https://doi.org/10.15388/vu.thesis.709
https://orcid.org/0000-0002-3968-3377

VILNIUS UNIVERSITY

Monika Tekorė

# Universality of Periodic Zeta-Functions with Multiplicative Coefficients

**DOCTORAL DISSERTATION**

Natural Sciences,
Mathematics (N 001)

VILNIUS 2024

This dissertation was written between 2020 and 2024 at Vilnius University.

**Academic supervisor:**
**Prof. Dr. Habil. Antanas Laurinčikas** (Vilnius University, Natural Sciences, Mathematics – N 001).
**Academic consultant:**
**Prof. Dr. Darius Šiaučiūnas** (Vilnius University, Natural Sciences, Mathematics – N 001).

**Dissertation defence council:**
**Chairman** – Prof. Dr. Habil. Artūras Dubickas (Vilnius University, Natural Sciences, Mathematics – N 001).

**Members:**
Prof. Dr. Igoris Belovas (Vilnius University, Natural Sciences, Mathematics – N 001),
Prof. Dr. Paulius Drungilas (Vilnius University, Natural Sciences, Mathematics – N 001),
Prof. Dr. Ramūnas Garunkštis (Vilnius University, Natural Sciences, Mathematics – N 001),
Prof. Dr. Arvet Pedas (University of Tartu (Estonia), Natural Sciences, Mathematics – N 001).

The dissertation will be defended at a public meeting of the Dissertation Defence Council at 3 pm on 16th December 2024 in room 102 of the Faculty of Mathematics and Informatics of Vilnius University.
Adress: Naugarduko str. 24, LT03225, Vilnius, Lithuania.
Phone: +37052193050, e-mail: mif@mif.vu.lt.

Dissertation can be reviewed in library of Vilnius University or on the website of Vilnius University:
www.vu.lt/lt/naujienos/ivykiu-kalendorius.

https://doi.org/10.15388/vu.thesis.709
https://orcid.org/0000-0002-3968-3377

VILNIAUS UNIVERSITETAS

Monika Tekorė

# Periodinių dzeta funkcijų su multiplikatyviaisiais koeficientais universalumas

**DAKTARO DISERTACIJA**

Gamtos mokslai,
matematika (N 001)

VILNIUS 2024

Disertacija rengta 2020–2024 metais Vilniaus universitete

**Mokslinis vadovas:**
**Prof. habil. dr. Antanas Laurinčikas** (Vilniaus universitetas, gamtos mokslai, matematika – N 001).
**Mokslinis konsultantas:**
**Prof. dr. Darius Šiaučiūnas** (Vilniaus universitetas, gamtos mokslai, matematika N 001).

**Gynimo taryba:**
**Pirminikas** – prof. habil. dr. Artūras Dubickas (Vilniaus universitetas, gamtos mokslai, matematika – N 001).

**Nariai:**
Prof. dr. Igoris Belovas (Vilniaus universitetas, gamtos mokslai, matematika – N 001),
Prof. dr. Paulius Drungilas (Vilniaus universitetas, gamtos mokslai, matematika – N 001),
Prof. dr. Ramūnas Garunkštis (Vilniaus universitetas, gamtos mokslai, matematika – N 001),
Prof. dr. Arvet Pedas (Tartu universitetas (Estija), gamtos mokslai, matematika – N 001).

Diertacija ginama viešame Gynimo tarybos posedyje 2024 m. gruodžio 16 d., 15 valandą, Vilniaus universiteto Matematikos ir informatikos fakultete, 102 auditorijoje.
Adresas: Naugarduko g. 24, LT03225, Vilnius, Lietuva.
Tel. +37052193050, el. paštas mif@mif.vu.lt.

Disertaciją galima peržiūrėti Vilniaus universiteto bibliotekoje ir Vilniaus universiteto interneto svetainėje adresu:
www.vu.lt/lt/naujienos/ivykiu-kalendorius.

# Table of Contents

# Notation

| | |
|---|---|
| $j, k, l, m, n$ | natural numbers |
| $p$ | prime number |
| $\mathbb{P}$ | set of all prime numbers |
| $\mathbb{N}$ | set of all natural numbers |
| $\mathbb{N}_0$ | $\mathbb{N} \cup \{0\}$ |
| $\mathbb{Z}$ | set of all integer numbers |
| $\mathbb{R}$ | set of all real numbers |
| $\mathbb{C}$ | set of all complex numbers |
| $i$ | imaginary unity: $i = \sqrt{-1}$ |
| $s = \sigma + it$, $\sigma, t \in \mathbb{R}$ | complex variable |
| $\bigoplus\limits_m A_m$ | direct sum of sets $A_m$ |
| $A \times B$ | Cartesian product of the sets $A$ and $B$ |
| $\prod\limits_m A_m$ | Cartesian product of sets $A_m$ |
| $A^m$ | Cartesian product of $m$ copies of the set $A$ |
| $\text{meas}A$ | Lebesgue measure of the set $A \subset \mathbb{R}$ |
| $\#A$ | cardinality of the set $A$ |
| $H(G)$ | space of analytic functions on $G$ |
| $\mathcal{B}(\mathbb{X})$ | class of Borel sets of the space $\mathbb{X}$ |
| $\mathbb{E}X$ | expectation of the random variable |
| $\xrightarrow{\mathcal{D}}$ | convergence in distribution |
| $\Gamma(s)$ | Euler gamma-function |
| $\zeta(s)$ | Riemann zeta-function |
| $\zeta(s, \alpha)$ | Hurwitz zeta-function |
| $a \ll_\eta b$, $b > 0$ | there exists a constant $C = C(\eta) > 0$ such that $\lvert a \rvert \leq Cb$ |

# CHAPTER 1

## Introduction

## 1.1 Research topic

In the dissertation, we consider the value distribution of periodic zeta-functions. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$, i. e., $a_{m+q} = a_m$ for all $m \in \mathbb{N}$. The periodic zeta-function $\zeta(s; \mathfrak{a})$, $s = \sigma + it$, is defined by the Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad \sigma > 1.$$

Clearly, the sequence $\mathfrak{a}$ is bounded, i. e.,

$$|a_m| \leqslant \max(|a_1|, \dots, |a_q|)$$

for all $m \in \mathbb{N}$. Therefore, the series for $\zeta(s; \mathfrak{a})$ is absolutely and uniformly convergent for $\sigma > 1 + \varepsilon$ with arbitrary $\varepsilon > 0$, hence, the function $\zeta(s; \mathfrak{a})$ is analytic in the half-plane $\sigma > 1$. To obtain the analytic continuation for the function $\zeta(s; \mathfrak{a})$ to the region $\sigma \leqslant 1$, the classical Hurwitz zeta-function is used. Let $0 < \alpha \leqslant 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$ was introduced and studied in [16], and, for $\sigma > 1$, is defined by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}.$$

Moreover, the function $\zeta(s, \alpha)$ has the analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1 [1, 41]. In other words, $\zeta(s, \alpha)$ is a meromorphic function.

In view of the periodicity of the sequence $\mathfrak{a}$, for $\sigma > 1$, the equality

$$\begin{aligned}\zeta(s;\mathfrak{a}) &= \sum_{l=1}^{q} \sum_{\substack{m=1\\ m\equiv l(\operatorname{mod} q)}}^{\infty} \frac{a_m}{m^s} = \sum_{l=1}^{q} a_l \sum_{k=0}^{\infty} \frac{1}{(kq+l)^s}\\ &= \frac{1}{q^s}\sum_{l=1}^{q} a_l \sum_{k=0}^{\infty} \frac{1}{(k+l/q)^s} = \frac{1}{q^s}\sum_{l=1}^{q} a_l \zeta\left(s, \frac{l}{q}\right) \end{aligned} \tag{I1}$$

is valid. Therefore, the mentioned analytic properties of the Hurwitz zeta-function give the analytic continuation to the whole complex plane for the function $\zeta(s, \mathfrak{a})$ with a possible unique simple pole at the point $s = 1$ with residue

$$\widehat{a} \stackrel{\text{def}}{=} \frac{1}{q} \sum_{l=1}^{q} a_l. \tag{I2}$$

If $\widehat{a} = 0$, then the periodic zeta-function $\zeta(s; \mathfrak{a})$ is entire one.

The second property of the sequence $\mathfrak{a}$ required in the dissertation is its multiplicativity. We recall that the sequence $\mathfrak{a}$ is called multiplicative if $a_1 = 1$, and $a_{m_1 m_2} = a_{m_1} a_{m_2}$ for all coprimes $m_1, m_2 \in \mathbb{N}$ ($(m_1, m_2) = 1$). Many arithmetic functions in number theory are multiplicative. Examples of multiplicative functions:

1. The divisor function
$$d(m) = \sum_{d|m} 1;$$

2. The Möbus function
$$\mu(m) = \begin{cases} 0 & \text{if } k^r \mid m,\ k \in \mathbb{N} \setminus \{1\}, \\ (-1)^k & \text{if } m = p_1 \cdots p_k,\ \ p_1, \dots, p_k \text{ are different} \\ & \text{prime numbers}, \\ 1 & \text{if } m = 1; \end{cases}$$

3. The Euler totient function
$$\varphi(m) = \#\{1 \leqslant k \leqslant m : (k, m) = 1\};$$

4. Suppose that $f : \mathbb{N} \to \mathbb{C}$ is an additive function, i. e.,
$$f(m_1 m_2) = f(m_1) + f(m_2), \quad \text{for all } (m_1, m_2) = 1.$$

Then the function $\mathrm{e}^{f(m)}$ is multiplicative.

Example 4 is widely used in probabilistic number theory, see, for example, [32], [8], [9], and [31], for investigation of limit distributions of real additive functions. It is well known that the characteristic function of the distribution function of $f(m)$

$$F_n(x) \stackrel{\text{def}}{=} \frac{1}{n}\#\{1 \leqslant m \leqslant n : f(m) < x\}$$

is

$$g_n(t) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{m=1}^{n} \mathrm{e}^{itf(m)}, \quad t \in \mathbb{R},$$

i.e, the problem is reduced to mean values of multiplicative functions. If $g_n(t)$, as $n \to \infty$, converges to $g(t)$ continuous at $t = 0$, then the distribution function $F_n(x)$, as $n \to \infty$, converges weakly to a distribution function with the characteristic function $g(t)$.

The simplest periodic and multiplicative arithmetic function is the Legendre symbol. Let $q > 2$ be a prime number. Then the Legendre symbol

$$\left(\frac{m}{q}\right) = \begin{cases} 0 & \text{if } m \equiv 0 (\operatorname{mod} q), \\ 1 & \text{if } m \text{ is a quadratic residue modulo } q \text{ and } m \not\equiv 0 (\operatorname{mod} q), \\ -1 & \text{if } m \text{ is a quadratic non-residue modulo } q\ . \end{cases}$$

The symbol $\left(\frac{m}{q}\right)$ is periodic with period $q$, and completely multiplicative, i. e.,

$$\left(\frac{m_1 m_2}{q}\right) = \left(\frac{m_1}{q}\right)\left(\frac{m_2}{q}\right) \quad \text{for all } m_1, m_2 \in \mathbb{N}.$$

Thus, the sequence $\mathfrak{a} = \left\{\left(\frac{m}{q}\right) : m \in \mathbb{N}\right\}$ is periodic and multiplicative.

A more general periodic multiplicative arithmetic function is every Dirichlet character modulo $q$. A full definition of Dirichlet characters is sufficiently long and complicated, however, it is well known that every arithmetic function $\chi(m)$ satisfying the hypotheses

1. $\chi(m)$ is completely multiplicative;

2. $\chi(m)$ is periodic with period $q$;

3. $\chi(m) = 0$ for $(m, q) > 1$;

4. $\chi(m) \neq 0$ for $(m, q) = 1$

coincides with a Dirichlet character modulo $q$.

Let $\chi(m)$ be a Dirichlet character modulo $q$. Then the series

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}, \quad \sigma>1,$$

is called a Dirichlet $L$-function, and used in the theory of distribution of prime numbers in arithmetic progressions.

In the dissertation, approximation properties of the function $\zeta(s;\mathfrak{a})$ with periodic multiplicative sequence $\mathfrak{a}$ are investigated. Note that in this case the function $\zeta(s;\mathfrak{a})$ has the Euler product

$$\zeta(s;\mathfrak{a})=\prod_{p\in\mathbb{P}}\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}}{p^{ls}}\right), \quad \sigma>1.$$

We also note that multiplicativity of the sequence $\mathfrak{a}$ describes a certain class of approximated analytic functions.

## 1.2 Aim and problems

The aim of the dissertation is an approximation of analytic functions defined in the strip $D=\{s\in\mathbb{C}:1/2<\sigma<1\}$ by shifts $\zeta(s+i\tau;\mathfrak{a})$, $\tau\in\mathbb{R}$. The problems are the following:

1. Approximation of a class of analytic functions by shifts $\zeta(s+iht_n;\mathfrak{a})$, $h>0$, where $\{t_n:n\in\mathbb{N}\}$ is a sequence of Gram points.

2. Joint approximation of a collection of analytic functions by shifts $(\zeta(s+ih_1\gamma_k;\mathfrak{a}_1),\dots,\zeta(s+ih_r\gamma_k;\mathfrak{a}_r))$, $h_j>0$, where $\{\gamma_k:k\in\mathbb{N}\}$ is a sequence of positive imaginary parts of non-trivial zeros of the Riemann zeta-function.

3. Approximation of a class of analytic functions by shifts $F(\zeta(s+ih_1\gamma_k;\mathfrak{a}_1),\dots,\zeta(s+ih_r\gamma_k;\mathfrak{a}_r))$ with certain continuous operators $F$ in the space of analytic functions.

4. Joint approximation of a class of analytic functions by generalized nonlinear shifts $(\zeta(s+i\varphi_1(\tau);\mathfrak{a}_1),\dots,\zeta(s+i\varphi_r(\tau);\mathfrak{a}_r))$.

## 1.3 Actuality

Analytic functions are not only an object of function theory. They play an important role in analytic and algebraic number theory, differential and integral equation problems, probability theory, functional analysis, mathematical physics and other branches of mathematics, and also occur in solving some problems of physics and other natural sciences. Therefore, problems arise in simplifying complicated analytic functions, and this leads to the approximation of analytic functions by simpler ones. It is well known that every analytic function continuous on a compact set $K$ with connected complement and analytic inside of that set can be approximated by a polynomial, uniformly on $K$. Thus, for every analytic function, there exists the corresponding approximating polynomial depending on the approximation function. Probably fifty years ago, it became known that there are comparatively simple functions, some zeta- and $L$-functions, having universal approximation property: shifts of one and the same function approximate the whole class of analytic functions. This unexpected progress in approximation theory opened new problems as the description of classes of universal functions, effectivization of the universality, description of classes of approximating shifts, and, of course, application of the universality phenomenon for solving mathematical and practical problems. In many scientific mathematical centers (Australia, Canada, France, Germany, India, Japan, S. Korea, Sweden, USA, etc.) groups investigating universality problems in approximation theory were created. The most numerous such a group is successfully working in Lithuania (I. Belovas, R. Garunkštis, R. Kačinskaitė, A. Laurinčikas, R. Macaitienė, etc.). The Lithuanian group obtained significant results in the effectivization of universality, extended the class of universal functions, introduced new types of universality, applied for approximation of new classes of shifts. However, as Professor A. Schinzel said, every proved theorem raises three new problems. Since universality problems are interesting and important for many fields of science, the development of universality is one of the modern directions of mathematics.

## 1.4 Methods

Probabilistic methods based on weak convergence of probability measures and their properties in the space of analytic functions occupy a central place in the proofs of universality theorems of the dissertation. Moreover, methods of analytic number theory, including various mean square estimates, Cauchy's integral formula, integral representations and properties of polynomials are applied.

## 1.5 Novelty

All results of the dissertation are new. Universality theorems for periodic zeta-functions with generalized shifts including Gram points, imaginary parts of non-trivial zeros of the Riemann zeta-function, and algebraic numbers linearly independent over $\mathbb{Q}$ earlier were not known.

## 1.6 History of the problem and the main results

Zeta- or $L$-functions usually are defined by ordinary Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a(m)}{m^s}$$

for $\sigma > \sigma_0$ with some $\sigma_0$, or general Dirichlet series

$$\sum_{m=1}^{\infty} b(m)\mathrm{e}^{-\lambda_m s}, \quad \sigma > \sigma_0,$$

where $\{a(m)\}$ and $\{b(m)\}$ are sequences of complex numbers, and $\{\lambda_m\} \subset \mathbb{R}$, $\lim_{m\to\infty} \lambda_m = +\infty$. For example, the Riemann zeta-function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

and by meromorphic continuation to the whole complex plane, except the point $s = 1$ which is its unique simple pole with residue 1. The definition of the Epstein zeta-function is more complicated. Let $Q$ be a positive definite quadratic $n \times n$ matrix, and [10], for $\underline{x} \in \mathbb{Z}^n$, $Q[\underline{x}] = \underline{x}^T Q \underline{x}$, where $\underline{x}^T$ denotes the transposition of $\underline{x}$. Then the Epstein zeta-function $\zeta(s; Q)$ for $\sigma > n/2$ is defined by the series

$$\zeta(s; Q) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q[\underline{x}])^{-s},$$

however, if $Q[\underline{x}] \in \mathbb{Z}$ for all $\underline{x} \in \mathbb{Z}^n$, then

$$\zeta(s;Q) = \sum_{m=1}^{\infty} \frac{r_Q(m)}{m^s}, \quad \sigma > \frac{n}{2},$$

where $r_Q(m)$ denotes the number of $\underline{x} \in \mathbb{Z}^n$ such that $Q[\underline{x}] = m$. Thus, we have again an ordinary Dirichlet series. By the way, the function $\zeta(s;Q)$ can be analytically continued to the whole complex plane, except for a simple pole at the point $s = n/2$, and

$$\operatorname*{Res}_{s=n/2} \zeta(s;Q) = \frac{\pi^{n/2}}{\Gamma(n/2)\sqrt{\det Q}},$$

here $\Gamma(s)$ is the Euler gamma-function.

The Hurwitz zeta-function $\zeta(s,\alpha)$ is defined by general Dirichlet series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \mathrm{e}^{-s\log(m+\alpha)}, \quad \sigma > 1,$$

i. e., with $\lambda_m = \log(m+\alpha)$ and $b(m) \equiv 1$.

Zeta-functions $Z(s)$ of the type

$$Z(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \quad \sigma > \sigma_0,$$

are generating functions of the sequence $\{a(m)\}$, they are applied to obtain information on the mean value

$$m(x) \stackrel{\text{def}}{=} \sum_{m \leqslant x} a(m), \quad x \to \infty.$$

There exist formulae of various forms involving a contour integral with the function $Z(s)$ which gives a representation for $m(x)$ with some error term. This can be clearly illustrated by the case of the function

$$\pi(x) = \sum_{p \leqslant x} 1, \quad x \to \infty.$$

Let $\Lambda(m)$ be the von Mangoldt function, i. e.,

$$\Lambda(m) = \begin{cases} \log p & \text{if } m = p^k,\, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The mean value

$$m_\Lambda(x) \stackrel{\text{def}}{=} \sum_{m \leqslant x} \Lambda(m), \quad x \to \infty,$$

is closely connected to $\pi(x)$; the asymptotics for $m_\Lambda(x)$ as $x \to \infty$ leads to that for $\pi(x)$. Therefore, the generating function

$$Z_\Lambda(s) \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s}, \quad \sigma > 1,$$

is considered. Moreover, it is easily seen that

$$Z_\Lambda(s) = -\frac{\zeta'(s)}{\zeta(s)}. \tag{I3}$$

Thus, we can use the described above representation

$$m_\Lambda(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) \mathrm{d}s + \text{error term}(T, x, \sigma),$$

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho} \left( \frac{1}{s-\rho} - \frac{1}{\rho} \right) + \text{simple term},$$

where $\rho$ are zeros of the function $\zeta(s)$ with $\mathrm{Re}\rho \geqslant 1/2$. This, together with (I3) gives the asymptotics for $m_\Lambda(x)$, and therefore, for $\pi(x)$ as $x \to \infty$. This example explains why it is important to study the zero-distribution of the function $\zeta(s)$.

The above remarks show that, first of all, we have to study the value distribution of zeta-function and then apply the obtained results to solve other problems.

Value distribution of zeta-functions includes analytic continuation, functional equations, various estimates, asymptotics, and estimates for moments

$$\int_0^T |Z(\sigma + it)|^{2k} \, \mathrm{d}t,$$

probabilistic limit theorems, zero-distribution, approximation properties, etc.

The zeta-function $\zeta(s;\mathfrak{a})$ is an attractive analytic object, and was investigated by numerous researchers. The first significant result for $\zeta(s;\mathfrak{a})$ is the functional equation obtained in [61]. For its statement, the notation

$$\mathfrak{b}=\{b_m : m\in\mathbb{Z}\},\quad b_m=\frac{1}{q}\sum_{l=0}^{q-1}a_l\mathrm{e}^{-2\pi il(m/q)},$$
$$\widehat{\mathfrak{b}}=\{\widehat{b}_m : \widehat{b}_m=b_{-m}, m\in\mathbb{Z}\}$$

is used. The sequences $\mathfrak{b}$ and $\widehat{\mathfrak{b}}$ are periodic with period $q$. Therefore, we have periodic zeta-functions $\zeta(s;\mathfrak{b})$ and $\zeta(s;\widehat{\mathfrak{b}})$ that appear in the functional equation [61]

$$\zeta(1-s;\mathfrak{a})=\left(\frac{q}{2\pi}\right)^s\Gamma(s)\left(\mathrm{e}^{\pi i(s/2)}\zeta(s;\mathfrak{b})+\mathrm{e}^{-\pi i(s/2)}\zeta(s;\widehat{\mathfrak{b}})\right).$$

The function $\zeta(s;\mathfrak{a})$ is connected in a certain sense to the Riemann zeta-function. Let $\widetilde{b}_m=\sqrt{q}\,b_m$, and $\mathfrak{c}_l=\{c_m : c_m=\mathrm{e}^{2\pi im(l/q)}, m\in\mathbb{N}\}$,

$$\zeta(s,\mathfrak{c}_l)=\sum_{m=1}^{\infty}\frac{c_m}{m^s},\quad \sigma>1.$$

Then in [18] it was proved that

$$\zeta(s;\mathfrak{a})=\frac{\widetilde{b}_q}{\sqrt{q}}\zeta(s)+\frac{1}{\sqrt{q}}\sum_{m=1}^{q-1}\widetilde{b}_m\zeta(s;\mathfrak{c}_l).$$

Important results for $\zeta(s;\mathfrak{a})$ were obtained by J. Steuding. He created [64] the theory of zero-distribution for $\zeta(s;\mathfrak{a})$. Denote by $\rho_{\mathfrak{a}}=\beta_{\mathfrak{a}}+i\gamma_{\mathfrak{a}}$ the zeros of $\zeta(s;\mathfrak{a})$, i. e., $\zeta(\rho_{\mathfrak{a}};\mathfrak{a})=0$, and let

$$c_{\mathfrak{a}}=\max\{|a_m| : 1\leqslant m\leqslant q\},\quad m_{\mathfrak{a}}=\min\{1\leqslant m\leqslant q : |a_m|\neq 0\}$$

and

$$A(\mathfrak{a})=\frac{c_{\mathfrak{a}}m_{\mathfrak{a}}}{|a_{m_{\mathfrak{a}}}|}.$$

Then J. Steuding proved [64] that $\zeta(s;\mathfrak{a})\neq 0$ in the half-plane $\sigma>1+A(\mathfrak{a})$. Moreover, in the above notation, let define the numbers $\widetilde{b}_m^{\pm}$, where $\widetilde{b}_m^{-}$ coincides with $\widetilde{b}_m$, and $\widetilde{b}_m^{+}$ is obtained from $\widetilde{b}_m^{-}$ taking "+" in place "-" in the exponent. Define $\widetilde{\mathfrak{b}}^{\pm}=\{\widetilde{b}_m^{\pm} : m\in\mathbb{N}\}$ and $B(\mathfrak{a})=\max\{A(\widetilde{\mathfrak{b}}^{+}),A(\widetilde{\mathfrak{b}}^{-})\}$.

Then the following assertion is valid. Suppose that $m_{\tilde{\mathfrak{b}}+} = m_{\tilde{\mathfrak{b}}-}$, then in [65] it was noted that the function $\zeta(s;\mathfrak{a})$ has zeros only close to negative real axis. If $m_{\tilde{\mathfrak{b}}+} \neq m_{\tilde{\mathfrak{b}}-}$, then $\zeta(s;\mathfrak{a})$ has zeros near the line

$$\sigma = 1 + \frac{\pi t}{\log(m_{\tilde{\mathfrak{b}}-}/m_{\tilde{\mathfrak{b}}+})}.$$

If $\beta < -B(\mathfrak{a})$, then the zeros $\rho_{\mathfrak{a}} = \beta_{\mathfrak{a}} + i\gamma_{\mathfrak{a}}$ are called trivial, otherwise non-trivial. Let $N(T;\mathfrak{a})$ be the number of non-trivial zeros of $\zeta(s;\mathfrak{a})$ with $|\gamma_{\mathfrak{a}}| \leqslant T$ counted with multiplicity. Then the formula

$$N(T;\mathfrak{a}) = \frac{T}{\pi}\log\frac{qT}{2\pi e m_{\mathfrak{a}}\sqrt{m_{\mathfrak{b}-}m_{\mathfrak{b}+}}} + O(\log T)$$

is valid [65]. The monograph [65] also contains other results on the distribution of zeros of $\zeta(s;\mathfrak{a})$.

In [67], see also [46], the approximate functional equation for $\zeta(s;\mathfrak{a})$ has been proved involving finite sums and connecting the variables $s$ and $1-s$. Moreover, the asymptotics for the mean square

$$\int\limits_0^T |\zeta(\sigma_T + it;\mathfrak{a})|^2\,\mathrm{d}t$$

with $\sigma_T = 1/2$ and $\sigma_T \to 1/2 + 0$ as $T \to \infty$ was obtained. Also, joint limit theorems for a collection of periodic zeta-functions $\zeta(s;\mathfrak{a}_1), \ldots, \zeta(s;\mathfrak{a}_r)$ were proved, i. e., the weak convergence for

$$\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : (\zeta(s+i\tau;\mathfrak{a}_1), \ldots, \zeta(s+i\tau;\mathfrak{a}_r)) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{X}(G)),$$

as $T \to \infty$ was considered. Here $\mathbb{X}(G)$ is the space of analytic or meromorphic on $G = \{s \in \mathbb{C} : \sigma > 1/2\}$ functions, and $\mathcal{B}(\mathbb{Y})$ denotes the Borel $\sigma$-field of a topological space $\mathbb{Y}$, and $\text{meas}A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Now, we focus on the approximation of classes of analytic functions by shifts of zeta-functions $Z(s+i\tau)$, in particular, by $\zeta(s+i\tau;\mathfrak{a})$. This property is called universality, and was discovered by S.M. Voronin in [72] for the Riemann zeta-function.

**Theorem 1.** *[72]. Suppose that $0 < r < 1/4$ is fixed, the function $f(s)$ is continuous and non-vanishing on $|s| \leqslant r$, and analytic in $|s| < r$. Then, for every $\varepsilon > 0$, there exists a number $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that*

$$\max_{|s| \leqslant r} \left| f(s) - \zeta\left(s + \frac{3}{4} + i\tau\right) \right| < \varepsilon.$$

Voronin also wrote in [72] that a similar assertion is true for all Dirichlet $L$-functions $L(s, \chi)$. Voronin's proof is based on the analogue of the Riemann theorem in the rearrangement of series with terms in Hilbert spaces.

Sufficiently soon, Theorem 1 was extended and improved in the theses [3] and [14]. At the moment, we have the following last version of Theorem 1.6. Recall that $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, $\mathcal{K}$ is the class of compact subsets of the strip $D$ with connected complements, and let $H_0(K)$, $K \in \mathcal{K}$, denote the class of continuous non-vanishing functions on $K$ that are analytic in the interior of $K$, and $\mathrm{meas}A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the following statement is known, see [20, 35, 65, 51].

**Theorem 2.** *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |f(s) - \zeta(s + i\tau)| < \varepsilon \right\} > 0.$$

*Moreover, the limit*

$$\lim_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |f(s) - \zeta(s + i\tau)| < \varepsilon \right\}$$

*exists and is positive for all but at most countably many $\varepsilon > 0$.*

The second assertion of the theorem is comparatively not old, it was obtained independently in [53] and [45].

Later, it turned out that some other zeta- and $L$-functions also are universal in the above sense (in the Voronin sense), however, it is not easy to prove that, and there are non-universal Dirichlet series, see, for example, [65].

The first universality theorem for the periodic zeta-function $\zeta(s; \mathfrak{a})$ has been proven by Steuding in [64], see also [65]. Let $H(K)$ with $K \in \mathcal{K}$ denote the class of continuous on $K$ functions that are analytic in the interior of $K$. Thus, $H(K)$ is an extension of the class $H_0(K)$.

**Theorem 3.** *Suppose that the period $q > 2$, $a_m$ not a multiple of Dirichlet character modulo $q$, and $a_m = 0$ for $(m, q) > 1$. Let $K \in \mathcal{K}$, and $f(s) \in$*

$H(K)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |f(s) - \zeta(s+i\tau;\mathfrak{a})| < \varepsilon\right\} > 0.$$

We observe that, by [64], a periodic sequence satisfying the conditions of Theorem 3 is not multiplicative.

J. Kaczorovski observed in [19] that not all Dirichlet series with periodic coefficients are universal in the Voronin sense. He obtained necessary and sufficient conditions that the function $\zeta(s;\mathfrak{a})$ would be universal with prime period $q$.

**Theorem 4.** *[19]. Let* $q$ *be a prime number and* $\mathfrak{a} \neq 0$, *the corresponding function* $\zeta(s;\mathfrak{a})$ *is universal in the Voronin sense if and only if one of the following possibilities holds:*

1. *Not all numbers* $a_1, \dots, a_{q-1}$ *are equal;*

2. *We have* $a_1 = \dots = a_{q-1} = 0$*;*

3. *We have* $a_1 = \dots = a_{q-1} \neq 0$ *and*

$$\left|1 - \frac{a_q}{a_1}\right| \leqslant \sqrt{q} \qquad \text{or} \qquad \left|1 - \frac{a_1}{a_q}\right| \geqslant q.$$

For example, suppose that $q = 3$, $a_1 = 1$, $a_2 = 4$, $a_3 = 3$. Then

$$\left|1 - \frac{a_3}{a_1}\right| = 2 > \sqrt{3} \quad \text{and} \quad \left|1 - \frac{a_1}{a_3}\right| = \frac{2}{3} < 3.$$

Therefore, the function $\zeta(s;\mathfrak{a})$, in view of Theorem 4, in this case, is not universal.

The first universality theorem for $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ has been obtained in [47].

**Theorem 5.** *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative. Let* $K \in \mathcal{K}$, *and* $f(s) \in H_0(K)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |f(s) - \zeta(s+i\tau;\mathfrak{a})| < \varepsilon\right\} > 0.$$

We note that, in [47], the additional requirement on the sequence $\mathfrak{a}$ that, for all $p \in \mathbb{P}$,

$$\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\alpha/2}} \leqslant c < 1,$$

was used, however, it is technical and can be easily removed. Thus, by Theorem 5, all functions $\zeta(s; \mathfrak{a})$ with multiplicative $\mathfrak{a}$ are universal, i. e., have the same approximation property as Dirichlet $L$-functions.

All the above-mentioned universality theorems are of continuous type because $\tau$ in shifts $\zeta(s+i\tau; \mathfrak{a})$ can take arbitrary real values. There exists another type of universality for zeta-functions, called the discrete universality, where $\tau$ in approximating shifts takes values from a certain discrete set, for example, from the arithmetical progression $\{hk\}$, $k \in \mathbb{N}_0$. Discrete universality theorems were proposed by A. Reich in [59]. He proved a discrete universality theorem for Dedekind zeta-functions $\zeta_{\mathbb{K}}(s)$ of algebraic number fields. When $\mathbb{K} = \mathbb{Q}$, then the class of Dedekind zeta-functions includes the Riemann zeta-function. Therefore, a corollary of Reich's universality theorem is the following statement. Denote by $\#A$ the cardinality of a set $A \subset \mathbb{R}$.

**Theorem 6.** *Let $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $h > 0$ be fixed. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s+ikh)| < \varepsilon \right\} > 0.$$

Discrete universality theorems for zeta-functions using another method were studied by B. Bagchi in [3].

The discrete universality for the function $\zeta(s; \mathfrak{a})$ is more complicated. The following result which is a consequence of a more general weighted universality theorem [50] is known. For $h > 0$, let

$$L(\mathbb{P}; h, \pi) = \{(h \log p : p \in \mathbb{P}), 2\pi\}.$$

**Theorem 7.** *[50]. Suppose that the set $\mathfrak{a}$ is multiplicative, and the set $L(\mathbb{P}; h, \pi)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s+ikh; \mathfrak{a})| < \varepsilon \right\} > 0.$$

One of the ways of extension of universality is using the so-called generalized shifts $\zeta(s + i\varphi(\tau))$ or $\zeta(s + i\varphi(k))$ with certain function $\varphi$. For the Riemann zeta-function and Dirichlet $L$-functions, this was done by Ł. Pańkowski in [57]. Let

$$b \in \begin{cases} \mathbb{R} & \text{if} \quad a \notin \mathbb{N}, \\ (-\infty, 0] \cup (1 + \infty) & \text{if} \quad a \in \mathbb{N}. \end{cases}$$

**Theorem 8.** *[57]. Suppose that $\alpha \in \mathbb{R}$, $a$ is a positive real number. Let $K \in \mathcal{K}$, $f(s) \in H_0(K)$. Then , for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [2, T] : \sup_{s\in K} |f(s) - \zeta(s + i\alpha\tau^a \log^b \tau)| < \varepsilon \right\} > 0,$$

*and*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 2 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + i\alpha k^a \log^b k)| < \varepsilon \right\} > 0.$$

Theorem 8 generalizes the works [7] and [43].

Chapter 2 of the dissertation is devoted to a discrete universality theorem for the function $\zeta(s; \mathfrak{a})$ with generalized shifts involving the Gram numbers that have an old and long history. The function $\zeta(s)$ satisfies, for all $s \in \mathbb{C}$, the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

The main ingredient of this equation is the function $g(s) \overset{\text{def}}{=} \pi^{-s/2}\Gamma(s/2)$. Denote by $\theta(t)$, $t \geqslant 0$, the increment of the argument of the function $g(s)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. It is known [15] that the function $\theta(t)$ is monotonically increasing and unbounded from above for $t > t^*$, $t^* = 6.289835\ldots$, therefore, the equation

$$\theta(t) = (n-1)\pi, \quad n \in \mathbb{N},$$

for $t > t^*$ has the unique solution $t_n$. The numbers $t_n$ are now called the Gram numbers because he considered them in connection with non-trivial zeros of $\zeta(s)$. Recall that zeros of $\zeta(s)$ lying in the strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$ are called non-trivial, while zeros $s = -2k$, $k \in \mathbb{N}$, are trivial. By the Riemann hypothesis, non-trivial zeros of $\zeta(s)$ are of the form $\rho_n = 1/2 + i\gamma_n$. All known non-trivial zeros at the moment support the Riemann hypothesis. J.-P. Gram in [15] observed that each interval $(t_{n-1}, t_n]$, $n = 1, \ldots, 15$, contains precisely one zero $1/2 + i\gamma_n$ of the function $\zeta(1/2 + it)$ such that $t_{n-1} < \gamma_n < t_n$.

Moreover, he conjectured that this is not true for $n > 15$. It turned out, that the Gram hypothesis is true. For example, it was found [70, 17] that

$$t_{127} < \gamma_{127} < \gamma_{128} < t_{128} \qquad \text{and} \qquad t_{134} < \gamma_{134} < \gamma_{135} < t_{135}.$$

This was proved and analytically in [70], namely, that the sequence

$$\frac{\gamma_n - t_n}{t_{n+1} - t_n}$$

is unbounded, therefore, the zero $\gamma_n$ can't lie in $(t_{n-1}, t_n]$ for sufficiently many $n$.

The Gram points also were considered by A. Selberg [62], and very deeply in [21] – [28].

In general, the points $t_n$ are very interesting object because it is known that

$$\lim_{n\to\infty} \frac{t_n}{\gamma_n} = 1.$$

For the first time, approximation of analytic functions by shifts involving Gram points was considered in [29].

**Theorem 9.** *Let $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $h > 0$ be fixed. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k)| < \varepsilon \right\} > 0.$$

*Moreover, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k)| < \varepsilon \right\}$$

*exists and is positive for all but at most countably many $\varepsilon > 0$.*

In [30], Theorem 9 was extended to short intervals. Namely, the assertion of Theorem 9 is valid for the quantity

$$\frac{1}{M+1} \# \left\{ N \leqslant k \leqslant N + M : \sup_{s\in K} |f(s) - \zeta(s + iht_k)| < \varepsilon \right\}$$

with $N \to \infty$, and $M$ restricted by

$$\left(\frac{3\pi N}{h^2}\right)^{1/3} (\log\{(h+1)N\})^{12/5} \leqslant M \leqslant N.$$

The main result of Chapter 2 of the dissertation extends Theorem 9 to the function $\zeta(s; \mathfrak{a})$ [69].

**Theorem 2.1.** *Suppose that the sequence $\mathfrak{a}$ is multiplicative. Let $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $h > 0$ be a fixed number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k; \mathfrak{a})| < \varepsilon \right\} > 0.$$

*Moreover, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k; \mathfrak{a})| < \varepsilon \right\}$$

*exists and is positive for all but at most countably many $\varepsilon > 0$.*

In Chapter 3, the generalized shifts $\zeta(s + ih\gamma_k; \mathfrak{a})$, where $0 < \gamma_1 < \gamma_2 < \cdots \leqslant \gamma_k \leqslant \cdots$ is the sequence of positive imaginary parts of non-trivial zeros of the Riemann zeta-function, are discussed. Therefore, we recall some results in this direction.

At the moment, it is known that more than five-twelfths of all non-trivial zeros of $\zeta(s)$ lie on the critical line [58]. Moreover, the Riemann hypothesis is supported by very big computer calculations. However, the Riemann hypothesis until our days remains one of the six open Millennium problems of mathematics [54].

In general, the distribution of the sequence $\{\gamma_k\}$ is very complicated, therefore, some hypotheses are proposed. One of them, the Mongomery pair correlation conjecture [55] asserts that

$$\sum_{\substack{\gamma_k, \gamma_l \leqslant T \\ (2\pi\alpha_1)/\log T \leqslant \gamma_k - \gamma_l \leqslant (2\pi\alpha_2)/\log T}} 1 \sim \left( \int_{\alpha_1}^{\alpha_2} \left( 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 \right) \mathrm{d}u + \delta(\alpha_1, \alpha_2) \right) \frac{T}{2\pi} \log T$$

as $T \to \infty$, where $\alpha_1 < \alpha_2$ are arbitrary real numbers, and

$$\delta(\alpha_1, \alpha_2) = \begin{cases} 1 & \text{if} \quad 0 \in [\alpha_1, \alpha_2], \\ 0 & \text{otherwise.} \end{cases}$$

In approximation by shifts $\zeta(s + ih\gamma_k)$, the condition inspired by the above Montgomery conjecture is used. Namely, in [13], the conjecture that, for $c > 0$,

$$\sum_{\substack{\gamma_k, \gamma_l \leqslant T \\ |\gamma_k - \gamma_l| < c/\log T}} 1 \ll T \log T \tag{I4}$$

was proposed.

**Theorem 10.** *[13]. suppose that estimate* (I4) *is valid. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $h > 0$ and $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} |f(s) - \zeta(s + ih\gamma_k)| < \varepsilon \right\} > 0.$$

*Moreover, for every $h > 0$, the limit*

$$\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} |f(s) - \zeta(s + ih\gamma_k)| < \varepsilon \right\}$$

*exists and is positive for all but at most countably many $\varepsilon > 0$.*

In [11] and [12], Theorem 10 was derived assuming the Riemann hypothesis instead (I4).

We note that recently in [63] the analogical result to Theorem 10 was obtained for some subset of the set $\{\gamma_k\}$ without using the Mongomery conjecture.

In [39], Theorem 10 was extended for the Hurwitz zeta-function $\zeta(s, \alpha)$ with $\alpha$ such that the set $\{\log(m + \alpha) : m \in \mathbb{N}_0\}$ is linearly independent over the field of rational numbers.

Universality theorems for zeta-functions have their joint versions. In this case, a collection of analytic functions is simultaneously approximated by a collection of shifts of zeta-functions. The first joint universality theorem was obtained by Voronin in [73] for Dirichlet $L$-functions.

The statement of the joint universality theorem involves the notion of non-equivalent Dirichlet characters. Let $\chi(m)$ be a Dirichlet character modulo $q$.

The character $\chi$ is called generated by a character $\chi_1(m)$ modulo $q_1 \mid q$ if

$$\chi(m) = \begin{cases} \chi_1(m) & \text{if } q_1 \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

The character $\chi$ is called primitive if it is not generated by any character modulo $q_1 \mid q$. Two characters $\chi_1$ and $\chi_2$ are called non-equivalent if they are not generated by the same primitive character.

Now, we state an improved version of the joint Voronin universality theorem for Dirichlet $L$-functions obtained in [37].

**Theorem 11.** *Suppose that the characters $\chi_1, \ldots, \chi_r$ are pairwise non-equivalent. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - L(s + i\tau, \chi_j)| < \varepsilon\right\} > 0.$$

In the joint case, the approximating shifts must be in some sense independent. In Theorem 11, this independence is realized by a pairwise non-equivalence of Dirichlet characters $\chi_1, \ldots, \chi_r$. There exists an another way which uses shifts $(L(s + i\varphi_1(\tau), \chi_1), \ldots, L(s + i\varphi_r(\tau), \chi_r))$ with arbitrary characters. This was proposed by Ł. Pańkowski in [57] with the functions $\varphi_j(\tau) = \tau^{\alpha_j} \log^{\beta_j} \tau$, where $\alpha_j$ and $\beta_j$ are certain real numbers, $j = 1, \ldots, r$.

Chapter 3 of the dissertation is devoted to a joint universality theorem for periodic zeta-functions with multiplicative coefficients by using shifts $\zeta(s + ih_j\gamma_k; \mathfrak{a}_j)$, $j = 1, \ldots, r$. The main result of the chapter is the following theorem [49]. Recall that a number $\alpha$ is algebraic if it is a root of a polynomial with rational coefficients.

**Theorem 3.1.** *Suppose that the sequences $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are multiplicative, $h_1, \ldots, h_r$ are positive algebraic numbers linearly independent over the field of rational numbers, and estimate* (I4) *is valid. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s + ih_j\gamma_k; \mathfrak{a}_j)| < \varepsilon\right\} > 0.$$

*Moreover, the limit*

$$\lim_{N\to\infty}\frac{1}{N}\#\left\{1\leqslant k\leqslant N:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s)-\zeta(s+ih_j\gamma_k;\mathfrak{a}_j)|<\varepsilon\right\}$$

*exists and is positive for all but at most countably many* $\varepsilon>0$.

Theorem 3.1 is a certain joint generalization of Theorem 2.1.

One of the ways to extend the class of universal functions in the Voronin sense is investigations of universality for some compositions. This way was proposed in [36] for the Riemann zeta-function. Earlier, it was obtained in [34] that the function $\zeta'(s)/\zeta(s)$ is universal, in this case the approximation function $f(s)$ is not necessarily non-vanishing, i. e., $f(s)\in H(K)$ for $K\in\mathcal{K}$. In [38], it was obtained that universality remains valid for $F(\zeta(s))$ for more general operators $F:H(D)\to H(D)$ of the Lipschitz type. More precisely, $F$ has the following properties:

1. For each polynomial $p=p(s)$ and every set $K\in\mathcal{K}$, there exists an element $q\in F^{-1}\{p\}\subset H(D)$ such that $q(s)\neq 0$ on $K$;

2. For every $K\in\mathcal{K}$, there exist a constant $c>0$, a set $K_1\in\mathcal{K}$ and $\alpha>0$ for which

$$\sup_{s\in K}|F(g_1(s))-F(g_2(s))|\leqslant c\sup_{s\in K_1}|g_1(s)-g_2(s)|^{\alpha}$$

holds for all $g_1,g_2\in H(D)$.

Denote the above class of operators $Lip(\alpha)$. Then the following statement is true [38].

**Theorem 12.** *Suppose that* $F\in Lip(\alpha)$. *Let* $K\in\mathcal{K}$ *and* $f(s)\in H(K)$. *Then, for every* $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|f(s)-F(\zeta(s+i\tau))|<\varepsilon\right\}>0. \quad (I5)$$

Now, let

$$S=\{g\in H(D):g(s)\neq 0 \text{ on } D, \text{ or } g(s)\equiv 0\}.$$

Then we have one more type of universal compositions [38].

**Theorem 13.** *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that for any open set $G \subset H(D)$,*

$$F^{-1}(G) \cap S \neq \varnothing.$$

*Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then inequality* (I5) *is true.*

Also, in [38], there is a theorem on approximation of analytic functions from some more narrow class than $H(K)$ by shifts $F(\zeta(s + i\tau))$.

In [44], the results of [38] were extended for zeta-functions of certain cusp forms. Universality of compositions $F(\underline{\zeta}(s + ikh, \alpha; \mathfrak{a}, \mathfrak{b}))$, where $F : H^r(D) \to H(D)$, and $\underline{\zeta}(s + ikh, \alpha; \mathfrak{a}, \mathfrak{b})$ is a collection consisting from periodic and periodic Hurwitz zeta-functions was obtained in [40].

In Chapter 4 of the dissertation, universality theorems for the compositions $F(\zeta(s; \mathfrak{a}_1), \dots \zeta(s; \mathfrak{a}_r))$, where $F : H^r(D) \to H(D)$, by using shifts $F(\zeta(s+ih_1\gamma_k; \mathfrak{a}_1), \dots, \zeta(s+ih_r\gamma_k; \mathfrak{a}_r))$ were obtained. Let $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_r)$, $\underline{g} = (g_1, \dots, g_r) \in H^r(D)$ and $\underline{h} = (h_1, \dots, h_r)$, $h_j > 0$, $j = 1, \dots, r$. For brevity, denote $\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) = (\zeta(s + ih_1\gamma_k; \mathfrak{a}_1), \dots, \zeta(s + ih_r\gamma_k; \mathfrak{a}_r))$, and say that the statement $A(\underline{\mathfrak{a}}, \underline{h}, \text{(I4)})$ holds if the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative, $h_1, \dots, h_r$ are positive algebraic numbers linearly independent over $\mathbb{Q}$, and statement (I4) is valid. For example, we may take $\underline{\mathfrak{a}} = (\chi_1(m), \dots, \chi_r(m))$, where $\chi_1(m), \dots, \chi_r(m)$ are Dirichlet characters modulo $q \in \mathbb{N}$, and $\underline{h} = (\sqrt{2}, \sqrt[3]{2}, \dots, \sqrt[r+1]{2})$ because the numbers $\sqrt{2}, \sqrt[3]{2}, \dots, \sqrt[r+1]{2}$ are linearly independent over $\mathbb{Q}$.

Define the class $Lip(\underline{\alpha})$ of operators $F : H^r(D) \to H(D)$. We say that $F \in Lip(\underline{\alpha})$, if:

1. For every polynomial $p = p(s)$ and sets $K_1, \dots, K_r \in \mathcal{K}$, there exists an element $\underline{g} \in F^{-1}\{p\} \subset H^r(D)$ such that $g_j(s) \neq 0$ on $K_j$, $j = 1, \dots, r$.

2. For every $K \subset \mathcal{K}$, there exist the sets $K_1, \dots, K_r \in \mathcal{K}$, a constant $c > 0$ and positive $\alpha_1, \dots, \alpha_r$ such that

$$\sup_{s \in K} \left| F(\underline{g}_1) - F(\underline{g}_2) \right| \leqslant c \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_{1j}(s) - g_{2j}(s)|^{\alpha_j}$$

   for all $\underline{g}_1, \underline{g}_2 \in H^r(D)$.

The first universality theorem for compositions $F(\underline{\zeta}(s; \underline{\mathfrak{a}}))$ is the following statement [68].

**Theorem 4.1.** *Suppose that $A(\underline{\mathfrak{a}}, \underline{h},$ (I4)) is valid, and $F : H^r(D) \to H(D)$ belongs to the class $Lip(\underline{\alpha})$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} \left| f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\right)\right| < \varepsilon \right\} > 0. \tag{I6}$$

For example, the operator

$$F(g_1, \dots, g_r) = c_1 g_1 + \dots + c_r g_r, \quad g_1, \dots, g_r \in H(D), \tag{I7}$$

with complex $c_j \neq 0$ belongs to the class $Lip(\underline{1})$.

The dissertation contains also other classes of operators $F$, see [68].

**Theorem 4.2.** *Suppose that $A(\underline{\mathfrak{a}}, \underline{h},$ (I4)) is valid, and $F : H^r(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the intersection $(F^{-1}G) \cap S^r$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the inequality* (I6) *is valid. Moreover, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} \left| f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\right)\right| < \varepsilon \right\} \tag{I8}$$

*exists and is positive for all but at most countably many $\varepsilon > 0$.*

The hypothesis $(F^{-1}G) \cap S^r$ can be modified.

**Theorem 4.3.** *Suppose that $A(\underline{\mathfrak{a}}, \underline{h},$ (I4)) is valid, and $F : H^r(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the intersection $(F^{-1}\{p\}) \cap S^r$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the inequality* (I6) *is valid, and the limit* (I8) *exists and is positive for all but at most countably many $\varepsilon > 0$.*

Obviously, the hypothesis $(F^{-1}\{p\} \cap S^r \neq \varnothing$ is more convenient than that $(F^{-1}G) \cap S^r \neq \varnothing$ for all open sets $G \subset H(D)$.

For some classes of approximated functions, the set $K \in \mathcal{K}$ can be replaced by an arbitrary compact set. This is given in the next theorem.

**Theorem 4.4.** *Suppose that $A(\underline{\mathfrak{a}}, \underline{h},$ (I4)) is valid, and $F : H^r(D) \to H(D)$ is a continuous operator. Let $K \subset D$ be a compact set, and $f(s) \in F(S^r)$. Then the assertion of Theorem 4.3 is true.*

Notice that it is difficult to describe the set $F(S^r)$. The problem becomes easier when it is known a certain simple set lying in $F(S^r)$. It is realized in the last theorem of the chapter. For distinct complex numbers $c_1, \dots, c_m$, define the set

$$H_{c_1,\dots,c_m}(D) = \{g \in H(D) : g(s) \neq c_j \text{ for all } s \in D,\ j = 1, \dots, m\}.$$

**Theorem 4.5.** *Suppose that $A(\underline{\mathfrak{a}}, \underline{h}, \text{(I4)})$ is valid, and $F : H^r(D) \to H(D)$ is a continuous operator such that $H_{c_1,\dots,c_m}(D) \subset F(S^r)$. For $m = 1$, let $K \subset \mathcal{K}$, $f(s) \in H(K)$ and $f(s) - c_1 \in H_0(K)$. For $m \geqslant 2$, let $K \subset D$ be arbitrary compact set, and $f(s) \in H_{c_1,\dots,c_m}(D)$. Then the assertion of Theorem 4.3 is true.*

For example, the operator (I7) satisfies the hypotheses of the theorem. Actually, if $g \in H_{c_1}(D)$, then $(g - c_1)/b_1 \in S$. Consequently, by the definition of $F$,

$$F\left(\frac{g - c_1}{b_1}, \frac{c_1}{b_2}, 0, \dots, 0\right) = g.$$

This shows that $g \in F(S^r)$.

Theorems 2.1 and 3.1 are of discrete type. The fifth chapter of the dissertation is devoted to joint continuous universality theorems for periodic zeta-functions. For this, continuous generalized shifts are used. Theorems of such a type were considered in [42], however, with ordinary shifts. For $j = 1, \dots, r$, let $\mathfrak{a}_j = \{a_{jm} : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q_j \in \mathbb{N}$. Denote by $q$ the least common multiple of the periods $q_1, \dots, q_r$, by $l_1, \dots, l_{r_1}$, where $r_1 = \varphi(q)$ is the Euler totient function, the reduced system modulo $q$, and define the matrix

$$A = \begin{pmatrix} a_{1l_1} & a_{2l_1} & \dots & a_{rl_1} \\ a_{1l_2} & a_{2l_2} & \dots & a_{rl_2} \\ \dots & \dots & \dots & \dots \\ a_{1l_{r_1}} & a_{2l_{r_1}} & \dots & a_{rl_{r_1}} \end{pmatrix}.$$

Then, in [42], we find the following result.

**Theorem 14.** *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative and* $\text{rank}(A) = r$. *For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for* *every* $\varepsilon > 0$*,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |f_j(s) - \zeta(s+i\tau;\mathfrak{a}_j)| < \varepsilon\right\} > 0.$$

We observe that in [42] a technical condition

$$\sum_{k=1}^{\infty} \frac{|a_{jp^k}|}{p^{k/2}} \leqslant c_j < 1, \quad j = 1,\dots,r,$$

is employed, however, it can be easily removed.

Also, there are more joint universality theorems for zeta-functions. They can be found in the survey paper [51].

In the dissertation, the condition $\mathrm{rank}(A) = r$ of Theorem 14 is replaced by using non-linear shifts $\zeta(s + i\varphi_j(\tau);\mathfrak{a}_j)$.

Denote by $U_1(T_0)$, $T_0 > 0$, the class of real increasing to $\infty$ continuously differentiable functions $\varphi(\tau)$ with monotonic derivative $\varphi'(\tau)$ on $[T_0,\infty)$ such that

$$\varphi(2\tau) \max_{\tau\leqslant u\leqslant 2\tau} \frac{1}{\varphi'(u)} \ll \tau, \quad \tau \to \infty.$$

Now we state the first theorem of Chapter 5 [48].

**Theorem 5.1.** *Suppose that the sequences* $\mathfrak{a}_1,\dots,\mathfrak{a}_r$ *are multiplicative,* $a_1$, $\dots, a_r$ *are real algebraic numbers linearly independent over the field of rational numbers* $\mathbb{Q}$*, and* $\varphi(\tau) \in U_1(T_0)$*. For* $j = 1,\dots,r$*, let* $K_j \in \mathcal{K}$ *and* $f_j(s) \in H_0(K_j)$*. Then, for every* $\varepsilon > 0$*,*

$$\liminf_{T\to\infty} \frac{1}{T-T_0}\mathrm{meas}\Bigg\{\tau \in [T_0,T] : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |f_j(s) - \zeta(s+ia_j\varphi(\tau);\mathfrak{a}_j)| < \varepsilon\Bigg\} > 0.$$

*Moreover, the limit*

$$\lim_{T\to\infty} \frac{1}{T-T_0}\mathrm{meas}\Bigg\{\tau \in [T_0,T] : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |f_j(s) - \zeta(s+ia_j\varphi(\tau);\mathfrak{a}_j)| < \varepsilon\Bigg\}$$

*exists and is positive for all but at most countably many* $\varepsilon > 0$*.*

The second joint universality theorem uses shifts $\zeta(s+i\varphi_j(\tau);\mathfrak{a}_j)$, $j=1,\dots,r$, which are more general as in Theorem 5.1. Denote by $U_r(T_0)$ the class of real increasing to infinity continuously differentiable functions $\varphi_1(\tau)$, $\dots,\varphi_r(\tau)$ on $[T_0,\infty)$ with derivatives

$$\varphi_j'(\tau)=\hat{\varphi}_j(\tau)(1+o(1)),$$

where $\hat{\varphi}_1(\tau),\dots,\hat{\varphi}_r(\tau)$ are monotonic and are compared in the sense that, for every subset $J\subset\{1,\dots,r\}$, $\#J\geqslant 2$, there exists $j_0=j_0(J)$ such that $\hat{\varphi}_j(\tau)=o(\hat{\varphi}_{j_0}(\tau))$ for $j\in J$, $j\neq j_0$, and

$$\hat{\varphi}_j(2\tau)\max_{\tau\leqslant u\leqslant 2\tau}\frac{1}{\hat{\varphi}_j(u)}\ll\tau,\quad j=1,\dots,r,\ \tau\to\infty.$$

**Theorem 5.2.** *Suppose that the sequences $\mathfrak{a}_1,\dots,\mathfrak{a}_r$ are multiplicative, and $(\varphi_1(\tau),\dots,\varphi_r(\tau))\in U_r(T_0)$. For $j=1,\dots,r$, let $K_j\in\mathcal{K}$ and $f_j(s)\in H_0(K_j)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T-T_0}\operatorname{meas}\left\{\tau\in[T_0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s)-\zeta(s+i\varphi_j(\tau);\mathfrak{a}_j)|<\varepsilon\right\}>0.$$

*Moreover, the limit*

$$\lim_{T\to\infty}\frac{1}{T-T_0}\operatorname{meas}\left\{\tau\in[T_0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s)-\zeta(s+i\varphi_j(\tau);\mathfrak{a}_j)|<\varepsilon\right\}$$

*exists and is positive for all but at most countably many $\varepsilon>0$.*

Denote by $p_r$ the $r$th prime number. Then in Theorem 5.1 we can take $(a_1,\dots,a_r)=(\sqrt{2},\sqrt{3},\sqrt{5},\dots,\sqrt{p_r})$, and $\varphi(\tau)=\tau\log\tau$, $\tau\geqslant 2$. In Theorem 5.2, we may use the functions $\varphi_1(\tau)=\tau\log\tau$, $\varphi_2=\tau^2\log\tau$, $\dots$, $\varphi_r(\tau)=\tau^r\log\tau$.

## 1.7 Approbation

The results of the dissertation were presented at:

- LXI Conference of Lithuanian Mathematical Society, December 4, 2020, Šiauliai, Lithuania;
- XIX International Conference "Algebra, number theory, discrete geometry and multiscale modeling: modern problems, applications and problems of history", dedicated to the bicentennial of the birth of academician P.L. Chebyshev, Tula, Russia, May 18–22, 2021;
- LXII Conference of Lithuanian Mathematical Society, June 16–17, 2021, Vilnius, Lithuania;
- XX International Conference "Algebra, Number Theory, Discrete Geometry and Multiscale Modeling: Modern Problems, Applications and Problems of History", Dedicated to 130 Anniversary of Academician I.M. Vinogradov, Tula, Russia, September 21–24, 2021;
- 25th International Conference Mathematical Modelling and Analysis, May 30 – June 2, 2022, Druskininkai, Lithuania;
- LXIII Conference of Lithuanian Mathematical Society, June 16–17, 2022, Kaunas, Lithuania;
- Lithuanian Conference on Probability Theory and Number Theory 2022, dedicated to the 100th anniversary of Lithuanian University and departments of Mathematical Analysis and Geometry of Vilnius University, September 5 – 10, 2022, Palanga, Lithuania;
- 26th International Conference Mathematical Modelling and Analysis, May 30 – June 2, 2023, Jūrmala, Latvia;
- LXIV Conference of Lithuanian Mathematical Society, June 21–22, 2023, Vilnius, Lithuania;
- International Conference on Probability Theory and Number Theory 2023, dedicated to commemorate the 100th anniversary of Professor Jonas Kubilius, to celebrate the 80th anniversary of Professor Donatas Surgailis, the 75th anniversary of Professors Antanas Laurinčikas and Eugenijus Manstavičius, and the 70th anniversary of Professors Kęstutis Kubilius and Alfredas Račkauskas, September 10–16, 2023, Palanga, Lithuania;

- 27th International Conference Mathematical Modelling and Analysis, May 28–31, 2024, Pärnu, Estonia;
- LXV Conference of Lithuanian Mathematical Society, June 27–28, 2024, Šiauliai, Lithuania;
- International Conference on Probability Theory and Number Theory 2024, dedicated to celebrate the 80th anniversary of Professor Vygantas Paulauskas, the 70th anniversary of Professors Konstantinas Pileckas and Gediminas Stepanauskas, and the 60th anniversary of Professors Artūras Dubickas and Olga Štikonienė, September 16–20, 2024, Palanga, Lithuania.

## 1.8 Main publications

The results of the dissertation are published in the following papers:

1. A. Laurinčikas, M. Tekorė, Joint universality of periodic zeta-functions with multiplicative coefficients, *Nonlinear Anal.: Modell. Control* **25** (2020), 860–883.
2. A. Laurinčikas, D. Šiaučiūnas, M. Tekorė, Joint universality of periodic zeta-functions with multiplicative coefficients. II, *Nonlinear Anal.: Modell. Control* **26** (2021), 550–564.
3. M. Tekorė, On joint universality of periodic zeta-functions with multiplicative coefficients, *Proceedings of the XIX International Conference "Algebra, number theory, discrete geometry and multiscale modeling: contemporary problems, applications and problems of history" dedicated to the bicentennial of the birth of Academician P.L. Chebyshev*, May 18 – 22, Tula, L. N. Tolstoi Tula State Pedagogical University, Tula, 2021, pp. 189–192.
4. D. Šiaučiūnas, R. Šimėnas and M. Tekorė, Approximation of analytic functions by shifts of certain compositions, *Mathematics* **9** (2021), 2583.
5. D. Šiaučiūnas and M. Tekorė, Gram points in the universality of the Dirichlet series with periodic coefficients, *Mathematics* **11** (2023), 4615.

Abstracts of conferences:

1. D. Šiaučiūnas and M. Tekorė, On joint universality of Dirichlet L-functions, 25th international conference on Mathematical Modelling and Analysis, May 30 – June 2, 2022, Druskininkai, Lithuania, abstracts, Vilnius Gediminas Technical University, Vilnius, 2022, pp. 20.

2. V. Garbaliauskienė and M. Tekorė, Universality of certain compositions, 26th international conference on Mathematical Modelling and Analysis, May 30 – June 2, 2023, Jurmala, Latvia, abstracts, University of Latvia, Riga, 2023, pp. 14.

3. M. Tekorė and D. Šiaučiūnas, On universality of periodic zeta-functions, 26th international conference on Mathematical Modelling and Analysis, May 30 – June 2, 2023, Jurmala, Latvia, abstracts, University of Latvia, Riga, 2023, pp. 65.

4. M. Tekorė, Jungtinis periodinių dzeta funkcijų universalumas, Lietuvos matematikų draugijos LXIV konferencijos santraukos, birželio 21–22, Vilnius, Vilniaus universitetas, Vilnius, 2023, pp. 18.

5. M. Tekorė, Approximation of analytic functions by shifts of the periodic zeta-functions, 27th international conference on Mathematical Modelling and Analysis, May 28–31, 2023, Pärnu, Estonia, abstracts, University of Tartu, Tartu, 2024, pp. 67.

6. M. Tekorė, Gramo taškai periodinės dzeta funkcijos universalume, Lietuvos matematikų draugijos LXV konferencijos santraukos, birželio 27–28, Šiauliai, Vilniaus universitetas, Šiauliai, 2024, pp. 33.

# CHAPTER 2

## Universality of periodic zeta-functions with shifts involving the Gram points

Let $t_n$ be a solution of the equation

$$\theta(t) = (n-1)\pi, \quad n \in \mathbb{N},$$

where $\theta(t)$, $t \geqslant 0$, is the increment of the argument of the product $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $1/2$ and $1/2 + it$. The points $t_n$ are called Gram points in honor of the Danish mathematician Jørgen Pedersen Gram who tried to apply these points for investigations of non-trivial zeros of the Riemann zeta-function, i. e., for zeros of $\zeta(s)$ lying in the critical strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$. In this chapter, we consider the approximation of analytic functions by shifts $\zeta(s + iht_k; \mathfrak{a})$, $h > 0$, of the periodic zeta-function

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad \sigma > 1,$$

with a periodic multiplicative sequence $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$.

## 2.1 Statement of the main theorem

We remind the objects that occur in the statements of universality theorems. Thus, $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ is the universality strip, $\mathcal{K}$ denotes the class of the compact subsets of the strip $D$ having connected complements, $H_0(K)$ with $K \in \mathcal{K}$ is the class of continuous non-vanishing functions on $K$ that are analytic in the interior of $K$. As usual, $\#A$ stands for the cardinality of a set $A$ of real numbers.

The main result of the chapter is the following theorem.

**Theorem 2.1.** *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative. Let* $K \in \mathcal{K}$, $f(s) \in H_0(K)$ *and* $h > 0$ *be a fixed number. Then, for all* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k; \mathfrak{a})| < \varepsilon\right\} > 0.$$

*Moreover, the limit*

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k; \mathfrak{a})| < \varepsilon\right\}$$

*exists and is positive for all but at most countably many* $\varepsilon > 0$.

The first assertion of Theorem 2.1 means that the set of shifts $\zeta(s+iht_k; \mathfrak{a})$ approximating a given function $f(s) \in H_0(K)$ has a positive lower density. Hence, it follows that this set is infinite. Thus, we have infinitely many shifts $\zeta(s + iht_k; \mathfrak{a})$ approximating the function $f(s)$. On the other hand, Theorem 2.1, as other universality theorems for zeta-functions, is not effective in the sense that any concrete shift $\zeta(s + iht_k; \mathfrak{a})$ with approximating property is not known.

The second assertion of Theorem 2.1 shows that the set of approximating shifts has a positive density, however, with the exception of a not wide set of values of approximation accuracy. We suspect that the above exception is conditioned by the used probabilistic proof method.

We notice that Theorem 2.1 covers an analogical statement obtained in [29] for the Riemann zeta-function.

Theorem 2.1 is stated in density terms, the ratio

$$\frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k; \mathfrak{a})| < \varepsilon\right\}$$

with respect to $\varepsilon$ is a probabilistic distribution function. Therefore, it is convenient to use a probabilistic approach. More precisely, for the proof of Theorem 2.1, we apply a limit theorem on weakly convergent probability measures in the space of analytic functions with an explicitly given probability limit measure.

## 2.2 Statement of a limit theorem

Let $\mathbb{X}$ be a topological space. Denote by $\mathcal{B}(\mathbb{X})$ the Borel $\sigma$-field of the space $\mathbb{X}$, i. e., a $\sigma$-field generated by open sets of the space $\mathbb{X}$. Let $P_n$, $n \in \mathbb{N}$, and $P$ be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. By the definition, $P_n$ converges weakly to $P$ as $n \to \infty$ if, for every real continuous bounded function $g$ on $\mathbb{X}$, the equality

$$\lim_{n\to\infty} \int_{\mathbb{X}} g \mathrm{d}P_n = \int_{\mathbb{X}} g \mathrm{d}P.$$

is valid.

It is difficult to prove weak convergence of probability measures by directly using the definition. There are equivalents of weak convergence in terms of some classes of sets of the space $\mathbb{X}$. We will use the classes of open and continuity sets. We remind that the set $A \in \mathcal{B}(\mathbb{X})$ is called a continuity set of the measure $P$ is $P(\partial A) = 0$, where $\partial A$ denotes the boundary of the set $A$.

**Lemma 2.1.** *The following statements are equivalent:*

*(i)* $P_n$ *converges weakly to* $P$ *as* $n \to \infty$*;*

*(ii) For every open set* $G \subset \mathbb{X}$*,*

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

*(iii) For all continuity sets* $A$ *of* $P$*,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is a part of Theorem 2.1 of the monograph [4].

In this dissertation, we are connected to the space $H(D)$ of analytic on $D$ functions endowed with the topology of uniform convergence on compacta. In this topology, a sequence $\{g_n(s)\} \subset H(D)$ converges to $g(s) \in H(D)$ if, for every compact set $K \subset D$,

$$\lim_{n\to\infty} \sup_{s\in K} |g_n(s) - g(s)| = 0.$$

The space $H(D)$ is metrisable. It is known that there exists a sequence $\{K_m : m \in \mathbb{N}\} \subset D$ of compact sets such that

$$D = \bigcup_{m=1}^{\infty} K_m,$$

$K_m \subset K_{m+1}$, for all $m \in \mathbb{N}$, and every compact set $K \subset D$ lies in a certain set $K_m$. Let $g_1, g_2 \in H(D)$. Then, denoting

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup_{s \in K_m} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_m} |g_1(s) - g_2(s)|},$$

we have that $\rho$ is a metric on $H(D)$ inducing its topology of uniform convergence on compacta.

In this chapter, we will consider weak convergence of the probability measure

$$P_{N,\mathfrak{a}}(A) \stackrel{\text{def}}{=} \frac{1}{N} \#\{1 \leqslant k \leqslant N : \zeta(s + iht_k; \mathfrak{a}) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

as $N \to \infty$. For the statement of a limit theorem for $P_{N,\mathfrak{a}}$, we need a definition of a limit measure. For this, we introduce a certain group defined by the Cartesian product of unit circles. Let $\mathbb{P}$ denote the set of all prime numbers, and $y = \{s \in \mathbb{C} : |s| = 1\}$. Define the set

$$\Omega = \prod_{p \in \mathbb{P}} y_p,$$

where $y_p = y$ for every $p \in \mathbb{P}$. The set $\Omega$ consists of all functions $\omega : \mathbb{P} \to y$. On $\Omega$, the operation of pointwise multiplication and product topology can be defined. Since $y$ is a compact set, by the Tikhonov theorem, see, for example, [60], $\Omega$ is a compact as well. Thus, we have that $\Omega$ is a compact topological Abelian group. Hence, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure $m_H$ exists. This leads to the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Note that the Haar measure $m_H$ is invariant with respect to elements from $\Omega$. This means that, for all $A \in \mathcal{B}(\Omega)$ and $\omega \in \Omega$,

$$m_H(A) = m_H(\omega A) = m_H(A\omega).$$

Let $\omega = (\omega(p) : p \in \mathbb{P})$. Now, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element

$$\zeta(s,\omega;\mathfrak{a}) = \prod_{p\in\mathbb{P}} \left(1 + \sum_{\alpha=1}^{\infty} \frac{a_{p^\alpha}\omega^\alpha(p)}{p^{\alpha s}}\right).$$

Note that the latter infinite product, for almost all $\omega \in \Omega$, is uniformly convergent on compact subsets of $D$, see Theorem 5.1.7 of [35]. Hence, it defines a $H(D)$-valued random element.

Denote by $P_{\zeta,\mathfrak{a}}$ the distribution of $\zeta(s,\omega;\mathfrak{a})$, i. e., for $A \in \mathcal{B}(H(D))$,

$$P_{\zeta,\mathfrak{a}}(A) = m_H \left\{\omega \in \Omega : \zeta(s,\omega;\mathfrak{a}) \in A\right\}.$$

Now we state the main ingredient of the proof of Theorem 2.1.

**Theorem 2.2.** *The measure $P_{N,\mathfrak{a}}$ converges weakly to $P_{\zeta,\mathfrak{a}}$ as $N \to \infty$.*

The proof of Theorem 2.2 is divided into parts. First, we present a limit lemma for probability measures on $(\Omega, \mathcal{B}(\Omega))$. This lemma is used for the proof of a limit lemma for a probability measure defined by means of an absolutely convergent Dirichlet series. The central place of the proof consists of approximation of $\zeta(s;\mathfrak{a})$ in the mean by an absolutely convergent Dirichlet series. The proof will be completed by application of a property of convergence of random elements in distribution.

## 2.3 A limit lemma on the group $\Omega$

For $A \in \mathcal{B}(\Omega)$, define

$$V_N(A) = \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \left(p^{-iht_k} : p \in \mathbb{P}\right) \in A\right\}.$$

**Lemma 2.2.** *The measure $V_N$ converges weakly to the Haar measure $m_H$ as $N \to \infty$.*

The lemma was already used in [29], therefore, we only remind some moments of its proof.

We start with the notion of uniform distribution modulo 1. Recall that a sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if, for every interval $[a,b) \subset [0,1)$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} I_{[a,b)}(\{x_k\}) = b-a,$$

where $I_{[a,b)}$ is the indicator function of $[a,b)$, and $\{x_k\}$ denotes the fractional part of $x_k$.

For uniformly distributed modulo 1 sequences, the following Weyl criterion is known.

**Lemma 2.3.** *A sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if and only if, for every $m \in \mathbb{Z}\setminus\{0\}$,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} e^{2\pi i m x_k} = 0.$$

Proof of the lemma is given, for example, in [33].

**Lemma 2.4.** *The sequence $\{at_k : k > k_0\}$ with every real $0 \neq a \in \mathbb{R}$ is uniformly distributed modulo 1.*

Proof of the lemma is given in [29], Lemma 2.3.

*Proof of Lemma 2.2.* Since $\Omega$ is a group, it is convenient to use the Fourier transform method. The characters $\chi$ of $\Omega$ have the representation

$$\chi(\omega) = \prod_{p\in\mathbb{P}} \omega^{k_p}(p), \quad \omega \in \Omega,$$

where only a finite number of $k_p \in \mathbb{Z}$ are distinct from zero. Therefore, the Fourier transform $F_N(\underline{k})$, $\underline{k} = \{k_p : p \in \mathbb{P}\}$, of the measure $V_N$ is given by

$$\begin{aligned} F_N(\underline{k}) &= \int_\Omega \prod_{p\in\mathbb{P}}{}^{*} \omega^{k_p}(p)\,\mathrm{d}Q_N \\ &= \frac{1}{N}\sum_{k=1}^{N} \exp\left\{-iht_k \sum_{p\in\mathbb{P}}{}^{*} k_p \log p\right\}, \end{aligned} \tag{2.3.1}$$

where the star $*$ means that only a finite number of $k_p \in \mathbb{Z}$ are non-zero. Let

$$A(\underline{k}) = \sum_{p\in\mathbb{P}}{}^{*} k_p \log p.$$

Consider two cases: $A(\underline{k}) = 0$ and $A(\underline{k}) \neq 0$. It is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Therefore, $A(\underline{k}) = 0$ if and only if $\underline{k} = \underline{0}$, where $\underline{0} = (0, 0, \dots)$. If $\underline{k} \neq \underline{0}$, then, in view of Lemma 2.4, we have that the sequence $\{ht_k A(\underline{k})\}$ is uniformly distributed modulo 1. Hence, by Lemma 2.3,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \exp\{-ht_k A(\underline{k})\} = 0.$$

Since, for $\underline{k} = \underline{0}$,

$$F_N(\underline{k}) = 1,$$

this and (2.3.1) prove that

$$\lim_{N\to\infty} F_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Thus, by the continuity theorem for the Fourier transform, we have that the measure $V_N$, as $N \to \infty$, converges weakly to the measure defined by the Fourier transform

$$F_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}, \end{cases}$$

i. e., to the Haar measure $m_H$. □

Lemma 2.2 is very important, it allows to prove a limit lemma in the space of analytic functions for absolutely convergent Dirichlet series.

## 2.4 Case of absolutely convergent series

Let $\beta > 1/2$ be a fixed number, and

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\beta} \right\}, \quad m, n \in \mathbb{N}.$$

Define the series

$$\zeta_n(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^s}.$$

Since $v_n(m)$ decreases with respect to $m$ exponentially and $a_m \ll 1$, the series for $\zeta_n(s; \mathfrak{a})$ is absolutely convergent for all $s \in \mathbb{C}$.

Extend the functions $\omega(p)$, $p \in \mathbb{P}$, to the set $\mathbb{N}$ by the formula

$$\omega(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N},$$

and define one more series

$$\zeta_n(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m) v_n(m)}{m^s}.$$

Since $|\omega(m)| = 1$, the latter series, as $\zeta_n(s; \mathfrak{a})$, is also absolutely convergent for all $s \in \mathbb{C}$.

In this section, we will consider weak convergence for probability measure

$$V_{N,n}(A) = \frac{1}{N} \# \{1 \leqslant k \leqslant N : \zeta_n(s + iht_k; \mathfrak{a}) \in A\}, \quad \mathcal{B}(H(D)),$$

as $N \to \infty$. For this, we will apply Lemma 2.2 and the principle of preservation of weak convergence under continuous mappings. Let $P$ be a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ and $u : \mathbb{X} \to \mathbb{X}_1$ a $(\mathcal{B}(\mathbb{X}), \mathcal{B}(\mathbb{X}_1))$-measurable mapping, i. e., $u^{-1}\mathcal{B}(\mathbb{X}_1) \subset \mathcal{B}(\mathbb{X})$. Then the measure $P$ defines on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$ the unique probability measure $Pu^{-1}$ by the equality

$$Pu^{-1}(A) = P(u^{-1}A), \quad A \in \mathcal{B}(\mathbb{X}_1).$$

It is known that under certain hypotheses on the mapping $u$, the weak convergence is preserved. More precisely, the following lemma is valid.

**Lemma 2.5.** *Suppose that $P_n$, $n \in \mathbb{N}$, and $P$ be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and $u : \mathbb{X} \to \mathbb{X}_1$ a continuous mapping. If $P_n$, as $n \to \infty$, converges weakly to $P$, then also $P_n u^{-1}$ converges weakly to $Pu^{-1}$ as $n \to \infty$.*

*Proof.* The lemma is a partial case of Theorem 5.1 from [4] which asserts that $P_n u^{-1}$ as $n \to \infty$ if $P(D_u) = 0$, where $D_u$ is the set of discontinuity points of the mapping $u$. □

Now we apply Lemma 2.5 for our aims. Define the mapping $u_n \to H(D)$ by the formula

$$u_n(\omega) = \zeta_n(s, \omega; \mathfrak{a}).$$

Since the series for $\zeta_n(s,\omega;\mathfrak{a})$ is absolutely convergent, the mapping $u_n$ is continuous. Moreover, by the definitions of $V_{N,\mathfrak{a}}$ and $V_{N,n,\mathfrak{a}}$, we have that, for every $A \in \mathcal{B}(H(D))$,

$$\begin{aligned} V_{N,n,\mathfrak{a}}(A) &= \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \zeta_n(s+iht_k;\mathfrak{a}) \in A\right\} \\ &= \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : u_n\left(p^{-iht_k} : p \in \mathbb{P}\right) \in A\right\} \\ &= \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \left(p^{-iht_k} : p \in \mathbb{P}\right) \in u_n^{-1}A\right\} \\ &= V_{N,\mathfrak{a}}(u_n^{-1}A). \end{aligned}$$

Hence, $V_{N,n,\mathfrak{a}} = V_{N,\mathfrak{a}}u_n^{-1}$. This, continuity of $u_n$ and Lemmas 2.5 and 2.2 prove the following statement.

**Lemma 2.6.** *The probability measure $V_{N,n,\mathfrak{a}}$ converges weakly to $m_H u_n^{-1}$ as $N \to \infty$.*

In order to prove Theorem 2.2, it remains to pass from the function $\zeta_n(s;\mathfrak{a})$ to $\zeta(s;\mathfrak{a})$. This place of the proof is the most complicated.

## 2.5 Approximation of $\zeta(s;\mathfrak{a})$ by $\zeta_n(s;\mathfrak{a})$

In this section, we will prove that $\zeta_n(s;\mathfrak{a})$ is close to $\zeta(s;\mathfrak{a})$ in the mean. Let $\rho$ be the metric in $H(D)$ defined in Section 2.2. More precisely, we will prove the following lemma.

**Lemma 2.7.** *For any fixed $h > 0$, the equality*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\rho\left(\zeta(s+iht_k;\mathfrak{a}),\zeta_n(s+iht_k;\mathfrak{a})\right) = 0$$

*holds.*

For the proof of Lemma 2.7, we need several auxiliary results. First of all, we use a continuous extension of the points $t_k$.

**Lemma 2.8.** *Suppose that $t_\tau$, $\tau \geqslant 0$, denotes the unique solution of the equation*

$$\theta(t) = (\tau - 1)\pi$$

*satisfying $\theta'(t_\tau) > 0$, and $\tau \to \infty$. Then*

$$t_\tau = \frac{2\pi\tau}{\log\tau}\left(1 + \frac{\log\log\tau}{\log\tau}(1+o(1))\right)$$

*and*

$$t'_\tau = \frac{2\pi}{\log\tau}\left(1 + \frac{\log\log\tau}{\log\tau}(1+o(1))\right).$$

The proof of lemma is given in [21].

The next lemma deals with generalized mean squares.

**Lemma 2.9.** *Suppose that $\sigma > 1/2$ is fixed and $h > 0$. Then, for $t \in \mathbb{R}$,*

$$\int_0^T |\zeta(\sigma + it + iht_\tau; \mathfrak{a})|^2 \, \mathrm{d}\tau \ll_{\sigma,\mathfrak{a},h} T(1+|t|)$$

*and*

$$\int_0^T \left|\zeta'(\sigma + it + iht_\tau; \mathfrak{a})\right|^2 \, \mathrm{d}\tau \ll_{\sigma,\mathfrak{a},h} T(1+|t|).$$

*Proof.* It is well known that, for the Hurwitz zeta-function $\zeta(s,\alpha)$, the estimate

$$\int_0^T |\zeta(\sigma + it, \alpha)|^2 \, \mathrm{d}t \ll_{\sigma,\alpha} T$$

is valid. This and equality (I1) show that

$$\int_0^T |\zeta(\sigma + it; \mathfrak{a})|^2 \, \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T. \tag{2.5.1}$$

Let $X \geqslant 1$. Then

$$\int_X^{2X} |\zeta(\sigma + it + iht_\tau; \mathfrak{a})|^2 \, \mathrm{d}\tau = \int_X^{2X} \frac{1}{t'_\tau} |\zeta(\sigma + it + iht_\tau; \mathfrak{a})|^2 \, \mathrm{d}t_\tau \tag{2.5.2}$$

By Lemma 2.8, the derivative $t'_\tau$ is decreasing. Therefore, by the second mean value theorem,

$$\int_X^{2X} \frac{1}{t'_\tau} \left|\zeta(\sigma+it+iht_\tau;\mathfrak{a})\right|^2 \,\mathrm{d}t_\tau = \frac{1}{t'_{2X}} \int_X^{\xi} \left|\zeta(\sigma+it+iht_\tau;\mathfrak{a})\right|^2 \,\mathrm{d}t_\tau$$
$$\ll \frac{1}{t'_{2X}} \int_X^{2X} \left|\zeta(\sigma+it+iht_\tau;\mathfrak{a})\right|^2 \,\mathrm{d}t_\tau, \tag{2.5.3}$$

where $X \leqslant \xi \leqslant 2X$. After change of variables, we have

$$\begin{aligned}\int_X^{2X} \left|\zeta(\sigma+it+iht_\tau;\mathfrak{a})\right|^2 \,\mathrm{d}t_\tau &= \int_{ht_X}^{ht_{2X}} \left|\zeta(\sigma+it+iu;\mathfrak{a})\right|^2 \,\mathrm{d}u \\ &= \int_{t+ht_X}^{t+ht_{2X}} \left|\zeta(\sigma+iu;\mathfrak{a})\right|^2 \,\mathrm{d}u \\ &\ll \int_{-|t|-ht_{2X}}^{|t|+ht_{2X}} \left|\zeta(\sigma+iu;\mathfrak{a})\right|^2 \,\mathrm{d}u \\ &\ll_{\sigma,\mathfrak{a}} |t| + ht_{2X}\end{aligned}$$

in virtue of (2.5.1). Thus, (2.5.2), (2.5.3) and Lemma 2.8 yield

$$\begin{aligned}\int_X^{2X} \left|\zeta(\sigma+it+iht_\tau;\mathfrak{a})\right|^2 \,\mathrm{d}\tau &\ll_{\sigma,\mathfrak{a}} \frac{1}{t'_{2X}}(|t| + ht_{2X}) \\ &\ll_{\sigma,\mathfrak{a}} \log X \left(|t| + \frac{hX}{\log X}\right) \\ &\ll_{\sigma,\mathfrak{a},h} X(1+|t|).\end{aligned}$$

Now, taking $X = T2^{-l-1}$ and summing over $l = 0, 1, \ldots$, we obtain the bound

$$\int_0^T \left|\zeta(\sigma+it+iht_\tau;\mathfrak{a})\right|^2 \,\mathrm{d}\tau \ll_{\sigma,\mathfrak{a},h} T(1+|t|).$$

For the proof of the second estimate of the lemma, we apply the Cauchy integral formula. We have

$$\zeta'(s;\mathfrak{a}) = \frac{1}{2\pi i}\oint\limits_L \frac{\zeta(z;\mathfrak{a})}{(z-s)^2}\,\mathrm{d}z,$$

where $L$ is a simple closed contour enclosing the point $s$ and lying in $D$. Hence,

$$\zeta'(\sigma+it;\mathfrak{a}) = \frac{1}{2\pi i}\oint\limits_L \frac{\zeta(z+it;\mathfrak{a})}{(z-\sigma)^2}\,\mathrm{d}z,$$

where $L$ is the corresponding contour. Therefore,

$$\zeta'(\sigma+it;\mathfrak{a}) \ll \oint\limits_L \frac{|\zeta(z+it;\mathfrak{a})|}{|z-\sigma|^2}\,|\mathrm{d}z|,$$

and

$$|\zeta'(\sigma+it;\mathfrak{a})|^2 \ll \oint\limits_L \frac{|\mathrm{d}z|}{|z-\sigma|^4}\oint\limits_L |\zeta(z+it;\mathfrak{a})|^2\,|\mathrm{d}z|.$$

From this, we find

$$\begin{aligned}\int\limits_0^T |\zeta'(\sigma+it;\mathfrak{a})|^2\,\mathrm{d}t &\ll_L \oint\limits_L \left(\int\limits_0^T |\zeta(z+it;\mathfrak{a})|^2\,\mathrm{d}t\right)|\mathrm{d}z| \\ &\ll_L \oint\limits_L \left(\int\limits_0^{T+\mathrm{Im}z} |\zeta(\mathrm{Re}z+it;\mathfrak{a})|^2\,\mathrm{d}t\right)|\mathrm{d}z| \\ &\ll_{\sigma,\mathfrak{a}} T\end{aligned}$$

by (2.5.1). Using the latter estimates, and repeating the arguments of the proof of the first bound of the lemma, we obtain the second estimate of the lemma. □

Estimates of Lemma 2.9 are of continuous type, while Lemma 2.7 deals with discrete shifts. To pass from continuous moments to discrete ones, we will apply the following Gallagher lemma.

**Lemma 2.10.** *Suppose that $T_0 \geqslant \delta$, $T \geqslant \delta$, $\delta > 0$, $\mathcal{T}$ is a non-empty finite set* *in the interval* $[T_0+\delta/2, T_0+T-\delta/2]$, *and*

$$N_\delta(\tau) = \sum_{\substack{t\in\mathcal{T} \\ |t-\tau|<\delta}} 1, \quad \tau \in \mathcal{T}.$$

*Moreover, let the complex-valued function* $S(t)$ *be continuous on* $[T_0, T_0+T]$, *and have a continuous derivative on* $(T_0, T_0+T)$. *Then*

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leqslant \frac{1}{\delta}\int\limits_{T_0}^{T_0+T} |S(t)|^2 \,\mathrm{d}t + \left(\int\limits_{T_0}^{T_0+T} |S(t)|^2 \,\mathrm{d}t \int\limits_{T_0}^{T_0+T} |S'(t)|^2 \,\mathrm{d}t\right)^{1/2}.$$

Proof of the lemma is given in [55], Lemma 1.4.

**Lemma 2.11.** *For fixed* $1/2 < \sigma < 1$, $h > 0$ *and* $t \in \mathbb{R}$, *the estimate*

$$\sum_{k=1}^{N} |\zeta(\sigma + it + iht_k; \mathfrak{a})|^2 \ll_{\sigma,\mathfrak{a},h} N(1+|t|)$$

*is valid.*

*Proof.* In Lemma 2.10, take $\delta = 1$, $T_0 = 1$, $T = N$ and $\mathcal{T} = \{3/2, 2, 3, \dots, N, N+1/2\}$. In this case, $N_\delta(\tau) = 1$. In view of Lemmas 2.10 and 2.9, we find

$$\sum_{k=1}^{N} |\zeta(\sigma + it + iht_k; \mathfrak{a})|^2 \ll_{\sigma,h,\mathfrak{a}} \int\limits_{3/2}^{N+1/2} |\zeta(\sigma + it + iht_u; \mathfrak{a})|^2 \,\mathrm{d}u$$
$$+ \left(\int\limits_{3/2}^{N+1/2} |\zeta(\sigma + it + iht_u, ; \mathfrak{a})|^2 \,\mathrm{d}u \int\limits_{3/2}^{N+1/2} |\zeta'(\sigma + it + iht_u; \mathfrak{a})|^2 \,\mathrm{d}u\right)^{1/2}$$
$$\ll_{\sigma,h,\mathfrak{a}} N(1+|t|).$$

□

For the proof of Lemma 2.7, we also need the integral representation for the function $\zeta_n(s;\mathfrak{a})$. We recall that

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\beta}\right\}, \quad m,n \in \mathbb{N},$$

with $\beta > 1/2$. Let

$$b_n(s) = \frac{1}{\beta}\Gamma\left(\frac{s}{\beta}\right) n^s.$$

**Lemma 2.12.** *Suppose that $\sigma > 1/2$. Then*

$$\zeta_n(s;\mathfrak{a}) = \frac{1}{2\pi i}\int\limits_{\beta-i\infty}^{\beta+i\infty} \zeta(s+z;\mathfrak{a}) b_n(z)\,\mathrm{d}z.$$

*Proof.* For arbitrary positive numbers $a$ and $b$, the Mellin formula, see, for example, [71],

$$\frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty} \Gamma(s)a^{-s}\,\mathrm{d}s = \mathrm{e}^{-a}$$

is valid. Thus, we have

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\beta-i\infty}^{\beta+i\infty} b_n(z)\frac{\mathrm{d}z}{m^z} &= \frac{1}{2\pi i}\int\limits_{\beta-i\infty}^{\beta+i\infty} \Gamma\left(\frac{z}{\beta}\right)\left(\frac{m}{n}\right)^{-(z\beta)/\beta}\mathrm{d}\left(\frac{z}{\beta}\right) \\ &= \exp\left\{-\left(\frac{m}{n}\right)^{\beta}\right\} = v_n(m).\end{aligned}$$

Hence,

$$\begin{aligned}\zeta_n(s;\mathfrak{a}) &= \sum_{m=1}^{\infty}\frac{a_m}{m^s}\left(\frac{1}{2\pi i}\int\limits_{\beta-i\infty}^{\beta+i\infty} b_n(z)\frac{\mathrm{d}z}{m^z}\right) \\ &= \frac{1}{2\pi i}\int\limits_{\beta-i\infty}^{\beta+i\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^{s+z}}\right) b_n(z)\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int\limits_{\beta-i\infty}^{\beta+i\infty} \zeta(s+z;\mathfrak{a}) b_n(z)\,\mathrm{d}z.\end{aligned}$$

□

*Proof of Lemma 2.7.* We observe that, in view of the definition of the metric $\rho$, it suffices to show that, for every compact set $K \subset D$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s\in K} |\zeta(s+iht_k;\mathfrak{a}) - \zeta_n(s+iht_k;\mathfrak{a})| = 0. \qquad (2.5.4)$$

Thus, let $K \subset D$ be an arbitrary compact set. Then there exists $\varepsilon > 0$ such that $1/2+2\varepsilon \leqslant \sigma \leqslant 1-\varepsilon$ for any point $s = \sigma+it \in K$. We take $\beta = 1/2+\varepsilon$ and $\theta = \sigma-\varepsilon-1/2$. Then $\theta > 0$. The integrand of the integral of Lemma 2.12 has a simple pole at the point $z = 0$ (a pole of $\Gamma(s)$), and a possible simple pole at $z = 1-s$ ( a pole of $\zeta(s+z;\mathfrak{a})$). We recall that

$$\widehat{a} = \operatorname*{Res}_{s=1} \zeta(s;\mathfrak{a}).$$

Therefore, by Lemma 2.12 and the residue theorem, we have

$$\begin{aligned}\zeta_n(s+iht_k;\mathfrak{a}) - \zeta(s+iht_k;\mathfrak{a}) =& \frac{1}{2\pi i} \int\limits_{-\theta-i\infty}^{-\theta+i\infty} \zeta(s+z;\mathfrak{a}) b_n(z)\,\mathrm{d}z \\ &+ \operatorname*{Res}_{z=1-s} \zeta(s+z;\mathfrak{a}) b_n(z).\end{aligned}$$

Therefore, for $s \in K$,

$$\begin{aligned}&\zeta_n(s+iht_k;\mathfrak{a}) - \zeta_n(s+iht_k;\mathfrak{a}) \\ &= \int\limits_{-\theta-i\infty}^{-\theta+i\infty} \zeta(s+iht_k+z;\mathfrak{a}) b_n(z)\,\mathrm{d}z + \widehat{a} b_n(1-s-iht_k) \\ &\ll \int\limits_{-\infty}^{+\infty} |\zeta(s+iht_k-\theta+i\tau;\mathfrak{a})|\,|b_n(-\theta+it)|\mathrm{d}\tau \\ &\quad + |\widehat{a}||b_n(1-s-iht_k)| \\ &= \int\limits_{-\infty}^{\infty} \left|\zeta\left(s+iht_k-\sigma+\frac{1}{2}+\varepsilon+i\tau;\mathfrak{a}\right)\right| \left|b_n\left(\frac{1}{2}+\varepsilon-\sigma+i\tau\right)\right| \mathrm{d}\tau \\ &\quad + |\widehat{a}||b_n(1-s-iht_k)|.\end{aligned}$$

Hence, shifting $t+\tau$ to $\tau$, we obtain, for all $s \in K$,

$$
\begin{aligned}
&|\zeta(s+iht_k;\mathfrak{a}) - \zeta_n(s+iht_k;\mathfrak{a})| \\
&\quad \ll \int_{-\infty}^{+\infty} \left|\zeta\left(\frac{1}{2}+\varepsilon+i(\tau+ht_k);\mathfrak{a}\right)\right| \sup_{s\in K}\left|b_n\left(\frac{1}{2}+\varepsilon-s+i\tau\right)\right| \mathrm{d}\tau \\
&\qquad + |\widehat{a}| \sup_{s\in K} |b_n(1-s-iht_k)|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{N}\sum_{k=1}^{N} \sup_{s\in K} |\zeta(s+iht_k;\mathfrak{a}) - \zeta_n(s+iht_k;\mathfrak{a})| \\
&\ll \int_{-\infty}^{\infty} \left(\frac{1}{N}\sum_{k=1}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+i(\tau+ht_k);\mathfrak{a}\right)\right|\right) \\
&\qquad \times \sup_{s\in K}\left|b_n\left(\frac{1}{2}+\varepsilon-s+i\tau\right)\right| \mathrm{d}\tau + |\widehat{a}|\sum_{k=1}^{N}\sup_{s\in K}|b_n(1-s-iht_k)| \\
&\overset{\text{def}}{=} I_1 + I_2.
\end{aligned}
\tag{2.5.5}
$$

It is well known (a corollary of the Stirling formula) that, uniformly in $\sigma \in [0,1]$, the estimate

$$
\Gamma(\sigma+it) \leqslant \exp\{-c|t|\}, \quad c>0, \tag{2.5.6}
$$

holds. Therefore, the definition of $b_n(s)$ shows that, for $s \in K$,

$$
\begin{aligned}
b_n\left(1/2+\varepsilon-s+i\tau\right) &\ll \frac{n^{1/2+\varepsilon-\sigma}}{\beta}\left|\Gamma\left(\frac{1/2+\varepsilon-\sigma}{\beta}+\frac{i(\tau-t)}{\beta}\right)\right| \\
&\ll_\beta n^{-\varepsilon}\exp\left\{-\frac{c}{\beta}|t-\tau|\right\} \\
&\ll_{\varepsilon,K} n^{-\varepsilon}\exp\{-c_1|\tau|\}
\end{aligned}
\tag{2.5.7}
$$

because $|t-\tau| \geqslant |\tau| - t(K)$, where $t(K) = \sup_{s\in K}|\operatorname{Im} s| + 1$. Moreover, the Cauchy-Schwarz inequality and Lemma 2.11 give

$$
\begin{aligned}
&\frac{1}{N}\sum_{k=1}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+i(ht_k+\tau);\mathfrak{a}\right)\right| \\
&\qquad \leqslant \left(\frac{1}{N}\sum_{k=1}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+i(ht_k+\tau);\mathfrak{a}\right)\right|^2\right)^{1/2} \\
&\qquad \ll_{\sigma,\mathfrak{a},h,K} (1+|\tau|)^{1/2}.
\end{aligned}
$$

This and (2.5.7) show that

$$
I_1 \ll_{\sigma,\mathfrak{a},h,K} n^{-\varepsilon}\int_{-\infty}^{\infty}(1+|\tau|)^{1/2}\exp\{-c_1|\tau|\}\,\mathrm{d}\tau \ll_{\sigma,\mathfrak{a},h,K} n^{-\varepsilon}. \tag{2.5.8}
$$

Similarly as above, using (2.5.6), we find, for all $s \in K$,

$$
b_n(1-s-iht_k) \ll_{\beta,K} n^{1-\sigma}\exp\{-c_2ht_k\} \ll_{K,h} n^{1/2-2\varepsilon}\exp\{-c_2ht_k\}.
$$

Then we have

$$
\begin{aligned}
I_2 &\ll_{K,h} n^{1/2-2\varepsilon}\frac{1}{N}\left(\sum_{k\leqslant \log N}+\sum_{\log N\leqslant k\leqslant N}\right)\exp\{-c_2ht_k\} \\
&\ll_{K,h,\mathfrak{a}} n^{1/2-2\varepsilon}\left(\frac{\log N}{N}+\frac{1}{N}\sum_{k\geqslant \log N}\exp\left\{-c_3h\frac{k}{\log k}\right\}\right) \\
&\ll_{K,h,\mathfrak{a}} n^{1/2-2\varepsilon}\left(\frac{\log N}{N}+\frac{1}{N}\right) \\
&\ll_{K,h,\mathfrak{a}} n^{1/2-2\varepsilon}\,\frac{\log N}{N}.
\end{aligned}
$$

This together with (2.5.5) and (2.5.8) show that

$$
\begin{aligned}
&\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|\zeta(s+iht_k;\mathfrak{a})-\zeta_n(s+iht_k;\mathfrak{a})| \\
&\qquad\qquad \ll_{K,h,\mathfrak{a}} n^{-\varepsilon}+n^{1/2-2\varepsilon}\,\frac{\log N}{N}.
\end{aligned} \tag{2.5.9}
$$

We note that the implied constant in the above estimates depends on $\varepsilon$. However, $\varepsilon$ depends on $K$. Therefore, we indicate the dependence on $K$, only.

Taking $N \to \infty$, and then $n \to \infty$ in (2.5.9), we obtain equality (2.5.4), and this completes the proof of the lemma. □

## 2.6 Proof of Theorem 2.2

We start with recalling the notation of convergence in distribution. Let $X_n$, $n \in \mathbb{N}$, and $X$ be $\mathbb{X}$-valued random elements on a certain probability space $(\widehat{\Omega}, \mathcal{B}, \mu)$ with distributions $P_n$ and $P$, respectively. We say that $X_n$ converges to $X$ as $n \to \infty$ in distribution, and write $X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$ if $P_n \xrightarrow[n\to\infty]{\text{w}} P$.

We will use the following limit lemma on convergence in distribution.

**Lemma 2.13.** *Suppose that the metric space* $(\mathbb{X}, d)$ *is the separable, and the* $\mathbb{X}$*-valued random elements* $X_{nk}$, $Y_n$, $n, k \in \mathbb{N}$, *are defined on the same probability space* $(\widehat{\Omega}, \mathcal{B}, \mu)$. *Let, for all* $k \in \mathbb{N}$,

$$X_{nk} \xrightarrow[n\to\infty]{\mathcal{D}} X_k,$$

*and*

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X.$$

*Moreover, let, for every* $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\left\{d(X_{kn}, Y_n) \geqslant \varepsilon\right\} = 0.$$

*Then*

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X.$$

Proof of the lemma is given in [4], Theorem 3.2.

Also, we remind the notions of tightness and relative compactness of families of probability measures.

The family of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is tight if, for every $\varepsilon > 0$, there exists a compact set $K = K_\varepsilon \subset \mathbb{X}$ such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$.

The family $\{P\}$ is relatively compact if every sequence $\{P_n\} \subset \{P\}$ contains a subsequence $\{P_{n_r}\}$ weakly convergent to a certain probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ as $r \to \infty$.

Relative compactness of families of probability measures is useful in investigations of weak convergence of probability measures. On the other hand, the proof of relative compactness is complicated as well. However, the Prokhorov theorem implies that a relative compactness can be replaced by a tightness. We state the Prokhorov theorem as a separate lemma.

**Lemma 2.14.** *If the family of probability measures $\{P\}$ is tight, then it is relatively compact.*

Proof of the lemma is given in [4], Theorem 5.1.

Now, we return to Lemma 2.6 and its limit measure $V_{n,\mathfrak{a}} \overset{\text{def}}{=} m_H u_n^{-1}$.

**Lemma 2.15.** *The sequence $\{V_{n,\mathfrak{a}} : n \in \mathbb{N}\}$ is relatively compact.*

*Proof.* In view of Lemma 2.14, it suffices to show that the sequence $\{V_{n,\mathfrak{a}}\}$ is tight.

Take a random variable $\theta_N$ defined on a certain probability space $(\widehat{\Omega}, \mathcal{B}, \mu)$, and having the distribution

$$\mu\{\theta_N = ht_k\} = \frac{1}{N}, \quad k = 1, \ldots, N.$$

Using $\theta_N$, define the $H(D)$-valued random element

$$X_{N,n,\mathfrak{a}} = X_{N,n,\mathfrak{a}}(s) = \zeta_n(s + i\theta_N; \mathfrak{a}).$$

Moreover, let $X_{n,\mathfrak{a}} = X_{n,\mathfrak{a}}(s)$ be the $H(D)$-valued random element with distribution $V_{n,\mathfrak{a}}$. Then, by Lemma 2.6, we have

$$X_{N,n,\mathfrak{a}} \xrightarrow[N\to\infty]{\mathcal{D}} X_{n,\mathfrak{a}}. \tag{2.6.1}$$

Since the series for $\zeta_n(s; \mathfrak{a})$ is absolutely convergent, by Lemma 2.8, we obtain that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma + iht_\tau; \mathfrak{a})|^2 \, \mathrm{d}\tau \\
&\leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{ht'_T} \frac{1}{T} \int_0^{ht_T} |\zeta_n(\sigma + iu; \mathfrak{a})|^2 \, \mathrm{d}u \\
&= \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \left( \frac{ht_T}{Tht'_T} \frac{1}{ht_T} \int_0^{ht_T} |\zeta_n(\sigma + iu; \mathfrak{a})|^2 \, \mathrm{d}u \right)^{1/2}
\end{aligned}$$

$$\ll_h \sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} \frac{|a_m|^2 v_n^2(m)}{m^{2\sigma}} \ll_h \sum_{m=1}^{\infty} \frac{|a_m|^2}{m^{2\sigma}} \leqslant C_{h,\mathfrak{a}} < \infty. \quad (2.6.2)$$

Using the Cauchy integral formula and (2.6.2), as in the proof of Lemma 2.9, we analogically find that

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left|\zeta'(\sigma + iht_\tau; \mathfrak{a})\right|^2 \, \mathrm{d}\tau \ll \widehat{C}_{h,\mathfrak{a}} < \infty. \quad (2.6.3)$$

However, we consider the discrete case. Therefore, we have to estimate

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s\in K} |\zeta_n(s + iht_k; \mathfrak{a})|,$$

with compact set $K$. Application of the Cauchy integral formula reduces problem to estimation for

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \left( \frac{1}{N} \sum_{k=1}^{N} |\zeta_n(\sigma + iht_k; \mathfrak{a})|^2 \right)^{1/2}$$

with a certain $\sigma > 1/2$. Applying Lemma 2.10 and taking into account (2.6.2) and (2.6.3), we see that

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} |\zeta_n(\sigma + iht_k; \mathfrak{a})|^2 \leqslant \widehat{\widehat{C}}_{h,\mathfrak{a}} < \infty.$$

From this, using the Cauchy integral formula once more, we deduce that, for every compact set $K \subset D$,

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s\in K} |\zeta_n(s + iht_k; \mathfrak{a})| \leqslant B_{h,\mathfrak{a},K} < \infty. \quad (2.6.4)$$

Since the topology in the space $H(D)$ is of uniform convergence on compacta, the mapping $u : H(D) \to \mathbb{R}$ given by

$$u(f) = \sup_{s\in K} |f(s)|, \quad f \in H(D),$$

with arbitrary compact set $K$, is continuous. Therefore, relation (2.6.1) and

Lemma 2.5 yield

$$\sup_{s\in K}|X_{N,n,\mathfrak{a}}(s)| \xrightarrow[N\to\infty]{\mathcal{D}} \sup_{s\in K}|X_{n,\mathfrak{a}}(s)|. \tag{2.6.5}$$

Now, let $K = K_m$, where $K_m$, $m \in \mathbb{N}$, are compact sets from the definition of the metric $\rho$. Fix $\varepsilon > 0$, and set $R_m = 2^m \varepsilon^{-1} \widehat{\widehat{C}}_{h,\mathfrak{a}} K_m$. Then, by (2.6.4) and Chebyshev's type inequality, we obtain

$$\begin{aligned}\limsup_{N\to\infty}\mu\left\{\sup_{s\in K_m}|X_{N,n,\mathfrak{a}}(s)| > R_m\right\}\\ \leqslant \sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{NR_m}\sum_{k=1}^{N}\sup_{s\in K_m}|X_{N,n,\mathfrak{a}}(s)| \leqslant \frac{\varepsilon}{2^m}.\end{aligned}$$

Hence, in view of (2.6.5),

$$\mu\left\{\sup_{s\in K_m}|X_{n,\mathfrak{a}}(s)| > R_m\right\} \leqslant \frac{\varepsilon}{2^m}. \tag{2.6.6}$$

Define the set

$$K = K(\varepsilon) = \left\{g \in H(D) : \sup_{s\in K_m}|g(s)| \leqslant R_m, m \in \mathbb{N}\right\}.$$

Then the set $K$ is compact in the space $H(D)$. Moreover, inequality (2.6.6) implies that

$$\mu\{X_{n,\mathfrak{a}} \in K(\varepsilon)\} = 1 - \mu\{X_{n,\mathfrak{a}} \notin K(\varepsilon)\} \geqslant 1 - \varepsilon\sum_{m=1}^{\infty}\frac{1}{2^m} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Since the measure $V_{n,\mathfrak{a}}$ is the distribution of the random element $X_{n,\mathfrak{a}}$, the latter inequality gives

$$V_{n,\mathfrak{a}}(K(\varepsilon)) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This means that the sequence $\{V_{n,\mathfrak{a}} : n \in \mathbb{N}\}$ is tight. Therefore, by Lemma 2.14, it is relatively compact. □

*Proof of Theorem 2.2.* Introduce one more $H(D)$-valued random element

$$X_{N,\mathfrak{a}} = X_{N,\mathfrak{a}}(s) = \zeta(s + i\theta_N; \mathfrak{a}).$$

Since, by Lemma 2.15, the sequence $\{V_{n,\mathfrak{a}} : n \in \mathbb{N}\}$ is relatively compact, there exists a sequence $\{V_{n_l,\mathfrak{a}}\} \subset \{V_{n,\mathfrak{a}}\}$ weakly convergent to a certain probability measure $P_\mathfrak{a}$ on $(H(D), \mathcal{B}(H(D)))$ as $l \to \infty$. Thus, the relation

$$X_{n_l,\mathfrak{a}} \xrightarrow[l\to\infty]{\mathcal{D}} P_\mathfrak{a} \tag{2.6.7}$$

is valid. Now, return to Lemma 2.7. Its application, for every $\varepsilon > 0$, gives

$$\begin{aligned}
&\lim_{l\to\infty} \limsup_{N\to\infty} \mu\left\{\rho(X_{N,n_l,\mathfrak{a}}(s), X_{N,\mathfrak{a}}(s)) \geqslant \varepsilon\right\} \\
&= \lim_{l\to\infty} \limsup_{N\to\infty} \frac{1}{N} \#\left\{1 \leqslant k \leqslant N : \rho(\zeta(s+iht_k;\mathfrak{a}), \zeta_{n_l}(s+iht_k;\mathfrak{a})) \geqslant \varepsilon\right\} \\
&\leqslant \lim_{l\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \rho(\zeta(s+iht_k;\mathfrak{a}), \zeta_{n_l}(s+iht_k;\mathfrak{a})) = 0.
\end{aligned}$$

This equality with relations (2.6.1) and (2.6.7) show that all hypotheses of Lemma 2.13 are fulfilled because the space $H(D)$ is separable. Thus, we obtain the relation

$$X_{N,\mathfrak{a}} \xrightarrow[N\to\infty]{\mathcal{D}} P_\mathfrak{a}. \tag{2.6.8}$$

This means that the measure $P_{N,\mathfrak{a}}$ converges weakly to $P_\mathfrak{a}$ as $N \to \infty$. The relation (2.6.8) also shows that the measure $P_\mathfrak{a}$ does not depend on the sequence $\{V_{n_l,\mathfrak{a}}\}$. From this, we have that $V_{n,\mathfrak{a}}$ converges weakly to $P_\mathfrak{a}$ as $n \to \infty$.

It remains to identify the measure $P_\mathfrak{a}$. Define, on $(H(D), \mathcal{B}(H(D)))$, the probability measure

$$Q_{T,\mathfrak{a}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta(s+i\tau;\mathfrak{a}) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Then, in [47], it was obtained that $Q_{T,\mathfrak{a}}$, as $T \to \infty$, also converges to the limit measure $P_\mathfrak{a}$ of $V_{n,\mathfrak{a}}$ as $n \to \infty$, and that this measure $P_\mathfrak{a}$ coincides with the measure $P_{\zeta,\mathfrak{a}}$. Thus, $P_{N,\mathfrak{a}}$ converges weakly to $P_{\zeta,\mathfrak{a}}$ as $N \to \infty$ as well. The lemma is proved. □

## 2.7 Proof of Theorem 2.1

For the proof of universality theorems, the Mergelyan theorem on approximation of analytic functions by polynomials [52] is usually applied. Therefore, it is needed also for the proof of Theorem 2.1. We state the Mergelyan theorem in a convenient form.

**Lemma 2.16.** *Let $K \subset \mathbb{C}$ be a compact set with connected complement, and $g(s)$ be a continuous function on $K$ that is analytic in the interior of $K$. Then, for every $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(s)$ such that*

$$\sup_{s \in K} |g(s) - p_\varepsilon(s)| < \varepsilon.$$

Proof of the lemma is given in [52], see also [75].

*Proof of Theorem 2.1.* Define the set

$$S = \{g(s) \in H(D) : g(s) \neq 0 \ \text{ or } \ g(s) \equiv 0\}.$$

In [35], it was obtained that the set $S$ is the support of the measure $P_{\zeta,\mathfrak{a}}$, i. e., $S$ is a minimal closed subset of $H(D)$ such that $P_{\zeta,\mathfrak{a}}(S) = 1$. Moreover, $S$ consists of all $g(s) \in H(D)$ such that the inequality $P_{\zeta,\mathfrak{a}}(G) > 0$ is satisfied for all open neighbourhoods $G$ of $g(s)$.

For some polynomial $p(s)$, define the set

$$G_\varepsilon = \left\{ g(s) \in H(D) : \sup_{s \in K} \left| g(s) - \mathrm{e}^{p(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

Since $\mathrm{e}^{p(s)} \neq 0$, the function $e^{p(s)}$ is an element of the support $S$. Thus, by a property of the support,

$$P_{\zeta,\mathfrak{a}}(G_\varepsilon) > 0.$$

Therefore, by Theorem 2.2 and (ii) of Lemma 2.1, we have

$$\liminf_{N \to \infty} P_{N,\mathfrak{a}}(G_\varepsilon) \geqslant P_{\zeta,\mathfrak{a}}(G_\varepsilon) > 0, \tag{2.7.1}$$

or, by the definitions of $P_{N,\mathfrak{a}}$ and $G_\varepsilon$,

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} \left| \mathrm{e}^{p(s)} - \zeta(s + iht_k; \mathfrak{a}) \right| < \frac{\varepsilon}{2} \right\} > 0. \tag{2.7.2}$$

In virtue of Lemma 2.16, we can choose the polynomial $p(s)$ satisfying

$$\sup_{s \in K} \left| f(s) - \mathrm{e}^{p(s)} \right| < \frac{\varepsilon}{2}. \tag{2.7.3}$$

Suppose that $k \in \mathbb{N}$ satisfies

$$\sup_{s\in K}\left|\mathrm{e}^{p(s)} - \zeta(s+iht_k;\mathfrak{a})\right| < \frac{\varepsilon}{2}.$$

Then, for such $k$,

$$\begin{aligned}
&\sup_{s\in K}|f(s) - \zeta(s+iht_k;\mathfrak{a})| \\
&\qquad \leqslant \sup_{s\in K}\left|\mathrm{e}^{p(s)} - \zeta(s+iht_k;\mathfrak{a})\right| + \sup_{s\in K}\left|f(s) - \mathrm{e}^{p(s)}\right| \\
&\qquad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

in view of (2.7.3). This shows that

$$\begin{aligned}
&\left\{1 \leqslant k \leqslant N : \sup_{s\in K}\left|\mathrm{e}^{p(s)} - \zeta(s+iht_k;\mathfrak{a})\right| < \frac{\varepsilon}{2}\right\} \\
&\qquad \subset \left\{1 \leqslant k \leqslant N : \sup_{s\in K}|f(s) - \zeta(s+iht_k;\mathfrak{a})| < \varepsilon\right\}.
\end{aligned}$$

This remark together with (2.7.2) proves the desired inequality

$$\liminf_{N\to\infty}\frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K}|f(s) - \zeta(s+iht_k;\mathfrak{a})| < \varepsilon\right\} > 0.$$

The first assertion of the theorem is proved.

For the proof of the second assertion of the theorem, we define one more set

$$\widehat{G}_\varepsilon = \left\{g(s) \in H(D) : \sup_{s\in K}|g(s) - f(s)| < \varepsilon\right\},$$

and observe that the boundary $\partial\widehat{G}_\varepsilon$ of the set $\widehat{G}_\varepsilon$ lies in the set

$$\left\{g(s) \in H(D) : \sup_{s\in K}|g(s) - f(s)| = \varepsilon\right\}.$$

Thus, $\partial\widehat{G}_{\varepsilon_1} \cap \partial\widehat{G}_{\varepsilon_2} = \varnothing$ for different positive $\varepsilon_1$ and $\varepsilon_2$. Therefore, we have that $P_{\zeta,\mathfrak{a}}(\partial\widehat{G}_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$. Hence, the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,\mathfrak{a}}$ for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 2.2 and (iii) of Lemma 2.1 give that

$$\lim_{N\to\infty} P_{N,\mathfrak{a}}(\widehat{G}_\varepsilon) = P_{\zeta,\mathfrak{a}}(\widehat{G}_\varepsilon)$$

for all but at most countably many $\varepsilon > 0$. This and the definitions of $P_{N,\mathfrak{a}}$ and $\widehat{G}_\varepsilon$ show that

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k; \mathfrak{a})| < \varepsilon \right\} = P_{\zeta,\mathfrak{a}}(\widehat{G}_\varepsilon)$$

for all but at most countably many $\varepsilon > 0$. It remains to show that $P_{\zeta,\mathfrak{a}}(\widehat{G}_\varepsilon) > 0$. Similarly as above, using (2.7.2), we obtain that $G_\varepsilon \subset \widehat{G}_\varepsilon$. Therefore, by (2.7.1) and monotonicity of measures, we have $P_{\zeta,\mathfrak{a}}(\widehat{G}_\varepsilon) > 0$. The theorem is proved. $\square$

# CHAPTER 3

## Joint universality of periodic zeta-functions with shifts involving imaginary parts of non-trivial zeros of the Riemann zeta-function

In this chapter, we consider a simultaneous approximation of a given collection $(f_1(s), \dots, f_r(s))$ of analytic in the strip $D$ functions by shifts $(\zeta(s + ih_1\gamma_k; \mathfrak{a}_1), \dots, \zeta(s + ih_r\gamma_k; \mathfrak{a}_r))$, where $h_1, \dots, h_r$ are certain positive numbers, and $\{\gamma_k : k \in \mathbb{N}\} = 0 < \gamma_1 < \dots \leqslant \gamma_k \leqslant \gamma_{k+1} \leqslant \cdots$ is a sequence of imaginary parts of non-trivial zeros of the Riemann zeta-function $\zeta(s)$, i. e., $\rho = \beta + i\gamma_k$, $0 \leqslant \beta \leqslant 1$, is a zeros of the Riemann zeta-function $\zeta(s)$:

$$\zeta(\beta + i\gamma_k) = 0.$$

It is well known that the sequence $\{\gamma_k\}$ is infinite. Let $N(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ lying in the rectangle $\{s \in \mathbb{C} : 0 \leqslant \sigma \leqslant 1,\ 0 < t \leqslant T\}$. Then the asymptotic formula

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \quad T \to \infty,$$

with an absolute constant in $O(\dots)$ is known, see, for example, [71], [65]. By the Riemann hypothesis, the real part of non-trivial zeros is $\beta = 1/2$, or, equivalently, that $\zeta(s) \neq 0$, for $\sigma > 1/2$.

The classical de la Valée Poussin zero-free region is of the form: there exists an absolute constant $c > 0$, such that, in the region

$$\sigma \geqslant 1 - \frac{c}{\log(|t| + 2)},$$

$\zeta(s) \neq 0$. The latter result was applied in [6] to prove the asymptotic distribution law of prime numbers that

$$\sum_{\substack{p \leqslant x \\ p \in \mathbb{P}}} 1 = \int_2^x \frac{\mathrm{d}u}{\log u}(1 + o(1)), \quad x \to \infty.$$

The best result on zero-free regions says [65] that there exists an absolute constant $c > 0$ such that $\zeta(s) \neq 0$ for

$$\sigma \geqslant 1 - \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}}, \quad t \geqslant t_0 > 0.$$

The first complete proof due to H.-E. Richert, and was published in [74]. Recently, it has been proved [56] that $c = 55.241$.

There are many equivalents of RH. One of them [3] is stated by terms of self-approximation by shifts $\zeta(s + i\tau)$. More precisely, RH is true if and only if, for every $K \in \mathcal{K}$ and $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas}\left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s) - \zeta(s + i\tau)| < \varepsilon\right\} > 0.$$

## 3.1 Statement of the main theorem

We recall definitions of some objects that occur in the statement of the main result of the chapter.

A complex number $a$ is called algebraic if there exists a polynomial $p(s) \not\equiv 0$ with rational coefficients such that $p(a) = 0$.

The numbers $h_1, \dots, h_r$ are linearly independent over $\mathbb{Q}$, if the equality

$$a_1 h_1 + \cdots + a_r h_r = 0$$

with arbitrary $a_1, \dots, a_r \in \mathbb{Q}$ implies that $a_1 = \cdots = a_r = 0$.

Moreover, we will use the conjecture (I4), i. e., that, for $c > 0$,

$$\sum_{\substack{\gamma_k, \gamma_l \leqslant T \\ |\gamma_k - \gamma_l| < c/\log T}} 1 \ll T \log T, \quad T \to \infty.$$

**Theorem 3.1.** *Suppose the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative, $h_1, \dots, h_r$ are positive algebraic numbers linearly independent over the field of rational* *numbers, and estimate* (I4) *is true. For* $j = 1, \dots, r$*, let* $K_j \in \mathcal{K}$ *and* $f_j(s) \in H_0(K_j)$*. Then, for every* $\varepsilon > 0$*,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s + ih_j\gamma_k; \mathfrak{a}_j)| < \varepsilon \right\} > 0.$$

*Moreover, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s + ih_j\gamma_k; \mathfrak{a}_j)| < \varepsilon \right\}$$

*exists and is positive for all but at most countably many* $\varepsilon > 0$*.*

The proof of Theorem 3.1 is based on properties of the sequence $\{\gamma_k\}$ and probabilistic behaviour of the collection $(\zeta(s; \mathfrak{a}_1), \dots, \zeta(s; \mathfrak{a}_r))$. We will derive Theorem 3.1 from a limit theorem on the weak convergence of probability measures in the $r$-dimensional space of analytic functions.

## 3.2 Statement of a discrete joint limit theorem

We preserve the notations $D$ and $H(D)$ used in previous chapters, and define

$$H^r(D) = \underbrace{H(D) \times \cdots \times H(D)}_{r}.$$

Let, as in Chapter 2, $\Omega$ stand for infinite-dimensional torus, and set

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then $\Omega^r$, as a Cartesian product of compact topological Abelian groups, is again a compact topological group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure $m_H^r$ can be defined, and this gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$. Denote by $\omega_j(p)$ the $p$th component, $p \in \mathbb{P}$, of an element $\omega_j \in \Omega_j$, $j = 1, \dots, r$. Moreover, for brevity, let $\omega = (\omega_1, \dots, \omega_r) \in \Omega^r$, $\omega_1 \in \Omega_1, \dots, \omega_r \in \Omega_r$, $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_r)$, $\mathfrak{a}_j = \{a_{jm} : m \in \mathbb{N}\}$, $j = 1, \dots, r$, and on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, define the $H^r(D)$-valued random element

$$\underline{\zeta}(s, \omega; \underline{\mathfrak{a}}) = (\zeta(s, \omega_1; \mathfrak{a}_1), \dots, \zeta(s, \omega_r; \mathfrak{a}_r)),$$

where

$$\zeta(s, \omega_j; \mathfrak{a}_j) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{l=1}^{\infty} \frac{a_{jp^l}\omega_j^l(p)}{p^{ls}}\right), \quad j = 1, \ldots, r.$$

All above products, for almost all $\omega_j$, are uniformly convergent on compact subsets of the strip $D$, see Section 2.2. Denote by $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ the distribution of the random element $\underline{\zeta}(s, \omega; \underline{\mathfrak{a}})$, i. e.,

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}(A) = m_H^r \left\{\omega \in \Omega^r : \underline{\zeta}(s, \omega; \underline{\mathfrak{a}}) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)).$$

Now, we will define a probability measure. Let $\underline{h} = (h_1, \ldots, h_r)$. For $A \in \mathcal{B}(H^r(D))$, define

$$P_{N,\underline{\mathfrak{a}}}(A) = \frac{1}{N} \# \left(1 \leqslant k \leqslant N : \underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) \in A\right),$$

where

$$\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) = \left(\zeta(s + ih_1\gamma_k; \mathfrak{a}_1), \ldots, \zeta(s + ih_r\gamma_k; \mathfrak{a}_r)\right).$$

Now, we state a joint limit theorem.

**Theorem 3.2.** *Suppose the sequences $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are multiplicative, $h_1, \ldots, h_r$ are positive algebraic numbers linearly independent over $\mathbb{Q}$, and estimate* (I4) *is valid. Then $P_{N,\underline{\mathfrak{a}}}$ converges weakly to $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ as $N \to \infty$.*

We divide the proof of Theorem 3.2 into several parts. The first of them is a limit lemma on the weak convergence of probability measures on $(\Omega^r, \mathcal{B}(\Omega^r))$. In general, we will develop a scheme of a proof of Theorem 2.2 for $r$-dimensional case.

## 3.3 A limit lemma on the group $\Omega^r$

First we recall some facts needed for the proof of a limit lemma on $\Omega^r$.

**Lemma 3.1.** *The sequence $\{a\gamma_k : k \in \mathbb{N}\}$ with every $a \in \mathbb{R} \setminus \{0\}$ is uniformly distributed modulo* 1.

Proof of the lemma is given in [66], and in the above form, was applied in [13].

We also recall one result of the Diophantine type.

**Lemma 3.2.** *Suppose that* $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ *are algebraic numbers such that the logarithms* $\log\lambda_1, \dots, \log\lambda_r$ *are linearly independent over* $\mathbb{Q}$. *Then, for any algebraic numbers* $\beta_0, \dots, \beta_r$, *not all zero, we have*

$$|\beta_0 + \beta_1 \log\lambda_1 + \dots + \beta_r \log\lambda_r| > H^{-C},$$

*where* $H$ *is the maximum of the heights of* $\beta_0, \beta_1, \dots, \beta_r$, *and* $C$ *is an effectively computable number depending on* $r$ *and the maximum of the degrees of* $\beta_0, \beta_1, \dots, \beta_r$.

The lemma is the well-known Baker theorem on logarithm forms, see, for example, [2].

For $A \in \mathcal{B}(\Omega^r)$, define

$$V_{n,r}(A) = \frac{1}{N}\#\Big\{1 \leqslant k \leqslant N : \\ \Big(\Big(p^{-ih_1\gamma_k} : p \in \mathbb{P}\Big), \dots, \Big(p^{-ih_r\gamma_k} : p \in \mathbb{P}\Big)\Big) \in A\Big\}.$$

**Lemma 3.3.** *Suppose that* $h_1, \dots, h_r$ *are positive algebraic numbers linearly independent over* $\mathbb{Q}$. *Then* $V_{N,r}$ *converges weakly to the Haar measure* $m_H^r$ *as* $N \to \infty$.

*Proof.* The character group of $\Omega^r$ is isomorphic to

$$\bigoplus_{j=1}^{r} \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_{jp},$$

where $\mathbb{Z}_{jp} = \mathbb{Z}$ for $j = 1, \dots, r$, and $p \in \mathbb{P}$. Therefore, the characters of the group $\Omega^r$ are of the form

$$\prod_{j=1}^{r} {\prod_{p\in\mathbb{P}}}^{*} \omega_j^{k_{jp}}(p),$$

where the star $*$ shows that only a finite number of integers $k_{jp}$ are distinct from zero. Hence, the Fourier transform $g_N(\underline{k}_1, \dots, \underline{k}_r)$ of the measure $V_{N,r}$ is

$$g_N(\underline{k}_1, \dots, \underline{k}_r) = \int\limits_{\Omega^r} \left( \prod_{j=1}^{r} {\prod_{p\in\mathbb{P}}}^{*} \omega_j^{k_{jp}}(p) \right) \mathrm{d}V_{N,r},$$

where $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \dots, r$. Thus, by the definition of

$V_{N,r}$,

$$g_N(\underline{k}_1,\dots,\underline{k}_r) = \frac{1}{N}\sum_{k=1}^{N}\prod_{j=1}^{r}\prod_{p\in\mathbb{P}}{}^{*}\, p^{-ih_jk_{jp}\gamma_k}$$
$$= \frac{1}{N}\sum_{k=1}^{N}\exp\left\{-i\gamma_k\sum_{j=1}^{r}h_j\sum_{p\in\mathbb{P}}{}^{*}\, k_{jp}\log p\right\}. \qquad (3.3.1)$$

It is obvious that

$$g_N(\underline{0},\dots,\underline{0}) = 1, \qquad (3.3.2)$$

where $\underline{0} = (0, 0, \dots)$.

Now, suppose that $\underline{k} = (\underline{k}_1,\dots,\underline{k}_r) \neq (\underline{0},\dots,\underline{0})$. Then there exists $j \in \{1,\dots,r\}$ such that $\underline{k}_j \neq \underline{0}$. Thus, there exists a prime number $p$ such that $k_{jp} \neq 0$. Define

$$a_p = \sum_{j=1}^{r} h_j k_{jp}.$$

Then, since the numbers $h_1,\dots,h_r$ are linearly independent, we have $a_p \neq 0$. The numbers $a_p$ are algebraic, and the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, in view of Lemma 3.2,

$$a_{\underline{k}_1,\dots,\underline{k}_r} \stackrel{\text{def}}{=} \sum_{j=1}^{r}h_j\sum_{p\in\mathbb{P}}{}^{*}\, k_{jp}\log p = \sum_{p\in\mathbb{P}}{}^{*}\, a_p\log p \neq 0.$$

Hence, in virtue of Lemma 3.1, the sequence

$$\left\{\frac{1}{2\pi}a_{\underline{k}_1,\dots,\underline{k}_r}\gamma_k : k \in \mathbb{N}\right\}$$

is uniformly distributed modulo 1. This together with (3.3.1) and Lemma 2.3 shows that, in the case $(\underline{k}_1,\dots,\underline{k}_r) \neq (\underline{0},\dots,\underline{0})$,

$$\lim_{N\to\infty} g_N(\underline{k}_1,\dots,\underline{k}_r) = 0.$$

Thus, in view of (3.3.2),

$$\lim_{N\to\infty} g_N(\underline{k}_1,\dots,\underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r) = (\underline{0},\dots,\underline{0}), \\ 0 & \text{if } (\underline{k}_1,\dots,\underline{k}_r) \neq (\underline{0},\dots,\underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the

Haar measure $m_H^r$, the lemma follows from the continuity theorem for Fourier transforms. □

Lemma 3.1 implies a limit lemma in the space $H^r(D)$ for absolutely convergent Dirichlet series. Let $v_n(m)$ be the same as in Section 2.4. Define

$$\zeta_n(s;\mathfrak{a}_j)=\sum_{m=1}^{\infty}\frac{a_{jm}v_n(m)}{m^s},\quad j=1,\dots,r.$$

For $j=1,\dots,r$, the series for $\zeta_n(s;\mathfrak{a}_j)$ is absolutely convergent in the whole complex plane. For $\mathcal{B}(H^r(D))$, define

$$V_{N,n,\underline{\mathfrak{a}}}(A)=\frac{1}{N}\#\left\{1\leqslant k\leqslant N:\underline{\zeta}_n(s+i\underline{h}\gamma_k;\underline{\mathfrak{a}})\in A\right\},$$

where

$$\underline{\zeta}_n(s;\underline{\mathfrak{a}})=\left(\zeta_n(s;\mathfrak{a}_1),\dots,\zeta_n(s;\mathfrak{a}_r)\right).$$

Moreover, let

$$\zeta_n(s,\omega_j;\mathfrak{a}_j)=\sum_{m=1}^{\infty}\frac{a_{jm}\omega_j(m)v_n(m)}{m^s},\quad j=1,\dots,r,$$

and

$$\underline{\zeta}_n(s,\omega;\underline{\mathfrak{a}})=\left(\zeta_n(s,\omega_1;\mathfrak{a}_1),\dots,\zeta_n(s,\omega_r;\mathfrak{a}_r)\right).$$

Define the mapping $\underline{u}_n:\Omega^r\to H^r(D)$ by the formula

$$\underline{u}_n(\omega)=\underline{\zeta}_n(s,\omega;\underline{\mathfrak{a}}).$$

Clearly, the series for $\zeta_n(s,\omega_j;\mathfrak{a}_j)$, $j=1,\dots,r$, are also absolutely convergent in the whole complex plane. Therefore, the mapping $\underline{u}_n$ is continuous, thus, $(\mathcal{B}(\Omega^r),\mathcal{B}(H^r(D)))$-measurable. Hence, we may define correctly the probability measure $V_{n,\underline{\mathfrak{a}}}=m_H^r\underline{u}_n^{-1}$, where

$$m_H^r\underline{u}_n^{-1}(A)=m_H^r(\underline{u}_n^{-1}A),\quad A\in\mathcal{B}(H^r(D)).$$

**Lemma 3.4.** *Suppose that $h_1,\dots,h_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then the measure $V_{N,n,\underline{\mathfrak{a}}}$ converges weakly to $V_{n,\underline{\mathfrak{a}}}$ as $N\to\infty$.*

*Proof.* The lemma is a consequence of Lemmas 3.3, continuity of $\underline{u}_n$ and

Lemma 2.5. □

We observe that the limit measure $V_{n,\underline{\mathfrak{a}}}$ in Lemma 3.4 is independent on $\underline{h}$ and the sequence $\{\gamma_k\}$. Moreover, the following statement is true.

**Lemma 3.5.** *Suppose that the sequences $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are multiplicative. Then $V_{n,\underline{\mathfrak{a}}}$ converges weakly to $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ as $n \to \infty$.*

*Proof.* In the paper [42], the weak convergence of the measure

$$\widehat{V}_{T,\underline{\mathfrak{a}}}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}(s+i\tau;\underline{\mathfrak{a}}) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)),$$

was considered, and it was obtained that $V_{n,\underline{\mathfrak{a}}}$ as $n \to \infty$, and $\widehat{V}_{T,\underline{\mathfrak{a}}}$ as $T \to \infty$ have the same limit measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$. Thus, we have that $V_{n,\underline{\mathfrak{a}}}$ converges weakly to $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ as $n \to \infty$. □

In view of Lemma 3.5, to prove Theorem 3.2, it suffices to show that $P_{N,\underline{\mathfrak{a}}}$, as $N \to \infty$, and $V_{n,\underline{\mathfrak{a}}}$ as $n \to \infty$, have the same limit measure. For this, a certain closeness among collections $\underline{\zeta}(s;\underline{\mathfrak{a}})$ and $\underline{\zeta}_n(s;\underline{\mathfrak{a}})$ is needed.

## 3.4 Approximation of $\underline{\zeta}(s;\underline{\mathfrak{a}})$ by $\underline{\zeta}_n(s;\underline{\mathfrak{a}})$

In Section 2.2, we introduced the metric $\rho$ in the space $H(D)$ inducing its topology of uniform convergence on compacta. Now, we need a metric in $H^r(D)$ inducing its product topology. For

$$\underline{g}_1 = (g_{11}, \ldots, g_{1r}),\ \underline{g}_2 = (g_{21}, \ldots, g_{2r}) \in H^r(D),$$

define

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho(g_{1j}, g_{2j}).$$

Then $\underline{\rho}$ is the desired metric in $H^r(D)$ inducing its product topology.

The aim of this section is to show that $\underline{\zeta}_n(s;\underline{\mathfrak{a}})$ is close to $\underline{\zeta}(s;\underline{\mathfrak{a}})$ in the mean. Note that, in this section, we do not use multiplicativity of the sequences $\mathfrak{a}_j$, $j = 1, \ldots, r$.

We start with recalling the analytic behavior of the sequence $\{\gamma_k : k \in \mathbb{N}\}$.

**Lemma 3.6.** *For $k \to \infty$,*

$$\gamma_k = \frac{2\pi k}{\log k}(1 + o(1)).$$

Proof of the asymptotic equality of the lemma can be found, for example, in the monograph [71].

Now we state the approximation lemma.

**Lemma 3.7.** *Suppose that the estimate* (I4) *is true. Then, for every positive numbers* $h_1, \dots, h_r$ *and sequences* $\mathfrak{a}_1, \dots, \mathfrak{a}_r$*, the equality*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \underline{\rho}\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}), \underline{\zeta}_n(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\right) = 0$$

*holds.*

*Proof.* By the definitions of the metrics $\underline{\rho}$ and $\rho$, it is sufficient to show, that for every compact set $K \subset D$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s\in K} |\zeta(s + ih_j\gamma_k; \mathfrak{a}_j) - \zeta_n(s + ih_j\gamma_k; \mathfrak{a}_j)| = 0 \quad (3.4.1)$$

for all $j = 1, \dots, r$.

Let $h > 0$ and a periodic sequence $\mathfrak{a}$ be arbitrary. Consider the approximation of $\zeta(s+ih\gamma_k; \mathfrak{a})$ by $\zeta_n(s+ih\gamma_k; \mathfrak{a})$. For this, we apply the representation of $\zeta_n(s; \mathfrak{a})$ given in Lemma 2.12. Thus, in notation of Lemma 2.12, we have

$$\zeta_n(s; \mathfrak{a}) - \zeta(s; \mathfrak{a}) = \frac{1}{2\pi i} \int_{-\theta-i\infty}^{-\theta+i\infty} \zeta(s + z; \mathfrak{a}) b_n(z) \mathrm{d}z + R_n(s; \mathfrak{a}), \quad (3.4.2)$$

where

$$R_n(s; \mathfrak{a}) = \widehat{a} b_n(1 - s), \qquad \widehat{a} = \operatorname*{Res}_{s=1} \zeta(s; \mathfrak{a}), \qquad \theta > 0.$$

Let $K \subset D$ be an arbitrary compact set, and $\varepsilon > 0$ such that $1/2 + \varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for $s = \sigma + it \in K$. Take $\theta = 2\sigma - \varepsilon - 1/2$. Then, in view of (3.4.2), for $s \in K$,

$$|\zeta_n(s; \mathfrak{a}) - \zeta(s; \mathfrak{a})| \ll \int_{-\infty}^{\infty} |\zeta(s - \theta + i\tau; \mathfrak{a})||b_n(-\theta + i\tau)| \, \mathrm{d}\tau + |R_n(s; \mathfrak{a})|.$$

From this, we derive that

$$\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|\zeta(s+ih\gamma_k;\mathfrak{a})-\zeta_n(s+ih\gamma_k;\mathfrak{a})|\ll I+Z, \tag{3.4.3}$$

where

$$I=\int_{-\infty}^{\infty}\left(\frac{1}{N}\sum_{k=1}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+ih\gamma_k+i\tau;\mathfrak{a}\right)\right|\right)\\ \times\sup_{s\in K}|b_n(1/2+\varepsilon-s+i\tau)|\,\mathrm{d}\tau$$

and

$$Z=\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|R_n(s+ih\gamma_k;\mathfrak{a})|\,.$$

From estimate (2.5.1), for $t\in\mathbb{R}$, we have

$$\int_{0}^{T}\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+it;\mathfrak{a}\right)\right|^2\,\mathrm{d}\tau\ll_{\varepsilon}T(1+|t|). \tag{3.4.4}$$

In view of integral Cauchy formula, analogical bound is valid for $\zeta'(s;\mathfrak{a})$, i. e.,

$$\int_{0}^{T}\left|\zeta'\left(\frac{1}{2}+\varepsilon+i\tau+it;\mathfrak{a}\right)\right|^2\,\mathrm{d}\tau\ll_{\varepsilon}T(1+|t|). \tag{3.4.5}$$

In the next step of the proof, we apply estimate (I4). Let $\delta=ch(\log\gamma_N)^{-1}$ and

$$N_\delta(h\gamma_k)=\sum_{\substack{\gamma_k,\gamma_l\leqslant\gamma_N\\|\gamma_l-\gamma_k|<\delta}}1\,.$$

Then, in view of (I4) and Lemma 3.6,

$$\sum_{k=1}^{N}N_\delta(h\gamma_k)=\sum_{\substack{\gamma_k,\gamma_l\leqslant\gamma_N\\|\gamma_k-\gamma_l|<c(\log\gamma_N)^{-1}}}\sum 1\ll\gamma_N\log\gamma_N\ll N. \tag{3.4.6}$$

Applying the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
&\sum_{k=1}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+ih\gamma_k+it;\mathfrak{a}\right)\right| \\
&\quad=\sum_{k=1}^{N}\left(N_\delta(h\gamma_k)N_\delta^{-1}(h\gamma_k)\right)^{1/2}\left|\zeta\left(\frac{1}{2}+\varepsilon+ih\gamma_k+it;\mathfrak{a}\right)\right| \\
&\quad\leqslant\left(\sum_{k=1}^{N}N_\delta(h\gamma_k)\sum_{k=1}^{N}N_\delta^{-1}(h\gamma_k)\left|\zeta\left(\frac{1}{2}+\varepsilon+ih\gamma_k+it;\mathfrak{a}\right)\right|^2\right)^{1/2}.
\end{aligned} \tag{3.4.7}$$

For the second sum in (3.4.7), we apply Lemma 2.10 with $\mathcal{T}=\{h\gamma_1\delta,\ldots,h\gamma_N\delta\}$, and $T_0=h\gamma_1-\delta/2$, $T=h\gamma_N-T_0+\delta/2$. This with estimates (3.4.4) and (3.4.5) yield

$$\begin{aligned}
&\sum_{k=1}^{N}N_\delta^{-1}(h\gamma_k)\left|\zeta\left(\frac{1}{2}+\varepsilon+ih\gamma_k+iu;\mathfrak{a}\right)\right|^2 \\
&\quad\ll_\varepsilon\frac{1}{\delta}\int_0^{2h\gamma_N}\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iu;\mathfrak{a}\right)\right|^2\,\mathrm{d}\tau \\
&\qquad+\left(\int_0^{2h\gamma_N}\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iu;\mathfrak{a}\right)\right|^2\,\mathrm{d}\tau\right. \\
&\qquad\times\left.\int_0^{2h\gamma_N}\left|\zeta'\left(\frac{1}{2}+\varepsilon+i\tau+iu;\mathfrak{a}\right)\right|^2\,\mathrm{d}\tau\right)^{1/2} \\
&\quad\ll_{\varepsilon,h,\mathfrak{a}}\log(\gamma_N)\gamma_N(1+|u|)+\gamma_N(1+|u|)\ll_{\varepsilon,h,\mathfrak{a}}N(1+|u|).
\end{aligned}$$

Therefore, (3.4.6) and (3.4.7) give the bound

$$\begin{aligned}
\sum_{k=1}^{N}N_\delta^{-1}(h\gamma_k)\left|\zeta\left(\frac{1}{2}+\varepsilon+ih\gamma_k+iu;\mathfrak{a}\right)\right|&\ll_{\varepsilon,h,\mathfrak{a}}N^{1/2}(N(1+|u|))^{1/2} \\
&\ll_{\varepsilon,h,\mathfrak{a}}N(1+|u|).
\end{aligned} \tag{3.4.8}$$

Using estimate (2.5.6) for the gamma-function, from the definition of $b_n(s)$, we obtain, that, for $s \in K$,

$$b_n\left(\frac{1}{2}+\varepsilon-s+iu\right) \ll n^{-\varepsilon}\exp\{-c|t-u|\}$$
$$\ll_K n^{-\varepsilon}\exp\{-c_1|u|\}, \quad c_1>0.$$

This and (3.4.8) give

$$I \ll_{\varepsilon,h,\mathfrak{a},K} n^{-\varepsilon}\int_{-\infty}^{\infty}(1+|u|)\exp\{-c_1|u|\}\mathrm{d}u \ll_{K,h,\varepsilon} n^{-\varepsilon}. \tag{3.4.9}$$

Similarly we estimate $b_n(1-s-ih\gamma_k)$. By (2.5.6), we have

$$b_n(1-s-ih\gamma_k) \ll n^{1-\sigma}\exp\{-c|t+h\gamma_k|\} \ll_K n^{1-\sigma}\exp\{-c_1h\gamma_k\}.$$

Thus, for $s \in K$,

$$b_n(1-s-ih\gamma_k) \ll_K n^{1/2-2\varepsilon}\exp\{-c_1h\gamma_k\}.$$

Hence, we obtain, by Lemma 3.6, that

$$\begin{aligned}
Z &\ll_K \frac{n^{1/2-2\varepsilon}}{N}\sum_{k=1}^{N}\exp\{-c_1h\gamma_k\}\\
&\ll_K \frac{n^{1/2-2\varepsilon}}{N}\left(\sum_{k\leqslant\log N}+\sum_{k\geqslant\log N}\right)\exp\{-c_1h\gamma_k\}\\
&\ll_K n^{1/2-2\varepsilon}\frac{\log N}{N}+n^{1/2-2\varepsilon}\sum_{k\geqslant\log N}\exp\left\{-\frac{c_1}{2}h\frac{k}{\log k}\right\}\\
&\ll_{K,h} n^{1/2-2\varepsilon}\frac{\log N}{N}+n^{1/2-2\varepsilon}\exp\left\{-\frac{c_1}{2}h\frac{\log N}{\log\log N}\right\}\\
&\ll_{K,h} n^{1/2-2\varepsilon}\exp\left\{-\frac{c_1}{2}h\frac{\log N}{\log\log N}\right\}.
\end{aligned}$$

This together with (3.4.9) show that

$$I+Z \ll_{K,h,\mathfrak{a}} n^{-\varepsilon}+n^{1/2-2\varepsilon}\exp\left\{-\frac{c_1}{2}h\frac{\log N}{\log\log N}\right\}$$

From this and (3.4.3), equality (3.4.1) follows. The lemma is proved. □

## 3.5 Proof of Theorem 3.2

As in proof of Theorem 2.2, we use the language of convergence in distribution ($\xrightarrow{\mathcal{D}}$).

*Proof of Theorem 3.2.* Remind that $V_{n,\underline{\mathfrak{a}}}$ is a limit measure in Lemma 3.4. Denote by $\underline{X}_{n,\underline{\mathfrak{a}}} = \underline{X}_{n,\underline{\mathfrak{a}}}(s)$ the $H^r(D)$-valued random element having the distribution $V_{n,\underline{\mathfrak{a}}}$. Then the statement of Lemma 3.5 can be written in the form

$$\underline{X}_{n,\underline{\mathfrak{a}}} \xrightarrow[n\to\infty]{\mathcal{D}} P_{\underline{\zeta},\underline{\mathfrak{a}}}. \tag{3.5.1}$$

Now, let the random variable $\eta_N$ be defined on a certain probability space $(\widetilde{\Omega}, \mathcal{A}, \mu)$, and having the distribution

$$\mu\{\eta_N = \gamma_k\} = \frac{1}{N}, \quad k = 1, \dots, N.$$

Define the $H^r(D)$-valued random element

$$\underline{X}_{N,n,\underline{\mathfrak{a}}} = \underline{X}_{N,n,\underline{\mathfrak{a}}}(s) = \underline{\zeta}_n(s + i\underline{h}\eta_N; \underline{\mathfrak{a}})$$

where

$$\underline{\zeta}_n(s + i\underline{h}\eta_N; \underline{\mathfrak{a}}) = (\zeta_n(s + ih_1\eta_N; \mathfrak{a}_1), \dots, \zeta_n(s + ih_r\eta_N; \mathfrak{a}_r)) .$$

Then the statement of Lemma 3.4 can be written as

$$\underline{X}_{N,n,\underline{\mathfrak{a}}} \xrightarrow[N\to\infty]{\mathcal{D}} \underline{X}_{n,\underline{\mathfrak{a}}}. \tag{3.5.2}$$

Define one more $H^r(D)$-valued random element

$$\underline{Y}_{N,\underline{\mathfrak{a}}} = \underline{Y}_{N,\underline{\mathfrak{a}}}(s) = \underline{\zeta}(s + i\underline{h}\eta_N; \underline{\mathfrak{a}}),$$

where

$$\underline{\zeta}(s + i\underline{h}\eta_N; \underline{\mathfrak{a}}) = (\zeta(s + ih_1\eta_N; \mathfrak{a}_1), \dots, \zeta(s + ih_r\eta_N; \mathfrak{a}_r)) .$$

Then Lemma 3.7 implies that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty}\limsup_{N\to\infty}\mu\left\{\underline{\rho}\left(\underline{Y}_{n,\underline{\mathfrak{a}}}(s),\underline{X}_{N,n,\underline{\mathfrak{a}}}(s)\right)\geqslant\varepsilon\right\}\\
&\quad=\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N}\#\Big\{1\leqslant k\leqslant N:\\
&\qquad\qquad\underline{\rho}\Big(\underline{\zeta}(s+i\underline{h}\gamma_k;\underline{\mathfrak{a}}),\,\underline{\zeta}_n(s+i\underline{h}\gamma_k;\underline{\mathfrak{a}})\Big)\geqslant\varepsilon\Big\}\\
&\quad\leqslant\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N\varepsilon}\sum_{k=1}^{N}\underline{\rho}\Big(\underline{\zeta}(s+i\underline{h}\gamma_k;\mathfrak{a}),\underline{\zeta}_n(s+i\underline{h}\gamma_k;\underline{\mathfrak{a}})\Big)=0.
\end{aligned}$$

This, (3.5.1) and (3.5.2) show that all hypotheses of Lemma 2.13 are satisfied by the $H^r(D)$-valued random elements $\underline{X}_{n,\underline{\mathfrak{a}}}$, $\underline{X}_{N,n,\underline{\mathfrak{a}}}$ and $Y_{N,\underline{\mathfrak{a}}}$. Therefore, the relation

$$\underline{Y}_{N,\underline{\mathfrak{a}}}\xrightarrow[N\to\infty]{\mathcal{D}}P_{\underline{\zeta},\underline{\mathfrak{a}}},$$

is valid, which is equivalent to weak convergence of the measure $P_{N,\underline{\mathfrak{a}}}$ to $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ as $N\to\infty$. □

From the proof, we see that the important role is played by an algebraic nature of the numbers $h_1,\dots,h_r$ (Lemmas 3.1 and 3.3) as well as by hypothesis (I4).

## 3.6 Proof of joint universality

We begin with the explicit form of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ in Theorem 3.2. Let $m_H$ be the Haar measure on $(\Omega,\mathcal{B}(\Omega))$, and

$$P_{\zeta,\mathfrak{a}}(A)=m_H\left\{\omega\in\Omega:\zeta(s,\omega;\mathfrak{a})\in A\right\},\quad A\in\mathcal{B}(H(D)),$$

where $\zeta(s,\omega;\mathfrak{a})$ is the $H(D)$-valued random element defined by

$$\zeta(s,\omega;\mathfrak{a})=\prod_{p\in\mathbb{P}}\left(1+\sum_{l=1}^{\infty}\frac{a_{p^l}\omega^l(p)}{p^{ls}}\right).$$

Then, in [47], the following lemma has been obtained.

**Lemma 3.8.** *Suppose that the sequence $\mathfrak{a}$ is multiplicative. Then the support of the measure $P_{\zeta,\mathfrak{a}}$ is the set*

$$S=\{g\in H(D):g(s)\neq 0\ \textit{or}\ g(s)\equiv 0\}.$$

Lemma 3.8 implies the form of the support of the limit measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ in Theorem 3.2.

**Lemma 3.9.** *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative. Then the support of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ is the set*

$$S^r = \underbrace{S \times \cdots \times S}_{r}.$$

*Proof.* Since the space $H^r(D)$ is separable it is known [4] that

$$\mathcal{B}(H^r(D)) = \underbrace{\mathcal{B}(H(D)) \times \cdots \times \mathcal{B}(H(D))}_{r}.$$

From this, it follows that it suffices to consider the measure $P_{\underline{\zeta}}$ on the rectangular sets

$$A = A_1 \times \cdots \times A_r, \quad A_1, \dots, A_r \in \mathcal{B}(H(D)). \tag{3.6.1}$$

Denote by $m_{jH}$ the Haar measure on $\Omega_j$, $j = 1, \dots, r$. Then the Haar measure $m_H^r$ is the product of the measures $m_{jH}$, $j = 1, \dots, r$, i. e.,

$$m_H^r(A) = m_{1H}(A_1) \cdots m_{rH}(A_r)$$

if $A$ is given by (3.6.1). Taking into account these remarks, we find

$$\begin{aligned} P_{\underline{\zeta},\underline{\mathfrak{a}}}(A) =& m_H^r \left\{\omega \in \Omega^r : \underline{\zeta}(s, \omega; \underline{\mathfrak{a}}) \in A\right\} \\ =& m_{1H} \left\{\omega_1 \in \Omega_1 : \zeta(s, \omega_1; \mathfrak{a}_1) \in A_1\right\} \times \cdots \\ & \times m_{rH} \left\{\omega_r \in \Omega_r : \zeta(s, \omega_r; \mathfrak{a}_r) \in A_r\right\}. \end{aligned} \tag{3.6.2}$$

From Lemma 3.8, we have that the support of the measure

$$P_{\zeta_j,\mathfrak{a}_j}(A_j) = m_{jH} \left\{\omega_j \in \Omega_j : \zeta(s, \omega_j; \mathfrak{a}_j) \in A_j\right\}, \quad j = 1, \dots r,$$

is the set $S$. Therefore, (3.6.2) and the minimality of the support prove the lemma. □

*Proof of Theorem 3.1.* By Lemma 2.16 (Mergelyan theorem), there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| f_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2}. \tag{3.6.3}$$

In view of Lemma 3.9, the set

$$\underline{G}_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\}$$

is an open neighbourhood of an element $(\mathrm{e}^{p(s)}, \dots, \mathrm{e}^{p_r(s)})$ of the support of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ because $(\mathrm{e}^{p(s)}, \dots, \mathrm{e}^{p_r(s)}) \in S^r$. Hence, by a property of the support,

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}(\underline{G}_\varepsilon) > 0. \tag{3.6.4}$$

Define one more set

$$\widehat{\underline{G}}_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Then, assuming that $(g_1, \dots, g_r) \in \underline{G}_\varepsilon$, we obtain that $(g_1, \dots, g_r) \in \widehat{\underline{G}}_\varepsilon$. Actually, if $(g_1, \dots, g_r) \in \underline{G}_\varepsilon$, then, in view of (3.6.3),

$$\begin{aligned} &\sup_{\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| \\ &\qquad \leqslant \sup_{\leqslant j\leqslant r} \sup_{s\in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| + \sup_{\leqslant j\leqslant r} \sup_{s\in K_j} \left| f_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $(g_1, \dots, g_r) \in \widehat{\underline{G}}_\varepsilon$. This shows that $\underline{G}_\varepsilon \subset \widehat{\underline{G}}_\varepsilon$. Hence, by (3.6.4),

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}(\widehat{\underline{G}}_\varepsilon) > 0. \tag{3.6.5}$$

Moreover, by Theorem 3.2 and Lemma 2.1,

$$\liminf_{N\to\infty} P_{N,\underline{\mathfrak{a}}}(\widehat{\underline{G}}_\varepsilon) \geqslant P_{\underline{\zeta},\underline{\mathfrak{a}}}(\widehat{\underline{G}}_\varepsilon) > 0.$$

This and the definitions of $P_{N,\underline{\mathfrak{a}}}$ and $\underline{G}_\varepsilon$ prove the first part of the theorem.

For the proof of the second statement of the theorem, we observe that the boundary $\partial\widehat{\underline{G}}_\varepsilon$ of $\widehat{\underline{G}}_\varepsilon$ lies in the set

$$\left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| = \varepsilon \right\}.$$

Hence, $\partial\widehat{\underline{G}}_{\varepsilon-1} \cap \partial\widehat{\underline{G}}_{\varepsilon_2} = \varnothing$ for different positive $\varepsilon_1$ and $\varepsilon_2$. Therefore, the set $\widehat{\underline{G}}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ for all but at most countably many

$\varepsilon > 0$. This, Theorem 3.2, Lemma 2.1 and (3.6.5) show that the limit

$$\lim_{N\to\infty} P_{N,\underline{\mathfrak{a}}}(\underline{G}_\varepsilon) = P_{\underline{\zeta},\underline{\mathfrak{a}}}(\underline{\widehat{G}}_\varepsilon)$$

exists and is positive for all but at most countably many $\varepsilon > 0$, and the definitions of $P_{N,\underline{\mathfrak{a}}}$ and $\underline{\widehat{G}}_\varepsilon$ prove the second statement of the theorem. The theorem is proved. □

# CHAPTER 4

## Universality of some compositions of periodic zeta-functions with shifts of Chapter 3

Let $F : H^r(D) \to H(D)$ be a certain operator. In this chapter, we extend Theorem 3.1 for compositions using the same shifts as in Chapter 3. More precisely, we approximate a given function $f(s) \in H(K)$, $K \in \mathcal{K}$, by shifts $F(\underline{\zeta}(s + i\underline{h}\gamma_k; \mathfrak{a}))$ for some classes of operators $F$. Recall that $H(K)$ with $K \in \mathcal{K}$ denotes the class of continuous functions on $K$ that are analytic in the interior of $K$. Clearly, we have the inclusion $H_0(K) \subset H(K)$, where $H_0(K)$ is the subclass of non-vanishing functions of $H(K)$.

Note that universality of compositions $F(\zeta(s; \mathfrak{a}_1), \ldots, \zeta(s; \mathfrak{a}_r))$ significantly extends the class of universal functions. We consider two types of universality theorems for compositions $F(\zeta(s; \mathfrak{a}_1), \ldots, \zeta(s; \mathfrak{a}_r))$. The theorem of the first type directly follows from Theorem 3.1 by using a property of the operator $F$ similar to Lipschitz's inequality. The second type of theorems are obtained by using a probabilistic approach developed in Chapters 2 and 3.

## 4.1 Composition of Lipschitz type

We preserve the notation of previous chapters. For brevity, denote $\underline{g} = (g_1, \ldots, g_r) \in H^r(D)$ and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in (\mathbb{R}^+)^r$, where $\mathbb{R}^+$ is the set of all positive numbers. $F^{-1}\{g\}$ denotes the preimage of $g$.

The operator $F : H^r(D) \to H(D)$ belongs to the class $Lip(\underline{\alpha})$ if the following hypotheses are satisfied:

1. For every polynomial $p = p(s)$ and sets $K_1, \ldots, K_r \in \mathcal{K}$, there exists an element $\underline{g} \in F^{-1}\{p\} \subset H^r(D)$ such that $g_j(s) \neq 0$ on $K_j$, $j = 1, \ldots, r$.

2. For every set $K \in \mathcal{K}$, there exist a constant $c > 0$ and the sets $K_1, \ldots,$ $K_r \in \mathcal{K}$ such that

$$\sup_{s \in K} \left| F(\underline{g}_1) - F(\underline{g}_2) \right| \leqslant c \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_{1j}(s) - g_{2j}(s)|^{\alpha_j}$$

for all $\underline{g}_1, \underline{g}_2 \in H^r(D)$.

For brevity, let the statement $A(\underline{\mathfrak{a}}, \underline{h}, (3.3.1))$ with $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ and $\underline{h} = (h_1, \ldots, h_r)$ be valid if the sequences $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are multiplicative, $h_1,$ $\ldots, h_r$ are positive algebraic numbers linearly independent over $\mathbb{Q}$, and estimate (3.3.1) is valid.

**Theorem 4.1.** *Suppose that $A(\underline{\mathfrak{a}}, \underline{h}, (3.3.1))$ is valid, and the operator $F :$ $H^r(D) \to H(D)$ belongs to the class $Lip(\underline{\alpha})$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} \left| f(s) - F\left( \underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) \right) \right| < \varepsilon \right\} > 0.$$

*Proof.* The theorem is a simple consequence of Lemma 2.16 (Mergelyan theorem) and the definition of the class $Lip(\underline{\alpha})$.

The function $f(s)$ is continuous on $K \in \mathcal{K}$ and analytic in the interior of $K$, therefore, by Lemma 2.16, there exists a polynomial $p_\varepsilon = p_\varepsilon(s)$ such that

$$\sup_{s \in K} |f(s) - p_\varepsilon(s)| < \frac{\varepsilon}{2}. \tag{4.1.1}$$

Now we will apply the properties of the class $Lip(\underline{\alpha})$. In view of hypothesis 1, we find an element $\underline{g} \in F^{-1}\{p_\varepsilon\}$ such that $g_j(s) \neq 0$ on a given set $K_j \in \mathcal{K}$, $j = 1, \ldots, r$. Let $\alpha = \min_{1 \leqslant j \leqslant r} \alpha_j$, and the sets $K_1, \ldots, K_r \in \mathcal{K}$ correspond the set $K$ in hypothesis 2 of the class $Lip(\underline{\alpha})$. Suppose that the number $k \in \mathbb{N}$ satisfies the inequality

$$\sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_j(s) - \zeta(s + ih_j\gamma_k; \mathfrak{a}_j)| < c^{-1/\alpha} \left( \frac{\varepsilon}{2} \right)^{1/\alpha}. \tag{4.1.2}$$

Then, by hypothesis 2 of the class $Lip(\underline{\alpha})$, for the above $k$,

$$\begin{aligned}
&\sup_{s\in K}\left|p_\varepsilon(s)-F\left(\underline{\zeta}(s+i\underline{h}\gamma_k;\underline{\mathfrak{a}})\right)\right| \\
&\quad=\sup_{s\in K}\left|F\left(g_1(s),\dots,g_r(s)\right)-F\left(\underline{\zeta}(s+i\underline{h}\gamma_k;\underline{\mathfrak{a}})\right)\right| \\
&\quad\leqslant c\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|g_j(s)-\zeta(s+ih_j\gamma_k;\mathfrak{a}_j)\right|^{\alpha_j} \\
&\quad\leqslant c\,c^{-1}\frac{\varepsilon}{2}=\frac{\varepsilon}{2}.
\end{aligned}$$

In view of Theorem 3.1, the set of $k\in\mathbb{N}$ satisfying inequality (4.1.2) has a positive lower density, i. e.,

$$\begin{aligned}
\liminf_{N\to\infty}\frac{1}{N}\#\Bigg\{&1\leqslant k\leqslant N: \\
&\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|g_j(s)-\zeta(s+ih_j\gamma_k;\mathfrak{a}_j)\right|<c^{-1/\alpha}\left(\frac{\varepsilon}{2}\right)^{1/\alpha}\Bigg\}>0.
\end{aligned}$$

Therefore, the set of $k\in\mathbb{N}$ satisfying the inequality

$$\sup_{s\in K}\left|p_\varepsilon(s)-F\left(\underline{\zeta}(s+i\underline{h}\gamma_k;\underline{\mathfrak{a}})\right)\right|<\frac{\varepsilon}{2} \tag{4.1.3}$$

has a positive lower density as well, i. e.,

$$\liminf_{N\to\infty}\frac{1}{N}\#\left\{1\leqslant k\leqslant N:\sup_{s\in K}\left|p_\varepsilon(s)-F\left(\underline{\zeta}(s+i\underline{h}\gamma_k;\mathfrak{a})\right)\right|<\frac{\varepsilon}{2}\right\}>0. \tag{4.1.4}$$

It remains to pass from the polynomial $p_\varepsilon(s)$ to the function $f(s)$. For this, we apply inequality (4.1.1). Suppose that $k\in\mathbb{N}$ satisfies (4.1.3). Then, by (4.1.1),

$$\begin{aligned}
&\sup_{s\in K}\left|f(s)-F\left(\underline{\zeta}(s+\underline{h}\gamma_k;\mathfrak{a})\right)\right| \\
&\quad\leqslant\sup_{s\in K}\left|F\left(\underline{\zeta}(s+\underline{h}\gamma_k;\mathfrak{a})\right)-p_\varepsilon(s)\right|+\sup_{s\in K}\left|f(s)-p_\varepsilon(s)\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

This shows that

$$\begin{aligned}
&\left\{1\leqslant k\leqslant N:\sup_{s\in K}\left|f(s)-F\left(\underline{\zeta}(s+\underline{h}\gamma_k;\mathfrak{a})\right)\right|<\varepsilon\right\} \\
&\quad\supset\left\{1\leqslant k\leqslant N:\sup_{s\in K}\left|F\left(\underline{\zeta}(s+\underline{h}\gamma_k;\mathfrak{a})\right)-p_\varepsilon(s)\right|<\frac{\varepsilon}{2}\right\}.
\end{aligned}$$

Therefore, in virtue of (4.1.4), we have

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K}\left|f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\right)\right| < \varepsilon\right\} > 0.$$

The theorem is proved. □

Unfortunately, Theorem 3.1 does not imply a version of Theorem 4.1 for density.

We will prove that the operator $F : H^r(D) \to H(D)$ given by

$$F(g_1, \ldots, g_r) = c_1 g_1 + \cdots + c_r g_r, \quad g_1, \ldots, g_r \in H(D),\ c_j \in \mathbb{C} \setminus \{0\},$$

is an element of the class $Lip(\underline{1})$. Actually, if $p(s)$ is a polynomial and $K_1, \ldots, K_r \in \mathcal{K}$, then there exist $a \in \mathbb{C}$ such that

$$p(s) - a - (c_1 + \cdots + c_{r-2}) \neq 0$$

for $s \in K_r$ because the polynomial $p(s)$ has only a finite number of roots. Let

$$\begin{aligned} g_1(s) &= 1, \quad \ldots, \quad g_{r-2}(s) = 1, \quad g_{r-1}(s) = \frac{a}{c_{r-1}}, \\ g_r(s) &= \frac{p(s) - a - (c_1 + \cdots + c_{r-2})}{c_r}. \end{aligned}$$

Then we have

$$F(g_1, \ldots, g_r) = p(s).$$

Thus, hypothesis 1 of the class $Lip(\underline{1})$ is satisfied.

Clearly, by the definition of $F$, we have

$$\begin{aligned} &\sup_{s\in K}\left|F(\underline{g}_1(s)) - F(\underline{g}_2(s))\right| \\ &\quad = \sup_{s\in K}|c_1(g_{11}(s) - g_{21}(s)) + \cdots + c_r(g_{1r}(s) - g_{2r}(s))| \\ &\quad \leqslant \sup_{s\in K}|c_1|\,|g_{11}(s) - g_{21}(s)| + \cdots + \sup_{s\in K}|c_r|\,|g_{1r}(s) - g_{2r}(s)| \\ &\quad \leqslant \sum_{j=1}^{r}|c_j| \sup_{s\in K}|g_{1j}(s) - g_{2j}(s)| \leqslant C \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|g_{1j}(s) - g_{2j}(s)| \end{aligned}$$

with $C = \max_{1\leqslant j\leqslant r}|c_j|$ and $K_j = K$ for $j = 1, \ldots, r$. The hypothesis 2 of the class $Lip(\underline{1})$ is also valid.

## 4.2 Convolutions of probabilistic type

In this section, we present universality theorems for the convolutions $F(\underline{\zeta}(s;\mathfrak{a}))$, $F : H^r(D) \to H(D)$, whose proofs are based on probabilistic arguments. Recall that

$$S \stackrel{\text{def}}{=} \{g \in H(D) : g(s) \neq 0 \text{ for all } s \in D \text{ or } g(s) \equiv 0\}.$$

**Theorem 4.2.** *Suppose that $A(\underline{\mathfrak{a}}, \underline{h}, (3.3.1))$ is valid, and $F : H^r(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the intersection $(F^{-1}G) \cap S^r$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K}\left|f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\right)\right| < \varepsilon\right\} > 0. \tag{4.2.1}$$

*Moreover, the limit*

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K}\left|f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\right)\right| < \varepsilon\right\} \tag{4.2.2}$$

*exists and is positive for all but at most countably many $\varepsilon > 0$.*

*Proof.* The main ingredient of the proof is Theorem 3.2. For $A \in \mathcal{B}(H(D))$, define

$$P_{N,F,\underline{\mathfrak{a}}}(A) = \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\right) \in A\right\}.$$

Let $P_{N,\underline{\mathfrak{a}}}$ be the measure of Theorem 3.2, and $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ its limit measure. Then, by Theorem 3.2, $P_{N,\underline{\mathfrak{a}}}$ converges weakly $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ as $N \to \infty$. By the definitions of $P_{N,F,\underline{\mathfrak{a}}}$ and $P_{N,\underline{\mathfrak{a}}}$ we have that, for every $A \in \mathcal{B}(H(D))$,

$$P_{n,F,\underline{\mathfrak{a}}}(A) = \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) \in F^{-1}A\right\} = P_{N,\underline{\mathfrak{a}}}(F^{-1}A).$$

Thus,

$$P_{N,F,\underline{\mathfrak{a}}} = P_{N,\underline{\mathfrak{a}}}F^{-1}.$$

Since $F : H^r(D) \to H(D)$ is a continuous operator, this, and Theorem 3.2 together with Lemma 2.5 imply that the measure $P_{N,F,\underline{\mathfrak{a}}}$ converges weakly to $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ as $N \to \infty$.

It remains to identify the support of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$. We will show that the support is the whole of the space $H(D)$. Actually, let $g$ be an arbitrary element of the space $H(D)$, and $G$ an arbitrary open neighborhood of $g$. Since $F$ is a continuous operator, the set $F^{-1}G$ (preimage of $G$) is open as well. By the hypothesis of the theorem that $(F^{-1}G) \cap S^r \neq \varnothing$, there exists an element $\underline{g}_1 \in F^{-1}G$ lying in $S^r$. By Lemma 3.9, the set $S^r$ is the support of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$. Therefore, the set $F^{-1}G$ is an open neighbourhood of an element $\underline{g}_1$ of the support of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$. Hence, by a property of the support,

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}(F^{-1}G) > 0.$$

Therefore,

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(G) = P_{\underline{\zeta},\underline{\mathfrak{a}}}(F^{-1}G) > 0.$$

Since $g$ and $G$ are arbitrary, we have that the support of $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ is the space $H(D)$.

For a polynomial $p(s)$, define the set

$$\mathcal{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Since $p(s) \in H(D)$, the set $\mathcal{G}_\varepsilon$ is an open neighbourhood of the support of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$. Therefore, by a property of a support,

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(\mathcal{G}_\varepsilon) > 0. \tag{4.2.3}$$

Thus, the weak convergence of $P_{N,F,\underline{\mathfrak{a}}}$ to $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ as $N \to \infty$, and Lemma 2.1 for open sets, give

$$\liminf_{N \to \infty} P_{N,F,\underline{\mathfrak{a}}}(\mathcal{G}_\varepsilon) \geqslant P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(\mathcal{G}_\varepsilon) > 0.$$

Hence, the definitions of $P_{N,F,\underline{\mathfrak{a}}}$ and $\mathcal{G}_\varepsilon$ yield

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} \left| p(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \mathfrak{a})\right) \right| < \frac{\varepsilon}{2} \right\} > 0. \tag{4.2.4}$$

Now, using Lemma 2.16, we choose the polynomial $p(s)$ satisfying

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{4.2.5}$$

Suppose that $k \in \mathbb{N}$ satisfies

$$\sup_{s\in K} \left|p(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \mathfrak{a})\right)\right| < \frac{\varepsilon}{2}.$$

Then, in view of (4.2.5), for such $k$,

$$\begin{aligned}
&\sup_{s\in K} \left|f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \mathfrak{a})\right)\right| \\
&\qquad \leqslant \sup_{s\in K} \left|p(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \mathfrak{a})\right)\right| + \sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This shows that

$$\begin{aligned}
&\left\{1 \leqslant k \leqslant N : \sup_{s\in K} \left|p(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \mathfrak{a})\right)\right| < \frac{\varepsilon}{2}\right\} \\
&\qquad\qquad \subset \left\{1 \leqslant k \leqslant N : \sup_{s\in K} \left|f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \mathfrak{a})\right)\right| < \varepsilon\right\}.
\end{aligned}$$

Therefore, by Lemma 2.1 again, (4.2.4) implies

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N} \# \left\{1 \leqslant k \leqslant N : \sup_{s\in K} \left|f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \mathfrak{a})\right)\right| < \varepsilon\right\} \\
&\quad \geqslant \liminf_{N\to\infty} \frac{1}{N} \# \left\{1 \leqslant k \leqslant N : \sup_{s\in K} \left|p(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \mathfrak{a})\right)\right| < \frac{\varepsilon}{2}\right\} > 0.
\end{aligned}$$

This proves the first statement of the theorem.

To prove the second statement of the theorem, introduce one more open set

$$\widehat{\mathcal{G}}_\varepsilon = \left\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\right\}.$$

Then the boundary $\partial\widehat{\mathcal{G}}_\varepsilon$ lies in the set

$$\left\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon\right\},$$

therefore, the boundaries $\partial\widehat{\mathcal{G}}_{\varepsilon_1}$ and $\partial\widehat{\mathcal{G}}_{\varepsilon_2}$ do not intersect for different $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Hence, in a standard way, it follows that

$$P_{\underline{\zeta},\underline{\mathfrak{a}}} F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0$$

for at most countably many $\varepsilon > 0$. In other words, the set $\widehat{\mathcal{G}}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, the weak convergence of $P_{N,F,\underline{\mathfrak{a}}}$, as $N \to \infty$, to $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ together with Lemma 2.1 for continuity sets implies the equality

$$\lim_{N\to\infty} P_{N,F,\underline{\mathfrak{a}}}(\widehat{\mathcal{G}}_\varepsilon) = P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(\widehat{\mathcal{G}}_\varepsilon) \tag{4.2.6}$$

for all but at most countably many $\varepsilon > 0$. By (4.2.4), we have

$$\sup_{s\in K}|g(s) - f(s)| \leqslant \sup_{s\in K}|g(s) - p(s)| + \sup_{s\in K}|f(s) - p(s)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\mathcal{G}_\varepsilon \subset \widehat{\mathcal{G}}_\varepsilon$. This, the monotonicity of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ and (4.2.3) show that

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0.$$

This, (4.2.6) and the definitions of $P_{N,F,\underline{\mathfrak{a}}}$ and $\widehat{\mathcal{G}}_\varepsilon$ prove that the limit

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K}\left|f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \mathfrak{a})\right)\right| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$. The theorem is proved. □

The hypothesis of Theorem 4.2 that $(F^{-1}G) \cap S^r \neq \varnothing$ is general but not convenient for applications. It turns out that the latter hypothesis can be replaced simpler one involving a polynomial in place of an open set $G$.

**Theorem 4.3.** *Suppose that* $A(\underline{\mathfrak{a}}, \underline{h}, (3.3.1))$ *is valid, and* $F : H^r(D) \to H(D)$ *is a continuous operator such that, for every polynomial* $p = p(s)$, *the intersection* $(F^{-1}\{p\}) \cap S^r$ *is non-empty. Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$. *Then inequality* (4.2.1) *is valid and limit* (4.2.2) *exists and is positive for all but at most countably many* $\varepsilon > 0$.

*Proof.* We will show that the hypothesis of the theorem $(F^{-1}\{p\}) \cap S^r \neq \varnothing$ for every polynomial $p = p(s)$ implies that of Theorem 4.2 that $(F^{-1}G) \cap S^r \neq \varnothing$ for arbitrary open set $G \subset H(D)$. Actually, let $G$ be an arbitrary non-empty open set of the space $H(D)$. Then there is $g \in G$. We will prove that there is a polynomial $p(s) \in G$. Fix $\varepsilon > 0$ such that

$$\sum_{m>m_0}^{\infty} 2^{-m} < \frac{\varepsilon}{2}. \tag{4.2.7}$$

We use the metric $\rho$ in the space $H(D)$ defined in the Section 2.2. Let $\{K_m : m \in \mathbb{N}\} \subset D$ be a sequence of compact sets in the definition of the metric $\rho$. The sets $K_m$ can be chosen with connected complements, for example, we can take the embedded rectangles. By Lemma 2.16, there exists a polynomial $p = p(s)$ such that

$$\sup_{s \in K_{m_0}} |g(s) - p(s)| < \frac{\varepsilon}{2}.$$

Since $K_m \subset K_{m+1}$, the latter inequality is valid for all $K_m$, $m \leqslant m_0 - 1$. Hence, by the definition of $\rho$ and (4.2.7),

$$\rho(g, p) < \sum_{m=1}^{m_0} 2^{-m} \frac{\sup_{s \in K_m} |g(s) - p(s)|}{1 + \sup_{s \in K_m} |g(s) - p(s)|} + \frac{\varepsilon}{2} < \varepsilon.$$

This shows that if $\varepsilon > 0$ is sufficiently small, the polynomial $p(s)$ lies in the set $G$. Therefore, $F^{-1}\{p\} \subset F^{-1}(G)$. Hence

$$(F^{-1}G) \cap S^r \supset F^{-1}\{p\} \cap S^r \neq \varnothing,$$

and the theorem follows from Theorem 4.3. $\square$

The requirement that $K \in \mathcal{K}$ in the above theorems is conditioned by applications in the proofs of the Mergelyan theorem on approximation of analytic functions by polynomials (Lemma 2.16). However, for some classes of approximated functions, universality theorems remain valid with uniform approximation on arbitrary compact sets. This situation is realized in the following theorem of the dissertation.

**Theorem 4.4.** *Suppose that $A(\underline{\mathfrak{a}}, \underline{h}, (3.3.1))$ is valid, and $F : H^r(D) \to H(D)$ is a continuous operator. Let $K \subset D$ be a compact set, and $f(s) \in F(S^r)$. Then the assertion of Theorem 4.3 is true.*

*Proof.* It is not difficult to see that the support of the limit measure $P_{\underline{\zeta},\underline{\mathfrak{a}}} F^{-1}$ is the set $F(S^r)$. Actually, let $g$ be an arbitrary element of $F(S^r)$, and $G$ be its any open neighborhood. Then $F^{-1}\{g\} \in S^r$, and lies in the open set $F^{-1}G$. Thus, by Lemma 3.9,

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}(F^{-1}G) > 0.$$

Hence,

$$P_{\underline{\zeta},\underline{\mathfrak{a}}} F^{-1}(G) = P_{\underline{\zeta},\underline{\mathfrak{a}}}(F^{-1}G) > 0.$$

Moreover,

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(F(S^r)) = P_{\underline{\zeta},\underline{\mathfrak{a}}}(F^{-1}F(S^r)) = P_{\underline{\zeta},\underline{\mathfrak{a}}}(S^r) = 1.$$

Since $g$ is arbitrary element of $F(S^r)$, we obtain that the support of $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ is the set $F(S^r)$.

Let $\widehat{\mathcal{G}}_\varepsilon$ be the same as in proof of Theorem 4.2. Since $f(s) \in F(S^r)$, and the support of $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ is $F(S^r)$, we have

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(\widehat{\mathcal{G}}) > 0.$$

Therefore, by Lemma 2.1 and weak convergence of $P_{N,F,\underline{\mathfrak{a}}}$ to $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ as $N \to \infty$, we obtain

$$\liminf_{N\to\infty} P_{N,F,\underline{\mathfrak{a}}}(\widehat{\mathcal{G}}_\varepsilon) \geqslant P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0.$$

The second inequality of the theorem with $\lim_{N\to\infty} P_{N,F,\underline{\mathfrak{a}}}(\widehat{\mathcal{G}}_\varepsilon)$ is obtained in the same way as in the proof of Theorem 4.2. □

The next theorem of the chapter deals with a special subset of the set $F(S^r)$. Let $c_1, \dots, c_m$ be distinct complex numbers, and

$$H_{c_1,\dots,c_m}(D) = \{g \in H(D) : g(s) \neq c_j \text{ for all } s \in D,\ j = 1, \dots, m\}.$$

**Theorem 4.5.** *Suppose that $A(\underline{\mathfrak{a}}, \underline{h}, (3.3.1))$ is valid, and $F : H^r(D) \to H(D)$ is a continuous operator such that $H_{c_1,\dots,c_m}(D) \subset F(S^r)$. For $m = 1$, let $K \subset \mathcal{K}$, $f(s) \in H(K)$ and $f(s) - c_1 \in H_0(K)$. For $m \geqslant 2$, let $K \subset D$ be arbitrary compact set, and $f(s) \in H_{c_1,\dots,c_m}(D)$. Then the assertion of Theorem 4.2 is true.*

*Proof.* In the proof of Theorem 4.4, it was obtained that the support of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ is the set $F(S^r)$. From this and the inclusion $H_{c_1,\dots,c_m}(D) \subset F(S^r)$, it follows that the support of $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ contains the set $H_{c_1,\dots,c_m}(D)$. Since the support is a closed set, hence, the support of $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ contains the closure of the set $H_{c_1,\dots,c_m}(D)$.

Further, we separate two cases.

1) $m = 1$. Since the function $f(s) \neq c_1$ on $K$, the function $f_1(s) \overset{\text{def}}{=} f(s) - c_1 \neq 0$ on $K$. Therefore, the principal branch of logarithm $\log f(s)$ satisfies on $K$ the hypotheses of Lemma 2.16. Thus, for every $\varepsilon_1 > 0$, there exists a

polynomial $p(s)$ such that

$$\sup_{s\in K}|\log f_1(s)-p(s)|<\varepsilon_1.$$

Hence, after a corresponding choosing of $\varepsilon_1=\varepsilon_1(\varepsilon)$, we find

$$\begin{aligned}\sup_{s\in K}\left|f_1(s)-\mathrm{e}^{p_{\varepsilon_1}(s)}\right|&=\sup_{s\in K}\left|\mathrm{e}^{\log f(s)}-\mathrm{e}^{p_{\varepsilon_1}(s)}\right|\\&<\sup_{s\in K}\left|\mathrm{e}^{p_{\varepsilon_1}(s)}\right|\left|\mathrm{e}^{\log f_1(s)-p_{\varepsilon_1}(s)}-1\right|<\frac{\varepsilon}{2}\end{aligned}\qquad(4.2.8)$$

after using the inequality

$$|\mathrm{e}^z-1|\leqslant|z|\mathrm{e}^{|z|},\quad z\in\mathbb{C}.$$

Obviously,

$$f_2(s)\stackrel{\text{def}}{=}c_1+\mathrm{e}^{p_{\varepsilon_1}(s)}\in H_{c_1}(D).$$

Therefore, by the above remark, $f_2(s)$ is an element of the support of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$. Let

$$\widehat{G}_\varepsilon=\left\{g\in H(D):\sup_{s\in K}|g(s)-f_2(s)|<\frac{\varepsilon}{2}\right\}.$$

Then, $\widehat{G}_\varepsilon$ is an open neighbourhood of element of the support of $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$, thus

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(\widehat{G}_\varepsilon)>0.\qquad(4.2.9)$$

Moreover, for $g\in\widehat{G}_\varepsilon$,

$$\begin{aligned}\sup_{s\in K}|g(s)-f(s)|&\leqslant\sup_{s\in K}|g(s)-f_2(s)|+\sup_{s\in K}|f(s)-f_2(s)|\\&<\frac{\varepsilon}{2}+\sup_{s\in K}\left|(f_1(s)+c_1)-\left(c_1+\mathrm{e}^{p_{\varepsilon_1}(s)}\right)\right|\\&=\frac{\varepsilon}{2}+\sup_{s\in K}\left|f_1(s)-\mathrm{e}^{p_{\varepsilon_1}(s)}\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

This shows that $\widehat{G}_\varepsilon\subset\widehat{\mathcal{G}}_\varepsilon$, where

$$\widehat{\mathcal{G}}_\varepsilon=\left\{g\in H(D):\sup_{s\in K}|g(s)-f(s)|<\varepsilon\right\}.$$

Therefore, by (4.2.9),

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0,$$

and weak convergence of $P_{N,F,\underline{\mathfrak{a}}}$ to $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ as $N \to \infty$ and Lemma 2.1 yield

$$\liminf_{N\to\infty} P_{N,F,\underline{\mathfrak{a}}}(\widehat{\mathcal{G}}_\varepsilon) \geqslant P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0.$$

The definitions of $P_{N,F,\underline{\mathfrak{a}}}$ and $\widehat{\mathcal{G}}_\varepsilon$ gives the first statement of the case $m = 1$. The second statement of the case $m = 1$ is obtained by repeating the arguments of the proof of the second statement of Theorem 4.2.

2) $m \geqslant 2$. Since $f(s) \in H_{c_1,\dots,c_m}(D)$, we have that $f(s)$ is an element of the support of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$. Therefore,

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0,$$

and it remains to apply weak convergence of $P_{N,F,\underline{\mathfrak{a}}}$ to $P_{\underline{\zeta},\underline{\mathfrak{a}}}F^{-1}$ as $N \to \infty$, as well as Lemma 2.1. □

We will give an example of the application of Theorem 4.5. We will prove that Theorem 4.5 implies the universality of the function

$$F(g_1,\dots,g_r) = \cos(g_1 + \dots + g_r), \quad g_1,\dots,g_r \in H(D).$$

Let $f \in H_{-1,1}(D)$. Consider the equation

$$\cos g = f,$$

i. e.,

$$\frac{\mathrm{e}^{ig} + \mathrm{e}^{-ig}}{2} = f.$$

From this, we find that

$$\mathrm{e}^{ig} = f \pm \sqrt{f^2 - 1}.$$

Since $f(s) \neq \pm 1$, $f + \sqrt{f^2-1}$ is well defined, and $f + \sqrt{f^2-1} \neq 0$ and $\neq 1$. Therefore,

$$g = \frac{1}{i}\log\left(f + \sqrt{f^2-1}\right) \in S.$$

Moreover,

$$F(g, 0, \dots, 0) = f. \tag{4.2.10}$$

Since $g(s) \in S$ and $0 \in S$, the collection $(g, 0, \dots, 0) \in S^r$. Consequently, by (4.2.10),

$$f(s) \in F(S^r).$$

Hence, since $f$ lies in $H_{-1,1}(D)$, we obtain that

$$H_{-1,1}(D) \subset F(S^r).$$

Therefore, by Theorem 4.5, the functions $f(s) \in H_{-1,1}(D)$ can be approximated by shifts

$$\cos\left(\zeta(s + ih_1\gamma_k; \mathfrak{a}_1) + \cdots + \zeta(s + ih_r\gamma_k; \mathfrak{a})\right),$$

where $\underline{h} = (h_1, \ldots, h_r)$ and $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ and the sequence $\{\gamma_k\}$ satisfy the hypotheses of Theorem 4.5.

Similar approximation properties are valid for

$$F(g_1, \ldots, g_r) = \sin(g_1 + \cdots + g_r),$$

$$F(g_1, \ldots, g_r) = \cosh(g_1 + \cdots + g_r)$$

and

$$F(g_1, \ldots, g_r) = \sinh(g_1 + \cdots g_r),$$

where $\sinh(s)$ and $\cosh(s)$ are sine and cosine hyperbolic functions, i. e.,

$$\sinh(s) = \frac{\mathrm{e}^s - \mathrm{e}^{-s}}{2}$$

and

$$\cosh(s) = \frac{\mathrm{e}^s + \mathrm{e}^{-s}}{2}.$$

We also observe that the operator

$$F(g_1, \ldots, g_r) = b_1 g_1 + \cdots + b_r g_r, \quad g_1, \ldots, g_r \in H(D),\, b_j \in \mathbb{C} \setminus \{0\},$$
$$j = 1, \ldots, r,$$

of the example of Theorem 4.1, also satisfies the hypotheses of Theorem 4.5 of the case $m = 1$. Actually, if $g \in H_{c_1}(D)$, then $(g - c_1)/b_1 \in S$. Consequently, by the definition of the operator $F$,

$$F\left(\frac{g - c_1}{b_1}, \frac{c_1}{b_2}, 0, \ldots, 0\right) = g.$$

This shows that $g \in F(S^r)$ because

$$\frac{g - c_1}{b_1} \in S, \quad \frac{c_1}{b_2} \in S, \quad \text{and } 0 \in S.$$

# CHAPTER 5

## Joint continuous universality of periodic zeta-functions with generalized shifts

In Chapters 2–4, we considered approximation of analytic functions by shifts of periodic zeta-functions $\zeta(s+i\varphi(k);\mathfrak{a})$ with some function $\varphi(k)$, where $k$ runs the set $\mathbb{N}$, and obtained that, under certain additional conditions on the function $\varphi(k)$ and sequence $\mathfrak{a}$, the set of approximating shifts is infinite. More precisely, this set has a positive lower density or even positive density. In this chapter, we will consider simultaneously approximation of a collection of analytic functions $(f_1(s),\dots,f_r(s))$ by continuous shifts $(\zeta(s+i\varphi_1(\tau);\mathfrak{a}_1),\dots,\zeta(s+\varphi_r(\tau);\mathfrak{a}_r))$ with a certain collection of functions $(\varphi_1(\tau),\dots,\varphi_r(\tau))$ of continuous variable $\tau\in[T_0,T]$.

It is well known that in joint universality theorems, the approximating shifts must be in a certain sense independent. In earlier researches, this independence was ensured by certain rank hypotheses for the sequences $\mathfrak{a}_1$, $\dots,\mathfrak{a}_r$, see, for example, [42]. However, the later rank hypothesis is difficultly verifiable. Therefore, in the dissertation, we propose, in place of rank condition, to use generalized shifts connected with certain simply verifiable conditions.

## 5.1 Statement of joint universality theorems

Let $T_0$ be a positive fixed number. Suppose that $\varphi(\tau)$ is a real increasing to $+\infty$ continuously differentiable function with monotonic derivative $\varphi'(\tau)$ on $[T_0,\infty)$ such that

$$\varphi(2\tau)\max_{\tau\leqslant u\leqslant 2\tau}\frac{1}{\varphi'(u)}\ll\tau,\quad \tau\to\infty.$$

The class of such functions is denoted by $U_1(T_0)$.

For $\mathcal{K}$ and $H_0(K)$, we preserve the notion of previous chapters. We will use two types of shifts.

**Theorem 5.1.** *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative, $a_1$, $\dots, a_r$ are real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$, and $\varphi(\tau) \in U_1(T_0)$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T-T_0} \text{meas}\Big\{\tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s + ia_j\varphi(\tau); \mathfrak{a}_j)| < \varepsilon\Big\} > 0.$$

*Moreover, the limit*

$$\lim_{T\to\infty} \frac{1}{T-T_0} \text{meas}\Big\{\tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s + ia_j\varphi(\tau); \mathfrak{a}_j)| < \varepsilon\Big\}$$

*exists and is positive for all but at most countably many $\varepsilon > 0$.*

In Theorem 5.1, the independence of approximating shifts is ensured by the linear independence of the numbers $a_1, \dots, a_r$, while the continuous independent $\varphi(\tau)$ for all shifts is the same. The next theorem deals with the second type of shifts with different functions $\varphi_1(\tau), \dots, \varphi_r(\tau)$ in shifts.

Define one more class of functions. Let $\varphi_1(\tau), \dots, \varphi_r(\tau)$ be real increasing to $+\infty$ continuously differentiable functions on $[T_0, \infty)$ with derivatives $\varphi_j'(\tau) = \widehat{\varphi}_j(\tau)(1 + o(1))$, where $\widehat{\varphi}_1(\tau), \dots, \widehat{\varphi}_r(\tau)$ are monotonic and are compared in the sense that, for every subset $J \subset \{1, \dots, r\}$, $\#J \geqslant 2$, there exists $j_0 = j_0(J)$ such that

$$\widehat{\varphi}_j(\tau) = o(\widehat{\varphi}_{j_0}(\tau))$$

for $j \in J$, $j \neq j_0$, and

$$\widehat{\varphi}_j(2\tau) \max_{\tau\leqslant u\leqslant 2\tau} \frac{1}{\widehat{\varphi}_j(u)} \ll \tau,$$

$j = 1, \dots, r$, as $\tau \to \infty$.

Denote the class of such functions $\varphi_1(\tau), \dots, \varphi_r(\tau)$ by $U_r(T_0)$. Now, we state the second joint continuous universality theorem on approximation of analytic functions by shifts $(\zeta(s+i\varphi_1(\tau); \mathfrak{a}_1), \dots, \zeta(s+i\varphi_r(\tau); \mathfrak{a}_r))$.

**Theorem 5.2.** *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative, and $(\varphi_1(\tau), \dots, \varphi_r(\tau)) \in U_r(T_0)$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T-T_0} \text{meas}\Big\{\tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s+i\varphi_j(\tau); \mathfrak{a}_j)| < \varepsilon\Big\} > 0.$$

*Moreover, the limit*

$$\lim_{T\to\infty} \frac{1}{T-T_0} \text{meas}\Big\{\tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s+i\varphi_j(\tau); \mathfrak{a}_j)| < \varepsilon\Big\}$$

*exists and is positive for all but at most countably many $\varepsilon > 0$.*

For the proof of Theorems 5.1 and 5.2, we will apply, as in Chapters 2–4, the probabilistic approach based on limit theorems for probability measures in the space of analytic functions $H^r(D)$. We will derive these theorems progressively, starting with comparatively simple spaces. For brevity, we will use the notation $\underline{a} = (a_1, \dots, a_r)$, $\underline{\varphi}(\tau) = (\varphi_1(\tau), \dots, \varphi_r(\tau))$. Other notations are the same as in previous chapters.

## 5.2 Statements of limit theorems

For $A \in \mathcal{B}(H^r(D))$, define two probability measures

$$P^1_{T,\underline{\mathfrak{a}}}(A) = \frac{1}{T-T_0} \text{meas}\left\{\tau \in [T_0, T] : \underline{\zeta}(s+i\underline{a}\varphi(\tau); \underline{\mathfrak{a}}) \in A\right\},$$

and

$$P^r_{T,\underline{\mathfrak{a}}}(A) = \frac{1}{T-T_0} \text{meas}\left\{\tau \in [T_0, T] : \underline{\zeta}(s+i\underline{\varphi}(\tau); \underline{\mathfrak{a}}) \in A\right\}.$$

We recall that

$$\underline{\zeta}(s+i\underline{a}\varphi(\tau);\underline{\mathfrak{a}}) = \big(\zeta(s+ia_1\varphi_(\tau);\mathfrak{a}_1),\dots,\zeta(s+ia_r\varphi(\tau);\mathfrak{a}_r)\big)$$

and

$$\underline{\zeta}(s+i\underline{\varphi}(\tau);\underline{\mathfrak{a}}) = (\zeta(s+i\varphi_1(\tau);\mathfrak{a}_1),\dots,\zeta(s+i\varphi_r(\tau);\mathfrak{a}_r)).$$

Now, define the limit measure for $P^1_{T,\underline{\mathfrak{a}}}$ and $P^r_{T,\underline{\mathfrak{a}}}$ as $T\to\infty$. Let $\Omega^r$ be the same compact topological group with elements $\omega=(\omega_1,\dots,\omega_r)$, and the probability Haar measure $m^r_H$ as in Section 3.2. On the probability space $(\Omega^r,\mathcal{B}(\Omega^r),m^r_H)$, define the $H^r(D)$-valued random element

$$\underline{\zeta}(s,\omega;\underline{\mathfrak{a}}) = (\zeta(s,\omega_1;\mathfrak{a}_1),\dots,\zeta(s,\omega_r;\mathfrak{a}_r)),$$

where

$$\zeta(s,\omega_j;\mathfrak{a}_j) = \prod_{p\in\mathbb{P}}\left(1+\sum_{l=1}^{\infty}\frac{a_{jp^l}\omega_j^l(p)}{p^{ls}}\right), \quad j=1,\dots,r.$$

Let, for $A\in\mathcal{B}(H^r((D)))$,

$$P_{\underline{\zeta},\underline{\mathfrak{a}}}(A) = m^r_H\left\{\omega\in\Omega^r : \underline{\zeta}(s,\omega;\underline{\mathfrak{a}})\in A\right\},$$

i. e., $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ is the distribution of the random element $\underline{\zeta}(s,\omega;\underline{\mathfrak{a}})$. Now, we state limit theorems for $P^1_{T,\underline{\mathfrak{a}}}$ and $P^r_{T,\underline{\mathfrak{a}}}$.

**Theorem 5.3.** *Suppose that the sequences $\mathfrak{a}_1,\dots,\mathfrak{a}_r$ are multiplicative, $a_1,\dots,a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$, and $\varphi(\tau)\in U_1(T_0)$. Then $P^1_{T,\underline{\mathfrak{a}}}$ converges weakly to the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ as $T\to\infty$.*

**Theorem 5.4.** *Suppose that the sequences $\mathfrak{a}_1,\dots,\mathfrak{a}_r$ are multiplicative, and $(\varphi_1(\tau),\dots\varphi_r(\tau))\in U_r(T_0)$. Then $P^r_{T,\underline{\mathfrak{a}}}$ converges weakly to the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ as $T\to\infty$.*

## 5.3 Limit lemmas on $\Omega^r$

We begin the proofs of Theorems 5.3 and 5.4 with the lemmas on the group $\Omega^r$.

For $A \in \mathcal{B}(\Omega^r)$, define

$$V_T^1(A) = \frac{1}{T-T_0}\text{meas}\Big\{\tau \in [T_0,T] : \Big(p^{-ia_1\varphi(\tau)} : p \in \mathbb{P}\Big), \dots, \Big(p^{-ia_r\varphi(\tau)} : p \in \mathbb{P}\Big) \in A\Big\}.$$

**Lemma 5.1.** *Suppose that $\underline{a}$ and $\varphi(\tau)$ satisfy the hypotheses of Theorem 5.3. Then $V_T^1$ converges weakly to the Haar measure $m_H^r$ as $T \to \infty$.*

*Proof.* As in the proof of Lemma 3.3, we have that the Fourier transform $F_T(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = \{k_{jp} : k_{jp} \in \mathbb{Z},\ p \in \mathbb{P}\}$, $j = 1, \dots, r$, of the measure $V_T^1$ has the representation

$$F_T(\underline{k}_1, \dots, \underline{k}_r) = \int\limits_{\Omega^r} \left(\prod_{j=1}^{r} {\prod_{p\in\mathbb{P}}}^{*} \omega_j^{k_{jp}}(p)\right) \mathrm{d}V_T^1,$$

where the star "$*$" indicates that only a finite number of integers $k_{jp}$ are distinct from zeros. Hence, the definition of $V_T^1$ yields

$$\begin{aligned} F_T(\underline{k}_1, \dots, \underline{k}_r) &= \frac{1}{T-T_0}\int\limits_{T_0}^{T} \prod_{j=1}^{r} {\prod_{p\in\mathbb{P}}}^{*} p^{-ia_j\varphi(\tau)k_{jp}}\, \mathrm{d}\tau \\ &= \frac{1}{T-T_0}\int\limits_{T_0}^{T} \exp\left\{-i\varphi(\tau)\sum_{j=1}^{r} a_j {\sum_{p\in\mathbb{P}}}^{*} k_{jp}\log p\right\} \mathrm{d}\tau. \end{aligned} \tag{5.3.1}$$

Clearly,

$$F_T^1((\underline{0}, \dots, \underline{0})) = 1. \tag{5.3.2}$$

Now, suppose that $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$. We have

$$A_{\underline{k}_1,\dots\underline{k}_r} \overset{\text{def}}{=} \sum_{j=1}^{r} a_j {\sum_{p\in\mathbb{P}}}^{*} k_{jp}\log p = {\sum_{p\in\mathbb{P}}}^{*} \log p \sum_{j=1}^{r} a_j k_{jp}.$$

Let

$$p_{min} = \min_{1\leqslant j\leqslant r}\min_{p}\{p : k_{jp} \in \underline{k}_j,\ k_{jp} \neq 0\}$$

and

$$p_{max} = \max_{1\leqslant j\leqslant r}\max_{p}\{p : k_{jp} \in \underline{k}_j,\ k_{jp} \neq 0\}.$$

Then there exists at least one $p \in [p_{min}, p_{max}]$ such that $k_{jp} \neq 0$ for some $j$. Thus, by the linear independence over $\mathbb{Q}$ of the numbers $a_1, \dots, a_r$,

$$\beta_p \stackrel{\text{def}}{=} \sum_{j=1}^{r} a_j k_{jp} \neq 0.$$

The numbers $\beta_p$ are algebraic, moreover, the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, in view of Lemma 3.2,

$$A_{\underline{k}_1,\dots,\underline{k}_r} = \sum_{p\in\mathbb{P}}{}^{*} \beta_p \log p \neq 0.$$

By the second mean value theorem and properties of $\varphi'(\tau)$, we find that

$$\begin{aligned}\int_{T_0}^{T} \cos\left(\varphi(\tau) A_{\underline{k}_1,\dots,\underline{k}_r}\right) \mathrm{d}\tau &= \frac{1}{A_{\underline{k}_1,\dots,\underline{k}_r}} \int_{T_0}^{T} \frac{1}{\varphi'(\tau)} \mathrm{d}\left(\sin\left(\varphi(\tau) A_{\underline{k}_1,\dots,\underline{k}_r}\right)\right) \\ &\ll \frac{1}{|A_{\underline{k}_1,\dots,\underline{k}_r}|} \max\left(\frac{1}{\varphi'(T)}, \frac{1}{\varphi'(T_0)}\right),\end{aligned}$$

and the same estimate holds for

$$\int_{T_0}^{T} \sin\left(\varphi(\tau) A_{\underline{k}_1,\dots,\underline{k}_r}\right) \mathrm{d}\tau.$$

Therefore, the equality (5.3.1) gives

$$F_T^1(\underline{k}_1, \dots, \underline{k}_r) \ll \frac{1}{|A(\underline{k})|T} \max\left(\frac{1}{\varphi'(T)}, \frac{1}{\varphi'(T_0)}\right). \tag{5.3.3}$$

Since $\varphi(\tau) \in U_1(T_0)$,

$$\frac{1}{\varphi'(T)} = o(T)$$

as $T \to \infty$. Hence, by (5.3.3), in the case $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$, we have

$$\lim_{T\to\infty} F_T^1(\underline{k}_1, \dots, \underline{k}_r) = 0.$$

This together with (5.3.2) shows that

$$\lim_{T\to\infty} F_T^1(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if } \underline{k} \neq (\underline{0}, \dots, \underline{0}). \end{cases}$$

Since the right-hand side of the above equality is the Fourier transform of the Haar measure $m_H^r$, the lemma is proved. $\square$

For $A \in \mathcal{B}(\Omega^r)$, define

$$V_T^r(A) = \frac{1}{T-T_0}\text{meas}\Big\{\tau \in [T_0, T] : \Big(p^{-i\varphi_1(\tau)} : p \in \mathbb{P}\Big), \dots, \\ \Big(p^{-i\varphi_r(\tau)} : p \in \mathbb{P}\Big) \in A\Big\}.$$

**Lemma 5.2.** *Suppose that* $(\varphi_1(\tau), \dots, \varphi_r(\tau)) \in U_r(T_0)$*. Then* $V_T^r$ *converges weakly to the Haar measure* $m_H^r$ *as* $T \to \infty$*.*

*Proof.* As in the proof of Lemma 5.1, we consider the Fourier transform $F_T^r(\underline{k}_1, \dots, \underline{k}_r)$ of the measure $V_T^r$, i. e.,

$$\begin{aligned} F_T^r(\underline{k}_1, \dots \underline{k}_r) &= \int\limits_{\Omega^r} \left( \prod_{j=1}^r \sideset{}{^*}\prod_{p\in\mathbb{P}} \omega_j^{k_{jp}}(p) \right) \mathrm{d}V_T^r \\ &= \frac{1}{T-T_0} \int\limits_{T_0}^{T} \exp\left\{ -i \sum_{j=1}^r \varphi_j(\tau) \sideset{}{^*}\sum_{p\in\mathbb{P}} k_{jp} \log p \right\} \mathrm{d}\tau. \end{aligned} \tag{5.3.4}$$

Obviously,

$$F_T^r(\underline{0}, \dots, \underline{0}) = 1. \tag{5.3.5}$$

Therefore, it remains to consider the case $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$. For brevity, let

$$b_j \stackrel{\text{def}}{=} \sideset{}{^*}\sum_{p\in\mathbb{P}} k_{jp} \log p.$$

Since, the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$, we have $b_j \neq 0$ for $\underline{k}_j \neq \underline{0}$, $j = 1, \dots, r$. Put

$$A(\tau) = \sum_{j=1}^r b_j \varphi_j(\tau).$$

Suppose that $\underline{k}_j \neq \underline{0}$ for $j \in J \subset \{1, \dots, r\}$, $\#J \geqslant 2$. Then there exists $j_0 \in J$ such that

$$\widehat{\varphi}_j(\tau) = o(\widehat{\varphi}_{j_0}(\tau)), \quad \tau \to \infty,$$

for $j \in J \setminus \{j_0\}$. Therefore,

$$A'(\tau) = \sum_{j\in J} b_j \varphi_j'(\tau) = \sum_{j\in J} b_j \widehat{\varphi}_j(\tau)(1+o(1)) = b_{j_0}\widehat{\varphi}_{j_0}(\tau)(1+o(1))$$

and

$$(A'(\tau))^{-1} = \frac{1}{b_{j_0}\widehat{\varphi}_{j_0}(\tau)(1+o(1))} = \frac{1}{b_{j_0}\widehat{\varphi}_{j_0}(\tau)}(1+o(1)), \quad \tau \to \infty,$$

in virtue of the identity

$$\frac{1}{1+a} = 1 - \frac{a}{1+a}, \quad a \neq -1.$$

Hence, using the monotonicity of $\widehat{\varphi}_{j_0}(\tau)$ and the second mean value theorem, we find

$$\begin{aligned}
\int_{T_0}^{T} \cos A(\tau)\,\mathrm{d}\tau &= \int_{\log T}^{T} \cos A(\tau)\,\mathrm{d}\tau + O(\log T) \\
&= \int_{\log T}^{T} \frac{1}{A'(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) + O(\log T) \\
&= \int_{\log T}^{T} \frac{1}{b_{j_0}\widehat{\varphi}_{j_0}(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) \\
&\quad + \int_{\log T}^{T} \frac{o(1)}{b_{j_0}\widehat{\varphi}_{j_0}(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) + O(\log T) \\
&= \int_{\log T}^{T} \frac{1}{b_{j_0}\widehat{\varphi}_{j_0}(\tau)} \,\mathrm{d}(\sin A(\tau)) \\
&\quad + \int_{\log T}^{T} \frac{o(1)(1+o(1))}{A'(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) + O(\log T) \\
&= o(T) + \int_{\log T}^{T} o(1)\cos A(\tau)\,\mathrm{d}\tau + O(\log T) \\
&= o(T), \quad T \to \infty, \qquad (5.3.6)
\end{aligned}$$

because $1/(\widehat{\varphi}'_{j_0}(\tau)) = o(\tau)$ as $\tau \to \infty$. By the same lines, we obtain

$$\int_{T_0}^{T} \sin A(\tau)\,\mathrm{d}\tau = o(T).$$

This, (5.3.6) and (5.3.4) show that, for $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$,

$$\lim_{T\to\infty} F_T^r(\underline{k}_1, \dots, \underline{k}_r) = 0, \tag{5.3.7}$$

because in the case $\#J = 1$, we have $A(\tau) = b_j\varphi_j(\tau)$ for some $j$. Thus, by (5.3.5) and (5.3.7),

$$\lim_{T\to\infty} F_T^r(\underline{k}_1, \dots, \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0}), \end{cases}$$

and the proof of the lemma is completed. □

## 5.4 Case of absolutely convergent series

Lemmas 5.1 and 5.2 allow us to prove limit lemmas in the space $H^r(D)$ for measures defined by means of an absolutely convergent Dirichlet series. We use the notation of Section 3.3, and, only for fullness, recall it. Thus,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\beta}\right\}, \quad \beta > \frac{1}{2},$$

for $m, n \in \mathbb{N}$,

$$\zeta_n(s; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm} v_n(m)}{m^s}, \quad j = 1, \dots, r,$$

the series being absolutely convergent in any half-plane $\sigma > \sigma_0$, and

$$\underline{\zeta}_n(s; \underline{\mathfrak{a}}) = (\zeta_n(s; \mathfrak{a}_1), \dots, \zeta_n(s; \mathfrak{a}_r)).$$

For $A \in \mathcal{B}(H^r(D))$,

$$V_{T,n}^1(A) = \frac{1}{T - T_0} \text{meas}\left\{\tau \in [T_0, T] : \underline{\zeta}_n(s + i\underline{a}\varphi(\tau); \underline{\mathfrak{a}}) \in A\right\}$$

and

$$V_{T,n}^{r}(A) = \frac{1}{T-T_0}\text{meas}\left\{\tau \in [T_0, T] : \underline{\zeta}_n(s + i\underline{\varphi}(\tau); \underline{\mathfrak{a}}) \in A\right\}.$$

Now, let, for $\omega_j \in \Omega_j$,

$$\zeta_n(s, \omega_j; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm}\omega_j(m)v_n(m)}{m^s}, \quad j = 1, \dots, r,$$

where

$$\omega_j(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad j = 1, \dots, r.$$

The latter series also converges absolutely in every half-plane $\sigma > \sigma_0$. For $\omega \in \Omega^r$, set

$$\underline{\zeta}_n(s, \omega; \underline{\mathfrak{a}}) = (\zeta_n(s, \omega_1; \mathfrak{a}_1), \dots, \zeta_n(s, \omega_r; \mathfrak{a}_r))$$

and, for $A \in \mathcal{B}(H^r(D))$, define

$$V_{T,n,\omega}^{1}(A) = \frac{1}{T-T_0}\text{meas}\left\{\tau \in [T_0, T] : \underline{\zeta}_n(s + i\underline{a}\varphi(\tau), \omega; \underline{\mathfrak{a}}) \in A\right\}$$

and

$$V_{T,n,\omega}^{r}(A) = \frac{1}{T-T_0}\text{meas}\left\{\tau \in [T_0, T] : \underline{\zeta}_n(s + i\underline{\varphi}(\tau), \omega; \underline{\mathfrak{a}}) \in A\right\}.$$

Moreover, let the mapping $u_n : \Omega^r \to H^r(D)$ be given by the formula

$$u_n(\omega) = \underline{\zeta}_n(s, \omega; \underline{\mathfrak{a}}).$$

Then the mapping $u_n$ is continuous because of the absolute convergence of the series for $\zeta_n(s, \omega_j; \mathfrak{a}_j)$, $j = 1, \dots, r$. Moreover, in view of definitions of $u_n$, $V_{T,n}^1$, $V_{T,n}^r$ and $V_T^1$, $V_T^r$, we have

$$V_{T,n}^{1} = V_T u_n^{-1}, \quad V_{T,n}^{r} = V_T^r u_n^{-1}.$$

Therefore, application of Lemma 2.5, and Lemmas 5.1 and 5.2, show that $V_{T,n}^1$ and $V_{T,n}^r$ converges weakly to the measure $m_H^r u_n^{-1}$ as $T \to \infty$.

Define one more mapping $\widehat{u}_n : \Omega^r \to H^r(D)$ by

$$\widehat{u}_n(\widehat{\omega}) = \underline{\zeta}_n(s, \omega\widehat{\omega}; \underline{\mathfrak{a}}), \quad \widehat{\omega} \in \Omega^r.$$

Then, similarly as above, we find that $V^1_{T,n,\omega}$ and $V^r_{T,n,\omega}$ converge weakly to the measure $m^r_H \widehat{u}_n^{-1}$. We will prove that $m^r_H \widehat{u}_n^{-1} = m^r_H u_n^{-1}$. Let $u : \Omega^r \to \Omega^r$ be given by $u(\omega) = \omega\widehat{\omega}$. Then $\widehat{u}_n(\widehat{\omega}) = u_n(u(\omega))$. Since the Haar measure $m^r_H$ is invariant with respect to shifts by elements from $\Omega^r$, we find that

$$m^r_H \widehat{u}_n^{-1} = m^r_H (u_n(u))^{-1} = (m^r_H u^{-1}) u_n^{-1} = m^r_H u_n^{-1}.$$

Thus, we have that the measures $V^1_{T,n,\omega}$ and $V^r_{T,n,\omega}$ also converge weakly to the measure $m_H u_n^{-1}$ as $T \to \infty$.

Summarising the above results, we have the following lemmas.

**Lemma 5.3.** *Suppose that $\underline{a}$ and $\varphi(\tau)$ satisfy the hypotheses of Theorem 5.1. Then $V^1_{T,n}$ and $V^1_{T,n,\omega}$ converge weakly to the measure $m^r_H u_n^{-1}$ as $T \to \infty$.*

**Lemma 5.4.** *Suppose that $(\varphi_1(\tau), \ldots, \varphi_r(\tau)) \in U_r(T_0)$. Then $P^r_{T,n}$ and $P^r_{T,n,\underline{\omega}}$ converge weakly to the measure $m^r_H u_n^{-1}$ as $T \to \infty$.*

## 5.5 Mean square estimates

To pass from weak convergence for $V^1_{T,n}$ and $V^r_{T,n}$ to for $V^1_T$ and $V^r_T$, respectively, as $T \to \infty$, a certain approximation of $\underline{\zeta}(s; \underline{\mathfrak{a}})$ by $\underline{\zeta}_n(s; \underline{\mathfrak{a}})$ is needed. This approximation is based on the mean square estimates for $\zeta(s, \mathfrak{a}_j)$.

Thus, let $\mathfrak{a}$ be an arbitrary periodic sequence of complex numbers, and $a \in \mathbb{R} \setminus \{0\}$.

**Lemma 5.5.** *Suppose that $\varphi(\tau) \in U_1(T_0)$. Then, for every fixed $\sigma$, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,*

$$\int_{T_0}^{T} |\zeta(\sigma + ia\varphi(\tau) + it; \mathfrak{a})|^2 \, d\tau \ll_{a,\sigma,\mathfrak{a}} T(1 + |t|).$$

*Proof.* From the estimate (2.5.1), it follows that

$$\int_{T_0}^{|t|+|a|\varphi(\tau)} |\zeta(\sigma + iu; \mathfrak{a})|^2 \, du \ll_{\sigma,\mathfrak{a}} (|t| + |a|\varphi(\tau)).$$

Therefore, for $X \geqslant T_0$, we have

$$\int\limits_X^{2X} |\zeta(\sigma + ia\varphi(\tau) + it; \mathfrak{a})|^2 \,\mathrm{d}\tau = \frac{1}{a} \int\limits_X^{2X} \frac{1}{\varphi'(\tau)} |\zeta(\sigma + ia\varphi(\tau) + it; \mathfrak{a})|^2 \,\mathrm{d}\varphi(\tau)$$
$$\ll_a \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\varphi'(\tau)} \left| \int\limits_X^{2X} \mathrm{d} \left( \int\limits_{T_0}^{t+a\varphi(\tau)} |\zeta(\sigma + iu; \mathfrak{a})|^2 \,\mathrm{d}u \right) \right|$$
$$\ll_{a,\sigma,\mathfrak{a}} (|t| + |a|\varphi(2X)) \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\varphi'(\tau)} \ll_{a,\sigma,\mathfrak{a}} X(1 + |t|)$$

because $\varphi(\tau) \in U_1(T_0)$. Taking $T = 2^{-k-1}$ and summing over $k \in \mathbb{N}_0$, give the estimate of the lemma. □

**Lemma 5.6.** *Suppose that* $(\varphi_1(\tau), \ldots, \varphi_r(\tau)) \in U_r(T_0)$. *Then, for every fixed* $\sigma$, $1/2 < \sigma < 1$, *and* $t \in \mathbb{R}$,

$$\int\limits_{T_0}^{T} |\zeta(\sigma + i\varphi_j(\tau) + it; \mathfrak{a})|^2 \,\mathrm{d}\tau \ll_{\sigma,\mathfrak{a}} T(1 + |t|), \quad j = 1, \ldots, r.$$

*Proof.* Using the notation of Lemma 5.5 and the class $U_r(T_0)$, we have

$$\int\limits_X^{2X} |\zeta(\sigma + i\varphi_j(\tau) + it; \mathfrak{a})|^2 \,\mathrm{d}\tau = \int\limits_X^{2X} \frac{1}{\varphi_j'(\tau)} |\zeta(\sigma + i\varphi_j(\tau) + it; \mathfrak{a})|^2 \,\mathrm{d}\varphi_j(\tau)$$
$$= \int\limits_X^{2X} \frac{(1 + o(1))}{\widehat{\varphi}_j(\tau)} \mathrm{d} \left( \int\limits_{T_0}^{t+\varphi_j(\tau)} |\zeta(\sigma + iu; \mathfrak{a})|^2 \,\mathrm{d}u \right)$$
$$= \int\limits_X^{2X} \frac{1)}{\widehat{\varphi}_j(\tau)} \mathrm{d} \left( \int\limits_{T_0}^{t+\varphi_j(\tau)} |\zeta(\sigma + iu; \mathfrak{a})|^2 \,\mathrm{d}u \right)$$
$$+ \int\limits_X^{2X} \frac{o(1)(1 + o(1))}{\varphi_j'(\tau)} \mathrm{d} \left( \int\limits_{T_0}^{t+\varphi_j(\tau)} |\zeta(\sigma + iu; \mathfrak{a})|^2 \,\mathrm{d}u \right)$$
$$\ll_{\sigma,\mathfrak{a}} \max_{X \leqslant \tau \leqslant 2X} (|t| + \widehat{\varphi}_j(2X)) + \int\limits_X^{2X} o(1) |\zeta(\sigma + i\varphi_j(\tau) + it; \mathfrak{a})|^2 \,\mathrm{d}\tau.$$

Hence,

$$\int_X^{2X} |\zeta(\sigma + i\varphi_j(\tau) + it; \mathfrak{a})|^2 \,\mathrm{d}\tau \ll_{\sigma,\mathfrak{a}} X(1+|t|)(1+r(X)) \ll_{\sigma,\mathfrak{a}} X(1+|t|),$$

where $r(X) \to 0$ as $X \to \infty$. From this, the lemma follows in the same way as Lemma 5.5. □

Lemmas 5.5 and 5.6 have their modifications for

$$\zeta(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \sigma > 1,$$

with $\omega \in \Omega$. We note that the latter series is uniformly convergent on compact subsets of the strip $D$ for almost all $\omega$ with respect to the Haar measure on $(\Omega, \mathcal{B}(\Omega))$, see, for example, Lemma 5.1.6 of [35].

**Lemma 5.7.** *Suppose that $a \in \mathbb{R} \setminus \{0\}$ and $\varphi(\tau) \in U_1(T_0)$. Then, for every fixed $\sigma$, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,*

$$\int_{T_0}^{T} |\zeta(\sigma + ia\varphi(\tau) + it, \omega; \mathfrak{a})|^2 \,\mathrm{d}\tau \ll_{a,\sigma,\mathfrak{a}} T(1+|t|)$$

*for almost all $\omega \in \Omega$.*

*Proof.* Since, for almost all $\omega \in \Omega$,

$$\int_{T_0}^{T} |\zeta(\sigma + it, \omega; \mathfrak{a})|^2 \,\mathrm{d}t \ll_{\sigma,\mathfrak{a}} T, \tag{5.5.1}$$

see [47], the proof coincides with that of Lemma 5.5. □

**Lemma 5.8.** *Let $(\varphi_1(\tau), \dots, \varphi_r(\tau)) \in U_r(T_0)$. Then, for every fixed $\sigma$, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,*

$$\int_{T_0}^{T} |\zeta(\sigma + ia\varphi_j(\tau) + it, \omega; \mathfrak{a})|^2 \,\mathrm{d}\tau \ll_{\sigma,\mathfrak{a}} T(1+|t|), \quad j = 1, \dots, r,$$

*for almost all $\omega \in \Omega$.*

*Proof.* We repeat the proof of Lemma 5.6 and apply the estimate (5.5.1). □

Now, we will apply Lemmas 5.5–5.8 for the approximation of $\underline{\zeta}(s;\underline{\mathfrak{a}})$ by $\underline{\zeta}_n(s;\underline{\mathfrak{a}})$. Let $\underline{\rho}$ be the metric in $H^r(D)$ introduced in Section 3.4.

**Lemma 5.9.** *Suppose that $a_1,\dots,a_r \in \mathbb{R}\setminus\{0\}$ and $\varphi(\tau)\in U_1(T_0)$. Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T-T_0}\int\limits_{T_0}^{T}\underline{\rho}\left(\underline{\zeta}(s+i\underline{a}\varphi(\tau);\underline{\mathfrak{a}}),\underline{\zeta}_n(s+i\underline{a}\varphi(\tau);\underline{\mathfrak{a}})\right)\,\mathrm{d}\tau=0. \tag{5.5.2}$$

*Moreover, for almost all $\omega\in\Omega^r$,*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T-T_0}\times\int\limits_{T_0}^{T}\underline{\rho}\left(\underline{\zeta}(s+i\underline{a}\varphi(\tau),\omega;\underline{\mathfrak{a}}),\underline{\zeta}_n(s+i\underline{a}\varphi(\tau),\omega;\underline{\mathfrak{a}})\right)\,\mathrm{d}\tau=0.$$

*Proof.* From the definitions of the metrics $\rho$ and $\underline{\rho}$, it follows that it is sufficient to prove that, for every compact set $K\subset D$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T-T_0}\times\int\limits_{T_0}^{T}\sup_{s\in K}|\zeta(s+ia_j\varphi(\tau);\mathfrak{a}_j)-\zeta_n(s+ia_j\varphi(\tau);\mathfrak{a}_j)|\,\mathrm{d}\tau=0$$

for all $j=1,\dots,r$.

Let $\mathfrak{a}$ and $a\neq 0$ be arbitrary. We use the integral representation of Lemma 2.12 for $\zeta_n(s;\mathfrak{a})$, namely

$$\zeta_n(s;\mathfrak{a})=\frac{1}{2\pi i}\int\limits_{\beta-i\infty}^{\beta+i\infty}\zeta(s+z;\mathfrak{a})b_n(z)\mathrm{d}z. \tag{5.5.3}$$

In what follows, we argue similarly to the proof of Lemma 2.7. Let $K\subset D$ be a compact set. Denote by $s=\sigma+it$ the points of the set $K$, and fix $\varepsilon>0$ such that $1/2+2\varepsilon\leqslant\sigma\leqslant 1-\varepsilon$. Let $\beta=1/2+\varepsilon$ and $\theta=\sigma-\varepsilon-1/2$. Then

(5.5.3) and the residue theorem give

$$\zeta_n(s;\mathfrak{a}) - \zeta(s;\mathfrak{a}) = \frac{1}{2\pi i} \int\limits_{-\theta-i\infty}^{-\theta+i\infty} \zeta(s+z;\mathfrak{a}) b_n(z) \mathrm{d}z + \widehat{a} b_n(1-s),$$

where $\widehat{a}$ is the residue of $\zeta(s;\mathfrak{a})$ at the point $s=1$. Hence, for $s \in K$,

$$\zeta(s+ia\varphi(\tau);\mathfrak{a}) - \zeta_n(s+ia\varphi(\tau);\mathfrak{a})$$
$$\ll \int\limits_{-\infty}^{\infty} |\zeta(s+ia\varphi(\tau)-\theta+iu;\mathfrak{a})| b_n(-\theta+iu)\,\mathrm{d}u + |b_n(1-s-ia\varphi(\tau))|.$$

Thus,

$$\frac{1}{T-T_0} \int\limits_{T_0}^{T} \sup_{s\in K} |\zeta(s+ia\varphi(\tau);\mathfrak{a}) - \zeta_n(s+ia\varphi(\tau);\mathfrak{a})|\,\mathrm{d}\tau \ll I_1 + I_2, \quad (5.5.4)$$

where

$$I_1 = \int\limits_{-\infty}^{\infty} \frac{1}{T-T_0} \int\limits_{T_0}^{T} \left( \left| \zeta\left(\frac{1}{2}+\varepsilon+i(u+a\varphi(\tau));\mathfrak{a}\right)\right| \mathrm{d}\tau \right)$$
$$\times \sup_{s\in K} \left| b_n\left(\frac{1}{2}+\varepsilon-s+iu\right)\right| \mathrm{d}u,$$

and

$$I_2 = \frac{1}{T-T_0} \int\limits_{T_0}^{T} \sup_{s\in K} |b_n(1-s-ia\varphi(\tau))|\,\mathrm{d}\tau.$$

For estimation of the function $b_n(s)$, we apply the estimate (2.5.6) for the gamma-function. Thus, for $s \in K$,

$$b_n\left(\frac{1}{2}+\varepsilon-s+iu\right) = \frac{n^{1/2+\varepsilon-\sigma}}{\beta} \Gamma\left(\frac{1/2+\varepsilon-\sigma}{\beta} + \frac{i(t-u)}{\beta}\right)$$
$$\ll_{\beta,K} n^{-\varepsilon} \exp\{-c_1|u|\}, \quad c_1>0.$$

This together with Lemma 5.5 yields

$$I_1 \ll_{\varepsilon,\mathfrak{a},K} n^{-\varepsilon} \int\limits_{-\infty}^{\infty} (1+|u|)\exp\{-c_1|u|\}\, \mathrm{d}u \ll_{\varepsilon,\mathfrak{a},K} n^{-\varepsilon} \tag{5.5.5}$$

because $\varepsilon$ depends on the set $K$. Similarly, we find that, for $s \in K$,

$$\begin{aligned} b_n(1-s-ia\varphi(\tau)) &\ll_{\beta,\mathfrak{a},K} n^{1-\sigma}\exp\left\{-\frac{c}{\beta}|t-a\varphi(\tau)|\right\} \\ &\ll_{a,\mathfrak{a},K} n^{1/2-2\varepsilon}\exp\{-c_2\varphi(\tau)\}, \quad c_2>0. \end{aligned}$$

Therefore, properties of the function $\varphi(\tau)$ show that

$$\begin{aligned} I_2 &\ll_{\varepsilon,\mathfrak{a},K} n^{1/2-2\varepsilon}\frac{1}{T-T_0}\int\limits_{T_0}^{T}\exp\{-c_2\varphi(\tau)\}\,\mathrm{d}\tau \\ &\ll_{\varepsilon,\mathfrak{a},K} n^{1/2-2\varepsilon}\left(\frac{\log T}{T}+\frac{1}{T}\int\limits_{\log T}^{T}\exp\{-c_2\varphi(\tau)\}\,\mathrm{d}\tau\right) \\ &\ll_{a,\mathfrak{a},K} n^{1/2-2\varepsilon}\left(\frac{\log T}{T}+\frac{1}{T}\exp\left\{-\frac{c_2}{2}\varphi(\log T)\right\}\right. \\ &\qquad\times\left.\int\limits_{\log T}^{T}\exp\left\{-\frac{c_2}{2}\varphi(\tau)\right\}\,\mathrm{d}\tau\right) = o(T) \end{aligned}$$

as $T\to\infty$. This, (5.5.5) and (5.5.4) prove (5.5.2).

For almost all $\omega\in\Omega$, the function $\zeta(s,\omega;\mathfrak{a})$ is analytic in the half-plane $\sigma>1/2$. Therefore, the second assertion of the lemma is obtained similarly to that of the first by using Lemma 5.7, and the representation

$$\zeta_n(s,\omega;\mathfrak{a}) = \frac{1}{2\pi i}\int\limits_{\beta-i\infty}^{\beta+i\infty}\zeta(s+z,\omega;\mathfrak{a})b_n(z)\,\mathrm{d}z$$

which is valid for almost all $\omega\in\Omega$. Also, we note, that in the case of the function $\zeta(s,\omega;\mathfrak{a})$, we do have not the integral $I_2$. □

**Lemma 5.10.** *Suppose that* $(\varphi_1(\tau), \dots, \varphi_r(\tau)) \in U_r(T_0)$. *Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \underline{\rho}\left(\underline{\zeta}(s+i\underline{\varphi}(\tau);\underline{\mathfrak{a}}), \underline{\zeta}_n(s+i\underline{\varphi}(\tau);\underline{\mathfrak{a}})\right) \, \mathrm{d}\tau = 0.$$

*Moreover, for almost all* $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \underline{\rho}\left(\underline{\zeta}(s+i\underline{\varphi}(\tau), \underline{\omega};\underline{\mathfrak{a}}), \underline{\zeta}_n(s+i\underline{\varphi}(\tau), \underline{\omega};\underline{\mathfrak{a}})\right) \, \mathrm{d}\tau = 0.$$

*Proof.* We use Lemmas 5.6 and 5.8 and apply similar arguments as in the proof of Lemma 5.9. □

## 5.6 Limit lemmas for $\underline{\zeta}(s;\underline{\mathfrak{a}})$

Let, for $A \in \mathcal{B}(H^r(D))$,

$$P^1_{T,\underline{\mathfrak{a}},\omega}(A) = \frac{1}{T-T_0} \text{meas}\left\{\tau \in [T_0, T] : \underline{\zeta}(s + i\underline{a}\varphi(\tau), \omega; \underline{\mathfrak{a}}) \in A\right\}$$

and

$$P^r_{T,\underline{\mathfrak{a}},\omega}(A) = \frac{1}{T-T_0} \text{meas}\left\{\tau \in [T_0, T] : \underline{\zeta}(s + i\underline{\varphi}(\tau), \omega; \underline{\mathfrak{a}}) \in A\right\}.$$

Together with $P^1_{T,\underline{\mathfrak{a}}}$ and $P^r_{T,\underline{\mathfrak{a}}}$, we will consider the weak convergence for $P^1_{T,\underline{\mathfrak{a}},\omega}$ and $P^r_{T,\underline{\mathfrak{a}},\omega}$ as $T \to \infty$.

**Lemma 5.11.** *Suppose that* $\underline{a}$ *and* $\varphi(\tau)$ *satisfy hypotheses of Theorem 5.1. Then, on* $(H^r(D), \mathcal{B}(H^r(D)))$, *there exists a probability measure* $P^1_{\underline{\mathfrak{a}}}$ *such that* $P^1_{T,\underline{\mathfrak{a}}}$ *and* $P^1_{T,\underline{\mathfrak{a}},\omega}$ *both converge weakly to* $P^1_{\underline{\mathfrak{a}}}$ *as* $T \to \infty$.

*Proof.* Let $\theta_T$ be a random variable defined on a certain probability space $(\Omega, \mathcal{B}, \mu)$, and uniformly distributed in the interval $[T_0, T]$, i. e., $\theta_T$ has the distribution density

$$p_T(x) = \begin{cases} 0, & x < T_0, \\ \frac{1}{T-T_0}, & T_0 \leqslant x \leqslant T, \\ 0, & x > T. \end{cases}$$

Define the $H^r(D)$-valued random element

$$\underline{X}^1_{T,n,\underline{\mathfrak{a}}} = \underline{X}^1_{T,n,\underline{\mathfrak{a}}}(s) = \underline{\zeta}_n(s + i\underline{a}\varphi(\theta_T); \underline{\mathfrak{a}}).$$

Moreover, denote by $\underline{X}^1_{n,\underline{\mathfrak{a}}} = \underline{X}^1_{n,\underline{\mathfrak{a}}}(s)$ the $H^r(D)$-valued random element with the distribution $V^1_n$, where $V^1_n = m_H u_n^{-1}$, and the mapping $u_n$ is from Lemma 5.3. Then the assertion of Lemma 5.3 can be written in the form

$$\underline{X}^1_{T,n,\underline{\mathfrak{a}}} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}^1_{n,\underline{\mathfrak{a}}}. \tag{5.6.1}$$

The series for $\zeta_n(s; \mathfrak{a}_j)$, $j = 1, \dots, r$, is absolutely convergent. Therefore, for $\sigma > 1/2$,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T |\zeta_n(\sigma + it; \mathfrak{a}_j)|^2 \, \mathrm{d}t &= \sum_{m=1}^{\infty} \frac{|a_{jm}|^2 v_n^2(m)}{m^{2\sigma}} \\ &\leqslant \sum_{m=1}^{\infty} \frac{|a_{jm}|^2}{m^{2\sigma}} \leqslant C_{\sigma,\mathfrak{a}_j} < \infty.\end{aligned}$$

From this, using properties of the function $\varphi(\tau)$, by the same arguments as in the proof of Lemma 2.15, we obtain that, for $\sigma > 1/2$,

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T - T_0} \int\limits_{T_0}^T |\zeta(\sigma + ia_j\varphi(\tau); \mathfrak{a}_j)|^2 \, \mathrm{d}\tau \ll C_{a,\sigma,\mathfrak{a}_j} < \infty. \tag{5.6.2}$$

Let $K \subset D$ be a compact set. then the application of the Cauchy integral formula and (5.6.2) lead to

$$\begin{aligned}&\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T - T_0} \int\limits_{T_0}^T \sup_{s\in K} |\zeta(s + ia_j\varphi(\tau); \mathfrak{a}_j)| \, \mathrm{d}\tau \\ &\quad \ll_K \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \left( \frac{1}{T - T_0} \int\limits_{T_0}^T \sup_{s\in K} |\zeta(s + ia_j\varphi(\tau); \mathfrak{a}_j)|^2 \, \mathrm{d}\tau \right)^{1/2} \\ &\quad \ll R_{a,K,\mathfrak{a}_j} < \infty.\end{aligned}$$

From this, repeating the proof of Lemma 2.15, we derive that the probability measure

$$V_{n,j}^1(A) = V_n^1\left(\underbrace{H(D) \times \cdots \times H(D)}_{j-1} \times A \times H(D) \times \cdots \times H(D)\right),$$
$$A \in H(D),\ j = 1, \ldots, r,$$

is tight. This means that, for every $\varepsilon > 0$, there exists a compact set $K_j \subset H(D)$ such that

$$V_{n,j}^1(K_j) > 1 - \frac{\varepsilon}{r}. \tag{5.6.3}$$

Moreover, let $K = K_1 \times \cdots \times K_r$. Then $K$ is a compact set in $H^r(D)$. Moreover, by (5.6.3),

$$\begin{aligned} V_n^1(H^r(D) \setminus K) = V_n^1\Bigg(\bigcup_{j=1}^{r} \Big(H(D) \times \cdots \times H(D) \\ \times (H(D) \setminus K_j) \times H(D) \times \cdots \times H(D)\Big)\Bigg) \\ \leqslant \sum_{j=1}^{r} V_{n,j}^1(H(D) \setminus K_j) < \frac{\varepsilon}{r} \cdot r = \varepsilon. \end{aligned}$$

Thus,

$$V_n^1(K) > 1 - \varepsilon,$$

i. e., the sequence $\{V_n^1 : n \in \mathbb{N}\}$ is tight. Therefore, by Lemma 2.14, this sequence is relatively compact. Hence, there exists a subsequence $\{V_{n_l}^1\} \subset \{V_n^1\}$ and the probability measure $P_{\underline{\mathfrak{a}}}^1$ on $(H^r(D), \mathcal{B}(H^r(D)))$ such that $V_{n_l}^1$, as $l \to \infty$, converges weakly to $P_{\underline{\mathfrak{a}}}^1$. In other words, we have

$$\underline{X}_{n_l,\underline{\mathfrak{a}}}^1 \xrightarrow[l\to\infty]{\mathcal{D}} P_{\underline{\mathfrak{a}}}^1. \tag{5.6.4}$$

Define one more $H^r(D)$-valued random element

$$\underline{X}_{T,\underline{\mathfrak{a}}}^1 = \underline{X}_{T,\underline{\mathfrak{a}}}^1(s) = \underline{\zeta}(s + i\underline{a}\varphi(\theta_T); \underline{\mathfrak{a}}).$$

Then, by the first assertion of Lemma 5.9, we find that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n_l\to\infty} \limsup_{T\to\infty} \mu\left\{\underline{\rho}(\underline{X}^1_{T,\underline{\mathfrak{a}}}(s), \underline{X}^1_{T,n_l,\underline{\mathfrak{a}}}(s)) \geqslant \varepsilon\right\} \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \underline{\rho}\left(\underline{\zeta}(s+i\underline{a}\varphi(\tau),\underline{\mathfrak{a}}), \underline{\zeta}_{n_l}(s+i\underline{a}\varphi(\tau),\underline{\mathfrak{a}})\right) \mathrm{d}\tau \\
&\quad = 0.
\end{aligned}$$

This, (5.6.1) and (5.6.4) show that all hypotheses of Lemma 2.13 are satisfied. Therefore, we have the relation

$$\underline{X}^1_{T,\underline{\mathfrak{a}}} \xrightarrow[T\to\infty]{\mathcal{D}} P^1_{\underline{\mathfrak{a}}}, \tag{5.6.5}$$

or that $P^1_{T,\underline{\mathfrak{a}}}$ converges weakly to $P^1_{\underline{\mathfrak{a}}}$ as $T \to \infty$. Moreover, in view of (5.6.5), the measure $P^1_{\underline{\mathfrak{a}}}$ is independent of the subsequence $\{V^1_{n_l}\}$. Therefore, we have

$$\underline{X}^1_{n,\underline{\mathfrak{a}}} \xrightarrow[n\to\infty]{\mathcal{D}} P^1_{\underline{\mathfrak{a}}}. \tag{5.6.6}$$

To obtain the weak convergence for $P^1_{T,\underline{\mathfrak{a}},\omega}$, introduce the $H^r(D)$-valued random elements

$$\underline{X}^1_{T,n,\underline{\mathfrak{a}},\omega} = \underline{X}^1_{T,n,\underline{\mathfrak{a}},\omega}(s) = \underline{\zeta}_n(s+i\underline{a}\varphi(\theta_T),\omega;\underline{\mathfrak{a}})$$

and

$$\underline{X}^1_{T,\underline{\mathfrak{a}},\omega} = \underline{X}^1_{T,\underline{\mathfrak{a}},\omega}(s) = \underline{\zeta}(s+i\underline{a}\varphi(\theta_T),\omega;\underline{\mathfrak{a}}).$$

Then, repeating the above arguments for $\underline{X}^1_{T,n,\underline{\mathfrak{a}},\omega}$ and $\underline{X}^1_{T,\underline{\mathfrak{a}},\omega}$, and using relation (5.6.6), we obtain the weak convergence of $P^1_{T,\underline{\mathfrak{a}},\omega}$ to $P^1_{\underline{\mathfrak{a}}}$ as $T \to \infty$. We also observe that, in this case, all relations are true for almost all $\omega \in \Omega^r$. The lemma is proved. □

**Lemma 5.12.** *Suppose that* $(\varphi_1(\tau), \ldots, \varphi_r(\tau)) \in U_r(T_0)$. *Then, on* $(H^r(D), \mathcal{B}(H^r(D)))$, *there exists a probability measure* $P^r_{\underline{\mathfrak{a}}}$ *such that* $P^r_{T,\underline{\mathfrak{a}}}$ *and* $P^r_{T,\underline{\mathfrak{a}},\omega}$ *both converge weakly to* $P^r_{\underline{\mathfrak{a}}}$ *as* $T \to \infty$.

*Proof.* We use arguments similar to those of the proof of Lemma 5.11 with application of Lemmas 5.4 and 5.10. □

It remains to identify the limit measure in Lemma 5.11. For this, elements of ergodic theory will be applied.

## 5.7 Identification of the limit measures

In this section, we identify the limit measures $P^1_{\underline{\mathfrak{a}}}$ and $P^r_{\underline{\mathfrak{a}}}$ in Lemmas 5.11 and 5.12. For this, we will use some results of ergodic theory.

For brevity, let, for $\tau \geqslant T_0$,

$$\underline{a}^1_\tau = \left(\left(p^{-ia_1\varphi(\tau)} : p \in \mathbb{P}\right), \dots, \left(p^{-ia_r\varphi(\tau)} : p \in \mathbb{P}\right)\right)$$

and

$$\underline{a}^r_\tau = \left(\left(p^{-i\varphi_1(\tau)} : p \in \mathbb{P}\right), \dots, \left(p^{-i\varphi_r(\tau)} : p \in \mathbb{P}\right)\right).$$

Clearly, $\underline{a}^1_\tau$ and $\underline{a}^r_\tau$ are elements of $\Omega^r$. Now, on $\Omega^r$, define the families of transformations $\{\Phi^1_\tau : \tau \geqslant T_0\}$ and $\{\Phi^r_\tau : \tau \geqslant T_0\}$, where

$$\Phi^1_\tau(\omega) = \underline{a}^1_\tau \omega \quad \text{and} \quad \Phi^r_\tau(\omega) = \underline{a}^r_\tau \omega, \quad \omega \in \Omega^r.$$

Then $\{\Phi^1_\tau\}$ and $\{\Phi^r_\tau\}$ are families of measurable measure preserving (because of invariance of the Haar measure $m^r_H$ on $(\Omega^r, \mathcal{B}(\Omega^r))$) transformations on $\Omega^r$. Recall that a set $A \in \mathcal{B}(\Omega^r)$ is called invariant with respect to $\{\Phi^k_\tau : \tau \geqslant T_0\}$ if, for every $\tau \geqslant T_0$, the sets $A$ and $A_\tau = \Phi^k_\tau(A)$ can differ one from other at most by a set of $m^r_H$-measure zero, $k = 1$ or $k = r$. All invariant sets forms a $\sigma$-field. The family $\{\Phi^k_\tau\}$ is called ergodic if its $\sigma$-field of invariant sets consists only from sets of $m^r_H$-measure zero or one.

**Lemma 5.13.** *The families $\{\Phi^1_\tau\}$ and $\{\Phi^r_\tau\}$ are ergodic.*

*Proof.* We consider only $\{\Phi^1_\tau\}$ because the case $\{\Phi^r_\tau\}$ is similar, and apply the Fourier transform method. In the proof of Lemma 5.1, we already have used that the characters $\chi$ of the group $\Omega^r$ are of the form

$$\chi(\omega) = \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p).$$

Therefore, if the character $\chi$ is non-trivial ($\chi(\omega) \not\equiv 1$), we have

$$\chi(\underline{a}^1_\tau) = \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} p^{-ia_jk_{jp}\varphi(\tau)} = \exp\left\{-i\varphi(\tau)\sum_{j=1}^{r} a_j \sum_{p\in\mathbb{P}}{}^{*} k_{jp}\log p\right\}.$$

Since the character $\chi$ is non-trivial, $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$. Thus, as in the proof of Lemma 5.1, we have

$$\sum_{j=1}^{r} a_j \sum_{p\in\mathbb{P}}^{*} k_{jp} \log p \neq 0.$$

Therefore, there exists a value $\tau_0 \geqslant T_0$ such that

$$\chi(\underline{a}^1_{\tau_0}) \neq 1. \tag{5.7.1}$$

Now, let $A$ be an invariant set with respect to $\{\Phi^1_\tau\}$, and let $I_A$ be its indicator function. Then, for almost all $\omega \in \Omega^r$,

$$I_A(\underline{a}^1_\tau \omega) = I_A(\omega).$$

Thus, in view of the invariance of the Haar measure $m^r_H$, the Fourier transform $\widehat{I}_A(\chi)$ is

$$\begin{aligned}\widehat{I}_A(\chi) &= \int\limits_{\Omega^r} \chi(\omega) I_A(\omega)\, \mathrm{d}m^r_H = \int\limits_{\Omega^r} \chi(\underline{a}^1_{\tau_0}\omega) I_A(\underline{a}^1_{\tau_0}\omega)\, \mathrm{d}m^r_H \\ &= \chi(\underline{a}^1_{\tau_0}) \int\limits_{\Omega^r} \chi(\omega) I_A(\omega)\, \mathrm{d}m^r_H = \chi(\underline{a}^1_{\tau_0}) \widehat{I}_A(\chi)\end{aligned}$$

because of the multiplicativity of characters. Therefore, taking into account (5.7.1), we obtain that

$$\widehat{I}_A(\chi) = 0 \tag{5.7.2}$$

for all non-trivial characters of $\Omega^r$.

Denote by $\chi_0$ the trivial character of $\Omega^r$ ($\chi_0(\omega) \equiv 1$), and suppose that $\widehat{I}(\chi_0) = c$. Then using the orthogonality of characters and (5.7.2) give the equality

$$\widehat{I}_A(\chi) = c \int\limits_{\Omega^r} \chi(\omega)\, \mathrm{d}m^r_H = c\widehat{1}(\chi) = \widehat{c}(\chi)$$

for every character $\chi$ of $\Omega^r$. This shows that $I_A(\omega) = c$ for almost all $\omega \in \Omega^r$. Since $c = 0$ or $c = 1$ ($I_A$ is the indicator function), we obtain that $m^r_H(A) = 0$, or $m^r_H(A) = 1$. The lemma is proved. □

Now we recall some other notions of ergodic theory. Let $(\widehat{\Omega}, \mathcal{B}, \mu)$ be a certain probability space, and $\mathcal{T}$ denote the parameter set. A finite real function $X(\tau, \omega)$, $\tau \in \mathcal{T}$, $\omega \in \Omega$, is said to be a random process if $X(\tau, \cdot)$ is a random variable for each fixed $\tau \in \mathcal{T}$. When $\omega \in \widehat{\Omega}$ is fixed, the function $X(\cdot, \omega)$ is called a sample path of the random process.

Let $\tau_1, \ldots, \tau_m$ be arbitrary values of $\tau$. Then the family of common distributions of random variables $X(\tau_1, \omega), \ldots, X(\tau_m, \omega)$, i. e.,

$$\mu \{X(\tau_1, \omega) < x_1, \ldots, X(\tau_m, \omega) < x_m\}$$

for all $m \in \mathbb{N}$ and all possible values of $\tau_j$, is called a family of finite-dimensional distributions of the random process $X(\tau, \omega)$.

A random process $X(\tau, \omega)$ is said to be a strongly stationary process if all its finite-dimensional distributions are invariant under the shifts $\tau \to \tau + u$, $u \in \mathbb{R}$.

Let $Y$ be the space of real finite functions $y(\tau)$, $\tau \in \mathcal{T}$. Then all finite-dimensional distribution of a random process define a probability measure $Q$ on $(Y, \mathcal{B}(Y))$. This gives the probability space $(Y, \mathcal{B}(Y), Q)$.

Let $A \in \mathcal{B}(Y)$ and $u \in \mathbb{R}$. Then the set $A_u$ is obtained from $A$ after shifting all $y(\tau) \in A$ to $y(\tau + u)$. A set $A \in \mathcal{B}(Y)$ is called an invariant set of the random process $X(\tau, \omega)$ if, for each $u \in \mathbb{R}$, the sets $A$ and $A_u$ differ one from another by a set of $Q$-measure zero.

A strongly stationary process $X(\tau, \omega)$ is called ergodic if its $\sigma$-field of invariant sets consists only from sets having $Q$-measure zero or one.

For ergodic processes, the following statement, called the Birkhoff-Khintchine theorem, is true. Denote by $\mathbb{E}X$ the expectation of $X$.

**Lemma 5.14.** *Suppose that the random process $X(\tau, \omega)$ is ergodic, and $\mathbb{E}|X(\tau, \omega)| < \infty$ with sample paths integrable almost surely in the Riemann sense over every finite interval. Then*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T X(\tau, \omega)\, \mathrm{d}\tau = \mathbb{E}X(0, \omega)$$

*for almost all $\omega \in \widehat{\Omega}$.*

Proof of lemma is given, for example, in [5].

*Proof of Theorem 5.3.* In view of Lemma 5.11, it suffices to show that $P^1_{\underline{\mathfrak{a}}} = P^1_{\underline{\zeta}, \underline{\mathfrak{a}}}$.

Let $A$ be a fixed continuity set of the measure $P^1_{\underline{\mathfrak{a}}}$. Then Lemmas 5.11

and 2.1 imply

$$\lim_{T\to\infty}\frac{1}{T-T_0}\text{meas}\left\{\tau\in[T_0,T]:\underline{\zeta}(s+i\underline{a}\varphi(\tau),\omega;\underline{\mathfrak{a}})\in A\right\}=P^1_{\underline{\mathfrak{a}}}(A). \tag{5.7.3}$$

On the probability space $(\Omega^r,\mathcal{B}(\Omega^r),m^r_H)$, define the random variable

$$\theta(\omega)=\begin{cases}1 & \text{if } \ \underline{\zeta}(s,\omega;\underline{\mathfrak{a}})\in A,\\ 0 & \text{otherwise.}\end{cases}$$

The expectation of $\theta(\omega)$ is

$$\mathbb{E}\theta=\int\limits_{\Omega^r}\theta\,\mathrm{d}m^r_H=m^r_H\left\{\omega\in\Omega^r:\underline{\zeta}(s,\omega;\underline{\mathfrak{a}})\in A\right\}=P_{\underline{\zeta},\underline{\mathfrak{a}}}(A). \tag{5.7.4}$$

Lemma 5.13 implies the ergodicity of the random process $\theta(\Phi^1_\tau(\omega))$. Therefore, by Lemma 5.14, for almost all $\omega\in\Omega^r$, we have

$$\lim_{T\to\infty}\frac{1}{T-T_0}\int\limits_{T_0}^{T}\theta(\Phi^1_\tau(\omega))\,\mathrm{d}\tau=\mathbb{E}\theta. \tag{5.7.5}$$

On the other hand, the definitions of $\theta(\omega)$ and $\Phi^1_\tau$ show that

$$\frac{1}{T-T_0}\int\limits_{T_0}^{T}\theta(\Phi^1_\tau(\omega))\,\mathrm{d}\tau$$
$$=\frac{1}{T-T_0}\text{meas}\left\{\tau\in[T_0,T]:\underline{\zeta}(s+i\underline{a}\varphi(\tau),\omega;\underline{\mathfrak{a}})\in A\right\}.$$

Thus, in virtue of (5.7.4) and (5.7.5),

$$\lim_{T\to\infty}\frac{1}{T-T_0}\text{meas}\left\{\tau\in[T_0,T]:\underline{\zeta}(s+i\underline{a}\varphi(\tau),\omega;\underline{\mathfrak{a}})\in A\right\}=P_{\underline{\zeta},\underline{\mathfrak{a}}}(A).$$

This together with (5.7.3) implies the equality $P^1_{\underline{\mathfrak{a}}}(A)=P_{\underline{\zeta},\underline{\mathfrak{a}}}(A)$ for all continuity sets $A$ of $P^1_{\underline{\mathfrak{a}}}$. Since all continuity sets of the measure constitute the determining class [4], hence, $P^1_{\underline{\mathfrak{a}}}(A)=P_{\underline{\zeta},\underline{\mathfrak{a}}}(A)$ for all $A\in\mathcal{B}(H^r(D))$. The theorem is proved. □

*Proof of Theorem 5.4.* We repeat the proof of Theorem 5.3 using other objects. Thus, let $A$ be a continuity set of the measure $P^r_{\underline{\mathfrak{a}}}$. Then, by Lemmas 5.12

and 2.1 we have

$$\lim_{T\to\infty}\frac{1}{T-T_0}\text{meas}\left\{\tau\in[T_0,T]:\underline{\zeta}(s+i\underline{\varphi}(\tau),\omega;\underline{\mathfrak{a}})\in A\right\}=P^r_{\underline{\mathfrak{a}}}(A). \tag{5.7.6}$$

By Lemma 5.13, the random process $\theta(\Phi^r_\tau(\omega))$ is ergodic, thus, in view of Lemma 5.14,

$$\lim_{T\to\infty}\frac{1}{T-T_0}\int\limits_{T_0}^{T}\theta(\Phi^r_\tau(\omega))\,\mathrm{d}\tau=\mathbb{E}\theta. \tag{5.7.7}$$

By the definitions of $\theta(\omega)$ and $\Phi^r_\tau$,

$$\frac{1}{T-T_0}\int\limits_{T_0}^{T}\theta(\Phi^r_\tau(\omega))\,\mathrm{d}\tau$$
$$=\frac{1}{T-T_0}\text{meas}\left\{\tau\in[T_0,T]:\underline{\zeta}(s+i\underline{\varphi}(\tau),\omega;\underline{\mathfrak{a}})\in A\right\}.$$

Hence, equalities (5.7.4) and (5.7.7) imply

$$\lim_{T\to\infty}\frac{1}{T-T_0}\text{meas}\left\{\tau\in[T_0,T]:\underline{\zeta}(s+i\underline{\varphi}(\tau),\omega;\underline{\mathfrak{a}})\in A\right\}=P_{\underline{\zeta},\underline{\mathfrak{a}}}(A).$$

This and (5.7.6) show that $P^r_{\underline{\mathfrak{a}}}(A)=P_{\underline{\zeta},\underline{\mathfrak{a}}}(A)$ for all continuity sets of the measure $P^r_{\underline{\mathfrak{a}}}$, thus, $P^r_{\underline{\mathfrak{a}}}=P_{\underline{\zeta},\underline{\mathfrak{a}}}$, and the theorem is proved. □

## 5.8 Proof of Theorems 5.1 and 5.2

Proofs of joint universality theorems 5.1 and 5.2 are based on limit theorems 5.3 and 5.4, and are standard. We give them only for fullness.

*Proof of Theorem 5.1.* By Lemma 2.16, there exists polynomials $p_1(s),\dots,p_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|f_j(s)-\mathrm{e}^{p_j(s)}\right|<\frac{\varepsilon}{2}. \tag{5.8.1}$$

In view of Lemma 3.9, the support of the measure $P_{\underline{\zeta},\underline{\mathfrak{a}}}$ is the set $S^r$, where

$$S=\left(\{g\in H(D):g(s)\neq 0 \text{ or } g(s)\equiv 0\}\right).$$

Define the set

$$G_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

By the above remark, the set $G_\varepsilon$ is an open neighbourhood of the element $\left(\mathrm{e}^{p_1(s)}, \ldots, \mathrm{e}^{p_r(s)}\right)$ of the support of the measure $P_{\underline{\zeta}, \underline{\mathfrak{a}}}$. Hence,

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{5.8.2}$$

Moreover, in view of (5.8.1), we have the inclusion $G_\varepsilon \subset \widehat{G}_\varepsilon$, where

$$\widehat{G}_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Therefore, by (5.8.2),

$$P_{\underline{\zeta}, \underline{\mathfrak{a}}}(\widehat{G}_\varepsilon) > 0,$$

and Theorem 5.3 and Lemma 2.1 give

$$\liminf_{T \to \infty} P^1_{T, \underline{\mathfrak{a}}}(\widehat{G}_\varepsilon) \geqslant P_{\underline{\zeta}, \underline{\mathfrak{a}}}(\widehat{G}_\varepsilon) > 0.$$

This, and the definitions of $P^1_{T, \underline{\mathfrak{a}}}$ and $\widehat{G}_\varepsilon$ prove the first statement of the theorem.

To prove the second statement of the theorem, we observe that the sets $\partial \widehat{G}_\varepsilon$ do not intersect for different $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Hence, the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}, \underline{\mathfrak{a}}}$ for all but at most countable many $\varepsilon > 0$. Therefore, by Theorem 5.3 and Lemma 2.1,

$$\lim_{T \to \infty} P^1_{T, \underline{\mathfrak{a}}}(\widehat{G}_\varepsilon) = P_{\underline{\zeta}, \underline{\mathfrak{a}}}(\widehat{G}_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. This, and the definitions of $P^1_{T, \underline{\mathfrak{a}}}$ and $\widehat{G}_\varepsilon$ prove the second statement of the theorem. □

*Proof of Theorem 5.2.* We repeat the proof of Theorem 5.1 using Theorem 5.4 in place of Theorem 5.3. □

# Conclusions

The results of the dissertation lead to the following conclusions:

1. For the periodic zeta-function $\zeta(s;\mathfrak{a})$ with multiplicative periodic sequence $\mathfrak{a}$, the discrete universality theorem on the approximation of analytic functions by shifts $\zeta(s+iht_k;\mathfrak{a})$, where $h>0$ and $\{t_k\}$ is a sequence of Gram points, is valid.

2. For periodic zeta-functions $\zeta(s;\mathfrak{a}_j)$ with multiplicative periodic sequences $\mathfrak{a}_j$, $j=1,\dots,r$, under weak Montgomery conjecture on correlation of imaginary parts of non-trivial zeros $\gamma_k$ of the Riemann zeta-function, the joint discrete universality theorem on approximation of a collection of analytic functions by shifts $\zeta(s+ih_j\gamma_k;\mathfrak{a}_j)$, where $h_1,\dots,h_r$ are positive algebraic numbers linearly independent over the field of rational numbers, is valid.

3. For some classes of operators $F$ in the multidimensional space of analytic functions, the compositions $F(\zeta(s+ih_1\gamma_k;\mathfrak{a}_1),\dots\zeta(s+ih_r\gamma_k;\mathfrak{a}_r))$ are universal.

4. For periodic zeta-functions $\zeta(s;\mathfrak{a}_j)$ with multiplicative periodic sequences $\mathfrak{a}_j$, $j=1,\dots,r$, joint universality theorems on the approximation of a collection of analytic functions by generalized non-linear shifts $\zeta(s+i\varphi_j(\tau);\mathfrak{a}_j)$ are valid without any independence hypotheses on the sequences $\mathfrak{a}_j$.

# Bibliography


[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, Heidelberg, Berlin, 1976.

[2] A. Baker, The theory of linear forms in logarithms, in: *Transcendence theory: advances and applications*, Proc. Conf. Univ. Cambridge, Cambridge 1976, Academic Press, Boston, Ma., pp. 1–27, 1977.

[3] B. Bagchi, *The Statistical Behaviuor and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series*, PhD Thesis, Indian Statistical Institute, Calcutta, 1981.

[4] P. Billingsley, *Convergence of Probability Measures*, 2nd edition, John Wiley & Son, New York, 1999.

[5] H. Cramér and M. Leadbetter, *Stationary and Related Stochastic Processes*, Wiley, New York, 1967.

[6] Ch. J. de la Vallée Poussin, Recherches analytiques sur la théorie des nombres premiers, I–III, *Ann. Soc. Sci. Bruxelles* **20** (1896), 183–256, 281–362, 363–397.

[7] A. Dubickas and A. Laurinčikas, Distribution modulo 1 and the discrete universality of the Riemann zeta-function, *Abh. Math. Semin. Univ. Hambg.* **86** (2016), 79–87.

[8] P. D. T. A. Elliott, *Probabilistic Number Theory I, Mean–Value Theorems*, Springer-Verlag, New York, Heidelberg, Berlin, 1979.

[9] P. D. T. A. Elliott, *Probabilistic Number Theory I, Central Limit Theorems*, Springer-Verlag, New York, Heidelberg, Berlin, 1980.

[10] P. Epstein, Zur Theorie allgemeiner Zetafunktionen, *Math. Ann.* **56** (1903), 615–644.

[11] R. Garunkštis and A. Laurinčikas, The Riemann hypothesis and universality of the Riemann zeta-function, *Math. Slovaca* **68** (2018), 741–748.

[12] R. Garunkštis and A. Laurinčikas, Discrete mean square of the Riemann zeta-function over imaginary parts of its zeros, *Periodica Math. Hung.* **76** (2018), 217-228.

[13] R. Garunkštis, A. Laurinčikas and R. Macaitienė, Zeros of the Riemann zeta-function and its universality, *Acta Arith.* **181** (2017), 127–142.

[14] S. M. Gonek, *Analytic properties of zeta and L-functions*, Ph.D. Thesis, University of Michigan, Ann Arbor, 1979.

[15] J.-P. Gram, Sur les zéros de la fonction $\zeta(s)$ de Riemann, *Acta Math.* **27** (1903), 289–304.

[16] A. Hurwitz, Einige Eigenschaften der Dirichletschen Funktionen $F(s) = \sum \left(\frac{D}{n}\right)\frac{1}{n^s}$, die bei der Bestimmung der Klassenzahlen binärer quadratischer Formen auftreten, *Z. Math. Phys.* **27** (1882), 86–101.

[17] J. I. Hutchinson, On the roots of the Riemann zeta-function, *Trans. Amer. Math. Soc.* **27** (1925), 49–60.

[18] M. Ishibashi, S. Kanemitsu, Dirichlet series with periodic coefficients, *Result. Math.* **35** (1999), 70–88.

[19] J. Kaczorowski, Some remarks on the universality of periodic *L*-functions, in: *New Directions in Value-Distribution Theory of Zeta and L-functions*, Proc. of the Würzburg Conf., R. Steuding and J. Steuding (Eds), Shaker Verlag, Aachen, pp. 113–120, 2009.

[20] A. A. Karatsuba and S. M. Voronin, *The Riemann Zeta-Function*, Walter de Gruyter, Berlin, 1992.

[21] M. A. Korolev, Gram's law and Selberg's conjecture on the distribution of the zeros of the Riemann zeta-function, *Izv. Math.* **74**(4) (2010), 743–780.

[22] M. A. Korolev, On Gram's law in the theory of the Riemann zeta-function, *Izv. Math.* **76**(2) (2012), 275–309.

[23] M. A. Korolev, Gram's law and the argument of the Riemann zeta-function, *Publ. Inst. Math. (Beograd) (N.S.)* **92**(106) (2012), 53–78.

[24] M. A. Korolev, On the Selberg formulas related to Gram's law, *Sb. Math.* **203**(11-12) (2012), 1808–1816.

[25] M. A. Korolev, On new results related to Gram's law *Izv. Math.* **77**(5) (2013), 917–940.

[26] M. A. Korolev, On small values of the Riemann zeta-function at the Gram points, *Sb. Math.* **205**(1-2) (2014), 63–82.

[27] M. A. Korolev, Gram's law in the theory of the Riemann zeta-function. Part 1, *Proc. Steklov Inst. Math.* **292**(2) (2016), 1–146.

[28] M. A. Korolev, Gram's law in the theory of the Riemann zeta-function. Part 2, *Proc. Steklov Inst. Math* **294**(1 Supplement) (2016), 1–78.

[29] M. Korolev and A. Laurinčikas, A new application of the Gram points, *Aequat. Math.* **93** (2019), 859–873.

[30] M. Korolev and A. Laurinčikas, A new application of the Gram points. II, *Aequat. Math.* **94** (2020), 1171–1187.

[31] E. Kowalski, *An Introduction to Probabilistic Number Theory*, Cambridge University Press, Cambridge, 2021.

[32] J. Kubilius, *Probabilistic Methods in the Theory of Numbers*, Transl. Math. Monogr., Vol. 11, Amer. Math. Soc., Providence, RI, 5th ed., 1997.

[33] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Pure and Applied Math., Wiley-Interscience, New York, London, Sydney, 1974.

[34] A. Laurinčikas, Zeros of the derivative of the Riemann zeta-function, *Lith. Math. J.* **25** (1985), 255–260.

[35] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1996.

[36] A. Laurinčikas, Universality of the Riemann zeta-function, *J. Number Th.* **130** (2010), 2323–2331.

[37] A. Laurinčikas, On joint universality of Dirichlet $L$-functions, *Chebysh. sb.* **12**(1) (2011), 124–139.

[38] A. Laurinčikas, Universality of composite functions, *RIMS Kôkyûroku Bessatsu* **B34** (2012), 191–204.

[39] A. Laurinčikas, Zeros of the Riemann zeta-function in the discrete universality of the Hurwitz zeta-function, *Stud. Sci. Math. Hung.* **57** (2020), 147–164.

[40] A. Laurinčikas, Joint discrete universality for periodic zeta-functions. II, *Quaest. Math.* **43** (2020), 1765–1779.

[41] A. Laurinčikas and R. Garunkštis, *The Lerch Zeta-Function*, Kluwer, Dordrecht, 2002.

[42] A. Laurinčikas and R. Macaitienė, On the joint universality for periodic zeta-functions, *Math. Notes* **85** (2009), 51–60.

[43] A. Laurinčikas, R. Macaitienė and D. Šiaučiūnas, Uniform distribution modulo 1 and the joint universality of Dirichlet $L$-functions, *Lith. Math. J.* **56** (2016), 529–539.

[44] A. Laurinčikas, K. Matsumoto and J Steuding, Universality of some functions related to zeta-functions of certain cusp forms, *Osaka J. Math.* **50** (2016), 1021–1037.

[45] A. Laurinčikas and L. Meška, Sharpening of the universality inequality, *Math. Notes* **96** (2014), 971–976.

[46] A. Laurinčikas and D. Šiaučiūnas, On the periodic zeta function. II, *Lith. Math. J.* **41**(4) (2001), 361–372.

[47] A. Laurinčikas and D. Šiaučiūnas, Remarks on the universality of the periodic zeta-functions, *Math. Notes* **80** (2006), 532–538.

[48] A. Laurinčikas, M. Tekorė, Joint universality of periodic zeta-functions with multiplicative coefficients, *Nonlinear Anal.: Modell. Control* **25** (2020), 860–883.

[49] A. Laurinčikas, D. Šiaučiūnas and M. Tekorė, Joint universality of periodic zeta-functions with multiplicative coefficients. II, *Nonlinear Anal.: Modell. Control* **26** (2021), 550–564.

[50] R. Macaitienė, M. Stoncelis and D. Šiaučiūnas, A weighted discrete universality theorem for periodic zeta-functions, in *Anal. Probab. Methods Number Theory*, Proc. 6th Int. Conf., Palanga, Lithuania, 11–17 September 2016, A. Dubickas, A. Laurinčikas, E. Manstavičius and G. Stepanauskas (eds.), VU Leidykla: Vilnius, Lithuania, 2017; pp. 97–107.

[51] K. Matsumoto, A survey on the theory of universality for zeta and $L$-functions, in *Number Theory: Plowing and Starring Through High Wave Forms, Proc. 7th China - Japan Seminar (Fukuoka 2013), Series on Number Theory and its Appl.*, M. Kaneko, Sh. Kanemitsu and J. Liu (Eds), World Scientific Publishing Co., New Jersey, London, Singapore, Bejing, Shanghai, Hong Kong, Taipei, Chennai, pp. 95–144, 2015.

[52] S. N. Mergelyan, Uniform approximations to functions of a complex variable, *Amer. Math. Soc. Translation* **1954**, (1954) no. 101, 99 pp.

[53] J.-L. Mauclaire, Universality of the Riemann zeta function: two remarks, *Ann. Univ. Sci. Budapest. Sect. Comput.* **39** (2013), 311–319.

[54] *The Millennium Prize Problems*, Clay Mathematics Institute of Cambridge, https://www.claymath.org/millennium-problems/ (žiūrėta: )

[55] H. L. Montgomery, The pair correlation of zeros of the zeta-function, in *Analytic Number Theory*, (St. Louis, MO, 1972), Proc. Sympos. Pure Math. 24, H.G. Diamond (Ed), Amer. Math. Soc., Providence, pp. 181–193, 1973.

[56] M. J. Mossinghoff, T. S. Trudgian and A. Yang, Explicit zero-free regions for the Riemann zeta-function, *Res. Number. Theory* **10** (2024), 11.

[57] Ł. Pańkowski, Joint universality for dependent $L$-functions, *Ramanujan J.* **45** (2018), 181–195.

[58] K. Pratt, N. Robles, A. Zaharescu and D. Zeindler, More than five-twelfths of the zeros of $\zeta(s)$v are on the critical line, *Res. Math. Sci.* **7** (2020), article no. 2.

[59] A. Reich, Werteverteilung von Zetafunktione, *Arch. Math.* **34** (1980), 440–451.

[60] W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill Science / Engineering / Math, New York, 1991.

[61] W. Schnee, Die Funktionalgleichung der Zetafunktion und der Dirichletschen Reiben mit periodishen Koeffizienten, *Math. Z.* **31** (1930), 378–390.

[62] A. Selberg, The zeta-function and the Riemann hypothesis, in: *C.R. Dixième Congrès Math. Scandinaves 1946*, Jul. Gjellerups Forlag, Copenhagen, 1947, pp. 187–200.

[63] A. Sourmelidis, T. Srichan and J. Steuding, On the vertical distribution of values of $L$-functions in the Selberg class, *Int. J. Number Th.* **18** (2022), 277–302.

[64] J. Steuding, On Dirichlet series with periodic coefficients, *Ramanujan J.* **6** (2002), 295–306.

[65] J. Steuding, *Value-Distribution of L-Functions*, Lecture Notes Math. vol. 1877, Springer, Berlin, Heidelberg, 2007.

[66] J. Steuding, The roots of the equation $\zeta(s) = a$ are uniformly distributed modulo one, in A. Laurinčikas, E. Manstavičius, G. Stepanauskas (Eds.), *Anal. Probab. Methods Number Theory*, TEV, Vilnius, 2012, pp. 243–249.

[67] D. Šiaučiūnas, *The Investigations of the Periodic Zeta-Function*, doctoral dissertation, Vilnius University, Vilnius, 2004.

[68] D. Šiaučiūnas, R. Šimėnas and M. Tekorė, Approximation of analytic functions by shifts of certain compositions, *Mathematics* **9** (2021), 2583.

[69] D. Šiaučiūnas and M. Tekorė, Gram points in the universality of the Dirichlet series with periodic coefficients, *Mathematics* **11** (2023), 4615.

[70] E. C. Titchmarsh, The zeros of the Riemann zeta-function, *Proc. Royal Soc. London. Ser. A* **151** (1935), 234–255.

[71] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Second edition revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.

[72] S. M. Voronin, Theorem on the “universality” of the Riemann zeta-function, *Izv. Akad. Nauk SSSR, Ser. Matem.* **39** (1975), 475–486 (in Russian).

[73] S. M. Voronin, On the functional independence of Dirichlet *L*-functions, *Acta Arith.* **27** (1975), 493–503.

[74] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, VEB Deutscher Verlag der Wissenschaften, 1963.

[75] J. L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, *Amer. Math. Soc. Colloq. Publ.* **20**, AMS, Providence, 1960.

# Santrauka
# (Summary in Lithuanian)

## Tyrimo objektas

Disertacijoje nagrinėjamas periodinių dzeta funkcijų reikšmių pasiskirstymas. Tegul $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ yra periodinė kompleksinių skaičių seka su minimaliuoju periodu $q \in \mathbb{N}$, t. y. $a_{m+q} = a_m$ su visais $m \in \mathbb{N}$. Periodinė dzeta funkcija $\zeta(s; \mathfrak{a})$, $s = \sigma + it$, pusplokštumėje $\sigma > 1$ apibrėžiama Dirichlė eilute

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

Kadangi seka $\mathfrak{a}$ yra aprėžta, eilutės, apibrėžiančios funkciją $\zeta(s; \mathfrak{a})$, absoliučiai ir tolygiai konverguoja pusplokštumėje $\sigma > 1 + \varepsilon$ su bet kokiu $\varepsilon > 0$. Vadinasi, $\zeta(s; \mathfrak{a})$ yra analizinė minėtoje pusplokštumėje $\sigma > 1$. Tam, kad gautume periodinės dzeta funkcijos $\zeta(s; \mathfrak{a})$ analizinį pratęsimą į sritį $\sigma \leqslant 1$, yra naudojama klasikinė Hurvico dzeta funkcija, kuri buvo pradėta nagrinėti [16] straipsnyje. Tegul $0 < \alpha \leqslant 1$ yra fiksuotas parametras. Hurvico dzeta funkcija pusplokštumėje $\sigma > 1$ apibrėžiama Dirichlė eilute

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}.$$

Be to, $\zeta(s, \alpha)$ yra analiziškai pratęsiama į visą kompleksinę plokštumą, išskyrus tašką $s = 1$, kuris yra jos paprastasis polius su reziduumu 1, žr. [1, 41]. Kitaip tariant, $\zeta(s, \alpha)$ yra meromorfinė funkcija.

Kadangi seka $\mathfrak{a}$ yra periodinė, su $\sigma > 1$, galioja lygybė

$$\zeta(s; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=1}^{q} a_l \zeta\left(s, \frac{l}{q}\right).$$

Dėl anksčiau paminėtų analizinių Hurvico dzeta funkcijos savybių, funkciją $\zeta(s, \mathfrak{a})$ galima pratęsti į visą kompleksinę plokštumą su tikėtinu paprastuoju poliumi taške $s = 1$ su reziduumu

$$\widehat{a} \stackrel{\text{def}}{=} \frac{1}{q} \sum_{l=1}^{q} a_l.$$

Jeigu $\widehat{a} = 0$, tai $\zeta(s; \mathfrak{a})$ yra sveikoji funkcija.

Kita sekos $\mathfrak{a}$ savybė, naudojama disertacijoje, yra jos multiplikatyvumas. Primename, kad seka yra multiplikatyvioji, jeigu $a_1 = 1$, o $a_{m_1 m_2} = a_{m_1} a_{m_2}$ su visais tarpusavyje pirminiais $m_1, m_2 \in \mathbb{N}$ ($(m_1, m_2) = 1$). Daugelis skaičių teorijos aritmetinių funkcijų pasižymi multiplikatyvumo savybe. Pavyzdžiui: daliklių funkcija, Miobuso funkcija, Oilerio funkcija, Dirichlė charakteriai ir kt.

Disertacijoje yra nagrinėjamos funkcijos $\zeta(s; \mathfrak{a})$ su periodiniais multiplikatyviaisiais koeficientais aproksimavimo savybės. Kadangi seka $\mathfrak{a}$ yra multiplikatyvi, todėl periodinė dzeta funkcija yra išreiškiama Oilerio sandauga pagal pirminius skaičius

$$\zeta(s; \mathfrak{a}) = \prod_{p \in \mathbb{P}} \left( 1 + \sum_{l=1}^{\infty} \frac{a_{p^l}}{p^{ls}} \right), \quad \sigma > 1.$$

Pažymime, kad sekos $\mathfrak{a}$ multiplikatyvumas apibrėžia tam tikrą aproksimuojamų funkcijų klasę.

## Tikslas ir uždaviniai

Disertacijos tikslas – analizinių funkcijų aproksimavimas postūmiais $\zeta(s + i\tau; \mathfrak{a})$, $\tau \in \mathbb{R}$, juostoje $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$.

Uždaviniai:

1. Analizinių funkcijų klasės aproksimavimas postūmiais $\zeta(s + iht_n; \mathfrak{a})$, $h > 0$, čia $\{t_n : n \in \mathbb{N}\}$ yra Rymano dzeta funkcijos Gramo taškų seka.

2. Analizinių funkcijų rinkinio jungtinis aproksimavimas postūmiais $(\zeta(s + ih_1\gamma_k; \mathfrak{a}_1), \dots, \zeta(s + ih_r\gamma_k; \mathfrak{a}_r))$, $h_j > 0$, čia $\{\gamma_k : k \in \mathbb{N}\}$ Rymano dzeta funkcijos netrivialiųjų nulių teigiamų menamųjų dalių seka.

3. Analizinių funkcijų klasės aproksimavimas sudėtinių funkcijų postū-

miais $F(\zeta(s+ih_1\gamma_k;\mathfrak{a}_1),\dots,\zeta(s+ih_r\gamma_k;\mathfrak{a}_r))$ su analizinių funkcijų erdvėje tolydžiaisiais operatoriais $F$.

4. Analizinių funkcijų klasės jungtinis aproksimavimas apibendrintais netiesiniais postūmiais $(\zeta(s+i\gamma_1(\tau);\mathfrak{a}_1),\dots,\zeta(s+i\gamma_r(\tau);\mathfrak{a}_r))$.

# Aktualumas

Analizinės funkcijos nėra vien funkcijų teorijos objektas. Jos yra svarbios analizinėje ir algebrinėje skaičių teorijoje, diferencialinių ir integralinių lygčių uždaviniuose, tikimybių teorijoje, funkcinėje analizėje, matematinėje fizikoje ir kitose matematikos srityse. Šios funkcijos taip pat yra naudojamos spendžiant fizikos ir kitų gamtos mokslų uždavinius. Todėl iškyla sudėtingų analizinių funkcijų supaprastinimo uždaviniai, vedantys prie analizinių funkcijų aproksimavimo paprastesnėmis funkcijomis. Gerai žinoma, kad kiekviena tolydžioji kompaktinėje aibėje $K$ su jungiuoju papildiniu ir analizinė tos aibės viduje funkcija aibėje $K$ gali būti tolygiai aproksimuojama polinomais. Vadinasi, kiekvieną analizinę funkciją atitinka aproksimuojantis polinomas priklausantis nuo aproksimuojamos funkcijos. Beveik prieš penkiasdešimt metų tapo žinoma, kad, palyginus paprastos, kai kurios dzeta ir $L$ funkcijos pasižymi universalia aproksimavimo savybe: vienos ir tos pačios funkcijos postūmiais aproksimuojame visą analizinių funkcijų klasę. Šis netikėtas aproksimavimo teorijos progresas atvėrė kelią platesniems uždaviniams: universalių funkcijų klasės apibrėžimui, universalumo efektyvizavimui, aproksimuojančių postūmių klasių apibūdinimui ir, be abejo, universalumo taikymui sprendžiant matematinius ir praktinius uždavinius. Daugelyje matematikos mokslo centrų (Australijoje, Indijoje, Japonijoje, JAV, Kanadoje, Pietų Korėjoje, Švedijoje, Vokietijoje ir kt.) buvo suburtos aproksimavimo teorijoje universalumo uždavinius tiriančios grupės. Gausiausia tokių grupių sėkmingai dirba Lietuvoje (I. Belovas, R. Garunkštis, R. Kačinskaitė, A. Laurinčikas, R. Macaitienė ir kt.). Ši grupė pasiekė reikšmingų universalumo efektyvizavimo rezultatų, išplėtė universalių funkcijų klasę, įvedė naujus universalumo tipus, aprašė naujas postūmių klases. Kaip pasakė profesorius A. Šincelis, kiekviena įrodyta teorema suformuluoja tris naujus uždavinius. Kadangi universalumo problemos yra įdomios ir svarbios daugeliui mokslo sričių, universalumo plėtra yra viena iš šiuolaikinių matematikos krypčių.

## Metodai

Disertacijoje suformuluotų teoremų įrodymuose pagrindinę vietą užima tikimybiniai metodai, paremti silpnuoju tikimybinių matų konvergavimu ir jo savybėmis analizinių funkcijų erdvėje. Be to, yra taikomi analizinės skaičių teorijos, įskaitant įvairius vidurkių įverčius, metodai, Koši integralinė formulė, intergralinės išraiškos ir polinomų savybės.

## Naujumas

Visi disertacijoje įrodyti rezultatai yra nauji. Periodinių dzeta funkcijų universalumo teoremos su apibendrintaisiais postūmiais, įskaitant Gramo taškus, Rymano dzeta funkcijos netrivialiųjų nulių menamasias dalis ir tiesiškai nepriklausomus virš racionaliųjų skaičių kūno $\mathbb{Q}$ algebrinius skaičius, anksčiau nebuvo žinomos.

## Problemos istorija ir rezultatai

Dzeta ir $L$ funkcijos įprastai yra apibrėžiamos paprastosiomis Dirichlė eilutėmis

$$\sum_{m=1}^{\infty} \frac{a(m)}{m^s}$$

pusplokštumėje $\sigma > \sigma_0$ su tam tikru $\sigma_0$, arba bendrosiomis Dirichlė eilutėmis

$$\sum_{m=1}^{\infty} b(m) e^{-\lambda_m s}, \quad \sigma > \sigma_0,$$

čia $\{a(m)\}$ ir $\{b(m)\}$ yra kompleksinių skaičių sekos, o $\{\lambda_m\} \subset \mathbb{R}$,

$$\lim_{m \to \infty} \lambda_m = +\infty.$$

Pavyzdžiui, Rymano dzeta funkcija $\zeta(s)$ yra apibrėžta paprastaja Dirichlė eilute

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

ir meromorfiškai pratęsiama į visą kompleksinę plokštumą, išskyrus tašką $s = 1$, kuris yra parastasis polius su reziduumu 1.

Hurvico dzeta funkcija $\zeta(s,\alpha)$ yra apibrėžiama bendrąja Dirichlė eilute

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \mathrm{e}^{-s\log(m+\alpha)}, \quad \sigma > 1,$$

t. y. $\lambda_m = \log(m+\alpha)$, o $b(m) \equiv 1$.

Dzeta funkcijos $Z(s)$, kurių pavidalas

$$Z(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \quad \sigma > \sigma_0,$$

yra sekos $\{a(m)\}$ generuojančios funkcijos. Jos yra taikomos gauti informaciją apie vidurkį

$$m(x) \stackrel{\text{def}}{=} \sum_{m\leqslant x} a(m), \quad x \to \infty.$$

Egzistuoja įvairios formulės, apimančios funkcijos $Z(s)$ kontūrinį integralą, kurios duoda vidurkio $m(x)$ išraišką su tam tikru liekamuoju nariu. Tai gali būti aiškiai iliustruota funkcijos

$$\pi(x) = \sum_{p\leqslant x} 1, \quad x \to \infty,$$

atveju. Tegul $\Lambda(m)$ yra Mangoldto funkcija, t. y.

$$\Lambda(m) = \begin{cases} \log p, & \text{jeigu } m = p^k,\ k \in \mathbb{N}, \\ 0, & \text{kitais atvejais.} \end{cases}$$

Vidurkis

$$m_\Lambda(x) \stackrel{\text{def}}{=} \sum_{m\leqslant x} \Lambda(m), \quad x \to \infty,$$

glaudžiai susijęs su funkcija $\pi(x)$. Vidurkio $m_\Lambda(x)$ asimptotika, kai $x \to \infty$, veda prie funkcijos $\pi(x)$ asimptotikos. Tuo tikslu yra įvedama generuojanti funkcija

$$Z_\Lambda(s) \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s}, \quad \sigma > 1.$$

Nesunku pastebėti, kad

$$Z_\Lambda(s) = -\frac{\zeta'(s)}{\zeta(s)}. \tag{S1}$$

Anksčiau paminėti faktai parodo, kad pirmiausia reikia nagrinėti dzeta funkcijų reikšmių pasiskirstymą, o tada gautus rezultatus taikyti kitiems uždaviniams spręsti.

Periodinė dzeta funkcija $\zeta(s;\mathfrak{a})$ yra įdomus analizinis objektas ir buvo nagrinėta daugelio matematikų. Pirmasis reikšmingas rezultatas yra funkcijos $\zeta(s;\mathfrak{a})$ funkcinė lygtis, kuri buvo įrodyta [61] straipsnyje.

Funkcija $\zeta(s;\mathfrak{a})$ tam tikra prasme yra susijusi su Rymano dzeta funkcija. Tegul

$$b_m = \frac{1}{q}\sum_{l=0}^{q-1} a_l \mathrm{e}^{-2\pi i l(m/q)}, \quad m \in \mathbb{N}, \qquad \widetilde{b}_m = \sqrt{q}\, b_m,$$

$$\mathfrak{c}_l = \{c_m : c_m = \mathrm{e}^{2\pi i m(l/q)}, m \in \mathbb{N}\}, \quad \zeta(s,\mathfrak{c}_l) = \sum_{m=1}^{\infty} \frac{c_m}{m^s}, \quad \sigma > 1.$$

Tuomet [18] straipsnyje buvo įrodyta, kad

$$\zeta(s;\mathfrak{a}) = \frac{\widetilde{b}_q}{\sqrt{q}}\zeta(s) + \frac{1}{\sqrt{q}}\sum_{m=1}^{q-1} \widetilde{b}_m \zeta(s;\mathfrak{c}_l).$$

Svarbių rezultatų periodinei dzeta funkcijai įrodė J. Štoidingas. Jis sukūrė funkcijos $\zeta(s;\mathfrak{a})$ nulių pasiskirstymo teoriją, kuri pateikta [64] straipsnyje bei [65] monografijoje.

Artutinė funkcinė lygtis funkcijai $\zeta(s;\mathfrak{a})$, įtraukiant baigtines sumas ir apjungiant kintamuosius $s$ ir $1-s$, buvo įrodyta [67] disertacijoje (taip pat galima žr. [46]). Be to, [67] buvo gauta kvadratinio vidurkio

$$\int_0^T |\zeta(\sigma_T + it;\mathfrak{a})|^2 \, \mathrm{d}t$$

asimptotika su $\sigma_T = 1/2$ ir $\sigma_T \to 1/2 + 0$, kai $T \to \infty$. Taip pat periodinių dzeta funkcijų rinkiniui $\zeta(s;\mathfrak{a}_1), \ldots, \zeta(s;\mathfrak{a}_r)$ įrodyta jungtinė ribinė teorema, t. y. mato

$$\frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : (\zeta(s+i\tau;\mathfrak{a}_1), \ldots, \zeta(s+i\tau;\mathfrak{a}_r)) \in A\right\}, \qquad A \in \mathcal{B}(\mathbb{X}(G)),$$

silpnasis konvergavimas, kai $T \to \infty$. Čia $\mathbb{X}(G)$ yra analizinių arba meromorfinių srityje $G = \{s \in \mathbb{C} : \sigma > 1/2\}$ funkcijų erdvė, $\mathcal{B}(\mathbb{Y})$ – topologinės erdvės $\mathbb{Y}$ Borelio $\sigma$ kūnas, o $\text{meas}A$ – mačiosios aibės $A \subset \mathbb{R}$ Lebego matas.

Toliau nagrinėsime analizinių funkcijų klasių aproksimavimą dzeta funkcijų postūmiais $Z(s+i\tau)$, jei tiksliau, postūmiais $\zeta(s+i\tau;\mathfrak{a})$. Ši savybė yra vadinama universalumu ir buvo atrasta Rymano dzeta funkcijai rusų matematiko S. M. Voronino 1975 m., žr. [72].

**1 teorema.** *[72]. Tarkime, kad $0 < r < 1/4$ yra fiksuotas skaičius, funkcija $f(s)$ yra tolydžioji, neįgyjanti nulių skritulyje $|s| \leqslant r$ ir analizinė skritulio $|s| < r$ viduje. Tuomet su kiekvienu $\varepsilon > 0$ egzistuoja toks skaičius $\tau = \tau(\varepsilon) \in \mathbb{R}$, kad*

$$\max_{|s|\leqslant r}\left|f(s) - \zeta\left(s + \frac{3}{4} + i\tau\right)\right| < \varepsilon.$$

Voroninas tame pačiame straipsnyje [72] parašė, kad analogiškas teiginys yra teisingas ir visoms Dirichlė $L$ funkcijoms $L(s,\chi)$. Pastarosios teoremos įrodymas remiasi Rymano teoremos analogu apie eilutės narių perstatymą Hilberto erdvėse.

Pakankamai greitai 1 teorema buvo išplėsta ir patobulinta [3] ir [14] daktaro disertacijose. Tam, kad suformuluotume šiuolaikinę Voronino teoremos versiją, mums reikia tam tikrų žymenų. Tegul $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ yra juosta kompleksinėje plokštumoje, $\mathcal{K}$ žymi juostos $D$ kompaktinių poaibių su jungiaisiais papildiniais klasę, $H_0(K)$ su $K \in \mathcal{K}$ yra tolydžiųjų, neįgyjančių nulių aibėje $K$ ir analizinių aibės $K$ viduje funkcijų klasė, o $\text{meas}A$ – mačiosios aibės $A \subset \mathbb{R}$ Lebego matas. Yra įrodyta tokia teorema, žr. [20, 35, 65, 51].

**2 teorema.** *Tarkime, kad $K \in \mathcal{K}$, o $f(s) \in H_0(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|f(s) - \zeta(s+i\tau)| < \varepsilon\right\} > 0.$$

*Be to, riba*

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|f(s) - \zeta(s+i\tau)| < \varepsilon\right\}$$

*egzistuoja ir yra teigiama su visais $\varepsilon > 0$, nebent išskyrus skaičiąją $\varepsilon > 0$ reikšmių aibę.*

Antrasis teoremos tvirtinimas yra palyginus nesenas ir nepriklausomai įrodytas [53] ir [45] straipsniuose.

Vėliau paaiškėjo, kad kai kurios kitos dzeta ir $L$ funkcijos taip pat Voronino prasme yra universalios, tačiau ne taip paprasta tai įrodyti. Be to, yra ir neuniversalių Dirichlė eilučių, žr, pvz., [65].

Pirmąją universalumo teoremą periodinėms dzeta funkcijoms $\zeta(s;\mathfrak{a})$ įrodė J. Štoidingas, žr. [64] arba [65]. Tegul $H(K)$ su $K \in \mathcal{K}$ žymi tolydžiųjų aibėje $K$ ir analizinių aibės $K$ viduje funkcijų klasę. Taigi, $H(K)$ yra klasės $H_0(K)$ plėtinys.

**3 teorema.** *Tarkime, kad periodas $q > 2$, $a_m$ nėra Dirichlė charakterio moduliu $q$ kartotinis, o $a_m = 0$ su $(m,q) > 1$. Tegul $K \in \mathcal{K}$, o $f(s) \in H(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |f(s) - \zeta(s+i\tau;\mathfrak{a})| < \varepsilon \right\} > 0.$$

Pažymėtina, žr. [64], kad periodinė seka tenkinanti 3 teoremos sąlygas nėra multiplikatyvi.

Lenkų matematikas J. Kačarovskis [19] straipsnyje įrodė, kad ne visos Dirichlė eilutės su periodiniais koeficientais yra universalios Voronino prasme. Jis suformulavo būtinas ir pakankamas sąlygas, kad funkcijos $\zeta(s;\mathfrak{a})$ būtų universalios su pirminiu periodu $q$.

Pirmoji universalumo teorema, su multiplikatyviaja seka $\mathfrak{a}$, periodinei dzeta funkcijai buvo įrodyta [47] straipsnyje.

**4 teorema.** *Tarkime, kad seka $\mathfrak{a}$ yra multiplikatyvioji ir su visais pirminiais skaičiais $p$ galioja nelygybė*

$$\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\alpha/2}} \leqslant c < 1.$$

*Tegul $K \in \mathcal{K}$, o $f(s) \in H_0(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |f(s) - \zeta(s+i\tau;\mathfrak{a})| < \varepsilon \right\} > 0.$$

Taigi, remiantis 4 teorema, visos funkcijos $\zeta(s;\mathfrak{a})$ su multiplikatyviaja periodine seka $\mathfrak{a}$ yra universalios, t. y. turi tą pačią aproksimavimo savybę kaip ir Dirichlė $L$ funkcijos.

Anksčiau santraukoje suformuluotos universalumo teoremos yra tolydžiojo tipo, nes $\tau$ postūmiuose $\zeta(s+i\tau;\mathfrak{a})$ gali įgyti visas realiasias reikšmes.

Yra žinomas kitas dzeta funkcijų universalumo tipas, vadinamas diskrečiuoju universalumu. Šiuo atveju, aproksimuojančiuose postūmiuose, $\tau$ gali įgyti reikšmes iš tam tikros diskrečiosios aibės, pavyzdžiui, iš aritmetinės progresijos $\{hk\}$, $k \in \mathbb{N}_0$. Diskretųjį universalumą pasiūlė nagrinėti A. Reichas. Jis [59] straipsnyje įrodė diskrečiojo universalumo teoremą algebrinių skaičių kūnų Dedekindo dzeta funkcijoms $\zeta_{\mathbb{K}}(s)$. Kai $\mathbb{K} = \mathbb{Q}$, gauname Rymano dzeta funkciją. Tegul $\#A$ žymi aibės $A \subset \mathbb{R}$ galią (elementų skaičių). Todėl Reicho teoremos išvada yra toks teiginys:

**5 teorema.** *Tegul $K \in \mathcal{K}$, $f(s) \in H_0(K)$, o $h > 0$ yra fiksuotas skaičius. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + ikh)| < \varepsilon \right\} > 0.$$

B. Bagčis dakataro disertacijoje [3] dzeta funkcijų diskrečiojo universalumo teoremoms įrodyti taikė kitus metodus.

Diskretusis funkcijų $\zeta(s; \mathfrak{a})$ universalumas yra daug sudėtingesnis. Kai $h > 0$, tegul

$$L(\mathbb{P}; h, \pi) = \{(h \log p : p \in \mathbb{P}), 2\pi\}.$$

Periodinių dzeta funkcijų diskrečiojo universalumo teorema yra gauta iš daug bendresnės svorinio universalumo teoremos, žr. [50].

**6 teorema.** *[50]. Tarkime, kad seka $\mathfrak{a}$ yra multiplikatyvioji, o $L(\mathbb{P}; h, \pi)$ yra tiesiškai nepriklausoma aibė virš racionaliųjų skaičių kūno $\mathbb{Q}$. Tegul $K \in \mathcal{K}$, o $f(s) \in H_0(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + ikh; \mathfrak{a})| < \varepsilon \right\} > 0.$$

Vienas iš būdų išplėsti universalumą yra naudoti, taip vadinamus, apibendrintuosius postūmius $\zeta(s + i\varphi(\tau))$ arba $\zeta(s + i\varphi(k))$ su tam tikra funkcija $\varphi$. Rymano dzeta ir Dirichlė $L$ funkcijoms tai padarė L. Pankovskis [57] straipsnyje. Tegul

$$b \in \begin{cases} \mathbb{R}, & \text{jeigu} \quad a \notin \mathbb{N}, \\ (-\infty, 0] \cup (1 + \infty), & \text{jeigu} \quad a \in \mathbb{N}. \end{cases}$$

**7 teorema.** *[57]. Tarkime, kad $\alpha \in \mathbb{R}$, o $a$ yra teigiamas realusis skaičius.*

*Tegul $K \in \mathcal{K}$, o $f(s) \in H_0(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [2,T] : \sup_{s\in K} |f(s) - \zeta(s + i\alpha\tau^a \log^b \tau)| < \varepsilon\right\} > 0$$

*ir*

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{2 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + i\alpha k^a \log^b k)| < \varepsilon\right\} > 0.$$

Ši teorema apibendrina [7] ir [43] straipsniuose gautus rezultatus.

Antras disertacijos skyrius yra skirtas įrodyti diskrečiojo universalumo teoremą funkcijai $\zeta(s; \mathfrak{a})$ postūmiuose naudojant Gramo taškus. Yra žinoma, kad funkcija $\zeta(s)$ su visais $s \in \mathbb{C}$ tenkina funkcinę lygtį

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Pagrindinis šios lygties narys yra funkcija $g(s) \stackrel{\text{def}}{=} \pi^{-s/2}\Gamma(s/2)$. Pažymėkime $\theta(t)$, $t \geqslant 0$, funkcijos $g(s)$ argumento pokytį išilgai atkarpos jungiančios taškus $s = 1/2$ ir $s = 1/2 + it$. Žinoma, žr. [15], kad funkcija $\theta(t)$ yra monotoniškai didėjanti ir neaprėžta iš viršaus, kai $t > t^*$, $t^* = 6.289836\ldots$. Todėl lygtis

$$\theta(t) = (n-1)\pi, \quad n \in \mathbb{N}$$

su $t > t^*$ turi vienintelį sprendinį $t_n$. Skaičiai $t_n$ yra vadinami Gramo taškais, kadangi jis įrodė šių skaičių ryšį su funkcijos $\zeta(s)$ netrivialiaisiais nuliais. Priminsime, kad Rymano dzeta funkcijos nuliai, esantys juostoje $\{s \in \mathbb{C} : 0 < \sigma < 1\}$, yra vadinami netrivialiaisiais, o nuliai $s = -2k$, $k \in \mathbb{N}$, – trivialiaisiais. Rymano hipotezė teigia, kad funkcijos $\zeta(s)$ netrivialiųjų nulių išraiška yra $\rho_n = 1/2 + i\gamma_n$. Visi šiuo metu žinomi netrivialieji nuliai patvirtina Rymano hipotezę. J.-P. Gramas [15] straipsnyje pastebėjo, kad kiekviename intervale $(t_{n-1}, t_n]$, $n = 1, \ldots, 15$, yra tik vienas funkcijos $\zeta(1/2+it)$ nulis $1/2+i\gamma_n$, kad $t_{n-1} < \gamma_n < t_n$. Be to, jis iškėlė prielaidą, kad tai nėra teisinga, kai $n > 15$. Vėliau paaiškėjo, kad Gramo hipotezė yra teisinga. Pavyzdžiui, žr. [70], [17], buvo gauta, kad

$$t_{127} < \gamma_{127} < \gamma_{128} < t_{128} \qquad \text{ir} \qquad t_{134} < \gamma_{134} < \gamma_{135} < t_{135}.$$

Pastarosios nelygybės analiziškai buvo įrodytos [70] straipsnyje, t. y., kad seka

$$\frac{\gamma_n - t_n}{t_{n+1} - t_n}$$

yra neaprėžta, todėl nulis $\gamma_n$, su pakankamai daug $n$ reikšmių, negali būti intervale $(t_{n-1}, t_n]$.

Taškai $t_n$ yra labai įdomus analizinės skaičių teorijos objektas, nes yra žinoma, kad

$$\lim_{n\to\infty} \frac{t_n}{\gamma_n} = 1.$$

Analizinių funkcijų aproksimavime Gramo taškai pirmą kartą buvo panaudoti [29] straipsnyje.

**8 teorema.** *Tegul $K \in \mathcal{K}$, $f(s) \in H_0(K)$, o $h > 0$ yra fiksuotas. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k)| < \varepsilon \right\} > 0.$$

*Be to, riba*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k) - | < \varepsilon \right\}$$

*egzistuoja ir yra teigiama su visais $\varepsilon > 0$, nebent išskyrus skaičiąją $\varepsilon > 0$ reikšmių aibę.*

Pastaroji teorema [30] straipsnyje buvo išplėsta į trumpuosius intervalus. Būtent, 8 teoremos tvirtinimas galioja dydžiui

$$\frac{1}{M+1} \# \left\{ N \leqslant k \leqslant N + M : \sup_{s\in K} |f(s) - \zeta(s + iht_k)| < \varepsilon \right\}$$

kai $N \to \infty$, o $M$ yra iš intervalo

$$\left(\frac{3\pi N}{h^2}\right)^{1/3} (\log\{(h+1)N\})^{12/5} \leqslant M \leqslant N.$$

Pagrindinis antrojo disertacijos skyriaus rezultatas funkcijai $\zeta(s; \mathfrak{a})$ praplečia 8 teoremą.

**2.1 teorema.** *Tarkime, kad seka $\mathfrak{a}$ yra multiplikatyvioji. Tegul $K \in \mathcal{K}$,*

*$f(s) \in H_0(K)$, o $h > 0$ yra fiksuotas skaičius. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k; \mathfrak{a})| < \varepsilon \right\} > 0.$$

*Be to, riba*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + iht_k; \mathfrak{a})| < \varepsilon \right\}$$

*egzistuoja ir yra teigiama su visais $\varepsilon > 0$, nebent išskyrus skaičiąją $\varepsilon > 0$ reikšmių aibę.*

Trečiajame disertacijos skyrius yra skirtas nagrinėti apibendrintuosius postūmius $\zeta(s + ih\gamma_k; \mathfrak{a})$, čia $0 < \gamma_1 < \gamma_2 < \cdots \leqslant \gamma_k \leqslant \cdots$ yra Rymano dzeta funkcijos netrivialiųjų nulių menamųjų dalių seka. Šios sekos pasiskirstymas yra labai sudėtingas, todėl yra suformuluotos kelios hipotezės. Viena jų yra Montgomerio prielaida, žr. [55], kuri teigia, kad

$$\sum_{\substack{\gamma_k, \gamma_l \leqslant T \\ (2\pi\alpha_1)/\log T \leqslant \gamma_k - \gamma_l \leqslant (2\pi\alpha_2)/\log T}} 1 \sim \left( \int_{\alpha_1}^{\alpha_2} \left( 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 \right) \mathrm{d}u + \delta(\alpha_1, \alpha_2) \right) \frac{T}{2\pi} \log T$$

kai $T \to \infty$, čia $\alpha_1 < \alpha_2$ yra bet kokie realieji skaičiai, o

$$\delta(\alpha_1, \alpha_2) = \begin{cases} 1, & \text{jeigu} \quad 0 \in [\alpha_1, \alpha_2], \\ 0, & \text{kitais atvejais.} \end{cases}$$

Aproksimuojančiuose postūmiuose $\zeta(s + ih\gamma_k)$ yra naudojama sąlyga susijusi su Montgomerio prielaida, žr. [13], kuri teigia, kad su $c > 0$

$$\sum_{\substack{\gamma_k, \gamma_l \leqslant T \\ |\gamma_k - \gamma_l| < c/\log T}} 1 \ll T \log T. \tag{S2}$$

**9 teorema.** *[13]. Tarkime, kad galioja* (S2) *įvertis. Tegul $K \in \mathcal{K}$, o*

*$f(s) \in H_0(K)$. Tuomet su kiekvienu $h > 0$ ir $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + ih\gamma_k)| < \varepsilon\right\} > 0.$$

*Be to, su kiekvienu $h > 0$ riba*

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K} |f(s) - \zeta(s + ih\gamma_k)| < \varepsilon\right\}$$

*egzistuoja ir yra teigiama su visais $\varepsilon > 0$, nebent išskyrus skaičiąją $\varepsilon > 0$ reikšmių aibę.*

Pastaroji teorema [11] ir [12] straipsniuose buvo įrodyta naudojant Rymano hipotezę. Pažymėsime, kad [63] straipsnyje buvo gautas analogiškas 9 teoremai rezultatas nenaudojant Montgomerio prielaidos. Taip pat ši teorema [39] straipsnyje buvo išplėsta Hurvico dzeta funkcijai $\zeta(s, \alpha)$ su parametru $\alpha$ tokiu, kad seka $\{\log(m + \alpha) : m \in \mathbb{N}_0\}$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$.

Dzeta funkcijų universalumo teoremos gali būti jungtinės. Šiuo atveju analizinių funkcijų rinkinys tuo pačiu metu gali būti aproksimuojamas dzeta funkcijų postūmių rinkiniu. Pirmąją jungtinio universalumo teoremą Dirichlė $L$ funkcijoms [73] straipsnyje įrodė Voroninas.

Trečiajame disertacijos skyriuje yra įrodyta jungtinio universalumo teorema periodinėms dzeta funkcijoms su multiplikatyviaisiais koeficientais naudojant postūmius $\zeta(s + ih_j\gamma_k; \mathfrak{a}_j)$, $j = 1, \dots, r$. Primename, kad skaičius $\alpha$ vadinamas algebriniu, jeigu jis yra polinomo su racionaliaisiais koeficientais šaknis. Pagrindinis 3 skyriaus rezultatas yra tokia teorema:

**3.1 teorema.** *Tarkime, kad sekos $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ yra multiplikatyvios, $h_1, \dots, h_r$ yra teigiami algebriniai skaičiai tiesiškai nepriklausomi virš $\mathbb{Q}$ ir galioja* (S2) *įvertis. Kai $j = 1, \dots, r$, tegul $K_j \in \mathcal{K}$, o $f_j(s) \in H_0(K_j)$. Tuomet su kiekvienu $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s + ih_j\gamma_k; \mathfrak{a}_j)| < \varepsilon\right\} > 0.$$

*Be to, riba*

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s + ih_j\gamma_k; \mathfrak{a}_j)| < \varepsilon\right\}$$

*egzistuoja ir yra teigiama su visais $\varepsilon > 0$, nebent išskyrus skaičiąją $\varepsilon > 0$*

*reikšmių aibę.*

3.1 teorema yra tam tikras 2.1 teoremos apibendrinimas.

Dar vienas būdas išplėsti universalių Voronino prasme funkcijų klasę yra tam tikrų sudėtinių funkcijų universalumas. Šis būdas Rymano dzeta funkcijos atveju buvo pasiūlytas [36] straipsnyje. Anksčiau, [34] straipsnyje buvo įrodytas funkcijos $\zeta'(s)/\zeta(s)$ universalumas. Šiuo atveju, aproksimuojama funkcija $f(s)$ nebūtinai yra neįgyjanti nulių, t. y. $f(s) \in H(K)$, $K \in \mathcal{K}$. [38] straipsnyje buvo gauta, kad funkcija $F(\zeta(s))$ yra universali su daug bendresniais operatoriais $F : H(D) \to H(D)$ iš Lipšico klasės. Tiksliau, operatorius $F$ tenkina sąlygas:

1. Su kiekvienu polinomu $p = p(s)$ ir kiekviena aibe $K \in \mathcal{K}$, egzistuoja elementas $q \in F^{-1}\{p\} \subset H(D)$, kad $q(s) \neq 0$ aibėje $K$;

2. Su kiekvienu $K \in \mathcal{K}$, egzistuoja konstanta $c > 0$, aibė $K_1 \in \mathcal{K}$, $\alpha > 0$ ir visiems $g_1, g_2 \in H(D)$ galioja nelygybė

$$\sup_{s\in K} |F(g_1(s)) - F(g_2(s))| \leqslant c \sup_{s\in K_1} |g_1(s) - g_2(s)|^\alpha.$$

Šią operatorių klasę pažymėkime $Lip(\alpha)$. Tuomet teisingas tvirtinimas, žr. [38].

**10 teorema.** *Tarkime, kad $F \in Lip(\alpha)$. Tegul $K \in \mathcal{K}$, o $f(s) \in H(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |f(s) - F(\zeta(s + i\tau))| < \varepsilon \right\} > 0. \quad \text{(S3)}$$

Ketvirtame disertacijos skyriuje įrodytos universalumo teoremos sudėtinėms funkcijoms $F(\zeta(s; \mathfrak{a}_1), \ldots \zeta(s; \mathfrak{a}_r))$, čia $F : H^r(D) \to H(D)$, naudojant postūmius $F(\zeta(s + ih_1\gamma_k; \mathfrak{a}_1), \ldots, \zeta(s + ih_r\gamma_k; \mathfrak{a}_r))$. Tegul, dėl trumpumo, $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$, $\underline{g} = (g_1, \ldots, g_r) \in H^r(D)$, $\underline{h} = (h_1, \ldots, h_r)$, $h_j > 0$, $j = 1, \ldots, r$, $\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) = (\zeta(s + ih_1\gamma_k; \mathfrak{a}_1), \ldots, \zeta(s + ih_r\gamma_k; \mathfrak{a}_r))$. Sakysime, kad galioja tvirtinimas $A(\underline{\mathfrak{a}}, \underline{h}, \text{(S2)})$, jeigu sekos $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ yra multiplikatyvios, $h_1, \ldots, h_r$ yra teigiami algebriniai skaičiai, tiesiškai nepriklausomi virš $\mathbb{Q}$, ir galioja (S2) įvertis.

Apibrėžkime operatorių $F : H^r(D) \to H(D)$ klasę $Lip(\underline{\alpha})$. Sakysime, kad $F \in Lip(\underline{\alpha})$, jeigu:

1. Su kiekvienu polinomu $p = p(s)$ ir aibėmis $K_1, \dots, K_r \in \mathcal{K}$, egzistuoja elementas $\underline{g} \in F^{-1}\{p\} \subset H^r(D)$, kad $g_j(s) \neq 0$ aibėse $K_j$, $j = 1, \dots, r$.

2. Su kiekviena $K \subset \mathcal{K}$, egzistuoja aibės $K_1, \dots, K_r \in \mathcal{K}$, konstanta $c > 0$ ir teigiami skaičiai $\alpha_1, \dots, \alpha_r$, kad

$$\sup_{s \in K} \left| F(\underline{g}_1) - F(\underline{g}_2) \right| \leqslant c \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_{1j}(s) - g_{2j}(s)|^{\alpha_j}$$

su visais $\underline{g}_1, \underline{g}_2 \in H^r(D)$.

Pirmoji universalumo teorema sudėtinei funkcijai $F(\underline{\zeta}(s; \underline{\mathfrak{a}}))$ yra toks tvirtinimas:

**4.1 teorema.** *Tarkime, kad galioja tvirtinimas $A(\underline{\mathfrak{a}}, \underline{h}, \text{(S2)})$, o operatorius $F : H^r(D) \to H(D)$ yra iš klasės $Lip(\underline{\alpha})$. Tegul $K \in \mathcal{K}$, o $f(s) \in H(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} \left| f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\right) \right| < \varepsilon \right\} > 0. \tag{S4}$$

Disertacijoje įrodytos universalumo teoremos ir kitoms operatorių $F$ klasėms. Tegul

$$S = \{g \in H(D) : g(s) \neq 0 \text{ juostoje } D, \text{ arba } g(s) \equiv 0\}.$$

**4.2 teorema.** *Tarkime, kad galioja tvirtinimas $A(\underline{\mathfrak{a}}, \underline{h}, \text{(S2)})$, o $F : H^r(D) \to H(D)$ yra toks tolydusis operatorius, kad su kiekviena atviraja aibe $G \subset H(D)$ sankirta $(F^{-1}G) \cap S^r$ nėra tuščioji aibė. Tegul $K \in \mathcal{K}$, o $f(s) \in H(K)$. Tuomet galioja* (S4) *nelygybė. Be to, riba*

$$\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} \left| f(s) - F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\right) \right| < \varepsilon \right\} \tag{S5}$$

*egzistuoja ir yra teigiama su visais $\varepsilon > 0$, nebent išskyrus skaičiąją $\varepsilon > 0$ reikšmių aibę.*

Sąlyga, kad $(F^{-1}G) \cap S^r$, gali būti pakeista.

**4.3 teorema.** *Tarkime, kad galioja tvirtinimas $A(\underline{\mathfrak{a}}, \underline{h}, \text{(S2)})$, o $F : H^r(D) \to H(D)$ yra toks tolydusis operatorius, kad su kiekvienu polinomu $p = p(s)$ sankirta $(F^{-1}\{p\}) \cap S^r$ nėra tuščioji aibė. Tegul $K \in \mathcal{K}$ ir*

$f(s) \in H(K)$. *Tuomet galioja* (S4) *nelygybė, o riba* (S5) *egzistuoja ir yra teigiama su visais* $\varepsilon > 0$*, nebent išskyrus skaičiąją* $\varepsilon > 0$ *reikšmių aibę.*

Akivaizdu, kad $G \subset H(D)$ sąlyga $(F^{-1}\{p\}) \cap S^r \neq \varnothing$ yra daug patogesnė, negu $(F^{-1}G) \cap S^r \neq \varnothing$, su visomis atvirosiomis aibėmis.

Kai kurioms aproksimuojamų funkcijų klasėms aibė $K \in \mathcal{K}$ gali būti pakeista bet kokia kompaktine aibe. Tai pateikta kitoje teoremoje.

**4.4 teorema.** *Tarkime, kad galioja tvirtinimas* $A(\underline{\mathfrak{a}}, \underline{h}, \text{(S2)})$*, o* $F : H^r(D) \to H(D)$ *yra tolydusis operatorius. Tegul* $K \subset D$ *yra kompaktinė aibė, o* $f(s) \in F(S^r)$*. Tuomet galioja 4.3 teoremos tvirtinimas.*

Atkreipsime dėmesį, kad aibę $F(S^r)$ sunku apibūdinti. Uždavinys tampa paprastesniu, kai yra žinoma tam tikra paprasta aibė, kuri yra aibės $F(S^r)$ poaibis. Tai yra pateikta paskutinėje 4 skyriaus teoremoje. Su skirtingais kompleksiniais skaičiais $c_1, \ldots, c_m$ apibrėžkime aibę

$$H_{c_1,\ldots,c_m}(D) = \{g \in H(D) : g(s) \neq c_j \text{ su visais } s \in D,\ j = 1, \ldots, m\}.$$

**4.5 teorema.** *Tarkime, kad galioja tvirtinimas* $A(\underline{\mathfrak{a}}, \underline{h}, \text{(S2)})$*, o* $F : H^r(D) \to H(D)$ *yra toks tolydusis operatorius, kad* $H_{c_1,\ldots,c_m}(D) \subset F(S^r)$*. Kai* $m = 1$*, tegul* $K \subset \mathcal{K}$*,* $f(s) \in H(K)$ *ir* $f(s) - c_1 \in H_0(K)$*. Kai* $m \geqslant 2$*, tegul* $K \subset D$ *yra bet kokia kompaktinė aibė, o* $f(s) \in H_{c_1,\ldots,c_m}(D)$*. Tuomet galioja 4.3 teoremos tvirtinimas.*

Visos anksčiau paminėtos disertacijoje įrodytos teoremos yra diskrečiojo tipo. Penktas disertacijos skyrius yra skirtas periodinių dzeta funkcijų tolydžiajam jungtiniam universalumui įrodyti. Tam yra naudojami tolydūs apibendrintieji postūmiai. Šio tipo teoremos buvo suformuluotos [42] straipsnyje, tačiau su paprastaisiais postūmiais. Kai $j = 1, \ldots, r$, tegul $\mathfrak{a}_j = \{a_{jm} : m \in \mathbb{N}\}$ yra periodinės kompleksinių skaičių sekos su minimaliuoju periodu $q_j \in \mathbb{N}$. Tegul $q$ yra periodų $q_1, \ldots, q_r$ mažiausias bendrasis kartotinis, o $l_1, \ldots, l_{r_1}$ – redukuotoji liekanų sistema moduliu $q$, čia $r_1 = \varphi(q)$ yra Oilerio funkcija. Apibrėžkime matricą

$$A = \begin{pmatrix} a_{1l_1} & a_{2l_1} & \ldots & a_{rl_1} \\ a_{1l_2} & a_{2l_2} & \ldots & a_{rl_2} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1l_{r_1}} & a_{2l_{r_1}} & \ldots & a_{rl_{r_1}} \end{pmatrix}.$$

Tuomet [42] straipsnyje yra įrodyta tokia teorema:

**11 teorema.** *Tarkime, kad sekos* $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ *yra multiplikatyvios, o rangas* $\operatorname{rank}(A) = r$. *Kai* $j = 1, \ldots, r$, *tegul* $K_j \in \mathcal{K}$, *o* $f_j(s) \in H_0(K_j)$. *Tuomet su kiekvienu* $\varepsilon > 0$

$$\liminf_{T\to\infty} \frac{1}{T}\operatorname{meas}\left\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s + i\tau; \mathfrak{a}_j)| < \varepsilon\right\} > 0.$$

Pastebėsime, kad [42] straipsnyje naudojama sąlyga

$$\sum_{k=1}^{\infty} \frac{|a_{jp^k}|}{p^{k/2}} \leqslant c_j < 1, \quad j = 1, \ldots, r,$$

gali būti lengvai pašalinama.

Disertacijoje 11 teoremos sąlyga $\operatorname{rank}(A) = r$ yra pakeista netiesiniais postūmiais $\zeta(s + i\varphi_j(\tau); \mathfrak{a}_j)$.

Pažymėkime $U_1(T_0)$, $T_0 > 0$, realiųjų, neaprėžtai didėjančių, tolydžiai diferencijuojamų funkcijų $\varphi(\tau)$ klasę su intervale $[T_0, \infty)$ turinčią monotoninę išvestinę $\varphi'(\tau)$, kad

$$\varphi(2\tau) \max_{\tau\leqslant u\leqslant 2\tau} \frac{1}{\varphi'(u)} \ll \tau, \quad \tau \to \infty.$$

Dabar suformuluosime pirmąją 5 skyriaus teoremą.

**5.1 teorema.** *Tarkime, kad sekos* $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ *yra multiplikatyvios,* $a_1, \ldots, a_r$ *yra realieji algebriniai skaičiai tiesiškai nepriklausomi virš* $\mathbb{Q}$, *o* $\varphi(\tau) \in U_1(T_0)$. *Kai* $j = 1, \ldots, r$, *tegul* $K_j \in \mathcal{K}$, *o* $f_j(s) \in H_0(K_j)$. *Tuomet su kiekvienu* $\varepsilon > 0$

$$\liminf_{T\to\infty} \frac{1}{T - T_0}\operatorname{meas}\left\{\tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s + ia_j\varphi(\tau); \mathfrak{a}_j)| < \varepsilon\right\} > 0.$$

*Be to, riba*

$$\lim_{T\to\infty} \frac{1}{T - T_0}\operatorname{meas}\left\{\tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - \zeta(s + ia_j\varphi(\tau); \mathfrak{a}_j)| < \varepsilon\right\}$$

*egzistuoja ir yra teigiama su visais $\varepsilon > 0$, nebent išskyrus skaičiąją $\varepsilon > 0$ reikšmių aibę.*

Antrojoje jungtinio universalumo teoremoje naudojami daug bendresni postūmiai $\zeta(s + i\varphi_j(\tau); \mathfrak{a}_j)$, $j = 1, \dots, r$, negu 5.1 teoremoje. Pažymėkime $U_r(T_0)$ realiųjų, neaprėžtai didėjančių, tolydžiai diferencijuojamų funkcijų $\varphi_1(\tau), \dots, \varphi_r(\tau)$ klasę su intervale $[T_0, \infty)$ turinčių išvestinę, kad

$$\varphi_j'(\tau) = \hat{\varphi}_j(\tau)(1 + o(1)),$$

čia $\hat{\varphi}_1(\tau), \dots, \hat{\varphi}_r(\tau)$ yra monotoninės funkcijos ir susietos ta prasme, kad kiekvienam poaibiui $J \subset \{1, \dots, r\}$, $\#J \geqslant 2$, egzistuoja $j_0 = j_0(J)$ toks, kad $\hat{\varphi}_j(\tau) = o(\hat{\varphi}_{j_0}(\tau))$, kai $j \in J$, $j \neq j_0$, ir

$$\varphi_j(2\tau) \max_{\tau \leqslant u \leqslant 2\tau} \frac{1}{\hat{\varphi}_j(u)} \ll \tau, \quad j = 1, \dots, r,$$

kai $\tau \to \infty$.

**5.2 teorema.** *Tarkime, kad sekos $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ yra multiplikatyvios, o $(\varphi_1(\tau), \dots, \varphi_r(\tau)) \in U_r(T_0)$. Kai $j = 1, \dots, r$, tegul $K_j \in \mathcal{K}$, o $f_j(s) \in H_0(K_j)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T \to \infty} \frac{1}{T - T_0} \operatorname{meas}\left\{ \tau \in [T_0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |f_j(s) - \zeta(s + i\varphi_j(\tau); \mathfrak{a}_j)| < \varepsilon \right\} > 0.$$

*Be to, riba*

$$\lim_{T \to \infty} \frac{1}{T - T_0} \operatorname{meas}\left\{ \tau \in [T_0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |f_j(s) - \zeta(s + i\varphi_j(\tau); \mathfrak{a}_j)| < \varepsilon \right\}$$

*egzistuoja ir yra teigiama su visais $\varepsilon > 0$, nebent išskyrus skaičiąją $\varepsilon > 0$ reikšmių aibę.*

# Aprobacija

Pagrindiniai disertacijos rezultatai buvo pristatyti šiose konferencijose:

- LXI Lietuvos matematikų draugijos konferencijoje, 2020 m. gruodžio 4 d., Šiauliuose;
- XIX tarptautinėje konferencijoje „Algebra, skaičių teorija, diskrečioji geometrija ir daugiamatis modeliavimas: šiuolaikinės problemos, taikymai ir problemų istorija", skirta akademiko P. L. Čebyševo 200 m. jubiliejui, 2021 m. gegužės 18–22 dienomis, Tuloje, Rusija;
- LXII Lietuvos matematikų draugijos konferencijoje, 2021 m. birželio 16–17 dienomis, Vilniuje;
- XX tarptautinėje konferencijoje „Algebra, skaičių teorija, diskrečioji geometrija ir daugiamatis modeliavimas: šiuolaikinės problemos, taikymai ir problemų istorija", skirta akademiko I. M. Vinogradovo 130 m. jubiliejui, 2021 m. rugsėjo 21–24 dienomis, Tuloje, Rusija;
- 25-tojoje tarptautinėje konferencijoje „Matematinis modeliavimas ir analizė", 2022 m. gegužės 30 d. – birželio 2 d., Druskininkuose;
- LXIII Lietuvos matematikų draugijos konferencijoje, 2022 m. birželio 16–17 dienomis, Kaune;
- Lietuvos tikimybių teorijos ir skaičių teorijos konferencijoje, skirtoje Lietuvos universiteto 100-mečiui, Matematinės analizės katedros 100-mečiui ir Geometrijos katedros 100-mečiui, 2022 m. rugsėjo 5–10 dienomis, Palangoje;
- 26-tojoje tarptautinėje konferencijoje „Matematinis modeliavimas ir analizė", 2023 m. gegužės 30 d. – birželio 2 d., Jūrmaloje, Latvija;
- LXIV Lietuvos matematikų draugijos konferencijoje, 2023 m. birželio 21–22 dienomis, Vilniuje;
- Tarptautinėje tikimybių teorijos ir skaičių teorijos konferencijoje, skirtoje profesoriaus Jono Kubiliaus 100 m. jubiliejui, profesoriaus Donato Surgailio 80 m. jubiliejui, profesorių Antano Laurinčiko ir Eugenijaus Manstavičiaus 75 m. jubiliejui, profesorių Kęstučio Kubiliaus ir Alfredo

Račkausko 70 m. jubiliejui paminėti, 2023 m. rugsėjo 10–16 dienomis, Palangoje;

- 27-tojoje tarptautinėje konferencijoje „Matematinis modeliavimas ir analizė", 2024 m. gegužės 28–31 dienomis, Parnu, Estija;
- LXV Lietuvos matematikų draugijos konferencijoje, 2024 m. birželio 27–28 dienomis, Šiauliuose;
- Tarptautinėje tikimybių teorijos ir skaičių teorijos konferencijoje, skirtoje profesoriaus Vyganto Paulausko 80 m. jubiliejui, profesorių Konstantino Pilecko ir Gedimino Stepanausko 70 m. jubiliejui, profesorių Artūro Dubicko ir Olgos Štikonienės 60 m. jubiliejui, 2024 m. rugsėjo 16–20 dienomis, Palangoje.

## Pagrindinės publikacijos

Disertacijos rezultatai buvo paskelbti šiuose straipsniuose:

1. A. Laurinčikas, M. Tekorė, Joint universality of periodic zeta-functions with multiplicative coefficients, *Nonlinear Anal.: Modell. Control* **25** (2020), 860–883;
2. A. Laurinčikas, D. Šiaučiūnas, M. Tekorė, Joint universality of periodic zeta-functions with multiplicative coefficients. II, *Nonlinear Anal.: Modell. Control* **26** (2021), 550–564;
3. M. Tekorė, On joint universality of periodic zeta-functions with multiplicative coefficients, *Proceedings of the XIX International Conference "Algebra, number theory, discrete geometry and multiscale modeling: contemporary problems, applications and problems of history" dedicated to the bicentennial of the birth of Academician P.L. Chebyshev*, May 18 – 22, Tula, L. N. Tolstoi Tula State Pedagogical University, Tula, 2021, pp. 181–184;
4. D. Šiaučiūnas, R. Šimėnas and M. Tekorė, Approximation of analytic functions by shifts of certain compositions, *Mathematics* **9** (2021), 2583;
5. D. Šiaučiūnas and M. Tekorė, Gram points in the universality of the Dirichlet series with periodic coefficients, *Mathematics* **11** (2023), 4615;

Konferencijų tezės:

1. D. Šiaučiūnas and M. Tekorė, On joint universality of Dirichlet L-functions, 25th international conference on Mathematical Modelling and Analysis, May 30 – June 2, 2022, Druskininkai, Lithuania, abstracts, Vilnius Gediminas Technical University, Vilnius, 2022, pp. 20.

2. V. Garbaliauskienė and M. Tekorė, Universality of certain compositions, 26th international conference on Mathematical Modelling and Analysis, May 30 – June 2, 2023, Jurmala, Latvia, abstracts, University of Latvia, Riga, 2023, pp. 14.

3. M. Tekorė and D. Šiaučiūnas, On universality of periodic zeta-functions, 26th international conference on Mathematical Modelling and Analysis, May 30 – June 2, 2023, Jurmala, Latvia, abstracts, University of Latvia, Riga, 2023, pp. 65.

4. M. Tekorė, Jungtinis periodinių dzeta funkcijų universalumas, Lietuvos matematikų draugijos LXIV konferencijos santraukos, birželio 21–22, Vilnius, Vilniaus universitetas, Vilnius, 2023, pp. 18.

5. M. Tekorė, Approximation of analytic functions by shifts of the periodic zeta-functions, 27th international conference on Mathematical Modelling and Analysis, May 28–31, 2024, Pärnu, Estonia, abstracts, University of Tartu, Tartu, 2024, pp. 67.

6. M. Tekorė, Gramo taškai periodinės dzeta funkcijos universalume, Lietuvos matematikų draugijos LXV konferencijos santraukos, birželio 27–28, Šiauliai, Vilniaus universitetas, Šiauliai, 2024, pp. 33.

# Išvados

Iš disertacijos išplaukia tokios išvados:

1. Periodinei dzeta funkcijai $\zeta(s;\mathfrak{a})$ su multiplikatyviaja periodine seka $\mathfrak{a}$ galioja universalumo teorema apie analizinių funkcijų aproksimavimą postūmiais $\zeta(s+iht_k;\mathfrak{a})$, čia $h>0$, o $\{t_k\}$ – Gramo taškų seka;

2. Periodinėms dzeta funkcijoms $\zeta(s;\mathfrak{a}_j)$ su multiplikatyviosiomis periodinėmis sekomis $\mathfrak{a}_j$, $j=1,\dots,r$, esant teisingai silpnajai Montgomerio prielaidai apie Rymano dzeta funkcijos netrivialiųjų nulių menamųjų

dalių $\gamma_k$ koreliaciją, galioja jungtinio universalumo teorema apie analizinių funkcijų rinkinio aproksimavimą postūmiais $\zeta(s + ih_j\gamma_k; \mathfrak{a}_j)$, čia $h_1, \ldots, h_r$ yra teigiami algebriniai skaičiai tiesiškai nepriklausomi virš racionaliųjų skaičių kūno;

3. Daugiamatėje analizinių funkcijų erdvėje sudėtinės funkcijos $F(\zeta(s + ih_1\gamma_k; \mathfrak{a}_1), \ldots \zeta(s + ih_r\gamma_k; \mathfrak{a}_r))$ yra universalios su tam tikromis operatorių $F$ klasėmis;

4. Periodinėms dzeta funkcijoms $\zeta(s; \mathfrak{a}_j)$ su multiplikatyviosiomis periodinėmis sekomis $\mathfrak{a}_j$, $j = 1, \ldots, r$, nenaudojant jokio sekos $\mathfrak{a}_j$ nepriklausomumo reikalavimo, galioja jungtinio universalumo teoremos apie analizinių funkcijų rinkinio aproksimavimą apibendrintais netiesiniais postūmiais $\zeta(s + i\varphi_j(\tau); \mathfrak{a}_j)$.

# Trumpos žinios apie autorę

**Gimimo data ir vieta:**

1987 m. rugpjūčio 13 d. Raudėnai, Šiaulių rajonas, Lietuva.

**Išsilavinimas:**

2006 m. Šiaulių rajono Kuršėnų Pavenčių vidurinė mokykla. Vidurinis išsilavinimas.

2010 m. Šiaulių universitetas. Matematikos bakalauro laipsnis.

2020 m. Šiaulių universitetas. Matematikos magistro laipsnis.

**Darbo patirtis:**

2008 m. Genijaus Vitkausko įmonė „Hobis", pardavėja-kasininkė.

2013 m. AB „Lietuvos paštas", klientų aptarnavimo specialistė (pašto kasininkė).

2015–2017 m. AB „Telia Lietuva", klientų aptarnavimo vadybininkė.

2019–2020 m. Aviva Lietuva, finansų konsultantė.

2019–2020 m. UAB „Studio Modema", pardavimų telefonu vadybininkė.

2021 m. AB „Telia Lietuva", klientų aptarnavimo vadybininkė.

nuo 2021 m. AB MV Group Production Gubernija, gamybos analitikė.

# Acknowledgments

I would like to thank the following people, without whom I would not have been able to complete this PhD dissertation.

First of all, I would like to thank my scientific supervisor prof. Antanas Laurinčikas for the thoughtful comments and recommendations, for support and patience, for wise guidance during my scientific work. I am honored to be a student of such a brilliant professor.

I am very grateful to my PhD study consultant prof. Darius Šiaučiūnas for his time, support, extra work, and encouragement to go the extra step, without his help, this dissertation would have not been the same.

I am very thankful to all Professors I had the honor to meet for their willingness to impart their knowledge and inspiration by their work.

To conclude, I cannot forget to thank my family and friends for all their unconditional support in this not easy academic travel. For my daughters: sorry for the time I didn‘t spend with you while writing my dissertation.

# 1st publication

## Joint universality of periodic zeta-functions with multiplicative coefficients

A. Laurinčikas, **M. Tekorė**

*Nonlinear Anal.: Modell. Control* **25** (2020), 860–883.

Link to the publication:
https://www.journals.vu.lt/nonlinear-analysis/article/view/19278/18444

# 2nd publication

# Joint universality of periodic zeta-functions with multiplicative coefficients. II

A. Laurinčikas, D. Šiaučiūnas, **M. Tekorė**

*Nonlinear Anal.: Modell. Control* **26** (2021), 550–564.

Link to the publication:
https://www.journals.vu.lt/nonlinear-analysis/article/view/23934/23238

# 3rd publication

# On joint universality of periodic zeta-functions with multiplicative coefficients

**M. Tekorė**

*Proceedings of the XIX International Conference "Algebra, number theory, discrete geometry and multiscale modeling: contemporary problems, applications and problems of history" dedicated to the bicentennial of the birth of Academician P.L. Chebyshev*, May 18–22, Tula, L. N. Tolstoi Tula State Pedagogical University, Tula, 2021, pp. 181–184.

Link to the publication:
https://poivs.tsput.ru/conf/international/XIX/files/Conference2021.pdf

# 4th publication

# Approximation of analytic functions by shifts of certain compositions

D. Šiaučiūnas, R. Šimėnas, **M. Tekorė**

*Mathematics* **9** (2021), 2583.

Link to the publication: https://www.mdpi.com/2227-7390/9/20/2583

# 5th publication

# Gram points in the universality of the Dirichlet series with periodic coefficients

D. Šiaučiūnas, **M. Tekorė**

*Mathematics* **11** (2023), 4615.

Link to the publication: https://www.mdpi.com/2227-7390/11/22/4615

# NOTES

# NOTES

Vilnius University Press
9 Saulėtekio Ave., Building III, LT-10222 Vilnius
Email: info@leidykla.vu.lt, www.leidykla.vu.lt
bookshop.vu.lt, journals.vu.lt
Print run 20