_{\xi_{(\eta)}} \in \mathcal{H}^c$.

TEOREMA 7.2.9. *Sakykime* $\{\xi_1, \xi_2, \ldots\}$ *nepriklausomų atsitiktinių dydžių, įgyjančių realias reikšmes seka, o* η *skaičiuojantis atsitiktinis dydis, nepriklausomas nuo sekos* $\{\xi_1, \xi_2, \ldots\}$.

(i) *Jei* $F_{\xi_1} \in \mathcal{H}$ *ir* $\min\limits_{1\leq k\leq\varkappa} \mathbb{P}(\xi_k \geq 0) > 0$, *su* $\varkappa = \min\{\operatorname{supp}(\eta) \setminus \{0\}\}$, *tai* $F_{S_{(\eta)}} \in \mathcal{H}$ *ir*

$$\overline{F}_{S_{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x). \tag{7.2.7}$$

(ii) *Jei* $F_{\xi_1} \in \mathcal{H}^c$, *tai* $F_{S_{(\eta)}} \in \mathcal{H}^c$ *bet kokiam atsitiktiniam dydžiui* η.

TEOREMA 7.2.10. *Sakykime* $\{\xi_1, \xi_2, \ldots\}$ *ir* η *teoremose 7.2.6-7.2.9 nusakyti atsitiktiniai dydžiai. Tada* $F_{S^{(\eta)}} \in \mathcal{H}$ *jei patenkinta bent viena iš sąlygų:*

(i) $\inf\limits_{k\geq 1} \mathbb{E}\, \mathrm{e}^{\lambda\xi_k} > 1$ *bet kuriam* $\lambda > 0$ *ir* $F_\eta \in \mathcal{H}$;

(ii) $\inf\limits_{k\geq 1} \mathbb{P}(\xi_k \geq a) = 1$ *kuriam nors* $a > 0$ *ir* $F_\eta \in \mathcal{H}$;

(iii) $F_{\xi_1} \in \mathcal{H}$;

(iv) $F_{\xi_\varkappa} \in \mathcal{H}$ *kuriam nors* $\varkappa \geq 1$ *tuo atveju, kai* $\operatorname{supp}(\eta)$ *yra begalinė, arba kuriam nors* $1 \leq \varkappa \leq \max\operatorname{supp}(\eta)$ *kai* $\operatorname{supp}(\eta)$ *yra baigtinė.*
Skirstinys $F_{S^{(\eta)}}$ *turi lengvą uodegą, jeigu:*
$\sup\limits_{k\geq 1} \mathbb{E}\, \mathrm{e}^{\lambda\xi_k} < \infty$ *kuriam nors* $\lambda > 0$ *ir* $F_\eta \in \mathcal{H}^c$.

Nepriklausomų, vienodai pasiskirsčiusių atsitiktinių dydžių atveju teorema 7.2.6 leidžia daryti sekančias išvadas. Atkreiptinas dėmesys, kad pirmas dvi išvadas galima rasti monografijoje [31] (Problema 2.12 ir 2.13).

IŠVADA 7.2.1. *Sakykime $\{\xi_1, \xi_2, \ldots\}$ nepriklausomi vienodai pasiskirstę atsitiktiniai dydžiai, įgyjantys realias reikšmes su bendra pasiskirstymo funkcija F_{ξ_1}, o η skaičiuojantis atsitiktinis dydis, nepriklausomas nuo $\{\xi_1, \xi_2, \ldots\}$. Jei $F_{\xi_1} \in \mathcal{H}^c$ ir $F_\eta \in \mathcal{H}^c$ tada $F_{S_\eta} \in \mathcal{H}^c$.*

IŠVADA 7.2.2. *Sakykime $\{\xi_1, \xi_2, \ldots\}$ nepriklausomi,vienodai pasiskirstę, nenei-giami, neišsigimę nulyje, o η skaičiuojantis atsitiktinis dydis, nepriklausomas nuo sekos $\{\xi_1, \xi_2, \ldots\}$. Jei $F_\eta \in \mathcal{H}$ tada $F_{S_\eta} \in \mathcal{H}$.*

IŠVADA 7.2.3. *Sakykime $\{\xi_1, \xi_2, \ldots\}$ nepriklausomi, vienodai pasiskirstę atsitiktiniai dydžiai, įgyjantys realias reikšmes su pasiskirstymo funkcija F_{ξ_1}, o η skaičiuojantis atsitiktinis dydis, nepriklausomas nuo sekos $\{\xi_1, \xi_2, \ldots\}$. Jeigu $F_{\xi_1} \in \mathcal{H}$ tada $F_{S_\eta} \in \mathcal{H}$.*

Analogiškos išvados gali būti suformuluotos ir atsitiktinai sustabdytiems minimumui, maksimumui.

7.3 Papildomos lemos

Šiame skyriuje pateiksime lemas, kuriomis buvo pasiremta įrodinėjant pagrin-dines teoremas. Visos lemos įrodytos skyriuose 3, 4, 5.

7.3.1 Papildomos lemos teoremoms 7.2.1-7.2.2

Šiame skyrelyje pateikiame lemas, kurios buvo pritaikytos teoremų 7.2.1–7.2.2 įrodymams. Pirmoji lema (7.3.1)- tai $\mathcal{OS}$ klasės pasiskirstymo funkcijų pagrindinių savybių rinkinys. Ši lema yra pilnai įrodyta skyriuje 3.3, o atskirų lemos dalių įrodymus galima rast [65, 76, 42, 75].

LEMA 7.3.1. *Tegul X ir Y du atsitiktiniai dydžiai, įgyjantys realias reikšmes su atitinkamomis pasiskirstymo funkcijomis F_X ir F_Y. Tada galioja šie teiginiai:*

(i) $F_X \in \mathcal{OS}$ *tada ir tik tada, kai* $\sup\limits_{x\in\mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty.$

(ii) *Jei $F_X \in \mathcal{OS}$ ir $\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$, tai $F_Y \in \mathcal{OS}$.*

(iii) *Jei $F_X \in \mathcal{OS}$ ir $F_Y \in \mathcal{OS}$, tai $F_X * F_Y \in \mathcal{OS}$.*

(iv) *Jei $F_X \in \mathcal{OS}$, tai $F_X \in \mathcal{OL}$ t.y.* $\limsup\limits_{x\to\infty} \frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} = 1.$

(v) *Jei $F_X \in \mathcal{OS}$ ir $\overline{F}_Y(x) = O\big(\overline{F}_X(x)\big)$, tai $F_X * F_Y \in \mathcal{OS}$ ir $\overline{F_X * F_Y}(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$.*

LEMA 7.3.2. *Tegul $\{X_1, X_2 ...\}$ nepriklausomų atsitiktinių dydžių seka, kurių $F_{X_1} \in \mathcal{OS}$, o kitiems indeksams $k \geqslant 2$ turi būti patenkinta viena iš dviejų sąlygų: arba $F_{X_k} \in \mathcal{OS}$ arba $\overline{F}_{X_k}(x) = O(\overline{F}_{X_1}(x))$. Tada $F_{S_n} \in \mathcal{OS}$ visiems $n \in \mathbb{N}$.*

LEMA 7.3.3. *Tegul $X_1, X_2, ...$ nepriklausomų atsitiktinių dydžių seka, kurių $F_{X_1} \in OS$ ir*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} < \infty \tag{7.3.1}$$

Tada egzistuoja tokia konstanta $\widehat{C}$, su kuria teisinga nelygybė

$$\overline{F}_{S_n}(x) \leqslant \widehat{C}^{\,n-1}\overline{F}_{X_1}(x) \tag{7.3.2}$$

visiems $x \in \mathbb{R}$ ir visiems $n \geqslant 2$.

7.3.2 Papildomos lemos teoremoms 7.2.6-7.2.8

Pirmoji šio skyrelio lema yra rafinuotesnė lemos 7.3.2 iš skyrelio 7.3.1 versija. Lema 7.3.4 su modifikuotu įrodymu pateikta skyriuje 4.3.

LEMA 7.3.4. *Tegul $\{X_1, X_2 \dots\}$ nepriklausomų atsitiktinių dydžių, įgyjančių realias reikšmes seka. Sakykime $F_{X_1} \in \mathcal{OS}$, kitiems indeksams $k \geqslant 2$, arba $F_{X_k} \in \mathcal{OS}$ arba $\overline{F}_{X_k}(x) = O(\overline{F}_{X_1}(x))$. Tada, $F_{S_n} \in \mathcal{OS}$ visiems $n \in \mathbb{N}$ ir*

$$\overline{F}_{S_n}(x) \underset{x\to\infty}{\asymp} \overline{\coprod_{k\in\mathcal{A}_n}^{*} F_{X_k}(x)} \tag{7.3.3}$$

su $\mathcal{A}_n := \{k \in \{1, 2, \dots, n\} : F_{X_k} \in \mathcal{OS}\}$.

Sumų maksimumui $S^{(n)} = \max\{S_1, S_2, \dots, S_n\}$ galioja toks teiginys.

LEMA 7.3.5. *Sakykime $\{X_1, X_2, \dots\}$ yra nepriklausomų atsitiktinių dydžių, įgyjančių realias reikšmes seka, kuriai pasiskirstymo funkcija $F_{X_1} \in \mathcal{OS}$ ir*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} < \infty.$$

Tada egzistuoja tokia konstanta c_4, su kuria

$$\overline{F}_{S^{(n)}}(x) \leqslant c_4^n \overline{F}_{X_1}(x)$$

visiems $x \in \mathbb{R}$ ir $n \in \mathbb{N}$.

Toliau pateikta lema yra apie nepriklausomų atsitiktinių dydžių $\{X_1, \dots, X_n\}$ minimumą $X_{(n)} = \min\{X_1, \dots, X_n\}$. Panašus teiginys yra pateiktas in Lemoje 3.1 [52]. Ten pateiktas įrodymas - tik neneigiamiems absoliučiai tolydiems atsitiktiniams dydžiams. Mūsų įrodymas yra detalesnis ir pateiktas skyrelyje 4.3.

LEMA 7.3.6. *Tegul $\{X_1, X_2, \dots, X_n\}$, $n \in \mathbb{N}$ yra rinkinys nepriklausomų atsitiktinių dydžių, įgyjančių realias reikšmes ir turinčių atitinkamas pasiskirstymo funkcijas $\{F_{X_1},\ F_{X_2}, \dots, F_{X_n}\}$. Jei $F_{X_k} \in \mathcal{OS}$ visiems $k \in \{1, 2, \dots n\}$, tada atsitiktinio dydžio $X_{(n)} = \min\{X_1, X_2, \dots, X_n\}$ pasiskirstymo funkcija $F_{X_{(n)}}$ priklauso klasei $\mathcal{OS}$.*

Lema 7.3.7. *Tegul* $\{X_1, X_2, \ldots\}$ *yra nepriklausomų atsitiktinių dydžių, įgyjančių realias reikšmes seka, kuriai galioja* $F_{X_1} \in \mathcal{OS}$ *ir*

$$\liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} > 0.$$

Tada,

$$\inf_{x\in\mathbb{R}} \frac{\overline{F}_{S^{(n)}}(x)}{\overline{F}_{X_1}(x)} \geqslant \inf_{x\in\mathbb{R}} \frac{\overline{F}_{S_n}(x)}{\overline{F}_{X_1}(x)} \geqslant c_7^{n-1} \tag{7.3.4}$$

visiems $n \geqslant 1$ *ir kažkokiam* $c_7 > 0$*, kai* $S_n = X_1 + \ldots + X_n$ *ir* $S^{(n)} = \max\{S_1, S_2, \ldots S_n\}$.

7.3.3 Papildomos lemos teoremoms 7.2.6-7.2.10

Mes suformuluosime dvi lemas, kurios naudojamos kai kurių pagrindinių teoremų įrodymams. Nors lemų rezultatai gerai žinomi ir gali būti randami [31], [50], [57], pateikiame jas ir jų įrodymus skyriuje 5.3 skaitytojo patogumui.

Lema 7.3.8. *Tarkime, kad* F *yra atsitiktinio dydžio* ξ*, įgyjančio realias reikš-mes pasiskirstymo funkcija. Tada žemiau pateikti teiginiai yra ekvivalentūs:*

(i) $\mathbb{E}\,e^{\lambda\xi} = \infty$ *bet kuriam* $\lambda > 0$,

(ii) $\limsup_{x\to\infty} e^{\lambda x}\overline{F}(x) = \infty$ *su visais* $\lambda > 0$,

(iii) $\limsup_{x\to\infty} x^{-1}\log\overline{F}(x) = 0$.

Taip pat ekvivalentūs bus ir šie teiginiai:

(i') $\mathbb{E}\,e^{\lambda\xi} < \infty$ *bet kuriam* $\lambda > 0$,

(ii') $\limsup_{x\to\infty} e^{\lambda x}\,\overline{F}(x) < \infty$ *kuriam nors* $\lambda > 0$,

(iii') $\limsup_{x\to\infty} x^{-1}\log\overline{F}(x) < 0$.

Sekanti lema reiškia, jog $\mathcal{H}$ ir $\mathcal{H}^c$ yra uždari silpno uodegos ekvivalentumo atžvilgiu.

Lema 7.3.9. *Tarkime, jog* F *ir* G *yra atsitiktinių dydžių, įgyjančių realias reikšmes pasiskirstymai*

(i) *Jeigu* $F \in \mathcal{H}$ *ir*

$$\liminf_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} > 0, \tag{7.3.5}$$

tai $G \in \mathcal{H}$.

(ii) *Jeigu* $F \in \mathcal{H}^c$, $\overline{G}(x) \leq \tilde{c}\,\overline{F}(x)$ *kažkokiai konstantai* $\tilde{c} > 0$ *ir dideliems* x $(x > x_{\tilde{c}})$*, tai* $G \in \mathcal{H}^c$.

7.4 Išvados

Šioje disertacijoje analizuojamos pasiskirstymo funkcijos, gautos atsitiktinai sustabdžius atsitiktinių dydžių sumą, minimumą, maksimumą, sumos minimumą, sumos maksimumą. Pirminiai atsitiktiniai dydžiai laikomi nepriklausomais, įgyjančiais realias reikšmes ir galimai skirtingai pasiskirsčiusiais. Atsitiktinis dydis, nusakantis stabdymo momentą yra sveikareikšmis, neneigiamas ir neišsigimęs nulyje. Randame sąlygas, kurioms esant atsitiktinai sustabdytų struktūrų pasiskirstymo funkcijos priklauso apibendrintų subeksponentinių skirstinių klasei. Atsitiktinai sustabdytų struktūrų skirstinių priklausymą apibendrintų subeksponentinių skirstinių klasei gali lemti arba pirminiai atsitiktiniai dydžiai, arba skaičiuojantis atsitiktinis dydis. Šioje disertacijoje mes pagrindinai svarstome atvejį, kada pirminių atsitiktinių dydžių rinkinys turi lemiamą reikšmę. Pagrindiniai rezultatai suformuluoti teoremose 4.1.1 ir 4.1.2. Pirminiai atsitiktiniai dydžiai šiose teoremose gali būti skirtingai pasiskirstę. Bet papildomos visų teoremų sąlygos yra tenkinamos ir tuo atveju, kai pirminiai atsitiktiniai dydžiai yra pasiskirstę vienodai. Ateityje būtų įdomu, kada kažkokios atsitiktinai sustabdytos struktūros priklauso apibendrintų sub-eksponentinių skirstinių klasei esant specifinėms skaičiuojančiojo atsitiktinio dydžio savybėms. Tuo atveju pirminių atsitiktinių dydžių uodegos galimai turės būti ženkliai lengvesnės už skaičiuojančiojo atsitiktinio dydžio uodegą. Iš kitos pusės būtų įdomu patyrinėti atsitiktinai sustabdytų struktūrų uždarumo savybes, kurios susijusios ne tik su suma ar maksimumu, bet ir su atsitiktinių dydžių sandauga, kas jau analizuota pavyzdžiui darbe [60]. Lyginti apibendrintus subeksponentinius skirstinius su kitomis klasėmis būtų lengviau naudojant straipsnio[55] rezultatus.

7.5 Rezultatų sklaida

Disertacijos rezultatai publikuoti šiuose moksliniuose straipsniuose

- Karasevičienė, J., Šiaulys, J. (2023). Randomly stopped sums with generalized subexponential distribution. *Axioms*, 12: 641.
- Karasevičienė, J., Šiaulys, J. (2024). Randomly stopped minimum, maximum, minimum of sums and maximum of sums with generalized subexponential distribution. *Axioms*, 13: 85.
- Leipus, R., Šiaulys, J., Danilenko, S., Karasevičienė , J. (2024). Randomly stopped sums, minima and maxima for heavy- tailed and light-tailed distribution *Axioms*, 13: 335.

Konferencijos, kuriose pristatyti pranešimai disertacijos tema:

- Atsitiktinai sustabdytos apibendrintų subeksponentinių skirstinių sumos. *Lietuvos matematikų draugijos 64-oji konferencija*, 2023 m. birželio 21-22 d., Vilnius.
- Randomly stopped sums with generalized subexponential distribution, properties of min, max and min, max of sums. *The international scientific conference dedicated to the 160th anniversary of Prof. Dr Hermann Minkowski*, 2024,birželio 20-22 d., Kaunas.
- Randomly stopped sums with generalized subexponential distribution, properties of min, max and min, max of sums. *11th Tartu Conference on Multivariate Statistics*, 2024, birželio 25-28 d., Tartu.

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