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Approximation of Analytic Functions by Absolutely Convergent Dirichlet Series

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Analizinių funkcijų aproksimavimas absoliučiai konverguojančiomis Dirichlė eilutėmis

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Notation

j, k, l, m, n	natural numbers
p	prime number
$\mathbb{P}$	set of all prime numbers
$\mathbb{N}$	set of all natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	set of all integer numbers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
i	imaginary unit: $i = \sqrt{-1}$
$s = \sigma + it, \sigma, t \in \mathbb{R}$	complex variable
$A \times B$	Cartesian product of the sets A and B
$\prod\limits_{m} A_m$	Cartesian product of sets A_m
A^m	Cartesian product of m copies of the set A
$\text{meas} A$	Lebesgue measure of the set $A \subset \mathbb{R}$
$\# A$	cardinality of the set A
$H(G)$	space of analytic functions on G
$\mathcal{B}(\mathbb{X})$	class of Borel sets of the space $\mathbb{X}$
$\xrightarrow{\mathcal{D}}$	convergence in distribution
$\Gamma(s)$	Euler gamma-function
$L(s, \chi)$	Dirichlet L function
$\zeta(s)$	Riemann zeta-function
$\zeta(s, \alpha)$	Hurwitz zeta-function
$\zeta(s, \mathfrak{a})$	periodic zeta-function
$\zeta(s, \alpha; \mathfrak{b})$	periodic Hurwitz zeta-function
(m, k)	largest common divisor of m and k
$a \ll b$	$a = O(b)$

$f(x) \sim g(x), x \to a$	$\lim_{x\to a} \frac{f(x)}{g(x)} = 1$
$a \ll_\eta b, b > 0$	there exists a constant $C = C(\eta) > 0$ such that $\lvert a\rvert \leqslant Cb$
$\exp\{s\}$	e^s

Chapter 1

Introduction

1.1 Research topic

The dissertation is devoted to approximation problems by certain Dirichlet series. Let $\{a_m : m \in \mathbb{N}\} \subset \mathbb{C}$ and $s = \sigma + i\tau$. We recall that an ordinary Dirichlet series is of the form

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

while a general Dirichlet series is

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s},$$

where $\{\lambda_m : m \in \mathbb{N}\} \subset \mathbb{R}$ is an increasing sequence, $\lim\limits_{m\to\infty} \lambda_m = +\infty$. The convergence region and absolute convergence region of all Dirichlet series are half-planes $\{s \in \mathbb{C} : \sigma > \sigma_0\}$ and $\{s \in \mathbb{C} : \sigma > \sigma_a\}$, respectively. Obviously, $\sigma_a \geqslant \sigma_0$.

Dirichlet series is an object of general analysis, however, they are also used in analytic number theory because the so-called zeta-functions are defined by Dirichlet series. For example, the famous Riemann zeta-function $\zeta(s)$ is given, for $\sigma > 1$, by an ordinary Dirichlet series with coefficients $a_m \equiv 1$, i.e.,

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.$$

The function $\zeta(s)$ can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. Note that the function $\zeta(s)$ is the main tool for investigation of the distribution of prime numbers in the set $\mathbb{N}$. Namely, using the function $\zeta(s)$, it was obtained [16], [54] that

$$\sum_{p\leqslant x} 1 \sim \int\limits_2^x \frac{du}{\log u}, x \to \infty.$$

The function $\zeta(s)$ also can be defined as an infinite product over prime numbers p

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \sigma > 1,$$

and this shows its link with distribution of prime numbers.

Let the function $\chi : \mathbb{N} \to \mathbb{C}$ have the following properties:

1° $\chi(m)$ is completely multiplicative, i.e., $\chi(m_1 m_2) = \chi(m_1)\chi(m_2)$ for all $m_1, m_2 \in \mathbb{N},$ and $\chi(1) = 1$;

2° $\chi(m)$ is periodic with a minimal period $q \in \mathbb{N}$, i.e., $\chi(m + q) = \chi(m)$ for all $m \in \mathbb{N}$;

3° $\chi(m) = 0$ for $(m, q) > 1$;

4° $\chi(m) \neq 0$ for $(m, q) = 1$.

Then $\chi(m)$ coincides with one of Dirichlet character modulo q. The Dirichlet $L-$function $L(s, \chi)$ with character χ, for $\sigma > 1$, is defined by the Dirichlet series

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

and has the meromorphic continuation to the whole complex plane with a possible simple pole at the point $s = 1$ (if the character χ is principal, i.e., $\chi(m) \equiv 1$ for all $(m, q) = 1$). Dirichlet $L-$functions were introduced [11] for investigations of prime numbers in arithmetic progressions $\{an + b : n \in \mathbb{N}\}, (a, b) = 1$. It is proved that

$$\sum_{\substack{p\leqslant x \\ p\equiv b \bmod a}} 1 \sim \frac{1}{\varphi(a)} \int_2^x \frac{du}{\log u}, x \to \infty,$$

where $\varphi(k) = \#\{1 \leqslant m \leqslant k : (m, k) = 1\}$ is the Euler totient function.

Let $0 < \alpha \leqslant 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$

[17] is an example of general Dirichlet series. For $\sigma > 1, \zeta(s, \alpha)$ is defined by

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s} = \sum_{m=0}^{\infty} e^{-s \log(m+\alpha)},$$

in this case, we have $\lambda_m = \log(m+\alpha)$. If $\alpha = 1$, then $\zeta(s, \alpha)$ becomes the Riemann zeta-function $\zeta(s)$. The function $\zeta(s, \alpha)$, as $\zeta(s)$, is meromorphic, it has a simple pole at the point $s = 1$ with residue 1.

In the dissertation, we consider Dirichlet series in connection to the so-called periodic zeta-functions. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0\}$ be two periodic sequences of complex numbers, with periods $q_1 \in \mathbb{N}$ and $q_2 \in \mathbb{N}$, respectively.

The periodic zeta-function $\zeta(s; \mathfrak{a})$ and periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{b})$ are defined, for $\sigma > 1$, by Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}$$

and

$$\zeta(s, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}.$$

The periodicity of the sequences $\mathfrak{a}$ and $\mathfrak{b}$ implies the equalities

$$\zeta(s; \mathfrak{a}) = \frac{1}{q_1^s} \sum_{k=1}^{q_1} a_k \zeta\Big(s, \frac{k}{q}\Big), \sigma > 1,$$

and

$$\zeta(s, \alpha; \mathfrak{b}) = \frac{1}{q_2^s} \sum_{k=0}^{q_2-1} b_k \zeta\Big(s, \frac{k+\alpha}{q_2}\Big), \sigma > 1.$$

Hence, in view of the mentioned above properties of the Hurwitz zeta-function, the functions $\zeta(s; \mathfrak{a})$ and $\zeta(s, \alpha; \mathfrak{b})$ have the analytic continuation to the whole complex plane, except for the point $s = 1$ that is a simple pole with residues

$$\frac{1}{q_1} \sum_{k=1}^{q_1} a_k$$

and
$$\frac{1}{q_2}\sum_{k=0}^{q_2-1} b_k,$$
respectively.

Let $\theta > \frac{1}{2}$ be a fixed number, and for $u > 0$,
$$v_u(m) = \exp\left\{-\left(\frac{m}{u}\right)^{\theta}\right\}, m \in \mathbb{N},$$
and
$$v_u(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{u}\right)^{\theta}\right\}, m \in \mathbb{N}_0.$$
In the dissertation, we consider the Dirichlet series
$$\zeta_u(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_u(m)}{m^s}$$
and
$$\zeta_u(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m v_u(m,\alpha)}{(m+\alpha)^s},$$
where, as above, $\mathfrak{a}$ and $\mathfrak{b}$ are periodic sequences of complex numbers. Since $v_u(m)$ and $v_u(m,\alpha)$ are decreasing with respect to m exponentially, the series for $\zeta_u(s;\mathfrak{a})$ and $\zeta_u(s,\alpha;\mathfrak{b})$ are absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0 for every $u > 0$. The dissertation is devoted to approximation of analytic functions by shifts $\zeta_{u_T}(s+i\tau;\mathfrak{a})$ and $\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b})$, $\tau \in \mathbb{R}$, with a certain function $u_T \to \infty$ as $T \to \infty$.

1.2 Aims and problems

The aims of the dissertation are approximation of analytic functions by shifts of absolutely convergent Dirichlet series $\zeta_{u_T}(s;\mathfrak{a})$ and $\zeta_{u_T}(s,\alpha;\mathfrak{b})$. The considered problems are the following:

1. Approximation of a class of analytic functions by continuous shifts $\zeta_{u_T}(s+i\tau;\mathfrak{a}), \tau \in \mathbb{R}$, with multiplicative sequence $\mathfrak{a}$.
2. Approximation of a class of analytic functions by discrete shifts $\zeta_{u_N}(s+ikh;\mathfrak{a}), h > 0, k \in \mathbb{N}_0$, with multiplicative sequence $\mathfrak{a}$.

3. Joint approximation of a class of pairs of analytic functions by continuous shifts $\Big(\zeta_{u_T}(s+i\tau;\mathfrak{a}), \zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b}\Big), \tau\in\mathbb{R}$, with multiplicative sequence $\mathfrak{a}$.

4. Joint approximation of a class of pairs of analytic functions by discrete shifts $\Big(\zeta_{u_N}(s+ikh_1;\mathfrak{a}), \zeta_{u_T}(s+ikh_2,\alpha;\mathfrak{b}\Big), h_1>0, h_2>0, k\in\mathbb{N}_0$, with multiplicative sequence $\mathfrak{a}$.

1.3 Actuality

Approximation of analytic functions is cultivated and applied in many branches of mathematics and other natural sciences. Many mathematical problems involve complicated analytic functions, therefore, a problem arises to replace them by simpler ones. This leads to approximation problems. It is well known (Mergelyan's theorem) that every analytic function, under certain conditions on approximation region, can be approximated by polynomials. Thus, for each analytic function, there exists a polynomial approximating that function. About 1970 – 1980, the number theorists (S. M. Voronin, B. Bagchi, S. M. Gonek) found new analytic objects which approximate the whole classes of analytic functions. These objects are given by Dirichlet series and include the majority of zeta – and L – functions studied in analytic number theory. New approximation objects have an universality property, the shifts of one and the same object approximate a wide class of analytic functions. The discovery of universal approximation objects raised new problems connected to effectivization, joint aproximation, using of generalized shifts, etc. Therefore, the theory of universality of Dirichlet series, including the zeta-functions, continues its development. Many attention to universality problem of Dirichlet series is devoted in Lithuania. It is important to extend the class of universal functions and simplify their structure. Therefore, in the dissertation, approximation of analytic functions by shifts of absolutely convergent Dirichlet series is investigated. The simplicity of such series allows to describe easier approximated functions.

1.4 Methods

We use a mix of analytic and probabilistic methods. Analytic methods include the Dirichlet series theory, integration, residue theory, application of the Cauchy integral formula and Mergelyan theorem. Application of probabilistic methods consists of weak convergence of probability measures, convergence in distribution of random elements and properties of supports of probability measures.

1.5 Novelty

All results of the dissertation are new. Universality of absolutely convergent Dirichlet series connected to periodic zeta-functions are considered in the dissertation for the first time.

1.6 History of the problem and the main results

Let $\{a(m) : m \in \mathbb{N}\}$ be a certain sequence of complex numbers. Suppose that we know a rule for calculation the numbers $a(m)$. However, often, the set of values $a(m)$ is chaotic and gives a little information on the whole sequence. Therefore, in place of individual values $a(m)$, the mean value

$$M(x) \stackrel{\text{def}}{=} \sum_{m \leqslant x} a(m), x \to \infty,$$

is considered. For investigation of $M(x)$, the generated Dirichlet series

$$Z(s) \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \sigma > \sigma_0,$$

can be applied. There is a formula (called Perron's formula) that connects $Z(s)$ and $M(x)$, i.e.,

$$M(x) = \text{integral operator involving } Z(s) + \text{error term.}$$

For example, suppose that the series for $f(s)$ is absolutely convergent for $\sigma > 1$, $|a(m)| \leqslant b(m)$ with monotonically increasing $b(m)$, and for $\sigma \to 1+0$,

$$\sum_{m=1}^{\infty} |a(m)| m^{-\sigma} = O((\sigma - 1)^{-\alpha}), \alpha > 0.$$

Then, for all $a_0 \geqslant a > 1$, $T \geqslant 1$ and $x = n + \frac{1}{2}$ the formula

$$M(x) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{x^a}{T(a-1)^{\alpha}}\right) + O\left(\frac{xb(2x)\log x}{T}\right)$$

is valid [57]. Thus, the problem of the behavior for $M(x)$ is reduced to analytic properties of the function $Z(s)$.

We present an example. Suppose that

$$d(m) = \sum_{d|m} 1$$

is the divisor function. The generating function of $d(m)$ is $\zeta^2(s)$, i.e.,

$$\zeta^2(s) = \sum_{m=1}^{\infty} \frac{d(m)}{m^s}, \ \sigma > 1.$$

Using properties of the function $\zeta(s)$ leads to the formula

$$\sum_{m \leqslant x} d(m) = x \log x + (2\hat{\gamma} - 1)x + O(x^{\theta}),$$

where

$$\hat{\gamma} = \lim_{n \to \infty} \Big(\sum_{m \leqslant n} \frac{1}{m} - \log m \Big) = 0.57721\ldots$$

is the Euler constant, and $\frac{1}{4} \leqslant \theta \leqslant \frac{1}{2}$. The greatest lower bound of the numbers θ is called the Dirichlet divisor problem. The last result belongs to M.Huxley (2003) and is equal to $\frac{131}{416} = 0.3149...$. This example shows the importance of the study of the Riemann zeta-function and other zeta-functions. By a work of H.Bohr and R.Courant [6], it is known that the set of values of the function $\zeta(s)$ is very rich, the set $\{\zeta(\sigma + it) : t \in \mathbb{R}\}$ with $\frac{1}{2} < \sigma \leqslant 1$ is dense everywhere in $\mathbb{C}$. This explains, in a certain sense, that the function $\zeta(s)$ appears in many problems not only arithmetic but also other branches of mathematics, and even some other natural sciences. For example, zeros of $\zeta(s)$ have relation to eigenvalues of some operators, $\zeta(s)$ has applications in cosmology [1] and even in music for tuning problems [50].

In the eight decade of the later century, one more important property of $\zeta(s)$ and other zeta-functions was found. It turned out that the shifts $\zeta(s + i\tau), \tau \in \mathbb{R}$, approximate a wide class of analytic functions. More precisely, S.M. Voronin proved [37], see also [26], the following property of $\zeta(s)$ and called it universality.

Theorem A. *Let $0 < r < \frac{1}{4}$ be a fixed number, $f(s)$ a continuos nonvanishing function on the disc $|s| \leqslant r$, and analytic in $|s| < r$. Then, for every $\varepsilon > 0$,* *there exists a number $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that*

$$\max_{|s|\leqslant r} \left|\zeta(s + \frac{3}{4} + i\tau) - f(s)\right| < \varepsilon.$$

Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. In this topology, $\{g_n(s)\} \subset H(D)$ converges to $g(s) \in H(D)$ as $n \to \infty$ if, for any compact set $K \subset D$,

$$\lim_{n\to\infty} \sup_{s\in K} |g_n(s) - g(s)| = 0.$$

The space $H(D)$ is infinite-dimensional, therefore, Theorem A is an infinite-dimensional version of the Bohr-Courant theorem [6].

The Voronin universality theorem was observed by members of the mathematical community. A new method for the proof of Theorem A was found, the assertion improved and extended for other zeta- and $L-$functions, for results, see [15], [16], [52], [28].

Universality of zeta-functions is widely studied in Japan, Poland, Germany, Canada, France, South Korea, Italy and other countries. New results appear every year. For example, the paper [42] on the universality of the general Dirichlet series was recently published. The last version of the Voronin theorem uses the following notation. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \subset \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Let $\mathrm{meas}A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the following statement is valid, see, for example, [16], [52], [28].

Theorem B. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right\} = 0.$$

Moreover, the lower limit can be replaced by the limit for all but at most countably many $\varepsilon > 0$.

Theorem B shows that there are infinitely many shifts $\zeta(s + i\tau)$ approximating with accuracy ε a given function $f(s) \in H_0(K)$. On the other hand, Theorem B is not effective in the sense that any approximating shift $\zeta(s + i\tau)$

is not known.

Partial solutions of effectivity for universality theorem can be found in [13] and [14], see also [12]. In these works, the effectively computed interval $[T_0, 2T_0]$ containing τ such that $\zeta(s+i\tau)$ is an approximating shift was indicated.

There exists the Linnik-Ibragimov conjecture, see, for example, [52], that all functions in some half-plane given by Dirichlet series, having analytic continuation to the left of this half-plane and satisfying some natural growth conditions are universal in the Voronin sense. However, until our days there are Dirichlet series whose universality is not proved. For example, the universality of periodic Hurwitz zeta-function with algebraic irrational parameter α is unknown.

Now, we recall some results on the function $\zeta(s;\mathfrak{a})$ with periodic sequence $\mathfrak{a}$ with minimal period $q_1 \in \mathbb{N}$. The value distribution of $\zeta(s;\mathfrak{a})$ is systematically considered in the monograph [52], Chapter 11. For example, we find there the functional equation

$$\zeta(1-s,\mathfrak{a}^{\pm}) = \left(\frac{q_1}{2\pi}\right)^s \frac{\Gamma(s)}{\sqrt{q_1}}\left(\exp\left\{\frac{\pi i s}{2}\right\}\zeta(s,\mathfrak{a}^{\pm}) + \exp\left\{-\frac{\pi i s}{2}\right\}\zeta(s,\mathfrak{a}^{\pm})\right)$$

for all $s \in \mathbb{C}$, where

$$\mathfrak{a}^{\pm} = \left\{a_m^{\pm} = \frac{1}{\sqrt{q_1}}\sum_{m=1}^{q_1} a_k \exp\left\{\pm 2\pi i m k\right\} : m \in \mathbb{N}\right\}.$$

The latter equation was first proved in [51].

In [40], the zero distribution of $\zeta(s,\mathfrak{a})$ also is described. It is obtained that there exists a constant $B(\mathfrak{a})$ such that, for $\sigma < -B(\mathfrak{a})$, the function $\zeta(s;\mathfrak{a})$ can only have zeros close to the negative real axis if $m_{\mathfrak{a}^+} = m_{\mathfrak{a}^-}$ with $m_{\mathfrak{a}_{\pm}} = \min\left\{m : 1 \leqslant m \leqslant q_1 : a_m^{\pm} \neq 0\right\}$, and close to the straight line

$$\sigma = 1 + \frac{\pi\tau}{\log\frac{m_{\mathfrak{a}^-}}{m_{\mathfrak{a}^+}}}$$

if $m_{a^+} \neq m_{a^-}$. Zeros $q = \beta + i\gamma$ of $\zeta(s,\mathfrak{a})$ are called trivial if $\beta < -B(\mathfrak{a})$. The non-trivial zeros lie in the strip $-B(\mathfrak{a}) \leqslant \sigma < 1 + A(\mathfrak{a})$ with some $A(\mathfrak{a}) > 0$.

Denote by $N(T;\mathfrak{a})$ the number of nontrivial zeros of $\zeta(s;\mathfrak{a})$ with $|\gamma| \leqslant T$.

Then it is proved in [52] that

$$N(T;\mathfrak{a}) = \frac{T}{\pi}\log\frac{q_1 T}{2\pi e m_{\mathfrak{a}}\sqrt{m_{\mathfrak{a}^-} m_{\mathfrak{a}^+}}} + O(\log T),$$

i.e., for the function $\zeta(s;\mathfrak{a})$ analogue of the von Mangoldt formula is valid.

The aim of the dissertation is approximation of analytic functions by shifts closely connected to $\zeta(s+i\tau;\mathfrak{a})$ and $\zeta(s+i\tau,\alpha;\mathfrak{b})$. Therefore, we recall some results on universality of $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$. The first result in this direction is contained in [52]. Denote by $H(K), K \in \mathcal{K}$, the class of continuous functions on K that are analytic in the interior of K. Thus, we have that $H_0(K) \subset H(K)$.

Theorem C. *Suppose that $q_1 > 2$, $\mathfrak{a}$ is not a multiple of a Dirichlet character modulo q_1, and $a_m = 0$ for $(m, q_1) > 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

Note that, in [40], it has been obtained that, under hypotheses of Theorem C, the sequence $\mathfrak{a}$ is not multiplicative. We recall that the sequence $\mathfrak{a} = \{a_m\}$ is called multiplicative if $a_1 = 1$ and $a_{m_1,m_2} = a_{m_1}a_{m_2}$ for all $(m_1, m_2) = 1$.

Universality of the function $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ has been given in [31].

Theorem D [31]. *Suppose that the periodic sequence $\mathfrak{a}$ is multiplicative. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau \in [0,T] \sup_{s\in k}|\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

Chapter 2 of the dissertation is devoted to universality of an absolutely convergent Dirichlet series

$$\zeta_u(s;\mathfrak{a}) = \sum_{m=1}^{\infty}\frac{a_m v_u(m)}{m^s}$$

with $v_u(m)$ defined in Section 1.1. For the statement of the theorem, we need some notation. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and $\mathbb{P}$ be the set of all prime numbers. Define the set

$$\Omega_1 = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. As the Cartesian product of compact sets γ_p, the infinite-dimensional torus Ω_1, by the classical Tikhonov theorem, is a compact topological Abelian group with respect to the product topology and operation of pointwise multiplication. Let $\mathcal{B}(\mathbb{X})$ stand for the Borel σ-field of the space $\mathbb{X}$. Then on $(\Omega_1, \mathcal{B}(\Omega_1))$, the probability Haar measure m_{1H} exists, and we have the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$. Let $\omega_1 = \{\omega_1(p) : p \in \mathbb{P}\}$ be elements of Ω_1. On the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$, define the $H(D)-$ valued random element $\zeta(s, \omega; \mathfrak{a})$ by

$$\zeta(s, \omega; \mathfrak{a}) = \prod_{p \in \mathbb{P}} \Big(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} \omega_1^l(p)}{p^{ls}}\Big).$$

Note that, for almost all $\omega_1 \in \Omega_1$, the last product converges uniformly on compact subsets of the strip D.

The main result of Chapter 2 is the following theorem.

Theorem 2.1. *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative, and that* $u_T \to \infty$ *and* $u_T \ll T^2$ *as* $T \to \infty$. *Let* $K \in \mathcal{K}$ *and* $f(s) \in H_0(K)$. *Then the limit*

$$\lim_{T \to \infty} \frac{1}{T} \text{meas}\Big\{\tau \in (0, T) : \sup_{s \in K} |\zeta_{u_T}(s + i\tau; \mathfrak{a}) - f(s)| < \varepsilon\Big\} =$$
$$m_{1H}\Big\{\omega \in \Omega : \sup_{s \in K} |\zeta(s, \omega; \mathfrak{a}) - f(s)| < \varepsilon\Big\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

It follows from Theorem 2.1. that there exists a number $T_0 = T_0(f, \chi, \varepsilon) > 0$ such that, for $T \geqslant T_0$, there are infinitely many shifts $\zeta_{u_T}(s + i\tau; \mathfrak{a})$ approximating a function $f(s) \in H_0(K)$ with accuracy ε.

For the proof of Theorem 2.1, a limit theorem in the space $H(D)$ for the function $\zeta_{u_T}(s; \mathfrak{a})$ is applied.

The results of Chapter 2 are published in [18].

The above discussed universality of functions $\zeta(s, \mathfrak{a})$ and $\zeta_{u_T}(s, \mathfrak{a})$ is called a continuous universality because τ in approximating shifts $\zeta(s + i\tau; \mathfrak{a})$ and $\zeta_{u_T}(s + i\tau; \mathfrak{a})$ can take arbitrary real values.

Parallelly to continuous universality of zeta-functions, discrete universality theorems are considered when τ in shifts $\zeta(s + i\tau; \mathfrak{a})$ takes values from a discrete set, for example, from $\{kh : k \in \mathbb{N}_0\}, h > 0$. Theorems of such a type were proposed by A.Reich in [49]. Denote by $\#A$ the cardinality (number of elements) of a set $A \subset \mathbb{R}$, and suppose that N runs over the set N_0. Reich in [49] considered Dedekind zeta-functions $\zeta_{\mathbb{K}}(s)$ of algebraic number fields $\mathbb{K}$, which are defined, for $\sigma > 1$, by

$$\zeta_{\mathbb{K}}(s) = \sum_{I} \frac{1}{(\mathcal{N}(I))^s},$$

where I runs over non-zeros ideals of the ring of integers of $\mathbb{K}$ and $\mathcal{N}(I)$ is the norm of I. If $\mathbb{K} = \mathbb{Q}$, then $\zeta_{\mathbb{K}}(s)$ becomes the Riemann zeta-function. Therefore, we state a discrete universality theorem for $\zeta(s)$.

Theorem E. *Let $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $h > 0$. Then, for any $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon\Big\} > 0.$$

We observe that Theorem E has a certain advantage against Theorem B because it is easier to detect discrete approximating shifts than continuous $\zeta(s+i\tau)$ ones.

We recall one corollary of a discrete weighted universality theorem from [55]. Define the set $L(\mathbb{P}, h, \pi) = \{(\log p : p \in \mathbb{P}), \frac{2\pi}{h}\}$.

Theorem F. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, and the set $L(\mathbb{P}; h, \pi), h \geqslant 1$, is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \#\Big\{1 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh; \mathfrak{a}) - f(s)| < \varepsilon\Big\} > 0.$$

In Chapter 3 of the dissertation, a discrete version of Theorem 2.1 is considered. We use the same notation as in Theorem 2.1.

Theorem 3.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the set $L(\mathbb{P}; h, \pi)$ is linearly independent over $\mathbb{Q}$, and $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta_{u_N}(s+ikh; \mathfrak{a}) - f(s)| < \varepsilon\Big\} = m_H\Big\{\omega \in \Omega : \sup_{s\in K} |\zeta(s,\omega; \mathfrak{a}) - f(s)| < \varepsilon\Big\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

The proof of Theorem 3.1 is based on a discrete limit theorem in the space $H(D)$ for the function $\zeta_{u_N}(s;\mathfrak{a})$.

The results of Chapter 3 are published in [19].

The next results of the dissertation are devoted to the joint universality of a pair $\Big(\zeta_{u_T}(s,\mathfrak{a}), \zeta_{u_T}(s,\alpha;\mathfrak{b})\Big)$ including its discrete version.

Joint universality of zeta$-$ and $L-$functions has a long and rich history. The first result in this direction , as a discovery of universality for zeta-functions in general, belongs to Voronin. He obtained in [56] the joint universality of Dirichlet $L-$functions. For the statement, we need some notion. The character $\chi \mod q$ is called generated by a character $\chi_1 \mod q_1, q_1|q$, if

$$\chi(m) = \begin{cases} \chi_1(m) & \text{if } (m,q)=1, \\ 0 & \text{if } (m,q)>0. \end{cases}$$

The character $\chi \mod q$ is called primitive if it is not generated by any character $\mathrm{mod} q_1, q_1 < q$. Two characters are called equivalent if they are generated by the same primitive character. The modern version of the Voronin theorem [56] has been presented in [35].

Theorem G. *Suppose that $\chi, \dots, \chi_r$ are pairwise nonequivalent Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\Big\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

It is easily seen that, in the case of joint universality, the approximating functions must be independent in a certain sense. In Theorem G, their independence is realised by nonequivalent characters.

Joint universality theorems are also known for other zeta-functions. Recall such a theorem for Hurwitz zeta-functions. Define the set

$$L(\alpha_1, \dots, \alpha_r) = \Big\{ \log(m+\alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r\Big\}.$$

In [33], the following joint universality theorem has been proved.

Theorem H. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over*

$\mathbb{Q}$. *For* $j = 1, \dots, r$, *let* $K_j \in \mathcal{K}$ *and* $f_j(s) \in H(K_j)$. *Then, for any* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta(s+i\tau,\alpha_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

The numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$ if there is no polynomial $p(s_1, \dots, s_r) \not\equiv 0$ with coefficients in $\mathbb{Q}$ such that $p(\alpha_1, \dots, \alpha_r) = 0$.

Note that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$ if the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$.

Theorems G and H are joint universality theorems for $L-$ and zeta$-$ functions of the same type (Dirichlet $L-$functions and Hurwitz zeta-functions). H. Mishou proposed in [46] a joint universality theorem for functions of two different types: for the Riemann zeta-function and Hurwitz zeta-function. The number α is transcendental if there is no any polynomial $p(s) \not\equiv 0$ with coefficients in $\mathbb{Q}$ such that $p(\alpha) = 0$.

Theorem I [46]. *Suppose that parameter α is transcendental. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) = H_0(K_1), f_2(s) = H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

The function $\zeta(s)$ has the Euler product over primes, while $\zeta(s,\alpha)$ with transcendent α has no such a product. Therefore, $f_1(s) \in H_0(K_1)$, while $f_2(s) \in H(K_2)$. Theorem I is called a mixed joint universality theorem for $\zeta(s)$ and $\zeta(s,\alpha)$.

Theorem I was generalized for $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ in [21].

Theorem J. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, and the parameter α is transcendental. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K_1), f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon\Big\} > 0.$$

Theorem J in [21] was applied to obtain the joint functional independence for the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$.

Obviously, Theorem J implies Theorem D, and universality of the function $\zeta(s,\alpha;\mathfrak{b})$ with transcendental α which, for the first time, was given in [20]. Also, there is a series of works on joint universality of the functions $\zeta(s;\mathfrak{a}_j), j = 1,\dots,r$, see, for example, [34], and of the functions $\zeta(s,\alpha_j;\mathfrak{b}), j = 1,\dots,r$, see, for example, [32]. Moreover, the functions $\zeta(s,\alpha_j;\mathfrak{b}_j)$ are involved in numerous works of R.Kačinskaitė and K.Matsumoto, see [22], [23], [24], [25].

Theorem I has a certain modification. Define

$$\Omega_2 = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. Then, as in the case of Ω_1, we have that Ω_2 is a compact topological Abelian group, and the probability Haar measure m_{2H}

in $(\Omega_2, \mathcal{B}(\Omega_2))$ can be defined. Define one more Cartesian product

$$\Omega = \Omega_1 \times \Omega_2.$$

Then again Ω is a compact topological Abelian group, and on $(\Omega, \mathcal{B}(\Omega)$, the probability Haar measure m_H exists. We observe that the measure m_H is the product of the measures m_{1H} and m_{2H}. This means that if $A = A_1 \times A_2 \in \mathcal{B}(\Omega)$ with $A_j \in \mathcal{B}(\Omega_j), j = 1, 2$, then

$$m_H(A) = m_{1H}(A_1) \cdot m_{2H}(A_2).$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D) = H(D) \times H(D)-$ valued random element

$$\zeta(s, \omega, \alpha) = \Big(\zeta(s, \omega_1), \zeta(s, \omega_2, \alpha)\Big),$$

where $\omega = (\omega_1, \omega_2) \in \Omega, \omega_1 \in \Omega_1, \omega_2 \in \Omega_2$, and

$$\zeta(s, \omega_1) = \prod_{p \in \mathbb{P}} \Big(1 - \frac{\omega_1(p)}{p^s}\Big)^{-1}$$

and

$$\zeta(s, \omega_2, \alpha) = \sum_{m \in \mathbb{N}_0} \frac{\omega_2(m)}{(m+\alpha)^s}, \omega_2 = \Big\{\omega_2(m) : m \in \mathbb{N}_0\Big\}.$$

In [39], the following version of Theorem I has been obtained.

Theorem K. *Suppose that the set* $L(\mathbb{P}, \alpha) \overset{def}{=} \Big\{(\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0)\Big\}$ *is linearly independent over* $\mathbb{Q}$*. Let* $K_1, K_2 \in \mathcal{K}$*, and* $f_1(s) \in H_0(K_1), f_2(s) \in H(K_2)$*. Then, for every* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$*, the limit*

$$\lim_{T \to \infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon_1,$$
$$\sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon_2\Big\} =$$
$$m_H\Big\{\omega \in \Omega : \sup_{s \in K_1} |\zeta(s; \omega_1) - f_1(s)| < \varepsilon_1,$$
$$\sup_{s \in K_2} |\zeta(s; \omega_2, \alpha) - f_2(s)| < \varepsilon_2\Big\}$$

exists and is positive for all but at most countably many $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$.

Theorem K for transcendental α was proved in [38]. In, [39],Theorem K was also extended to absolutely convergent Dirichlet series. Define

$$\zeta_u(s) = \sum_{m=1}^{\infty} \frac{v_u(m)}{m^s}$$

and

$$\zeta_u(s,\alpha) = \sum_{m=0}^{\infty} \frac{v_u(m,\alpha)}{(m+\alpha)^s}.$$

Theorem L. *Suppose that the set $L(\mathbb{P},\alpha)$ is linearly independent over $\mathbb{Q}$ and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta_{u_T}(s+i\tau) - f_1(s)| < \varepsilon_1, \sup_{s\in K_2} |\zeta_{u_T}(s+i\tau,\alpha) - f_2(s)| < \varepsilon_2\Big\} = m_H\Big\{\omega \in \Omega : \sup_{s\in K_1} |\zeta(s;\omega_1) - f_1(s)| < \varepsilon_1, \sup_{s\in K_2} |\zeta(s;\omega_2,\alpha) - f_2(s)| < \varepsilon_2\Big\}$$

exists and is positive for all but at most countably many $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

In Chapter 4 of the dissertation, Theorem K is extended to the functions $\zeta_{u_T}(s;\mathfrak{a})$ and $\zeta_{u_T}(s,\alpha;\mathfrak{b})$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D)$ – valued random element

$$\zeta(s,\alpha,\omega;\mathfrak{a},\mathfrak{b}) = (\zeta(s,\omega_1;\mathfrak{a}), \zeta(s,\alpha,\omega_2;\mathfrak{b})),$$

where $\zeta(s,\omega_1;\mathfrak{a})$ is the same $H(D)-$valued random element as in Theorem 1.1, and

$$\zeta(s,\alpha,\omega_2;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m+\alpha)^s}.$$

Theorem 4.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, parameter α is transcendental, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Let $K_1, K_2 \in \mathcal{K}$,* *and $f_1(s) \in H_0(K_1), f_2(s) \in H(K_2)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta_{u_T}(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon_1,$$

$$\sup_{s\in K_2} |\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon_2\Big\} =$$

$$m_H\Big\{\omega \in \Omega : \sup_{s\in K_1} |\zeta(s,\omega_1,\mathfrak{a}) - f_1(s)| < \varepsilon_1,$$

$$\sup_{s\in K_2} |\zeta(s,\omega_2,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon_2\Big\}$$

exists and is positive for all but at most countably many $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

The results of Chapter 4 are published in [3].

The last chapter of the dissertation, Chapter 5, deals with a discrete version of Theorem 4.1., therefore, we present the main results on a discrete version of the Mishou theorem. The first result of such a type was obtained in [9]. Let $h > 0$, and

$$L(\mathbb{P};\alpha,h,\pi) = \Big\{(\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0), \frac{2\pi}{h}\Big\}.$$

Theorem M [9]. *Suppose that the set $L(\mathbb{P};\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1), f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant n : \sup_{s\in K_1} |\zeta(s+ikh) - f_1(s)| < \varepsilon,$$

$$\sup_{s\in K_2} |\zeta(s+ikh,\alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

In [8], Theorem M was generalized by using shifts $\zeta(s+ikh_1)$ and $\zeta(s+ikh_2,\alpha)$ with different $h_1 > 0$ and $h_2 > 0$. Let $L(\mathbb{P};\alpha,h_1,h_2,\pi) = \Big\{(h_1 \log p : p \in \mathbb{P}), (h_2 \log(m+\alpha) : m \in \mathbb{N}_0), 2\pi\Big\}$.

Theorem N [8]. *Suppose that the set $L(\mathbb{P};\alpha,h_1,h_2)$ is linearly independent over $\mathbb{Q}$. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1), f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K_1} |\zeta(s+ikh_1) - f_1(s)| < \varepsilon,$$

$$\sup_{s\in K_2} |\zeta(s+ikh_2,\alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

Chapter 5 of the dissertation is devoted to a generalization of Theorem N, for the functions $\zeta_{u_N}(s;\mathfrak{a})$ and $\zeta_{u_N}(s,\alpha;\mathfrak{b})$. The statement of the theorem is the following.

Results of Chapter 5 are published in [4].

Theorem 5.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$, and $u_N\to\infty$ and $u_N\ll N^2$ as $N\to\infty$. Let $K_1,K_2\in\mathcal{K}$, and $f_1(s)\in H_0(K_1), f_2(s)\in H(K_2)$. Then the limit*

$$\lim_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K_1}|\zeta_{u_T}(s+ikh_1;\mathfrak{a})-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta_{u_T}(s+ikh_2,\alpha;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}=$$
$$m_H\Big\{(\omega_1,\omega_2)\in\Omega:\sup_{s\in K_1}|\zeta(s,\omega_1;\mathfrak{a})-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta(s,\alpha,\omega_2;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}$$

exists and is positive for all but at most countably many ε_1 and ε_2.

We observe that $L(\mathbb{P},\alpha,h,\pi)$ and $L(\mathbb{P},\alpha,h_1,h_2,\pi)$ are multisets, i.e., if they contain two or more equal elements, these elements remain in the set.

The requirements $u_T\ll T^2(u_N\ll N^2)$ appeared in the above theorems are used in approximation in the mean $\zeta_{u_T}(s;\mathfrak{a}),\zeta_{u_T}(s,\alpha;\mathfrak{b})$ $(\zeta_{u_N}(s;\mathfrak{a}),\zeta_{u_N}(s,\alpha;\mathfrak{b}))$ by $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$, respectively. In the case, when $\zeta(s;\mathfrak{a})$ or $\zeta(s,\alpha;\mathfrak{b})$ have no pole at $s=1$, these requirements are not needed.

1.7 Approbation

The results of the dissertation were presented at the International MMA (Mathematical Modelling and Analysis) conferences (MMA2023, May 30 – June 2, 2023, Jurmala, Latvia), (MMA2024, May 28 – 31 d., 2024 Pärnu, Estonia), at the international conference of probability theory and the number theory (ICPTNT2024, September 16 – 20, Palanga) at the conferences of Lithuanian Mathematical Society (LMS2021, June 16 – 17, 2021, Vilnius), (LMS2022, June 16 – 17, 2022, Kaunas), (LMS2023, June 21 – 22, 2023, Vilnius), (LMS2024, June 27 – 28, Šiauliai, 2024).

1.8 Main publications

The results of the dissertation are published in the following papers:

1. M. Jasas, A. Laurinčikas, D. Šiaučiūnas, On the approximation of analytic functions by shifts of an absolutely convergent Dirichlet series, *Math. Notes* **109** (2021), 832–841.

2. M. Jasas, A. Laurinčikas, M.Stoncelis, D. Šiaučiūnas, Discrete universality of absolutely convergent Dirichlet series, *Mathematical Modelling and Analysis*, **27** (2022), 78–87.

3. A. Balčiūnas, M. Jasas, R. Macaitienė, D. Šiaučiūnas, On the Mishou Theorem for Zeta-Functions with Periodic Coefficients, *Mathematics* (2023), https://doi.org/10.3390/math11092042.

4. A. Balčiūnas, M. Jasas, A. Rimkevičienė, A Discrete version of the Mishou theorem related to periodic Zeta-Functions, *Mathematical Modelling and Analysis*, **29** (2024), 331–346.

Abstracts for conferences:

1. M. Jasas, R. Macaitienė. On the Mishou theorem for periodic zeta-functions. Abstracts of MMA2023, May 30 – June 2, 2023, Jurmala, Latvia, pp. 21.

2. M. Jasas. Approximation of analytic functions by shifts of absolutely convergent Dirichlet series related to periodic zeta-functions. Abstracts of MMA2024, May 28 – 31, 2024, Pärnu, Estonia, pp. 30.

Chapter 2

Approximation of analytic functions by continuos shifts of an absolutely convergent Dirichlet series related to the periodic zeta-function

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q_1 \in \mathbb{N}$, i.e., $a_{m+q_1} = a_m$ for all $m \in \mathbb{N}$. We recall that the periodic zeta-function $\zeta(s, \mathfrak{a})$, $s = \sigma + it$, is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and, in view of the representation

$$\zeta(s; \mathfrak{a}) = \frac{1}{q_1^s} \sum_{k=1}^{q_1} a_k \, \zeta\left(s, \frac{k}{q_1}\right), \tag{2.1}$$

where $\zeta(s, \frac{k}{q_1})$ is the Hurwitz zeta-function with parameter $\frac{k}{q_1}$, has the meromorphic continuation to the whole complex plane with unique simple pole at the point $s = 1$ with residue $\frac{1}{q_1} \sum_{k=1}^{q_1} a_k \overset{\text{def}}{=} \widehat{a}_1$. If $\widehat{a}_1 = 0$, then the function $\zeta(s; \mathfrak{a})$ is analytic in the whole complex plane, i.e., it is an entire function.

In this chapter, we consider an absolutely convergent Dirichlet series connected to the function $\zeta(s;\mathfrak{a})$. Thus, let $\theta > \frac{1}{2}$ be a fixed parameter, and for $u > 0$,

$$v_u(m) = \exp\Big\{ - \Big(\frac{m}{u}\Big)^{\theta} \Big\}, m \in \mathbb{N}.$$

Define the Dirichlet series

$$\zeta_u(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_u(m)}{m^s}. \tag{2.2}$$

Since the sequence $\mathfrak{a}$ is periodic, it is bounded. Thus, there exists a constant $C = C(a_1, \dots, a_{q_1}) > 0$ such that $|a_m| \leqslant C$ for all $m \in \mathbb{N}$. The sequence $\{v_u(m) : m \in \mathbb{N}\}$ is exponentially decreasing for any fixed u. Therefore, the Dirichlet series (2.2) is absolutely convergent in any half-plane $\sigma > \sigma_0$ with arbitrary fixed finite σ_0. Hence, the function $\zeta_u(s;\mathfrak{a})$ is entire.

This chapter is devoted to approximation of analytic functions by continuous shifts $\zeta_{u_T}(s + i\tau;\mathfrak{a})$, $\tau \in \mathbb{R}$, with multiplicative sequence $\mathfrak{a}$ and a certain function u_T. We recall that the sequence $\mathfrak{a}$ is multiplicative if $a_1 = 1$ and $a_{m_1 m_2} = a_{m_1} a_{m_2}$ for all coprimes $m_1, m_2 \in \mathbb{N}$. Also, we notice that, for absolute convergence, it suffices to take $\theta > 0$, however, for further aims, we need $\theta > \frac{1}{2}$.

2.1 Statement of the main theorem

For the statement of a theorem on approximation of analytic functions by shifts $\zeta_{u_T}(s + i\tau;\mathfrak{a})$, we need some notation and definitions. We start with one topological group. Let γ be the unit circle on the complex plane, i.e., $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and $\mathbb{P}$ the set of all prime numbers. Define the Cartesian product

$$\Omega_1 = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. The infinite-dimensional torus Ω_1, by the classical Tikhonov theorem, see, for example [29], with the product topology and operation of pointwise multiplication, is a compact topological Abelian group. The compactness implies the existence of the probability Haar measure m_{1H} on $(\Omega_1, \mathcal{B}(\Omega_1))$, where $\mathcal{B}(\mathbb{X})$ denotes the Borel σ - field of the space $\mathbb{X}$. This gives the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$. Let $D = \big\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\big\}$. Denote by $H(D)$ the space of analytic on the strip D functions endowed with the topology of uniform convergence on compacta. Let $\omega_1(p)$ stand for the

pth component of an element $\omega_1 \in \Omega_1, p \in \mathbb{P}$. Now, on the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$, define the $H(D)$ - valued random element

$$\zeta(s, \omega_1; \mathfrak{a}) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{l=1}^{\infty} \frac{a_p \omega_1^l(p)}{p^{ls}}\right).$$

The latter infinite product is uniformly convergent on compact sets of the strip D for almost all $\omega_1 \in \Omega_1$ with respect to the measure m_{1H}. This can be found, for example, in [16], [28]. Let K be the class of compact sets of the strip D with connected complements, and $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K.

We recall that $a \ll_\xi b, b > 0$, means that there exists a constant $c = c(\xi) > 0$ such that $|a| \leqslant cb$.

The main result of this chapter is the following theorem.

Theorem 2.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_{u_T}(s + i\tau; \mathfrak{a}) - f(s)| < \varepsilon \right\} =$$

$$m_{1H} \left\{ \omega_1 \in \Omega_1 : \sup_{s \in K} |\zeta(s, \omega_1; \mathfrak{a}) - f(s)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Since the density of shifts $\zeta_{u_T}(s + i\tau; \mathfrak{a})$, approximating the function $f(s)$ is positive, we have from Theorem 2.1 that there is a number $T_0 = T_0(f, K, \varepsilon) > 0$ such that we have infinitely many shifts $\zeta_{u_T}(s + i\tau; \mathfrak{a})$ for $T_0 \geqslant T_0$ with approximating property.

Since τ takes arbitrary real values, Theorem 2.1 is of continuous type. Moreover, Theorem 2.1 shows that the function $\zeta_{u_T}(s; \mathfrak{a})$ has a similar approximation property as the function $\zeta(s; \mathfrak{a})$ given by analytic continuation. On the other hand, for $s \in D$, the equality

$$\lim_{T \to \infty} \zeta_{u_T}(s; \mathfrak{a}) = \zeta(s; \mathfrak{a})$$

is not known.

2.2 Integral representation

A bit later, we will prove that $\zeta_{u_T}(s;\mathfrak{a})$ is close to $\zeta(s;\mathfrak{a})$ in the mean for $s \in D$. For this, a certain integral representation for $\zeta_{u_T}(s;\mathfrak{a})$ is needed.

First of all, we recall some properties of the classical Euler gamma-function $\Gamma(s)$. For $\sigma > 0$, the function $\Gamma(s)$ is defined by the integral

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} du.$$

Moreover, $\Gamma(s)$ is analytically continued to the whole complex plane, except for simple poles $s = -k$, $k \in \mathbb{N}_0$, with residues

$$\operatorname*{Res}_{s=-k} \Gamma(s) = \frac{(-1)^{k-1}}{k!}.$$

Lemma 2.1. *Suppose that a and b are positive numbers. Then*

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \Gamma(s) a^{-s} ds = e^{-a}.$$

Proof. The equality of the lemma is called the Mellin formula. Its proof can be found, for example, in [53].

Lemma 2.2. *Uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ with arbitrary $\sigma_1 < \sigma_2$, the estimate*

$$\Gamma(s) \ll \exp\{-c|t|\}, c > 0.$$

is valid.

Proof of the lemma can be found, for example, in [29].

Lemma 2.3. *Let θ be the number from definition of $v_u(m)$. Then, for $\sigma > \frac{1}{2}$, the representation*

$$\zeta_u(s;\mathfrak{a}) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z;\mathfrak{a})\, l_u(z)\frac{dz}{z},$$

where

$$l_u(s) = \frac{s}{\theta}\Gamma(\frac{s}{\theta})u^s$$

is valid.

Proof. Using Lemma 2.1, we find

$$v_u(m) = \exp\left\{-\left(\frac{m}{u}\right)^\theta\right\} = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma(z)\left(\left(\frac{m}{u}\right)^\theta\right)^{-z} dz =$$

$$\frac{1}{2\pi i}\int_{1-i\infty}^{\theta+i\infty} \Gamma\left(\frac{z}{\theta}\right)\left(\left(\frac{m}{u}\right)^\theta\right)^{-\frac{z}{\theta}} d\left(\frac{z}{\theta}\right).$$

Since $\theta > \frac{1}{2}$ and $\sigma > \frac{1}{2}$, hence we have

$$\zeta_u(s;\mathfrak{a}) = \frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{a_m}{m^s}\int_{\theta-i\infty}^{\theta+i\infty}\frac{z}{\theta}\Gamma\left(\frac{z}{\theta}\right)\left(\frac{m}{u}\right)^{-z}\frac{dz}{z} =$$

$$\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\left(\frac{l_u(z)}{z}\sum_{m=1}^{\infty}\frac{a_m}{m^{s+z}}\right)dz = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z;\mathfrak{a})\, l_u(z)dz.$$

The interchange of summation and integration is justified by an estimate of Lemma 2.2.

2.3 Approximation in the mean

In this section, we will prove that the functions $\zeta(s;\mathfrak{a})$ and $\zeta_{u_T}(s;\mathfrak{a})$ are close in some sense in the strip D. For this, we will use the mean square estimate for the function $\zeta(s;\mathfrak{a})$.

Lemma 2.4. *Suppose that* σ, $\frac{1}{2} < \sigma < 1$, *is fixed. Then the estimate*

$$\int_{-T}^{T}|\zeta(\sigma+it;\mathfrak{a})|^2 dt \ll_{\sigma,\mathfrak{a}} T$$

is valid.

Proof. It is well known, see, for example, [29], that, for the Hurwitz zeta-function $\zeta(s,\alpha)$ and a fixed σ, $\frac{1}{2} < \sigma < 1$, the estimate

$$\int_{-T}^{T}|\zeta(\sigma+it,\alpha)|^2 dt \ll_{\sigma,\alpha} T$$

holds. Hence, in virtue of equality (2.1), for these values of σ, we have

$$\int_{-T}^{T}|\zeta(\sigma+it;\mathfrak{a})|^2 dt \ll_{q_1} \int_{-T}^{T}\left|\sum_{k=1}^{q_1} a_k\zeta\left(\sigma+it,\frac{k}{q_1}\right)\right|^2 dt \ll_{\sigma,\mathfrak{a}}$$

$$\sum_{k=1}^{q_1}\int_{-T}^{T}\left|\zeta\left(\sigma+it,\frac{k}{q_1}\right)\right|^2 dt \ll_{\sigma,\mathfrak{a}} T.$$

Lemma 2.5. *Suppose that $K\subset D$ is a compact set, and $u_T\to\infty$ and $u_T\ll T^2$. Then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a})-\zeta_{u_T}(s+i\tau;\mathfrak{a})|dt=0.$$

Proof. By Lemma 2.3, we have

$$\zeta_u(s;\mathfrak{a})=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z;\mathfrak{a})\, l_{u_T}(z)\frac{dz}{z}. \tag{2.3}$$

Let K be an arbitrary fixed set in the strip D. We fix $\varepsilon>0$ such that $\frac{1}{2}+2\varepsilon\leqslant\sigma\leqslant 1-\varepsilon$ for all $s=\sigma+it\in K$. We take $\theta=\frac{1}{2}+\varepsilon$ and $\theta_1=\sigma-\frac{1}{2}-\varepsilon$. Then, $\theta_1>0$. By properties of the functions $\Gamma(s)$ and $\zeta(s;\mathfrak{a})$, the integrand in (2.3) has simple poles $z=0$ and $z=1-s$ lying in the strip $-\theta_1<Rez<\theta$. Therefore, moving the line of integration to the left, we obtain from the residue theorem and (2.3) that, for $s\in K$,

$$\zeta_{u_T}(s;\mathfrak{a})-\zeta(s;\mathfrak{a})=\frac{1}{2\pi i}\int_{-\theta_1-i\infty}^{-\theta_1+i\infty}\zeta(s+z;\mathfrak{a})\, l_{u_T}(z)\frac{dz}{z}+\widehat{a}\,\frac{l_{u_T}(1-s)}{1-s}. \tag{2.4}$$

Here $\zeta(s;\mathfrak{a})$ is the residue at $z=0$, and $\widehat{a}\frac{l_{u_T}(1-s)}{1-s}$ the residue at $z=1-s$. The equality (2.4) and the definition of θ_1 imply

$$\zeta_{u_T}(s;\mathfrak{a})-\zeta(s;\mathfrak{a})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\zeta(\sigma+it+iv-\sigma+\frac{1}{2}+\varepsilon;\mathfrak{a})\times$$

$$\frac{l_{u_T}(\frac{1}{2}+\varepsilon-\sigma+iv)}{\frac{1}{2}+\varepsilon-\sigma+iv}dv+\widehat{a}\frac{l_{u_T}(1-s)}{1-s}$$

Thus, for $s\in K$,

$$\zeta_{u_T}(s,\mathfrak{a})-\zeta(s,\mathfrak{a})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\zeta(it+iv+\frac{1}{2}+\varepsilon;\mathfrak{a})\times$$

$$\frac{l_{u_T}(\frac{1}{2}+\varepsilon-\sigma+iv)}{\frac{1}{2}+\varepsilon-\sigma+iv}dv+\widehat{a}\frac{l_{u_T}(1-s)}{1-s}.$$

Hence, replacing $v+t$ by v, we find that, for $s \in K$,

$$\zeta_{u_T}(s+i\tau,\mathfrak{a}) - \zeta(s+i\tau,\mathfrak{a}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta(\frac{1}{2}+\varepsilon+i\tau+iv;\mathfrak{a})\times$$

$$\frac{l_{u_T}(\frac{1}{2}+\varepsilon-s+iv)}{\frac{1}{2}+\varepsilon-s+iv}dv + \widehat{a}\frac{l_{u_T}(1-s-i\tau)}{1-s-i\tau}.$$

Therefore, for all $s \in K$,

$$\zeta_{u_T}(s+i\tau,\mathfrak{a}) - \zeta(s+i\tau,\mathfrak{a}) \ll \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv;\mathfrak{a}\right)\right| \times$$

$$\sup_{s\in K}\left|\frac{l_{u_T}(\frac{1}{2}+\varepsilon-s+iv)}{\frac{1}{2}+\varepsilon-s+iv}\right| dv + \sup_{s\in K}|\widehat{a}|\left|\frac{l_{u_T}(1-s-i\tau)}{1-s-i\tau}\right|. \quad (2.5)$$

Taking into account Lemma 2.2 again, we can change the order of integration, therefore, by estimate (2.5), we have

$$\frac{1}{T}\int_0^T \sup_{s\in K}|\zeta_{u_T}(s+i\tau;\mathfrak{a}) - \zeta(s+i\tau;\mathfrak{a})|d\tau \ll_{\mathfrak{a}} I_1(T) + I_2(T), \quad (2.6)$$

where

$$I_1(T) = \int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv;\mathfrak{a}\right)\right| d\tau\right)\sup_{s\in K}\left|\frac{l_{u_T}(\frac{1}{2}+\varepsilon-s+iv)}{\frac{1}{2}+\varepsilon-s+iv}\right| dv$$

and

$$I_2(T) = \frac{1}{T}\int_0^T \sup_{s\in K}\left|\frac{l_{u_T}(1-s-i\tau)}{1-s-i\tau}\right| d\tau.$$

Using Lemma 2.4, we find that, for all $v \in \mathbb{R}$ and $T \geqslant 1$,

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv;\mathfrak{a}\right)\right| d\tau \leqslant \left(\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv;\mathfrak{a}\right)\right|^2 d\tau\right)^2 \ll$$

$$\left(\frac{1}{T}\int_{-|v|}^{T+|v|}\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau;\mathfrak{a}\right)\right|^2 d\tau\right)^{\frac{1}{2}} \ll_{\varepsilon,\mathfrak{a}} \left(\frac{T+|v|}{T}\right)^{\frac{1}{2}} \ll_{\varepsilon,\mathfrak{a}} (1+|v|)^{\frac{1}{2}}. \quad (2.7)$$

Moreover, applying Lemma 2.2 once more, we see that, for all $s \in K$,

$$\frac{l_{u_T}\left(\frac{1}{2}+\varepsilon-s+iv\right)}{\frac{1}{2}+\varepsilon-s+iv} \ll_\theta n_T^{\frac{1}{2}+\varepsilon-\sigma}\left|\Gamma\left(\frac{1}{\theta}\left(\frac{1}{2}+\varepsilon-\sigma-it+iv\right)\right)\right| \ll_\varepsilon$$

$$n_T^{-\varepsilon} \exp\left\{-\frac{c}{\theta}|t-v|\right\} \ll_{\varepsilon,K} u_T^{-\varepsilon} \exp\{-c_1|v|\}, c_1 > 0.$$

This estimate together with estimate (2.7) yields the estimate

$$I_1(T) \ll_{\mathfrak{a},\varepsilon,K} u_T^{-\varepsilon} \int_{-\infty}^{\infty} (1+|v|)^{\frac{1}{2}} \exp\{-c_1|v|\} dv \ll_{\mathfrak{a},\varepsilon,K} u_T^{-\varepsilon}. \qquad (2.8)$$

Similarly, as above, we find that, for all $s \in K$,

$$\frac{l_{u_T}(1-s-i\tau)}{1-s-i\tau} \ll_{\varepsilon,K} u_T^{1-\sigma} \exp\{-c_1|\tau|\} \ll_{\varepsilon,K} u_T^{\frac{1}{2}-2\varepsilon} \exp\{-c_1|\tau|\}.$$

Hence, we conclude that

$$I_2(T) \ll_{\varepsilon,K} u_T^{\frac{1}{2}-2\varepsilon} \frac{1}{T} \int_0^T \exp\{-c_1|\tau|\} dt \ll_{\varepsilon,K},$$

$$u_T^{\frac{1}{2}-2\varepsilon} \frac{1}{T} \int_0^\infty \exp\{-c_1|\tau|\} dt \ll_{\varepsilon,K} \frac{u_T^{\frac{1}{2}-2\varepsilon}}{T}.$$

This, (2.8) and (2.6) show that

$$\frac{1}{T} \int_0^T \sup_{s \in K} |\zeta_u(s+i\tau;\mathfrak{a}) - \zeta(s+i\tau;\mathfrak{a})| d\tau \ll_{\mathfrak{a},\varepsilon,K} \left(u_T^{-\varepsilon} + \frac{u_T^{\frac{1}{2}-2\varepsilon}}{T} \right).$$

Since $u_T \to \infty$ and $u_T \ll T^2$, the latter estimate proves the lemma.

We observe that the restriction $u_T \ll T^2$ comes from the pole of the function $\zeta(s;\mathfrak{a})$ at the point $s = 1$. If $\widehat{a} = 0$, then $u_T \to \infty$ can be arbitrary.

2.4 Limit lemmas

Theorem 2.1 follows from a limit theorem on the weak convergence of probability measures in the space of analytic functions.

Let P and $P_n, n \in \mathbb{N}$, be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We recall that P_n converges weakly to P as $n \to \infty$ if for every real continuous bounded function g on $\mathbb{X}$

$$\lim_{n\to\infty} \int_{\mathbb{X}} g dP_n = \int_{\mathbb{X}} g dP.$$

The definition of weak convergence of probability measures has some equivalents. We recall these equivalents in terms of open and continuity sets. A set $A \in \mathcal{B}(\mathbb{X})$ is called a continuity set of the measure P if $P(\partial A) = 0$, where ∂A is the boundary of the set A.

For the proof of universality for zeta-functions, usually the equivalent in the terms of open sets is applied. By this equivalent, P_n, as $n \to \infty$, converges weakly to P if and only if, for every open set $G \subset \mathbb{X}$,

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G).$$

For example, this method is used for the proof of Theorem D.

Also, we recall the definition of the support of a probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Suppose that the space $\mathbb{X}$ is separable. Then the support S_P of the measure P is a minimal closed set of $\mathbb{X}$ such that $P(S_P) = 1$. The set S_P consists of all $x \in \mathbb{X}$ such that, for any open neighbourhood G of x, the inequality $P(G) > 0$ is satisfied.

For $A \in \mathcal{B}(H(D))$, define

$$P_{T,\mathfrak{a}}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \zeta(s+i\tau;\mathfrak{a}) \in A\right\}.$$

Moreover let $P_{\zeta,\mathfrak{a}}$ be the distribution of the $H(D)$ - valued random element $\zeta(s,\omega_1;\mathfrak{a})$ defined in Section 2.1, i.e.,

$$P_{\zeta,\mathfrak{a}}(A) = m_{1H}\{\omega_1 \in \Omega_1 : \zeta(s,\omega_1;\mathfrak{a}) \in A\}, A \in \mathcal{B}(H(D)).$$

Then, in [18], the following statement was obtained.

Lemma 2.6. *Suppose that the sequence $\mathfrak{a}$ is multiplicative. Then $P_{T,\mathfrak{a}}$ converges weakly to $P_{\zeta,\mathfrak{a}}$ as $T \to \infty$. Moreover, the support of the measure $P_{\zeta,\mathfrak{a}}$ is the set*

$$S \overset{def}{=} \{g \in H(D) : g(s) \neq 0 \textit{ or } g(s) \equiv 0\}.$$

We note that Theorem D can be complemented by the following statement by using Lemma 2.6 and the equivalent of weak convergence of probability measures in terms of continuity sets.

Proposition 2.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\{\tau \in [0,T] : \sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\} =$$
$$m_{1H}\{\omega_1 \in \Omega_1 : \sup_{s\in K}|\zeta(s,\omega_1;\mathfrak{a}) - f(s)| < \varepsilon\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

For convenience, we recall the mentioned equivalent of weak convergence in terms of continuity sets.

Lemma 2.7. *Let $P_n, n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then the weak convergence of P_n to P as $n \to \infty$ is equivalent to the equality*

$$\lim_{n\to\infty} P_n(A) = P(A)$$

for each continuity set A of the measure P.

Proof. The lemma is a part of Theorem 2.1 from [19], where its proof is given.

We also need the Mergelyan theorem on the approximation of analytic functions by polynomials, see [43].

Lemma 2.8. *Let $K \in \mathbb{C}$ be a compact set with connected complement, and $g(s)$ a continuos function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s) = p_{\varepsilon,g,K}(s)$ such that*

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon.$$

Proof of Proposition 2.1. Since $f(s) \neq 0$ on K, by Lemma 2.8, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - e^{p(s)}| < \frac{\varepsilon}{2}. \tag{2.9}$$

To obtain the latter inequality, it suffices to apply Lemma 2.8 for $\log f(s)$. Consider the set

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - e^{p(s)}| < \frac{\varepsilon}{2} \right\}.$$

It is obvious that $e^{p(s)} \neq 0$, hence $e^{p(s)} \in S$. Therefore, by Lemma 2.6, the set G_ε is an open neighbourhood of an element of the support of the measure $P_{\zeta,\mathfrak{a}}$. Thus, in virtue of properties of the support, we have

$$P_{\zeta,\mathfrak{a}}(G_\varepsilon) > 0. \tag{2.10}$$

Define one more set

$$\widehat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

The boundary $\partial\widehat{G}_\varepsilon$ lies in the set

$$\left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\}.$$

Therefore, $\partial\widehat{G}_{\varepsilon_1} \cap \partial\widehat{G}_{\varepsilon_2} = \varnothing$ for different positive ε_1 and ε_2. From this, it follows easily that the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,\mathfrak{a}}$ for all but at most countably many $\varepsilon > 0$. Actually, for any $m \in \mathbb{N}$, there are at most $m-1$ sets $\widehat{G}_\varepsilon$ for which

$$P_{\zeta,\mathfrak{a}}(\partial\widehat{G}_\varepsilon) > \frac{1}{m}.$$

From this, we obtain that there are at most countably many sets $\partial\widehat{G}_\varepsilon$ such that

$$P_{\zeta,\mathfrak{a}}(\partial\widehat{G}_\varepsilon) > 0.$$

Hence, $P_{\zeta,\mathfrak{a}}(\partial\widehat{G}_\varepsilon) = 0$ for all but at most countably many $\varepsilon > 0$, i.e., by the definition, the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,\mathfrak{a}}$ for all but at most countably many $\varepsilon > 0$.

The latter remark and Lemmas 2.6 and 2.7 show that

$$\lim_{T \to \infty} P_{T,\mathfrak{a}}(\widehat{G}_\varepsilon) = P_{\zeta,\mathfrak{a}}(\widehat{G}_\varepsilon) \tag{2.11}$$

for all but at most countably many $\varepsilon > 0$. It remains to prove that $P_{\zeta,\mathfrak{a}}(\widehat{G}_\varepsilon) > 0$. Suppose that $g \in G_\varepsilon$. Then, by (2.9),

$$\sup_{s \in K} |g(s) - f(s)| \leqslant \sup_{s \in K} |g(s) - e^{p(s)}| + \sup_{s \in K} |f(s) - e^{p(s)}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $g \in \widehat{G}_\varepsilon$. Consequently, $G_\varepsilon \subset \widehat{G}_\varepsilon$. Therefore, in view of (2.10), we have $P_{\zeta,\mathfrak{a}}(\widehat{G}_\varepsilon) > 0$. This inequality, (2.11) and the definition of the set $\widehat{G}_\varepsilon$ prove the proposition.

2.5 Proof of the main theorem

This section is devoted to the proof of Theorem 2.1. For this, we will apply a continuity correspondence between distribution and characteristic functions.

Let ξ be a random variable defined on a certain probability space with a measure P. Then the distribution function $F(x), x \in \mathbb{R}$, of ξ is defined by

$$F(X) = P\{\xi < x\}.$$

It is well known that every nondecreasing and left continuous function $F(x)$ such that $F(+\infty) = 1$ and $F(-\infty) = 0$ is a distribution function of a certain random variable.

The characteristic function $\phi(u), u \in \mathbb{R}$, of the distribution function $F(x)$ is defined by

$$\phi(u) = \int_{-\infty}^{\infty} e^{iux} dF(x).$$

Let $F_n(x), n \in \mathbb{N}$, and $F(x)$ be distribution functions. We recall that $F_n(x)$ converges weakly to $F(x)$ as $n \to \infty$ if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for all continuity points x of the function $F(x)$.

The following continuity theorem between distribution and characteristic functions is very useful, see, for example, [27].

Lemma 2.9. *Let $F_n(x), n \in \mathbb{N}$, be the distribution function, and $\phi_n(u)$ the corresponding characteristic function. If $F_n(x)$, as $n \to \infty$, converges weakly to a certain distribution function, then $\phi_n(u)$ converges to the characteristic function $\phi(u)$ of $F(x)$ as $n \to \infty$. This convergence is uniform in any finite interval. On the other hand, if, for any $u \in \mathbb{R}$, the limit* $\lim_{n\to\infty} \phi_n(u) = \phi(u)$ *exists, and the function $\phi(u)$ is continuous at $u = 0$, then $F_n(x)$ converges weakly to the distribution function $F(x)$, and $\phi(u)$ is the characteristic function of F(x).*

Proof of Theorem 2.1. For $\varepsilon > 0$, define two functions

$$F_{T,\mathfrak{a}}(\varepsilon) = P_{T,\mathfrak{a}}(\widehat{G}_\varepsilon) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\}$$

and

$$F_{\zeta,\mathfrak{a}}(\varepsilon) = P_{\zeta,\mathfrak{a}}(\widehat{G}_\varepsilon) = m_{1H}\left\{\omega_1 \in \Omega_1 : \sup_{s\in K} |\zeta(s,\omega;\mathfrak{a}) - f(s)| < \varepsilon\right\}.$$

These functions are nondecreasing, left continuous, and take the values 0 at $\varepsilon = -\infty$ and 1 at $\varepsilon = +\infty$. Therefore, $F_{T,\mathfrak{a}}(\varepsilon)$ and $F_{\zeta,\mathfrak{a}}(\varepsilon)$ are the distribution functions. In addition, a point ε is a continuity point of the function $F_{\zeta,\mathfrak{a}}(\varepsilon)$ if and only if the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,\mathfrak{a}}$. Indeed, since

$$\partial\widehat{G}_\varepsilon = \{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| \leqslant \varepsilon\}\backslash\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\}$$

and

$$\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\} \subset \{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| \leqslant \varepsilon\},$$

we have

$$P_{\zeta,\mathfrak{a}}(\widehat{G}_\varepsilon) = F_{\zeta,\mathfrak{a}}(\varepsilon + 0) - F_{\zeta,\mathfrak{a}}(\varepsilon). \tag{2.12}$$

Hence, if $P_{\zeta,\mathfrak{a}}(\partial\widehat{G}_\varepsilon) = 0$, then $F_{\zeta,\mathfrak{a}}(\varepsilon+0) = F_{\zeta,\mathfrak{a}}(\varepsilon)$, i.e., the function $F_{\zeta,\mathfrak{a}}(\varepsilon)$ is right continuous at the point ε. Since $F_{\zeta,\mathfrak{a}}(\varepsilon)$ is left continuous at this point, $F_{\zeta,\mathfrak{a}}(\varepsilon)$ is continuous at the point ε. If $F_{\zeta,\mathfrak{a}}(\varepsilon)$ is continuous at the point ε, then $F_{\zeta,\mathfrak{a}}(\varepsilon + 0) = F_{\zeta,\mathfrak{a}}(\varepsilon)$, and by (2.12), $P_{\zeta,\mathfrak{a}}(\partial\widehat{G}_\varepsilon) = 0$, i.e., the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,\mathfrak{a}}$.

Let $\phi_{T,\mathfrak{a}}(u)$ and $\phi_{\zeta,\mathfrak{a}}(u)$ be the characteristic functions of the distribution functions $F_{T,\mathfrak{a}}(\varepsilon)$ and $F_{\zeta,\mathfrak{a}}(\varepsilon)$, respectively. Then, in view of (2.11) and above remark, we obtain that $F_{T,\mathfrak{a}}(\varepsilon)$, as $T \to \infty$, converges weakly to $F_{\zeta,\mathfrak{a}}(\varepsilon)$. Therefore, by Lemma 2.9, the convergence of the corresponding characteristic functions follows, i.e.,

$$\lim_{t\to\infty} \phi_{T,\mathfrak{a}}(u) = \phi_{\zeta,\mathfrak{a}}(u) \tag{2.13}$$

for all $u \in \mathbb{R}$, and uniformly for $|u| \leqslant C$ with arbitrary $C > 0$.

Define one more distribution function

$$\widehat{F}_{T,\mathfrak{a}}(\varepsilon) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \sup_{s\in K} |\zeta_{u_T}(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\},$$

and let $\widehat{\phi}_{T,\mathfrak{a}}(u)$ denote the characteristic function of $\widehat{F}_{T,\mathfrak{a}}(\varepsilon)$. We will estimate the quantity

$$\triangle_{T,\mathfrak{a}}(u) \stackrel{\text{def}}{=} |\widehat{\phi}_{T,\mathfrak{a}}(u) - \phi_{T,\mathfrak{a}}(u)|.$$

By the definitions of $\widehat{F}_{T,\mathfrak{a}}(\varepsilon)$ and $F_{T,\mathfrak{a}}(\varepsilon)$, we find

$$\triangle_{T,\mathfrak{a}}(u) = \left| \int_{-\infty}^{\infty} e^{iu\varepsilon} d\widehat{F}_{T,\mathfrak{a}}(\varepsilon) - \int_{-\infty}^{\infty} e^{iu\varepsilon} dF_{T,\mathfrak{a}}(\varepsilon) \right| =$$

$$\frac{1}{T}\int_0^T \Big(\exp\big\{iu \sup_{s\in K} |\zeta_{u_T}(s+i\tau;\mathfrak{a}) - f(s)|\big\} -$$

$$\exp\big\{iu \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)|\big\}\Big) d\tau =$$

$$\frac{1}{T}\int_0^T \exp\{iu \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)|\}\Big(\exp\big\{iu\big(\sup_{s\in K} |\zeta_{u_T}(s+i\tau;\mathfrak{a}) - f(s)| -$$

$$\sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a}) - f(s)|\big)\big\} - 1\Big) d\tau.$$

Hence, using the inequality $|e^{ia} - 1| \leqslant |a|$, $a \in \mathbb{R}$, we obtain

$$\triangle_{T,\varepsilon}(u) \leqslant \frac{|u|}{T}\int_0^T \left| \sup_{s\in K} |\zeta_{u_T}(s+i\tau;\mathfrak{a}) - f(s)| - |\zeta(s+i\tau;\mathfrak{a}) - f(s)| \right| dt,$$

and, by the triangle inequality

$$|\sup a - \sup b| \leqslant \sup |a-b|,$$

we find that

$$\triangle_{T,\varepsilon}(u) \leqslant \frac{|u|}{T}\int_0^T \sup_{s\in K} |\zeta_{u_T}(s+i\tau;\mathfrak{a}) - \zeta(s+i\tau;\mathfrak{a})| d\tau.$$

Therefore, Lemma 2.5 implies, that uniformly in $|u| \leqslant C$ for any $C > 0$,

$$\lim_{n\to\infty} \triangle_{T,\varepsilon}(u) = 0.$$

This, together with (2.13) shows that, for all $u \in \mathbb{R}$,

$$\lim_{T\to\infty} \widehat{\phi}_{T,\mathfrak{a}}(u) = \phi_{\zeta,u}(u).$$

An application of Lemma 2.9 and the weak convergence of $F_{T,\mathfrak{a}}(\varepsilon)$ to $F_{\zeta,\mathfrak{a}}(\varepsilon)$ as $T \to \infty$ gives the relation

$$\lim_{T\to\infty} \widehat{F}_{T,\mathfrak{a}}(\varepsilon) = F_{\zeta,u}(\varepsilon)$$

for all continuity points ε of $F_{\zeta,\mathfrak{a}}(\varepsilon)$. It is well known that the set of discontinuity points of a distribution function is at most countable. This and Proposition 2.1 show that the limit

$$\lim_{t\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta_{u_T}(s+i\tau;\mathfrak{a})-f(s)|<\varepsilon\right\}=$$

$$m_{1H}\left\{\omega_1\in\Omega_1:\sup_{s\in K}|\zeta(s,\omega_1;\mathfrak{a})-f(s)|<\varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon>0$.

The theorem is proved.

Chapter 3

Approximation of analytic functions by discrete shifts of an absolutely convergent Dirichlet series related to the periodic - zeta function

This chapter is devoted to a discrete analogue of Theorem 2.1. In this case, analytic functions are approximated by shifts $\zeta_{u_N}(s + ikh; \mathfrak{a})$, where $h > 0$ is a fixed number, $k \in \mathbb{N}_0$, and u_N is a certain increasing sequence of real numbers. We notice that approximation of analytic functions by disrete shifts of zeta-functions is sometimes more convenient because of a possible easier detection of approximating shifts.

We preserve the notation of Chapter 2. Recall that $\#A$ denotes the number of elements of a discrete set $A \subset \mathbb{R}$.

3.1 Statement of the main theorem

For $h > 0$, define the set

$$L(\mathbb{P}; h, \pi) = \left\{ (\log p : p \in \mathbb{P}), \frac{2\pi}{h} \right\}.$$

Theorem 3.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the set $L(\mathbb{P}; h, \pi)$*

is linearly independent over $\mathbb{Q}$, and $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit

$$\lim_{N\to\infty} \frac{1}{N+1} \#\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta_{u_N}(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\} =$$

$$m_{1H}\{\omega_1 \in \Omega_1 : \sup_{s\in K} |\zeta(s,\omega_1;\mathfrak{a}) - f(s)| < \varepsilon\}$$

and is positive for all but at most countably many $\varepsilon > 0$.

Theorem 3.1 implies that there exists a number $N_0 = N_0(f, K, \varepsilon) \in \mathbb{N}$ such that there are infinitely many shifts $\zeta_{u_N}(s + ikh; \mathfrak{a})$ for $N \geqslant N_0$ approximating with accuracy ε a given analytic function $f(s)$.

We observe that the linear independence of the set $L(\mathbb{P}; h, \pi)$ can't be replaced by that of the set

$$\widehat{L}(\mathbb{P}; h, \pi) \stackrel{\text{def}}{=} \{\log p : p \in \mathbb{P}\} \cup \left\{\frac{2\pi}{h}\right\}.$$

Indeed, if $h = \frac{2\pi}{\log p_0}, p_0 \in \mathbb{P}$, then we have

$$\widehat{L}(\mathbb{P}; h, \pi) = \{\log p : p \in \mathbb{P}\}$$

It is well known that the set of logarithms of primes is linearly independent over $\mathbb{Q}$, i.e. the set $\widehat{L}(\mathbb{P}; h, \pi)$ is linearly independent over $\mathbb{Q}$, while the set $L(\mathbb{P}; h, \pi)$ contains two equal elements, and is linearly dependant over $\mathbb{Q}$. Thus, we consider $L(\mathbb{P}; h, \pi)$ as a multiset.

A proof of Theorem 3.1 is based on a discrete probabilistic limit theorem for weakly convergent probabilistic measures in the space of analytic functions $H(D)$, and a discrete universality theorem for $\zeta(s; \mathfrak{a})$.

We recall that a number a is transcendental, if it is not algebraic, i.e., is not a root of any polynomial $p(s) \not\equiv 0$ with rational coefficients. For example, the numbers e, π and e^π are transcendental.

For example, we can take $h = 2$. If we suppose that $k_1, \dots, k_r, \widehat{k} \in \mathbb{Z}$ are not all zeros, and

$$k_1 \log p_1 + \ldots + k_r \log p_r + \widehat{k}\pi = 0,$$

then we obtain

$$p_1^{k_1} \ldots p_r^{k_r} e^{\widehat{k}\pi} = 1$$

which contradicts the transcendence of the number e^{π}.

Also, if $h = \pi$ and,

$$k_1 \log_{p_1} + \ldots + k_r \log_{p_r} + \widehat{k} = 0,$$

then $p_1^{k_1} \ldots p_r^{k_r} p^{\widehat{K}} = 1$, and this contradicts the transcendence of e^a because the Lindemann-Weierstrass theorem asserts that the number e^a with algebraic $a \neq 0$ is transcendental.

3.2 Discrete approximation in the mean

This section is devoted to discrete estimation in the mean between the functions $\zeta(s;\mathfrak{a})$ and $\zeta_u(s;\mathfrak{a})$. The estimate will be used for the proof of the universality for $\zeta(s;\mathfrak{a})$ and Theorem 3.1.

Proposition 3.1. *Suppose that $K \subset D$ is a compact set, $h > 0, \mathfrak{a}$ and $u \geqslant 1$. Then there exists $\varepsilon = \varepsilon(K) > 0$ such that the estimate*

$$\frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} |\zeta(s+ikh;\mathfrak{a}) - \zeta_u(s+ikh;\mathfrak{a})| \ll_{\varepsilon,\mathfrak{a},h,K} u^{-\varepsilon} + u^{\frac{1}{2}-2\varepsilon} \frac{\log N}{N}$$

holds.

For the proof of Proposition 3.1, discrete mean square estimates are needed. For this, the Gallagher lemma on connection between discrete and continuous mean squares for some differentiable functions usually is applied.

Lemma 3.1. *Suppose that $\delta > 0, T_0 > \delta, T > \delta$, $A \neq \varnothing$ is a finite set lying in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$, and*

$$N_\delta(t) = \sum_{\substack{x \in A \\ |t-x| < \sigma}} 1.$$

Let a complex-valued function $g(t)$ be continuous in $[T_0, T_0+T]$ and have a continuous derivative $g'(t)$ in (T_0, T_0+T). Then the inequality

$$\sum_{t \in A} N_\sigma^{-1}(t) |g(t)|^2 \leqslant \frac{1}{\delta} \int_{T_0}^{T_0+T} |g(x)|^2 dx + \left(\int_{T_0}^{T_0+T} |g(x)|^2 dx \int_{T_0}^{T_0+T} |g'(x)|^2 dx \right)^{\frac{1}{2}}$$

holds.

A proof of the lemma can be found in [44], Lemma 1.4. Also we recall the Cauchy integral formula.

Lemma 3.2. *Suppose that a function $g(s)$ is analytic in a finite region $G \subset \mathbb{C}$ and on its boundary ∂G which is a simple closed curve. Then, for every $s \in G$, the formula*

$$g'(s) = \frac{1}{2\pi i} \int_{\partial G} \frac{g(z)}{(z-s)^2} dz$$

is valid.

A proof of the lemma is given in all textbooks on the theory of function of complex variable, see, for example, [53].

Lemma 3.3. *Let $\frac{1}{2} < \sigma < 1$ and $h > 0$ be fixed. Then, for $v \in \mathbb{R}$, the mean square estimate*

$$\sum_{k=0}^{N} |\zeta(\sigma + ikh + iv; \mathfrak{a})|^2 \ll_{\sigma,\mathfrak{a},h} N(1 + |v|)$$

holds.

Proof. Repeating the proof of estimate (2.7), we obtain

$$\int_0^T |\zeta(\sigma + i\tau + iu; \mathfrak{a})|^2 d\tau \ll_{\sigma,\mathfrak{a}} T(1 + |v|). \tag{3.1}$$

Let L be a simple closed contour lying in D and enclosing the point s. Then, by Lemma 3.1

$$\zeta'(s; \mathfrak{a}) = \frac{1}{2\pi i} \int_L \frac{\zeta(z; \mathfrak{a})}{(z-s)^2} dz.$$

Hence, for L_1 enclosing σ,

$$\zeta'(\sigma + i\tau; \mathfrak{a}) = \frac{1}{2\pi i} \int_{L_1} \frac{\zeta(z + i\tau; \mathfrak{a})}{(z-\sigma)^2} dz$$

and

$$\zeta'(\sigma + i\tau; \mathfrak{a}) \ll \int_{L_1} \frac{|dz|}{|z-\sigma|^4} \int_{L_1} |\zeta(z + i\tau; \mathfrak{a})|^2 |dz| \ll_{\sigma,L_1}$$

$$\int_{L_1} |\zeta(z + i\tau; \mathfrak{a})|^2 |dz|.$$

Therefore, by (2.5) we find

$$\int_{-T}^{T}|\zeta'(\sigma+i\tau;\mathfrak{a})|^2 d\tau \ll_{\sigma,L_1} \int_{L_1}|dz|\int_{-T}^{T}|\zeta(z+i\tau;\mathfrak{a})|^2 d\tau \ll_{\sigma,L_1}$$

$$\int_{L_1}|dz|\int_{-T}^{T}|\zeta(Rez+Imz+i\tau;\mathfrak{a})|^2 d\tau \ll_{\sigma} T$$

because L_1 depends on σ. From this, we obtain that

$$\int_{0}^{T}|\zeta'(\sigma+i\tau+iv;\mathfrak{a})|^2 d\tau \ll \int_{-|v|}^{T+|v|}|\zeta'(\sigma+i\tau;\mathfrak{a})|^2 d\tau \ll_{\sigma,\mathfrak{a}} T(1+v). \quad (3.2)$$

Now we apply Lemma 2.1. We take $\delta = h, T_0 = \frac{3h}{2}, T = Nh - h$. Then we have $N_\delta(t) = 1$, and Lemma 3.1 together with (3.1) and (3.2) gives

$$\sum_{k=2}^{N}|\zeta(\sigma+ikh+iv;\mathfrak{a})|^2 \ll_{\sigma,\mathfrak{a},h} \int_{\frac{3h}{2}}^{(N+1)h}|\zeta(\sigma+i\tau+iv;\mathfrak{a})|^2 d\tau + \quad (3.3)$$

$$\left(\int_{\frac{3h}{2}}^{(N+1)h}|\zeta(\sigma+it+iv;\mathfrak{a})|^2 d\tau \int_{\frac{3h}{2}}^{(N+1)h}|\zeta'(\sigma+it+iv;\mathfrak{a})|^2\right)^{\frac{1}{2}} \ll_{\sigma,\mathfrak{a},h}$$

$$N(1+|v|).$$

Since $\zeta(\sigma+it,\alpha) \ll_\alpha 1+|t|$ [48], by (2.1), we have that also

$$\zeta(\sigma+it;\mathfrak{a}) \ll_{\sigma,\mathfrak{a}} (i+|t|).$$

Therefore,

$$\sum_{k=0}|\zeta(\sigma+ikh+iv;\mathfrak{a})|^2 \ll_{\sigma,\mathfrak{a},h} (1+|v|).$$

This and (3.3) prove the lemma.

Now, we are ready to prove Proposition 3.1.

Proof of proposition 3.1. As in the proof of Lemma 2.5, we use the integral representation of Lemma 2.3. Let $K \subset D$ be an arbitrary compact set. Then K is bounded, thus, there exists $\varepsilon > 0$ such that K lies in the strip $\{s \in \mathbb{C} : \frac{1}{2}+2\varepsilon \leqslant \sigma \leqslant 1-\varepsilon\}$. For s from that strip, we have $\theta_1 \overset{\text{def}}{=} \sigma - \frac{1}{2} - \varepsilon > 0$. Therefore, by Lemma (2.3) and the residue theorem, we obtain, for $s \in K$,

$$\zeta_u(s;\mathfrak{a}) - \zeta(s;\mathfrak{a}) = \frac{1}{2\pi i}\int_{-\theta_1-i\infty}^{-\theta_1+i\infty}\zeta(s+z;\mathfrak{a})\, l_n(z)\frac{dz}{z} + \frac{\widehat{a}\, l_n(1-s)}{1-s}.$$

Therefore, for $s \in K$,

$$\zeta(s+ikh;\mathfrak{a}) - \zeta_u(s+ikh;\mathfrak{a}) \ll \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+it+iv;\mathfrak{a}\right)\right| \times$$

$$\left|\frac{l_u(\frac{1}{2}+\varepsilon-\sigma+iv)}{\frac{1}{2}+\varepsilon-\sigma+iv}\right| dv + \frac{|\widehat{a}||l_u(1-s-ikh)|}{|1-s-ikh|} \ll$$

$$\int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+iv;\mathfrak{a}\right)\right| \sup_{s\in K}\left|\frac{l_u(\frac{1}{2}+\varepsilon-s-iv)}{\frac{1}{2}+\varepsilon-s-iv}\right| dv +$$

$$\sup_{s\in K}\frac{|\widehat{a}|l_u(1-s-ikh)}{|1-s-ikh|}.$$

Hence,

$$\frac{1}{N+1}\sum_{k=0}^{N}|\zeta(s+ikh;\mathfrak{a})-\zeta_u(s+ikh;\mathfrak{a}))| \ll \tag{3.4}$$

$$\int_{-\infty}^{\infty}\left(\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+iv;\mathfrak{a}\right)\right|\right)\sup_{s\in K}\left|\frac{l_u(\frac{1}{2}+\varepsilon-s+iv)}{\frac{1}{2}+\varepsilon-s+iv}\right| dv +$$

$$|\widehat{a}|\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\left|\frac{l_u(1-s-ikh)}{1-s-ikh}\right| \overset{\text{def}}{=} I+Z.$$

The Cauchy inequality and Lemma 3.2 show that

$$\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+iv;\mathfrak{a}\right)\right| \ll \tag{3.5}$$

$$\left(\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+iv;\mathfrak{a}\right)\right|\right)^{\frac{1}{2}} \ll_{\varepsilon,\mathfrak{a},h} (1+|v|)^{\frac{1}{2}}.$$

By the definition of $l_u(s)$ and Lemma 2.2, for $s \in K$,

$$\frac{l_u\left(\frac{1}{2}+\varepsilon-s+iv\right)}{\frac{1}{2}+\varepsilon-s+iv} \ll_\theta u^{\frac{1}{2}+\varepsilon-\sigma}\left|\Gamma\left(\frac{1}{\theta}\left(\frac{1}{2}+\varepsilon-it+iv\right)\right)\right| \ll_\varepsilon$$

$$u^{-\varepsilon}\exp\left\{-\frac{c}{\theta}|v-t|\right\} \ll_{\varepsilon,K} u^{-\varepsilon}\exp\left\{-\frac{c}{\theta}|v|\right\}.$$

This together with (3.5) implies the bound

$$I \ll_{\varepsilon,\mathfrak{a},h,K} u^{-\varepsilon}\int_{-\infty}^{\infty}(1+|v|)^{\frac{1}{2}}\exp\left\{-\frac{c}{\theta}|v|\right\} dv \ll_{\varepsilon,\mathfrak{a},h,K} u^{-\varepsilon}. \tag{3.6}$$

Similarly, using Lemma 2.2, we obtain that, for all $s \in K$,

$$\frac{l_u(1-\sigma-it+ikh)}{1-\sigma-it+ikh} \ll_{\varepsilon,K} u^{1-\sigma} \exp\left\{-\frac{ch}{\theta}k\right\}.$$

Therefore,

$$Z \ll_{\varepsilon,K,\mathfrak{a}} \frac{u^{\frac{1}{2}-2\varepsilon}}{N} \sum_{k=0}^{N} \exp\left\{-\frac{chk}{\theta}\right\} \ll_{\varepsilon,K,\mathfrak{a}}$$

$$\frac{1}{N} u^{\frac{1}{2}-2\varepsilon} \left(\sum_{k \leqslant \log T} + \sum_{k \geqslant \log T} \right) \exp\left\{-\frac{chk}{\theta}\right\} \ll_{\varepsilon,K,\mathfrak{a},h} u^{\frac{1}{2}-2\varepsilon} \frac{\log N}{N}.$$

The latter estimate together with (3.6) shows that

$$I + Z \ll_{\varepsilon,K,\mathfrak{a},u^{-q},h} u^{\frac{1}{2}-2\varepsilon} \frac{\log N}{N}.$$

This and (3.4) prove the proposition.

3.3 Discrete universality of the function $\zeta(s;\mathfrak{a})$

The proof of the Theorem 3.1, as Theorem 2.1, uses a limit theorem and universality theorem for the function $\zeta(s;\mathfrak{a})$. The discrete versions of such theorems in a more general form were obtained in [41].

Let $w(t)$ be a non-increasing positive for $t \geqslant 1$ function, which has a continuous derivative satisfying the requirements: for $h > 0$, $w(t) \ll_h w(ht)$ and $(w'(t))^2 \ll_h w(t)$. Define

$$V(N,w) = \sum_{k=0}^{N} w(k)$$

and suppose that

$$\lim_{t\to\infty} V(N,w) = +\infty.$$

Denote by V the class of the above functions $w(t)$. Moreover, let $I(A)$ be the indicator function of the set $A = \{k\}$, i.e.,

$$I_k(A) = \begin{cases} 1, & \text{if } k \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then, in [41], the following weighted universality theorem for the function

$\zeta(s;\mathfrak{a})$ has been proved.

Lemma 3.4. *Suppose that $w \in V$, the sequence $\mathfrak{a}$ is multiplicative and the set $L(\mathbb{P}; h, \pi)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=0}^{N} w(k) I_k\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a})-f(s)| < \varepsilon\} > 0.$$

Moreover, "$\liminf$" can be replaced by "$\lim$" for all but at most countably many $\varepsilon > 0$.

Clearly, the function $w(t) \equiv 1$ belongs to the class V. Thus, from Lemma 3.4 we have the following statement.

Lemma 3.5. *Suppose that the sequence $\mathfrak{a}$ is multiplicative and the set $L(\mathbb{P}; h, \pi)$ linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} N+1\#\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\} =$$

$$m_{1H}\{\omega_1 \in \Omega_1 : \sup_{s\in K} |\zeta(s,\omega_1;\mathfrak{a}) - f(s)| < \varepsilon\} > 0.$$

Moreover, the limit

$$\liminf_{N\to\infty} \#\{1 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\} =$$

$$m_{1H}\{\omega_1 \in \Omega_1 : \sup_{s\in K} |\zeta(s,\omega_1;\mathfrak{a}) - f(s)| < \varepsilon\} > 0.$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Proof. Clearly, $W(N,1) = N, N+1$ and

$$\sum_{k=1}^{N} I_k\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\} =$$

$$\#\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\}.$$

Therefore, the lemma is a specific case of Lemma 3.4. The left-hand sides of equalities of the theorems appear from the proof of Lemma 3.4, where these sides are

$$P_{\zeta,\mathfrak{a}} = \left(\{\omega_1 \in \Omega_1 : \sup_{s\in K} |\zeta(s,\omega_1;\mathfrak{a}) - f(s)| < \varepsilon\}\right).$$

3.4 Discrete limit theorem

For $A \in \mathcal{B}(H(D))$, define

$$P_{N,\mathfrak{a},h}(A) = \frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \zeta(s+ikh;\mathfrak{a}) \in A\}.$$

The following limit theorem will be useful in the sequel.

Theorem 3.2. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, and the set $L(\mathbb{P};h,\pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,\mathfrak{a},h}$ converges weakly to $P_{\zeta,\mathfrak{a}}$ as $N \to \infty$.*

We divide a proof of Theorem 3.2 into lemmas. We use a similar but simpler method as for a weighted limit theorem in [41]. We start with a limit lemma for probability measures on $(\Omega_1, \mathcal{B}(\Omega_1))$. For $A \in \mathcal{B}(\Omega_1)$, set

$$Q_{N,h}(A) = \frac{1}{N+1}\#\{0 \leqslant N : (p^{-ikh} : p \in \mathbb{P}) \in A\}. \tag{3.7}$$

Lemma 3.6. *Suppose that the set $L(\mathbb{P},h,\pi)$ is linearly independent over $\mathbb{Q}$. Then, $Q_{N,h}$ converges weakly to the measure m_{1H} as $N \to \infty$.*

Proof. We apply the Fourier transform method. Let $g_{N,h}(\underline{k}), \underline{k} = (k_p : k_p \in Z, p \in \mathbb{P})$ be the Fourier transform of the measure $Q_{N,h}$, i.e.,

$$g_{N,k}(\underline{k}) = \int_{\Omega_1} \left(\prod_{p\in\mathbb{P}}^{*} \omega_1^{k_p}(p)\right) dQ_{N,h},$$

where the star $*$ indicates that only a finite number of integer numbers k_p are not zeros. Thus, in view of (3.7),

$$g_{N,h}(\underline{k}) = \frac{1}{N+1}\sum_{k=0}^{N}\prod_{p\in\mathbb{P}}^{*} p^{-ikk_p h} =$$

$$\frac{1}{N+1}\sum_{k=0}^{N} \exp\{-ikh\sum_{p\in\mathbb{P}}^{*} k_p \log p\}. \tag{3.8}$$

Since, for $\underline{k} = \underline{0}$,

$$A_{\underline{k}}^{k}(h) = 1, A_{\underline{k}}(h) = \exp\{-ih \sum_{p\in\mathbb{P}}^{*} k_p \log p\},$$

we have, by (3.2),

$$g_{N,h}(\underline{0}) = 1. \tag{3.9}$$

In the case $\underline{k} \neq \underline{0}$, we apply the linear independence of the set $L(\mathbb{P}; h, \pi)$. We observe that, in this case,

$$A_{\underline{k}}(h) \neq 1. \tag{3.10}$$

Indeed, since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over Q, we have

$$\sum_{p\in\mathbb{P}}^{*} k_p \log p \neq 0. \tag{3.11}$$

Moreover, if $A_{\underline{k}}(h) = 1$, then there exists $r \in \mathbb{Z}$ such that

$$A_{\underline{k}}(h) = e^{2\pi i r}.$$

Hence, by the definition of $A_{\underline{k}}(h)$,

$$h \sum_{p\in\mathbb{P}}^{*} k_p \log p = 2\pi r_1, r_1 \in \mathbb{Z},$$

and

$$\sum_{p\in\mathbb{P}}^{*} k_p \log p + \frac{2\pi r_2}{h} = 0, r_2 \in \mathbb{Z}.$$

However, the latter equality contradicts the linear independence over $\mathbb{Q}$ of the set $L(\mathbb{P}; h, \pi)$. This and (3.11) show that (3.10) is valid.

Now, a simple application of the formula for the sum of geometric progression, and (3.10) and (3.8) give, for $\underline{k} \neq \underline{0}$, that

$$g_{N,h}(\underline{k}) = \frac{1 - A_{\underline{k}}^{N+1}(h)}{(N+1)(1 - A_k(h))}.$$

This together with (3.9) shows that

$$\lim_{N\to\infty} g_{N,h}(\underline{k}) = \begin{cases} 1, & \text{if } \underline{k} = \underline{0} \\ 0, & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

i.e., the Fourier transform of $Q_{N,h}$ converges to that of the Haar measure m_{1H} as $N \to \infty$. This proves the lemma.

The next step of the proof of Theorem 2.2 is devoted to a limit lemma for absolutely convergent Dirichlet series. We consider the Dirichlet series (2.2) with $u = n \in \mathbb{N}$. For $A \in \beta(H(D))$, define

$$P_{N,n,h}(A) = \frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \zeta_n(s + ikh; \mathfrak{a}) \in A\}.$$

Moreover, let $u_{n,\mathfrak{a}} : \Omega_1 \to H(D)$ be given by

$$u_{n,\mathfrak{a}}(\omega_1) = \sum_{m=1}^{\infty} \frac{a_m \omega_1(m) v_n(m)}{m^s} \stackrel{\text{def}}{=} \zeta_n(s, \omega_1; \mathfrak{a}),$$

where

$$\omega_1(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

The absolute convergence of the series for $\zeta_n(s; \mathfrak{a})$ ensures the continuity of $u_{n,\mathfrak{a}}$. Moreover, the definition of $u_{n,\mathfrak{a}}$ implies that

$$P_{N,n,h} = Q_{N,h} u_{n,\mathfrak{a}}^{-1},$$

where

$$Q_{N,h} u_{n,\mathfrak{a}}(A) = Q_{N,h}(u_{n,\mathfrak{a}}^{-1} A, A \in \mathcal{B}(H(D)).$$

Those remarks and a standard way of preservation of weak convergence under continuous mappings, Theorem 5.1 of [5], gives the following lemma.

Lemma 3.7. *Suppose that the set $L(\mathbb{P}; h, \pi)$ is lineraly idependent over $\mathbb{Q}$. Then $P_{N,n,h}$ converges weakly to the measure $R_{n,\mathfrak{a}} \stackrel{def}{=} m_{1H} u_{n,\mathfrak{a}}^{-1}$ as $N \to \infty$.*

The weak convergence of $R_{n,\mathfrak{a}}$ as $n \to \infty$ is very important for the proof of Theorem 3.2

Lemma 3.8. *$R_{n,\mathfrak{a}}$ converges weakly to $P_{\zeta,\mathfrak{a}}$ as $n \to \infty$*

Proof. The measure $R_{n,\mathfrak{a}}$ depends only on n and sequence $\mathfrak{a}$. The same measure is involved in the proof of Lemma 2.6. It is proved that the measure $P_{T,\mathfrak{a}}$ (in the notation of section 2.4), as $T \to \infty$, and $R_{n,\mathfrak{a}}$ as $n \to \infty$ converge weakly to the same limit measure, and this limit measure is $P_{\zeta,\mathfrak{a}}$.

Now we recall the metric in the space $H(D)$ which induces its topology of uniform convergence on compacta. The desired metric $\rho(g_1, g_2)$, $g_1, g_2 \in H(D)$, is defined by the formula

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|}.$$

Here $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact embedded sets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

and if $K \subset D$ is arbitrary compact set, then K lies in a certain set K_l.

Lemma 3.9. *For every $h > 0$ and $\mathfrak{a}$, the equality*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\left(\zeta(s + ikh; \mathfrak{a}) - \zeta_n(s + ikh; \mathfrak{a})\right) = 0.$$

Proof. We use Proposition 3.1 with $u = n$. For every compact set $K \in D$, this gives

$$\frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+ikh; \mathfrak{a}) - \zeta_n(s+ikh; \mathfrak{a})| \ll_{\varepsilon,\mathfrak{a},h,K} n^{-\varepsilon} + n^{\frac{1}{2}-2\varepsilon} \frac{\log N}{N}.$$

Hence, for every $n \in \mathbb{N}$,

$$\limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s + ikh; \mathfrak{a}) - \zeta_n(s + ikh; \mathfrak{a})| \ll_{\varepsilon,\mathfrak{a},h,K} n^{-\varepsilon}.$$

Now, letting $n \to \infty$, we obtain the lemma.

The next lemma is a probabilistic statement on convergence in distribution $\xrightarrow{\mathcal{D}}$. Let X_n, $n \in \mathbb{N}$, and X be $\mathbb{X}-$valued random element defined on a certain probability space $(\widehat{\Omega}, \mathcal{A}, \mu)$, and P_n and P their distributions, respectively, i.e., for every $A \in \mathcal{B}(\mathbb{X})$,

$$P_n(A) = \{\widehat{\omega} \in \widehat{\Omega} : X_n(\widehat{\omega}) \in A\}$$

and

$$P(A) = \{\widehat{\omega} \in \widehat{\Omega} : X(\omega) \in A\}.$$

We say that $X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$ (converges in distribution) if the measure P_n converges weakly to P as $n \to \infty$.

Lemma 3.10. *Suppose that the metric space* $(\mathbb{X}, d)$ *is a separable,* $\mathbb{X}-$*valued random elements* X_{kn} *and* $Y_n, k \in \mathbb{N}, n \in \mathbb{N}$*, are defined on the probability space* $(\widehat{\Omega}, \mathcal{A}, \mu)$*, and*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k, \forall k \in \mathbb{N},$$

and

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X.$$

If, for every $\varepsilon > 0$*,*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\{\widehat{\omega} \in \widehat{\Omega} : d(X_{kn}(\widehat{\omega}), Y_n(\widehat{\omega})) \geqslant \varepsilon\} = 0,$$

then $Y_n \xrightarrow{\mathcal{D}} X$*.*

Proof of the lemma is given in [5], Theorem 4.2.

Proof of Theorem 3.2. We will apply Lemma 3.9 for $H(D)-$ valued random elements because it is well known that the space $H(D)$ is separable. On the probability space $(\widehat{\Omega}, \mathcal{A}, \mu)$, define a random variable ζ_N having the distribution

$$\mu\{\xi_N = kh\} = \frac{1}{N+1}, k = 0, \ldots, N.$$

Using the random variable ξ_N, define two $H(D)-$ valued random elements

$$X_{N,n,\mathfrak{a}} = X_{N,n,\mathfrak{a}}(s) = \zeta_n(s + i\xi_n; \mathfrak{a})$$

and

$$X_{N,\mathfrak{a}} = X_{N,\mathfrak{a}}(s) = \zeta(s + i\xi_n; \mathfrak{a}).$$

Moreover, let $X_{n,\mathfrak{a}}$ be the $H(D)-$valued random element with the distribution $R_{N,\mathfrak{a}}$. Then, in view of Lemma 3.7, we have

$$X_{N,n,\mathfrak{a}} \xrightarrow[N\to\infty]{\mathcal{D}} X_{n,\mathfrak{a}}, \tag{3.12}$$

while Lemma 3.8 implies

$$X_{n,\mathfrak{a}} \xrightarrow[n\to\infty]{\mathcal{D}} P_{z,\mathfrak{a}}. \tag{3.13}$$

Application of Lemma 3.8 and definitions of the above random elements give, for $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu \left\{ \rho \left(X_{N,a}(s), X_{N,n,\mathfrak{a}}(s) \right) \geqslant \varepsilon \right\} =$$

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \#\{0 \leqslant k \leqslant N : \rho(\zeta(s+ikh;\mathfrak{a}), \zeta_n(s+ikh;\mathfrak{a}) \geqslant \varepsilon\} \leqslant$$

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{k=0}^{N} \rho(\zeta(s+ikh;\mathfrak{a}), \zeta_n(s+ikh;\mathfrak{a}) = 0.$$

The latter equality together with (3.12) and (3.13) shows that all hypotheses of Lemma 3.10 are satisfied. Thus, we have

$$X_{N,\mathfrak{a}} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\zeta,\mathfrak{a}},$$

and this is equivalent to the assertion of the theorem.

3.5 Proof of Theorem 3.1

First we state a lemma on the distance between $\zeta(s;\mathfrak{a})$ and $\zeta_{u_N}(s;\mathfrak{a})$.

Lemma 3.11. *Suppose that $K \subset D$ is a compact set, and $u_N \to \infty$ and $u_N \ll N^2$. Then, for every $h > 0$, the equality*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - \zeta_{u_N}(s+ikh;\mathfrak{a})| = 0$$

holds.

Proof. We take $u = u_N$ in Proposition 3.1. This gives the estimate

$$\frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - \zeta_{u_N}(s+ikh;\mathfrak{a})| \ll_{\varepsilon,\mathfrak{a},h,K} u_N^{-\varepsilon} + u_N^{\frac{1}{2}-2\varepsilon} \frac{\log N}{N}.$$

Since $u_N \ll N^2$, this gives the equality of the lemma.

Proof of Theorem 3.1. Define the set

$$G_\varepsilon = \{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\},$$

where K and $f(s)$ are from Theorem 3.1. Consider the distribution functions

$$F_{N,\mathfrak{a},h}(\varepsilon) \stackrel{\text{def}}{=} P_{N,\mathfrak{a},h}(G_\varepsilon) = \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K}|\zeta(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\right\},$$

$$\widehat{F}_{N,\mathfrak{a},h}(\varepsilon) \stackrel{\text{def}}{=} \frac{1}{N+1}\#\left\{0 \leqslant K \leqslant N : \sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\right\}$$

and

$$F_{\zeta;\mathfrak{a}}(\varepsilon) \stackrel{\text{def}}{=} P_{\zeta,\mathfrak{a}}(G_\varepsilon) = m_{1H}\{\omega_1 \in \Omega_1 : \sup_{s\in K}|\zeta(s,\omega_1;\mathfrak{a}) - f(s)| < \varepsilon\}.$$

Since

$$\partial G_\varepsilon = \{g \in H(D) : \sup_{s\in K}|g(s)-f(s)| \leqslant \varepsilon\}\backslash\{g \in H(D) : \sup_{s\in K}|g(s)-f(s)| < \varepsilon\},$$

we have

$$P_{\zeta,\mathfrak{a}}(\partial G_\varepsilon) = m_{1H}\{\omega_1 \in \Omega_1 : \sup_{s\in K}|\zeta(s,\omega_1;\mathfrak{a}) - f(s)| \leqslant \varepsilon\} -$$

$$m_{1H}\{\omega_1 \in \Omega_1 : \sup_{s\in K}|\zeta(s,\omega_1;\mathfrak{a}) - f(s)| < \varepsilon\} = F_{\zeta,\mathfrak{a}}(\varepsilon+0) - F_{\zeta,\mathfrak{a}}(\varepsilon).$$

This shows that $F_{\zeta,\mathfrak{a}}(\varepsilon+0) = F_{\zeta,\mathfrak{a}}(\varepsilon)$ if and only if $P_{\zeta,\mathfrak{a}}(\partial G_\varepsilon) = 0$. Therefore, the point ε is a continuity point of the distribution function $F_{\zeta,\mathfrak{a}}$ if and only if the set G_ε is a continuity set of the measure $P_{\zeta,\mathfrak{a}}$. By Theorem 3.2, the measure $P_{N,\mathfrak{a},h}$ converges weakly to $P_{\zeta,\mathfrak{a}}$ as $N \to \infty$. Therefore, in view of Lemma 2.7, $P_{N,\mathfrak{a},h}$ converges to $P_{\zeta,\mathfrak{a}}(_\varepsilon)$, as $N \to \infty$, for all continuity sets G_ε of $P_{\zeta,\mathfrak{a}}$. This and the above remark on continuity of the set G_ε show that $F_{N,\mathfrak{a},h}$, as $N \to \infty$, converges to the distribution function $F_{\zeta,\mathfrak{a}}(\varepsilon)$ at all its continuity points ε, in other words, $F_{N,\mathfrak{a},h}$ converges weakly to $F_{\zeta;\mathfrak{a}}$ as $N \to \infty$.

Denote by $\phi_{N,\mathfrak{a},h}(u)$, $\widehat{\phi}_{N,\mathfrak{a},h}(u)$ and $\phi_{\zeta,\mathfrak{a}}(u)$, $u \in \mathbb{R}$, the characteristic functions of the distribution functions $F_{N,\mathfrak{a},h}(\varepsilon)$, $\widehat{F}_{N,\mathfrak{a},h}(\varepsilon)$ and $F_{\zeta,\mathfrak{a}}(u)$, respectively. Since $F_{N,\mathfrak{a},h}(\varepsilon)$ converges weakly to $F_{\zeta;\mathfrak{a}}(\varepsilon)$ as $N \to \infty$, by Lemma 2.9, we have

$$\lim_{n\to\infty}\phi_{N;\mathfrak{a},h}(u) = \phi_{\zeta;\mathfrak{a}}(u) \tag{3.14}$$

uniformly in u in every finite interval. We have to show that $\widehat{\phi}_{N,\mathfrak{a},h}(u)$ also converges to $\phi_{\zeta;\mathfrak{a}}(u)$ as $N \to \infty$. By the definitions of characteristic functions,

$$\widehat{\phi}_{N,\mathfrak{a},h}(u) - \phi_{N,\mathfrak{a},h}(u) = \int_{-\infty}^{\infty} e^{iu\varepsilon} d\left(\widehat{F}_{N,\mathfrak{a},h}(\varepsilon) - F_{N,\mathfrak{a},h}(\varepsilon)\right) =$$

$$\frac{1}{N+1}\sum_{k=0}^{N}\Big(\exp\big\{iu\sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a})-f(s)|\big\}-$$

$$\exp\big\{iu\sup_{s\in K}|\zeta(s+ikh;\mathfrak{a})-f(s)|\big\}\Big).$$

Hence, in the virtue of the equality $|e^{ix}-1|\leqslant|x|, \forall x\in R$, we find

$$|\widehat{\phi}_{N,\mathfrak{a},h}(u)-\phi_{N,\mathfrak{a},h}(u)|\leqslant\frac{1}{N+1}\sum_{k=0}^{N}\Big|\exp\Big\{iu\big(\sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a})-f(s)|-$$

$$\sup_{s\in K}|\zeta(s+ikh;\mathfrak{a}-f(s)|\big)\Big\}-1\Big|\leqslant\frac{|u|}{N+1}\sum_{k=0}^{N}\Big|\sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a})-f(s)|-$$

$$\sup_{s\in K}|\zeta(s+ikh;\mathfrak{a})-f(s)|\Big|\leqslant\frac{|u|}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a})-\zeta(s+ikh;\mathfrak{a})|.$$

Here we have applied the triangle inequality. This and Lemma 3.11 show that

$$\widehat{\phi}_{N,\mathfrak{a},h}(u)-\phi_{N,\mathfrak{a},h}(u)=o(1)$$

as $N\to\infty$ uniformly in u in every finite interval. This together with (3.14) implies that

$$\lim_{N\to\infty}\widehat{\phi}_{N,\mathfrak{a},h}(u)=\phi_{\zeta,\mathfrak{a}}(u)$$

uniformly in u in every finite interval. Therefore, by Lemma 2.9 again, we obtain that the distribution function $\widehat{F}_{N,\mathfrak{a},h}(\varepsilon)$ converges weakly to $F_{\zeta,\mathfrak{a}}(\varepsilon)$ as $N\to\infty$. Thus, $\widehat{F}_{N,\mathfrak{a},h}(\varepsilon)$ converges weakly to $F_{\zeta,\mathfrak{a}}(\varepsilon)$ as $N\to\infty$ in all continuity points ε of the function $F_{\zeta,\mathfrak{a}}(\varepsilon)$. However, the distribution function has at most countably many of discontinuity points. This and the definitions of the distribution functions $\widehat{F}_{N,\mathfrak{a},h}(\varepsilon)$ and $F_{\zeta,\mathfrak{a}}(\varepsilon)$ show that the limit

$$\lim_{n\to\infty}\frac{1}{N+1}\#\{0\leqslant k\leqslant N:\sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a})-f(s)|<\varepsilon\}=$$

$$m_{1H}\{\omega_1\in\Omega_1:\sup_{s\in K}|\zeta(s+ikh;\mathfrak{a})-f(s)|<\varepsilon\}$$

exists and is positive for all but at most countably many ε. The positivity of the limit follows from Lemma 3.5. The theorem is proved.

Chapter 4

Joint approximation of analytic functions by continuous shifts of absolutely convergent Dirichlet series related to periodic and periodic Hurwitz zeta - functions

In Chapters 2 and 3, we approximated analytic functions by continuous shifts $\zeta_{u_T}(s+i\tau;\mathfrak{a})$ and discrete shifts $\zeta_{u_N}(s+ikh;\mathfrak{a})$ of absolutely convergent Dirichlet series

$$\zeta_{u_T}(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_{u_T}(m)}{m^s}$$

and

$$\zeta_{u_N}(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_{u_N}(m)}{m^s},$$

respectively, where a_m are coefficients of the periodic zeta-function

$$\zeta_{u_N}(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \sigma > 1.$$

We obtained (Theorems 2.1 and 3.1) that the above shifts, under restrictions that u_T and u_N increases to $+\infty$ not too fast $(u_T \ll T^2, u_N \ll N^2)$, and under requirement, that the set $P(\mathbb{P}; h, \pi)$ is linearly independent over $\mathbb{Q}$, approximate a wide class of analytic functions, in other words, the functions $\zeta_{u_T}(s; \mathfrak{a})$ and $\zeta_{u_N}(s; \mathfrak{a})$ are universal in approximating sense.

In this chapter, we consider a more complicated problems on simultaneous approximation of a pair of analytic functions by a pair of continuous shifts of absolutely convergent Dirichlet series. For this, we introduce for consideration one more zeta-function $-$ the periodic Hurwitz zeta-function.

Let $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $q_2 \in \mathbb{N}$, i.e., $b_{m+q} = b_m$ for all $m \in \mathbb{N}_0$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{b})$, $s = \sigma + i\tau$, with parameter α, $0 < \alpha \leqslant 1$, is defined, for $\sigma > 0$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}.$$

The periodicity of the sequence $\mathfrak{b}$ implies the equality

$$\zeta(s, \alpha; \mathfrak{b}) = \frac{1}{q_2^s} \sum_{k=0}^{q_2-1} b_k \zeta\left(s, \frac{k+\alpha}{q_2}\right). \tag{4.1}$$

Since the Hurwitz zeta-function $\zeta(s, \frac{k+\alpha}{q_2})$ is analytic in the whole complex plane, except for a simple pole at the point $s = 1$, equality (4.1) gives analytic continuation for the function $\zeta(s, \alpha; \mathfrak{b})$ to the whole complex plane, except a possible simple pole at the point $s = 1$ with residue

$$\frac{1}{q_2} \sum_{k=0}^{q_2-1} b_k \stackrel{\text{def}}{=} \widehat{b}.$$

If $\widehat{b} = 0$, then the function $\zeta(s, \alpha; \mathfrak{b})$ is entire. A traditional example of the function $\zeta(s, \alpha; \mathfrak{b})$ is the Lerch zeta-function

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}, \sigma > 1,$$

with rational parameter $\lambda = \frac{a}{b}$, $a, b \in \mathbb{N}$, because $e^{2\pi i \frac{a}{b} m}$ is periodic with period b. The theory of the Lerch zeta-function is given in [29]. The periodic Hurwitz zeta-function, as classical Hurwitz zeta-function, has no Euler's product over primes. Therefore, investigation of approximation of a pair of analytic functions by shifts of a pair $\big(\zeta(s;\mathfrak{a}), \zeta(s,\alpha;\mathfrak{b})\big)$ is an extension of the Mishou theorem [46] proved for a pair $\big(\zeta(s), \zeta(s,\alpha)\big)$ with transcendental α. For $\theta > \frac{1}{2}$, and $u > 0$, define

$$v_u(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{u}\right)^{\theta}\right\},$$

and

$$\zeta_u(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m v_u(m,\alpha)}{(m+\alpha)^s}.$$

Since the sequence $\{v_u(m,\alpha) : m \in \mathbb{N}_0\}$ is exponentially decreasing with respect to m, and the coefficients b_m are bounded, the Dirichlet series for $\zeta_u(s,\alpha;\mathfrak{b})$, as for $\zeta_u(s;\mathfrak{a})$, is absolutely convergent for every fixed $u > 0$ in any half-plane $\sigma > \sigma_0$ with arbitrary fixed finite σ_0.

In this chapter, we consider approximation of a pair of analytic functions by continuous shifts $\zeta_{u_T}(s+i\tau;\mathfrak{a}), \zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b})$ with some $u_T \to \infty$, multiplicative sequence $\mathfrak{a}$ and transcendental parameter α.

4.1 Statement of the main theorem

We preserve the notation of Chapter 2 concerning the function $\zeta_u(s;\mathfrak{a})$. Additionally, we define one more torus

$$\Omega_2 = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. The finite-dimensional torus Ω_2, as the torus Ω_1, by the Tikhonov theorem, with product topology and operation of pointwise multiplication, is a compact topological Abelian group. Therefore, on $(\Omega_2, \mathcal{B}(\Omega)_2))$, the probability Haar measure m_{2H} can be defined. Denote elements of Ω_2 by ω_2, and by $\omega_2(m)$ the mth component of $\omega_2 \in \Omega_2, m \in \mathbb{N}_0$.

Define one more product

$$\Omega = \Omega_1 \times \Omega_2.$$

Then, by the Tikhonov theorem again, Ω is a compact topological group because Ω_1 and Ω_2 are compact groups. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability

Haar measure m_H exists, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. We note that the measure m_H is the product of the Haar measures m_{1H} and m_{2H}. This means that if

$$A = A_1 \times A_2, A_1 \in \mathcal{B}(\Omega_1), A_2 \in \mathcal{B}(\Omega_2)$$

then $m_H(A) = m_{1H}(A_1) \cdot m_{2H}(A_2)$. By $\omega = (\omega_1, \omega_2), \omega_1 \in \Omega_1, \omega_2 \in \Omega_2$, denote elements of the group Ω. Denote $H^2(D) = H(D) \times H(D)$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D)-$valued random element

$$\underline{\zeta}(s, \alpha, \omega_1, \omega_2; \mathfrak{a}, \mathfrak{b}) = \left(\zeta(s, \omega_1; \mathfrak{a}), \zeta(s, \alpha, \omega_2; \mathfrak{b})\right),$$

where

$$\zeta(s, \omega_1; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega_1(m)}{m^s}$$

and

$$\zeta(s, \alpha, \omega_2; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m+\alpha)^s}$$

The main result of this chapter is the following statement.

Theorem 4.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the parameter α is transcendental, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, the limit*

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T}\text{meas}\{\tau \in [0,T] : &\sup_{s\in K_1} |\zeta_{u_T}(s+i\tau; \mathfrak{a}) - f_1(s)| < \varepsilon_1, \\ &\sup_{s\in K_2} |\zeta_{u_T}(s+i\tau, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon_2\} \\ m_H\{\omega \in \Omega : &\sup_{s\in K_1} |\zeta(s_1, \omega_1; \mathfrak{a}) - f_1(s)| < \varepsilon_1, \\ &\sup_{s\in K_2} |\zeta(s, \omega_2, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon_2\} \end{aligned}$$

exists and is positive for all but at most countable many $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

Theorem 4.1 implies that there is a number

$$T_0 = T_0(f_1, f_2, K_1, K_2, \varepsilon_1, \varepsilon_2) > 0$$

such that, for $T \geqslant T_0$, there are infinitely many shifts $\big(\zeta_{u_T}(s+i\tau; \mathfrak{a}), \zeta_{u_T}(s+i\tau, \alpha; \mathfrak{b})\big)$ having approximating property.

Theorem 4.1 extends the previous results on universality of Dirichlet series with period coefficients because a pair of analytic functions in this theorem is approximated simultaneously by shifts of absolutely convergent series. This property is a certain advantage in the estimation of approximated functions because it is more convenient to deal with absolutely convergent series.

The idea of the proof of Theorem 4.1 is similar to that of Theorem 2.1, a difference is that we consider a pair of functions in place of one function in Theorem 2.1.

Clearly, from Theorem 4.1, the following corollary follows.

Corollary 4.1. *Suppose that the parameter α is transcendental, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\{\tau \in [0,T] : \sup_{s\in K} |\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b}) - f(s)| < \varepsilon\} =$$
$$m_{2H}\{\omega_2 \in \Omega_2 : \sup_{s\in K} |\zeta_{u_T}(s,\omega_2,\alpha;\mathfrak{b}) - f(s)| < \varepsilon$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

For the proof of the above corollary, it suffices to take $\varepsilon = \infty$ in Theorem 2.1, and change a notation.

Theorem 4.1 also implies Theorem 2.1.

4.2 Integral representation

In this section, we recall the integral representation for the function $\zeta_u(s,\alpha;\mathfrak{b})$.

Lemma 4.1. *Let θ be the number from the definition of $v_u(m,\alpha)$. Then, for $\sigma > \frac{1}{2}$, the representation*

$$\zeta_u(s,\alpha;\mathfrak{b}) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z,\alpha;\mathfrak{b})\, l_u(z)\frac{dz}{z},$$

where

$$l_u(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right) u^s,$$

is valid.

Proof. Using of Lemma 2.1 gives

$$v_u(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{u}\right)^\theta\right\} = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \Gamma\left(\frac{z}{\theta}\right)\left(\left(\frac{m+\alpha}{u}\right)^\theta\right)^{(-\frac{z}{\theta})} d\left(\frac{z}{\theta}\right).$$

Hence,

$$\zeta_u(s;\alpha;\mathfrak{b}) = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\frac{b_m}{(m+\alpha)^s}\int_{\theta-i\infty}^{\theta+i\infty}\frac{z}{\theta}\Gamma\left(\frac{z}{\theta}\right)\left(\frac{m+\alpha}{u}\right)^{-z}dz =$$

$$\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\left(\frac{l_u(z)}{z}\sum_{m=0}^{\infty}\frac{b_m}{(m+\alpha)^{s+z}}\right)dz. \tag{4.2}$$

Since $\theta > \frac{1}{2}$ and $\sigma > \frac{1}{2}$, we have $Re(s+z) > 1$. Therefore,

$$\sum_{m=0}^{\infty}\frac{b_m}{(m+\alpha)^{s+z}} = \zeta(s+z,\alpha;\mathfrak{b}).$$

This and (4.2) show that

$$\zeta_u(s,\alpha;\mathfrak{a}) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z,\alpha;\mathfrak{b})\, l_u(z)\frac{dz}{z}.$$

4.3 Approximation in the mean

Let, for brevity,

$$\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b}) = \big(\zeta(s;\mathfrak{a}),\zeta(s,\alpha;\mathfrak{b})\big)$$

and

$$\underline{\zeta}_{u_T}(s,\alpha;\mathfrak{a},\mathfrak{b}) = \big(\zeta_{u_T}(s;\mathfrak{a}),\zeta_{u_T}(s,\alpha;\mathfrak{b})\big).$$

In this section, we will approximate $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$ by $\underline{\zeta}_{u_T}(s,\alpha;\mathfrak{a},\mathfrak{b})$ in the mean.

We start with recalling the metrics in the spaces $H(D)$ and $H^2(D)$. It is known that there exists a sequence $\{K_l : l\in\mathbb{N}\}$ of compact subsets of the strip D such that:

1° $D = \bigcup_{l=1}^{\infty} K_l$;

2° $K_l \subset K_{l+1}, l\in\mathbb{N}$;

3° if $K\subset D$ is a compact set, then K lies in a certain set K_l.

For example, we can take the sequence of embedded rectangles with edges parallel to the axis.

For $g_1, g_2 \in H(D)$, define

$$\rho_1(g_1,g_2) = \sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}.$$

Then ρ_1 is a metric on $H(D)$ which induces its topology of uniform convergence on compact sets.

Now, let $\underline{g}_1 = (g_{11}, g_{12})$, $\underline{g}_2 = (g_{21}, g_{22}) \in H^2(D)$. Then

$$\rho_2(\underline{g}_1, \underline{g}_2) = \max_{j=1,2} \rho(g_{1j}, g_{2j})$$

is a metric in $H^2(D)$ which induces its product topology.

Lemma 4.2. *Suppose that $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Then the equality*

$$\lim_{\to\infty} \frac{1}{T} \int_0^T \rho_2\big(\underline{\zeta}(s+i\tau, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_{u_T}(s+i\tau, \alpha; \mathfrak{a}, \mathfrak{b})\big) = 0$$

holds.

Proof. By the definition of the metric ρ_2, it suffices to show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(\zeta(s+i\tau; \mathfrak{a}), \zeta_{u_T}(s+i\tau; \mathfrak{a})\big) = 0$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(\zeta(s+i\tau, \alpha; \mathfrak{b}), \zeta_{u_T}(s+i\tau, \alpha; \mathfrak{b})\big) = 0.$$

The first of these equalities follows from Lemma 2.5 which states that, for every compact set $K \subset D$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} \big(\zeta(s+i\tau; \mathfrak{a}) - \zeta_{u_T}(s+i\tau; \mathfrak{a})\big) dt = 0,$$

and from the definition of the metric ρ. To prove the second equality, we have to show that, for every compact set $K \subset D$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} \big(\zeta(s+i\tau, \alpha; \mathfrak{b}) - \zeta_{u_T}(s+i\tau, \alpha; \mathfrak{b})\big) dt = 0. \tag{4.3}$$

A proof of equality (4.3) is similar to that of Lemma 2.5 for the function $\zeta(s; \mathfrak{a})$. In view of Lemma 4.1, for $\sigma > \frac{1}{2}$, we have

$$\zeta(s, \alpha; \mathfrak{b}) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s+z, \alpha; \mathfrak{b})\, l_{u_T}(z) \frac{dz}{z}. \tag{4.4}$$

We fix an arbitrary compact set $K \subset D$, and $\varepsilon > 0$ such that $\frac{1}{2} + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$

for all $s = \sigma + it \in K$. Let $\theta = \frac{1}{2} + \varepsilon$ and $\theta_1 = \sigma - \frac{1}{2} - \varepsilon > 0$. The function $\Gamma(s)$ has simple poles $s = -k$ with residues,

$$\frac{(-1)^{k-1}}{k!}$$

and the function $\zeta(s, \alpha; \mathfrak{b})$ has a possible simple pole at the point $s = 1$ with residue $\widehat{b}$. Therefore, the function $\Gamma\left(\frac{z}{\theta}\right)$ in the strip $-\theta_1 < Rez < \theta$ has only one simple pole at the point $z = 0$ with residue

$$\lim_{z\to 0} \Gamma\left(\frac{z}{\theta}\right) z = \lim_{z\to 0} \Gamma\left(\frac{z}{\theta}\right) \frac{z}{\theta} \cdot \theta = \theta$$

Actually, since $\theta_1 > \frac{1}{2} + \varepsilon - 1 + \varepsilon = -\frac{1}{2} + 2\varepsilon$, the pole $z = -\theta = -\frac{1}{2} - \varepsilon$ does not lie in the mentioned strip. Therefore, by the residue theorem and representation (4.4), we find, for $s \in K$,

$$\zeta_{u_T}(s, \alpha; \mathfrak{b}) - \zeta(s, \alpha; \mathfrak{b}) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s+z, \alpha; \mathfrak{b})\, l_{u_T}(z) \frac{dz}{z} + \widehat{b} \frac{l_{u_T}(1-s)}{1-s}.$$

Here $\zeta(s, \alpha; \mathfrak{b})$ is the residue of the integrand in (4.4) at the (4.5) pole $z = 0$, and

$$\frac{\widehat{b}\, l_{u_T}(1-s)}{1-s} = \mathrm{Res}_{z=1-s}\, \zeta(s+z, \alpha; \mathfrak{b})\, l_{u_T}(z) \frac{1}{z} =$$
$$\lim_{z=1-s} \frac{\zeta(s+z, \alpha; \mathfrak{b})(z-(1-s)) l_n(z)}{z} = \widehat{b}\, \frac{l_{u_T}(1-s)}{1-s}.$$

From (4.5), we obtain that for all $s \in K$,

$$\zeta_{u_T}(s+i\tau, \alpha; \mathfrak{b}) - \zeta(s+i\tau, \alpha; \mathfrak{b}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(\sigma + it + iv + i\tau + \frac{1}{2} + \varepsilon, \alpha; \mathfrak{b}) \times$$
$$\frac{l_{u_T}(\frac{1}{2} + \varepsilon - \sigma + iv)}{\frac{1}{2} + \varepsilon - \sigma + iv} dv + \widehat{b} \frac{l_{u_T}(1-s)}{1-s}.$$

Now, shifting $v + t$ to v, gives, for $s \in K$,

$$\zeta_{u_T}(s+i\tau, \alpha; \mathfrak{b}) - \zeta\left(s+i\tau, \alpha; \mathfrak{b}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2} + \varepsilon + i\tau + iv, \alpha; \mathfrak{b}\right) \times$$
$$\frac{l_{u_T}(\frac{1}{2} + \varepsilon - s + iv)}{\frac{1}{2} + \varepsilon - s + iv} dv + \frac{\widehat{b} l_{u_T}(1-s)}{1-s}.$$

The latter equality, for $s \in K$, leads to the estimate

$$\zeta(s+i\tau,\alpha;\mathfrak{b})-\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b}) \ll \int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv,\alpha;\mathfrak{b}\right)\right|\times$$

$$\sup_{s\in K}\left|\frac{l_{u_T}(\frac{1}{2}+\varepsilon-s+iv)}{\frac{1}{2}+\varepsilon-s+iv}\right|dv+\sup_{s\in K}\left|\frac{l_{u_T}(1-s)}{1-s}\right|.$$

From this, we find

$$\frac{1}{T}\int_0^T\sup_{s\in K}|\zeta(s+i\tau,\alpha;\mathfrak{b})-\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b})|\,dt \ll J_1(T)+J_2(T), \quad (4.6)$$

where

$$J_1(T)=\int_{-\infty}^{\infty}\left(\frac{1}{2}\int_0^T\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv,\alpha;\mathfrak{b}\right)\right|d\tau\right)\sup_{s\in K}\left|\frac{l_{u_T}(\frac{1}{2}+\varepsilon-s+iv)}{\frac{1}{2}+\varepsilon-s+iv}\right|dv$$

and

$$J_2(T)=\frac{1}{2}\int_0^T\sup_{s\in K}\left|\frac{l_{u_T}(1-s-i\tau)}{1-s-i\tau}\right|d\tau.$$

Here, as in section 2.3, we used the estimate of Lemma 2.2 for the gamma-function.

Now, we need a mean square estimate for the function $\zeta(s,\alpha;\mathfrak{b})$ in the strip $\frac{1}{2}<\sigma<1$. Using the estimate for the Hurwitz zeta-function,

$$\int_{-T}^{T}|\zeta(\sigma+i\tau,\alpha)|^2 \ll_{\sigma,\alpha} T,\ \frac{1}{2}<\sigma<1,$$

and the equality (3.1), we obtain that, for fixed $\frac{1}{2}<\sigma<1$,

$$\int_{-T}^{T}|\zeta(\sigma+i\tau,\alpha;\mathfrak{b})|^2 \ll_{q_2}\int_{-T}^{T}\left|\sum_{k=0}^{q_2-1}b_k\zeta\left(\sigma+it,\frac{k+\alpha}{q_2}\right)\right|^2dt \ll_{\sigma,\alpha,\mathfrak{b}}$$

$$\sum_{k=0}^{q_2-1}\int_{-T}^{T}\left|\zeta\left(\sigma+it,\frac{k+\alpha}{q_2}\right)\right|^2dt \ll_{\sigma,\mathfrak{b}} T.$$

From this and the Cauchy-Schwarz inequality, we find that, for all $v\in\mathbb{R}$ and $T>1$,

$$\frac{1}{T}\int_0^T\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv,\alpha;\mathfrak{b}\right)\right|d\tau \ll$$

$$\left(\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+iv,\alpha;\mathfrak{b}\right)\right|^2\right)^{\frac{1}{2}} \ll$$

$$\left(\frac{1}{T}\int_{-|v|}^{T+|v|} \left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau,\alpha;\mathfrak{b}\right)\right|^2 d\tau\right)^{\frac{1}{2}} \ll_{\varepsilon,\mathfrak{b}} \left(\frac{T+|v|}{T}\right)^{\frac{1}{2}} \ll_{\varepsilon,\mathfrak{b}} (1+|v|)^{\frac{1}{2}}. \tag{4.7}$$

For the second multiplier of integrand of J_1, the estimate

$$\frac{l_{u_T}\left(\frac{1}{2}+\varepsilon-s+iv\right)}{\frac{1}{2}+\varepsilon-s+iv} \ll_{\varepsilon,\alpha,\mathfrak{b},K} u_T^{-\varepsilon}\exp\{-c_1|v|\}, c_1>0,$$

obtained in Section 2.3, is valid for all $s\in K$. This and (4.7) implies the estimate

$$J_1(T) \ll_{\varepsilon,K} u_T^{\varepsilon}. \tag{4.8}$$

In section 2.3, also it was obtained that, for all $s\in K$, the bound

$$\frac{l_{u_T(1-s-i\tau)}}{1-s-i\tau} \ll \varepsilon K^{\frac{1}{2}-2\varepsilon}\exp\{-c_2|\tau|\}, c_2>0,$$

is true. Thus,

$$I_2 \ll_{\varepsilon,K} \frac{u^{\frac{1}{2}-2\varepsilon}}{T}.$$

This, and (4.8) and (4.6) show that

$$\frac{1}{T}\int_0^T \sup_{s\in K}|\zeta(s+i\tau,\alpha,\mathfrak{b})-\zeta_{u_T}(s+i\tau,\alpha,\mathfrak{b})|dt \ll_{\varepsilon,u,\mathfrak{b},K} \left(u_T^{-\varepsilon}+\frac{u_T^{\frac{1}{2}-2\varepsilon}}{T}\right).$$

Since $u_T \ll T^2$, hence equality (4.3) follows, and lemma is proved.

4.4 Limit theorem

We will apply a limit theorem for $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$ in the space $H^2(D)$. For $A\in\mathcal{B}(H^2(D))$, define

$$P_{T,\alpha;\mathfrak{a},\mathfrak{b}}(A)=\frac{1}{T}\text{meas}\{\tau\in[0,T]:\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})\in A\}.$$

Moreover, let $P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}$ be the distribution of the $H^2(D)-$ valued random element $\underline{\zeta}(s,\alpha,\omega;\underline{\mathfrak{a}},\mathfrak{b})$, i.e.,

$$P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}(A) = m_H\{\omega \in \Omega : \underline{\zeta}(s,\alpha,\omega;\underline{\mathfrak{a}},\mathfrak{b}) \in A\}.$$

We state a result as the following lemma.

Lemma 4.3. *Suppose that the sequence $\mathfrak{a}$ is multiplicative and the parameter α is transcendental. Then $P_{T,\alpha;\mathfrak{a},\mathfrak{b}}$ converges weakly to $P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}$ as $T \to \infty$. Moreover, the support of the limit measure $P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}$ is the set*

$$\{g \in H(D) : \text{ either } g(s) \neq 0 \text{ on } D, \text{ or } g(s) \equiv 0\} \times H(D).$$

Proof. The lemma is the union of Theorem C and Lemma 12 from [21].

Now we consider a limit theorem for $\underline{\zeta}_{u_T}(s,\alpha;\mathfrak{a},\mathfrak{b})$. For $A \in \mathcal{B}(H^2(D))$, define

$$\widehat{P}_{T,\alpha;\mathfrak{a},\mathfrak{b}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta_{u_T}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}) \in A\}.$$

Theorem 4.2. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the parameter α is transcendental, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Then $\widehat{P}_{T,\alpha;\mathfrak{a},\mathfrak{b}}$ converges weakly to $P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}$ as $T \to \infty$.*

Before the proof of Theorem 4.2, we recall the equivalent of weak convergence of probability measures in terms of open sets.

Lemma 4.4. *Let $P_n, n \in \mathbb{N}$, and P be the probability measures on $\mathbb{X}, \mathcal{B}(\mathbb{X})$. The weak convergence of P_n to P as $n \to \infty$ is equivalent to the inequality*

$$\limsup_{n\to\infty} P_n(F) \leqslant P(F)$$

for each closed set of the space $\mathbb{X}$.

Proof. The lemma is a part of Theorem 3.1 from [5], where its proof can be found.

Proof of Theorem 4.2. Let θ_T be a random variable defined on a certain probability space $(\widehat{\Omega}, A, \mu)$, and uniformly distributed in $[0,T]$, i.e. its distribution function is

$$\mu\{\theta_T < x\} = \begin{cases} 0 & \text{if } x \geqslant 0, \\ \frac{x}{T} & \text{if } 0 \leqslant x \leqslant T, \\ 1 & \text{if } x > T. \end{cases} \qquad x \in \mathbb{R},$$

Define the $H^2(D)-$ valued random elements

$$\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}} = \underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}}(s) = (X_{T,\mathfrak{a}}(s), X_{T,\alpha,\mathfrak{b}}(s)),$$

where

$$X_{T,\mathfrak{a}}(s) = \zeta(s + i\theta_T; \mathfrak{a}), \ \ X_{T,\alpha,\mathfrak{b}}(s) = \zeta(s + i\theta_T, \alpha, \mathfrak{b}),$$

and

$$\underline{\widehat{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}} = \underline{\widehat{X}}_{T,\alpha;\mathfrak{a},\mathfrak{b}}(s) = \left(\widehat{X}_{T,\mathfrak{a}}(s), \widehat{X}_{T,\alpha;\mathfrak{b}}(s)\right),$$

where

$$\widehat{X}_{T,\mathfrak{a}}(s) = \zeta_{u_T}(s + i\theta_T; \mathfrak{a}), \ \ \widehat{X}_{T,\alpha,\mathfrak{b}}(s) = \zeta_{u_T}(s + i\theta_T, \alpha; \mathfrak{b}).$$

By the definitions of $\theta_T, \underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}}$ and $\underline{\widehat{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}}$, for $A \in \mathcal{B}(H^2(D))$, we have

$$\mu\{\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in A\} = P_{T,\alpha;\mathfrak{a},\mathfrak{b}}(A) \tag{4.9}$$

and

$$\mu\{\underline{\widehat{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in A\} = \widehat{P}_{T,\alpha;\mathfrak{a},\mathfrak{b}}(A). \tag{4.10}$$

Let $F \subset H^2(D)$ be a fixed closed set, and

$$\rho_2(\underline{g}, F) = \inf_{g_1 \in F} \rho_2(\underline{g_1}, \underline{g}).$$

Moreover, for fixed $\varepsilon > 0$, define

$$F_\varepsilon = \{\underline{g} \in H^2(D) : \rho_2(\underline{g}, F) \leqslant \varepsilon\}.$$

Then the set F_ε is closed as well. Then Lemmas 4.3 and 4.4, together with (4.9) imply

$$\limsup_{T\to\infty} P_{T,\alpha;\mathfrak{a},\mathfrak{b}}(F_\varepsilon) = \limsup_{T\to\infty} \mu\{X_{T,\alpha;\mathfrak{a},\mathfrak{b}} \in F_\varepsilon\} \leqslant P_{\zeta,\alpha;\mathfrak{a},\mathfrak{b}}(F_\varepsilon). \tag{4.11}$$

It is easily seen that

$$\{\underline{\widehat{X}}_{T,\alpha;\mathfrak{a},\mathfrak{b}} \in F\} \subset \{\underline{X}_{T,\alpha;\mathfrak{a},\mathfrak{b}} \in F_\varepsilon\} \cup \{\rho_2(\underline{X}_{T,\alpha;\mathfrak{a},\mathfrak{b}}, \underline{\widehat{X}}_{T,\alpha;\mathfrak{a},\mathfrak{b}}) \geqslant \varepsilon\}. \quad (4.12)$$

We observe that $\rho_2(\underline{X}_{T,\alpha;\mathfrak{a},\mathfrak{b}}, \overline{X}_{T,\alpha;\mathfrak{a},\mathfrak{b}})$ is a random variable, and, by the definition of θ_T, its expectation is

$$\frac{1}{T}\int_0^T \rho_2(\underline{\zeta}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}), \zeta_{u_T}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}))d\tau. \quad (4.13)$$

Therefore, by the inclusion (4.12),

$$\mu\{\underline{\widehat{X}}_{T,\alpha;\mathfrak{a},\mathfrak{b}} \in F\} \leqslant \mu\{\underline{X}_{T,\alpha;\mathfrak{a},\mathfrak{b}} \in F_\varepsilon\} + \mu\{\rho_2(X_{T,\alpha;\mathfrak{a},\mathfrak{b}}, \underline{\widehat{X}}_{T,\alpha;\mathfrak{a},\mathfrak{b}}) \geqslant \varepsilon\},$$

and, in view of Chebyshev's type inequality,

$$\text{meas}\left\{\tau \in [0,T] : \rho_2\left(\underline{\zeta}(s+i\tau,\alpha,\mathfrak{a},\mathfrak{b}), \underline{\zeta}_{u_T}(s+i\tau,\alpha,\mathfrak{a},\mathfrak{b})\right) \geqslant \varepsilon\right\} \leqslant$$

$$\frac{1}{\varepsilon}\int_0^T \rho_2\Big(\underline{\zeta}(s+i\tau,\alpha,\mathfrak{a},\mathfrak{b}), \zeta_{u_T}(s+i\tau,\alpha,\mathfrak{a},\mathfrak{b})\Big)d\tau.$$

Thus, Lemma 4.2 implies

$$\mu\{\rho_2(\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}}, \underline{\widehat{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}}) \geqslant \varepsilon\} = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \rho_2(\underline{\zeta}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b})),$$

$$\underline{\zeta}_{u_T}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b})) \geqslant \varepsilon\} \leqslant$$

$$\frac{1}{\varepsilon T}\int_{0,T} \rho_2(\underline{\zeta}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}), \zeta_{u_T}(s+i\tau,\alpha,\mathfrak{a},\mathfrak{b}))d\tau = o(1)$$

as $T \to \infty$. Therefore, this and (4.13) give

$$\limsup_{T\to\infty} \mu\{\underline{\widehat{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in F\} \leqslant \limsup_{T\to\infty} \mu\{\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in F_\varepsilon\},$$

and, by (4.9), (4.10) and (4.11), we have

$$\limsup_{T\to\infty} \widehat{P}_{T,\alpha;\mathfrak{a},\mathfrak{b}}(F) \leqslant P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}(F_\varepsilon).$$

Because $F_\varepsilon \to F$ as $\varepsilon \to 0+$, and the left-hand side of the latter inequality is independent on ε, we obtain that

$$\limsup_{T\to\infty} \widehat{P}_{T,\alpha;\mathfrak{a},\mathfrak{b}}(F) \leqslant P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}(F).$$

This together with Lemma 4.4 proves that $\widehat{P}_{T,\alpha;\mathfrak{a},\mathfrak{b}}$ converges weakly to $P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}$ as $T \to \infty$. The theorem is proved.

Let $K_1, K_2, \in K$ and $f_1(s) \in H_0(K_1), f_2(s) \in H(K)$. For $A \in \mathcal{B}(R^2)$, define

$$Q_{T,\alpha;\mathfrak{a},\mathfrak{b}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : (\sup_{s\in K_1} |\zeta_{u_T}(s+i\tau;\mathfrak{a}) - f_1(s)|,$$

$$\sup_{s\in K_2} |\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)|) \in A\}.$$

Corollary 4.2. *Under hypotheses of Theorem 4.2, $Q_{T,\alpha;\mathfrak{a},\mathfrak{b}}$ converges weakly to the measure*

$$m_H\Big\{\omega_1, \omega_2 \in \Omega : \Big(\sup_{s\in K_1} |\zeta(s,\omega_1,\mathfrak{a}) - f_1(s)|,$$

$$\sup_{s\in K_2} |\zeta(s,\alpha,\omega_2;\mathfrak{b}) - f_2(s)|\Big) \in A\Big\}, A \in \mathcal{B}(\mathbb{R}^2),$$

as $T \to \infty$.

Proof. Define the function $v : H^2(D) \to \mathbb{R}^2$ by the formula

$$v(g_1, g_2) = (\sup_{s\in K_1} |g_1(s) - f_1(s)|, \sup_{s\in K_2} |g_2(s) - g_2(s)|).$$

Because the space $H(D)$ is equipped with the topology of uniform convergence on compact sets, the function v is continuous. Actually, let (g_{n1}, g_{n2}) converges to (g_1, g_2) in $H^2(D)$ as $n \to \infty$. Then we have

$$\lim_{n\to\infty} \sup_{s\in K_j} |g_{ij}(s) - g_j(s)| = 0, j = 1, 2.$$

Since, by the triangle inequality,

$$\Big| \sup_{s\in K_j} |g_{nj}(s) - f_j(s)| - | \sup_{s\in K_j} |g_j(s) - f_j(s)|\Big| \leqslant$$

$$\sup_{s\in K} |(g_{nj}(s) - f_j(s)) - (g_j(s) - f_j(s))| = \sup_{s\in K_j} |g_{nj}(s) - g_j(s)|, j = 1, 2,$$

from this, it follows that

$$v(g_{n1}, g_{n2}) \xrightarrow[n\to\infty]{} v(g_1, g_2)$$

in the space $\mathbb{R}^2$, and the function u is continuous.

Moreover, by the definitions of v, and $\widehat{P}_{T,\alpha;\mathfrak{a},\mathfrak{b}}$ and $Q_{T,\alpha;\mathfrak{a},\mathfrak{b}}$, for $A \in \mathcal{B}(\mathbb{R}^2)$, we have

$$\widehat{P}_{T,\alpha;\mathfrak{a},\mathfrak{b}} v^{-1}(A) = \widehat{P}_{T,\alpha;\mathfrak{a},\mathfrak{b}}(v^{-1}A) =$$

$$\frac{1}{T}\text{meas}\{\tau \in [0,T] : \underline{\zeta}_{u_T}(s+i\tau, \alpha; \mathfrak{a}, \mathfrak{b}) \in v^{-1}A\} =$$

$$\frac{1}{T}\text{meas}\{\tau \in [0,T] : v\big(\zeta_{u_T}(s+i\tau, \alpha; \mathfrak{a}), \zeta_{u_T}(s+i\tau, \alpha; \mathfrak{b})\big) \in A\} =$$

$$Q_{T,\alpha,\mathfrak{a},\mathfrak{b}}(A),$$

and

$$P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}} v^{-1}(A) = P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}(v^{-1}A) =$$

$$m_H\{(\omega_1, \omega_2) \in \Omega : \underline{\zeta}(s, \alpha, \omega_1, \omega_2; \mathfrak{a}, \mathfrak{b}) \in v^{-1}A\} =$$

$$m_H\{(\omega_1, \omega_2) \in \Omega : \big(\sup_{s \in K_1} |\zeta(s, \omega_1, \mathfrak{a}) - f_1(s)|, \sup_{s \in K_2} |\zeta(s, \alpha, \omega_2, \mathfrak{b}) - f_2(s)|\big) \in A\}.$$

Two latter equalities, Theorem 4.2, continuity of the function v, and the property of preservation of weak convergence under continuous mapping (Theorem 5.1 of [5]) prove the corollary.

Let P be a probability measure on $\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)$. To this measure, the distribution function

$$F(x_1, x_2) = P\{(y_1, y_2) \in \mathbb{R}^2 : y_1 < x_1, y_2 > x_2\}, x_1, x_2 \in \mathbb{R},$$

can be attached.

Let $P_n, n \in \mathbb{N}$, and P be probability measures on $\big(\mathbb{R}^2, \mathcal{B}(\mathbb{R})\big)$, and $F_n(x_1, x_2)$ and $F(x_1, x_2)$ be the corresponding distribution functions. $F_n(x_1, x_2)$ converges weakly to F_{x_1,x_2} as $n \to \infty$, if

$$\lim_{n \to \infty} F_n(x_1, x_2) = F(x_1, x_2)$$

for all continuity points (x_1, x_2) of $F(x_1, x_2)$.

Lemma 4.5. *The weak convergence of P_n to P as $n \to \infty$ is equivalent to the weak convergence of $F_n(x_1, x_2)$ to $F(x_1, x_2)$ as $n \to \infty$.*

Proof. The lemma is proved in a general case on the space $\mathbb{R}^k$, $k \in \mathbb{N}$, in § 3 of [5].

Now, define the distribution functions corresponding the probability measure $Q_{T,\alpha;\mathfrak{a},\mathfrak{b}}$ and its limit measure in Corollary 4.2, namely,

$$F_{T,\alpha;\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)=\frac{1}{T}\text{meas}\Big\{\tau\in[0,T]:\sup_{s\in K_1}|\zeta_{u_T}(s+i\tau;\mathfrak{a})-f_1(s)|<\varepsilon_1,$$

$$\sup_{s\in K_2}|\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}$$

and

$$F_{\underline{\zeta}}(\varepsilon_1,\varepsilon_2)=m_H\big\{(\omega_1,\omega_2)\in\Omega:\sup_{s\in K_1}|\zeta(s,\omega_1;\mathfrak{a})-f_1(s)|<\varepsilon_1$$

$$\sup_{s\in K_2}|\zeta(s,\alpha,\omega_2;\mathfrak{b})-f_2(s)|<\varepsilon_2\big\}.$$

Then Corollary 4.2 and Lemma 4.5 imply the following convenient for us corollary.

Corollary 4.3. *Under hypotheses of Theorem 4.2, the distribution function* $F_{T,\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$ *converges weakly to distribution function* $F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$ *as* $T\to\infty$.

4.5 Proof of Theorem 4.1

Since the set of distribution points of a distribution function is at most countable, by Corollary 4.3, the limit

$$\lim_{T\to\infty}F_{T,\alpha;\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)=F_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$$

exists for all but at most countable many $\varepsilon_1>0$ and $\varepsilon_2>0$. Thus, it remains to prove the positivity of $F_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$.

In view of Lemma 2.8 (Mergelyan's theorem), there exists polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1}\left|f_1(s)-e^{p_1(s)}\right|<\frac{\varepsilon_1}{2}\tag{4.14}$$

and

$$\sup_{s\in K_2}|f_2(s)-p_2(s)|<\frac{\varepsilon_2}{2}.\tag{4.15}$$

By Lemma 4.3, the support S of the measure $P_{\underline{\zeta}}$ is the set $\{g\in H(D):$ either $g(s)\neq 0$ on D, or $g(s)\equiv 0\}\times H(D)$. Therefore, $\big(e^{p_1(s)},p_2(s)\big)$ is an element of S because $e^{p_1(s)} \neq 0$. Hence, by a support property,

$$P_{\underline{\zeta},\alpha;\mathfrak{a},\mathfrak{b}}(G_{\varepsilon_1,\varepsilon_2}) > 0, \tag{4.16}$$

where

$$G_{\varepsilon_1,\varepsilon_2} = \Big\{(g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - e^{p_1(s)}| < \frac{\varepsilon_1}{2}, \sup_{s\in K_2} |g_2(s) - p_2(s)| < \frac{\varepsilon_2}{2}\Big\}$$

is an open neighbourhood of $(e^{p_1}(s), p_2(s))$. Define one more set

$$\widehat{G}_{\varepsilon_1,\varepsilon_2} = \left\{(g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| < \varepsilon_1, \sup_{s\in K_2} |g_2(s) - f_2(s)| < \varepsilon_2\right\}.$$

Suppose that $(g_1, g_2) \in G_{\varepsilon_1,\varepsilon_2}$. Then, (4.14) implies

$$\sup_{s\in K_1} |g_1(s)-f_1(s)| \leqslant \sup_{s\in K_1} |g_1(s)-e^{p_1(s)}|+\sup_{s\in K_1} |f_1(s)-e^{p_1(s)}| < \frac{\varepsilon_1}{2}+\frac{\varepsilon_1}{2} = \varepsilon_1,$$

and (4.15) gives

$$\sup_{s\in K_2} |g_2(s)-f_2(s)| \leqslant \sup_{s\in K_2} |g_2(s)-p_2(s)|+\sup_{s\in K_2} |f_2(s)-p_2(s)| < \frac{\varepsilon_2}{2}+\frac{\varepsilon_2}{2} = \varepsilon_2.$$

This shows that if $(g_1, g_2) \in G_{\varepsilon_1,\varepsilon_2}$, then also $(g_1, g_2) \in \widehat{G}_{\varepsilon_1,\varepsilon_2}$. Thus, we have the inclusion $G_{\varepsilon_1,\varepsilon_2} \subset \widehat{G}_{\varepsilon_1,\varepsilon_2}$. This, together with (4.16) leads to inequality

$$P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\widehat{G}_{\varepsilon_1,\varepsilon_2}) > 0,$$

or

$$F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1, \varepsilon_2) > 0.$$

The theorem is proved.

Proof of Corollary 4.1. It suffices to take $\varepsilon_1 = +\infty$ in Theorem 4.1.

Chapter 5

Joint approximation of analytic functions by discrete shifts of absolutely convergent Dirichlet series related to periodic and Hurwitz zeta-functions

In this chapter, we give a discrete version of Theorem 4.1. More precisely, we consider the simultaneous approximation of a pair of analytic functions by discrete shifts $\zeta_{u_N}(s+ikh_1;\mathfrak{a})$ and $\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})$ with certain positive h_1 and h_2, and $k \in \mathbb{N}_0$. As in previous chapters, we suppose that the sequence $\mathfrak{a}$ is multiplicative. Therefore, as in Chapter 4, we continue investigations connected to the Mishou theorem on the approximation of a pair of analytic functions by shifts of one zeta-function having the Euler product over primes, and by shifts of another zeta-function without Euler's product. In discrete approximation of a pair of analytic functions, one or two additional parameters – the differences of arithmetical progressions appear. Therefore, some hypotheses connecting the numbers h_1, h_2 and other characteristics of the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ are needed. The most general result has been obtained in [36]. For positive h_1 and h_2, define the set

$$L(\mathbb{P};\alpha,h_1,h_2,\pi)=\{(h_1\log p : p\in\mathbb{P}),(h_2\log(m+\alpha) : m\in\mathbb{N}_0),2\pi\}.$$

Using the linear independence over $\mathbb{Q}$ of the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$, in [36] it was proved that, for $K_1,K_2\in\mathcal{K}$, $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$ and all $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\{0\leqslant k\leqslant N:\sup_{s\in K_1}|\zeta(s+ikh_1;\mathfrak{a})-f_1(s)|<\varepsilon,$$

$$\sup_{s\in K_2}|\zeta(s+ikh_2,\alpha;\mathfrak{b})-f_2(s)|<\varepsilon\}>0.$$

In this chapter, we use the notation of previous chapters.

5.1 Statement of the main theorem

Let Ω_1 and Ω_2 be the same infinite dimensional tori as in Chapter 4, and $\Omega=\Omega_1\times\Omega_2$. Elements of Ω are denoted by $\omega=(\omega_1,\omega_2)$ with $\omega_1\in\Omega_1$, and $\omega_2\in\Omega_2$. We use the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$ with probability Haar measure m_H. On that probability space, the $H^2(D)-$ valued random element

$$\underline{\zeta}(s,\alpha,\omega_1,\omega_2;\mathfrak{a},\mathfrak{b})=\big(\zeta(s,\omega_1;\mathfrak{a}),\zeta(s,\alpha,\omega_2;\mathfrak{b})\big),$$

where

$$\zeta(s,\omega_1;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m\omega_1(m)}{m^s}$$

and

$$\zeta(s,\alpha,\omega_2;\mathfrak{b})=\sum_{m=0}^{\infty}\frac{b_m\omega_2(m)}{(m+\alpha)^s},$$

is defined. The main result of the chapter is the following theorem.

Theorem 5.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$, and $u_N\to\infty$ and $u_N\ll N^2$ as $N\to\infty$. Let $K_1,K_2\in\mathcal{K}$ and $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$. Then* *the limit*

$$\lim_{N\to\infty}\frac{1}{N+1}\#\{0\leqslant k\leqslant N:\sup_{s\in K_1}|\zeta_{u_N}(s+ikh_1;\mathfrak{a})-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})-f_2(s)|<\varepsilon_2\}=$$
$$m_H\{(\omega_1,\omega_2)\in\Omega:\sup_{s\in K_1}|\zeta_{u_N}(s,\omega_1;\mathfrak{a})-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta_{u_N}(s,\alpha,\omega_2;\mathfrak{b})-f_2(s)|<\varepsilon_2\}$$

From Theorem 5.1, it follows that there exists a natural number $N_0 = N_0(f_1, f_2, K_1, K_2, h_1, h_2, \varepsilon_1, \varepsilon_2)$ such that, for $N \geqslant N_0$, there are infinitely many shifts $(\zeta_{u_N}(s+ikh_1;\mathfrak{a}), \zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b}))$ that approximate a given pair $(f_1(s), f_2(s))$.

Theorem 5.1 implies the following discrete statement on approximation by shifts $\zeta_{u_T}(s+ikh,\alpha;\mathfrak{b})$. Let, for $h>0$,

$$L(\alpha,h,\pi)=\left\{(\log(m+\alpha):m\in\mathbb{N}_0)\,,\frac{2\pi}{h}\right\}.$$

Corollary 5.1. *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over Q, and $u_N\to\infty$ and $u_N\ll N^2$ as $N\to\infty$. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then the limit*

$$\lim_{N\to\infty}\frac{1}{N+1}\#\{0\leqslant k\leqslant N:\sup_{s\in K}|\zeta_{u_N}(s+ikh,\alpha;\mathfrak{b})-f(s)|<\varepsilon\}=$$
$$m_{2H}\{\omega_2\in\Omega_2:\sup_{s\in K}|\zeta(s,\alpha,\omega_2;\mathfrak{b})-f(s)|<\varepsilon\}$$

exists and is positive for all but at most countably many $\varepsilon>0$.

We note that discrete theorems on approximation of analytic functions have a certain advantage against similar continuous theorems because it is easier to detect a discrete approximating shifts in the set $\{k:0\leqslant k\leqslant N\}$ than a continuous shift in the interval $[0,T]$.

We give an example of a linearly independent set $L(\mathbb{P},\alpha,h_1,h_2,\pi)$. Consider the set $L(\mathbb{P};\frac{1}{\pi},h_1,h_2,\pi)$ with positive rational h_1 and h_2. Suppose, on the contrary, that the latter set is linearly dependent over $\mathbb{Q}$. By the Nesterenko theorem [45], mentioned in Introduction, the numbers π and e^{π} are algebraically independent. Hence, they are transcendental. Then there exist integers $k_1, \dots, k_r$, $\widehat{k}_1, \dots, \widehat{k}_{r_2}$ and $\widetilde{k}$, not all zeros, such that

$$k_1 h_1 \log p_1 + \ldots + k_{r_1} h_1 \log p_{r_1} + \widehat{k_1} h_2 \log\left(m_1 + \frac{1}{\pi}\right) + \ldots +$$

$$kr_2 h_2 \log\left(m_{r_2} + \frac{1}{\pi}\right) + 2\widehat{k}\pi = 0,$$

where $p_1, \dots, p_{r_1}$ are some prime numbers and $m_1, \dots, m_r$ are some non-negative integers. Hence,

$$p_1^{l_1} \ldots p_{r_1}^{l_{r_1}} \left(m_1 + \frac{1}{\pi}\right)^{\widehat{l}_1} \ldots \left(m_{r_2} + \frac{1}{\pi}\right)^{\widehat{l}_{r_2}} e^{\widetilde{l}\pi} = 1$$

with some integers $l_1, \dots, l_{r_1}, \widehat{l_1}, \dots, \widehat{l}_{r_2}$ and $\widetilde{l}$, and this contradicts the algebraic independence of the numbers π and e^{π}. Similarly, the equalities

$$k_1 h_1 \log p_1 + \ldots + k_{r_1} h_1 \log_{p_{r_1}} + \widehat{k}_1 h_2 \log\left(m_1 + \frac{1}{\pi}\right) +$$

$$\ldots + \widehat{k}_{r_2} h_2 \log\left(m_{r_2} + \frac{1}{\pi}\right) = 0$$

and

$$k_1 h_1 \log_{p_1} + \ldots + k_{r_1} h_1 \log p_{r_1} + \widetilde{k}\pi = 0$$

contradict the transcendence of the numbers π and e^{π}, respectively. Moreover, it is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$.

A proof of Theorem 5.1, as other approximation theorems of the dissertation, is probabilistic, it is based on a joint discrete limit theorem in the space $H^2(D)$ for the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ obtained in [36].

5.2 The main equality

In this section, we consider the mean value of the distance between $\underline{\zeta}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b})$ and $\underline{\zeta}_{u_N}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b})$, where

$$\underline{\zeta}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}) = (\zeta(s + ikh_1; \mathfrak{a}), \zeta(s + ikh_2, \alpha; \mathfrak{b})),$$

$$\underline{\zeta}_{u_N}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}) = (\zeta_{u_N}(s + ikh_1; \mathfrak{a}), \zeta_{u_N}(s + ikh_2, \alpha; \mathfrak{b}))$$

and $\underline{h} = (h_1, h_2)$.

Recall that ρ is a metric in $H(D)$ inducing its topology of uniform convergence on compact sets.

Lemma 5.1. *Suppose that $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$. Then, for every $h_1 > 0$,*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\big(\zeta(s+ikh_1;\mathfrak{a}), \zeta_{u_N}(s+ikh_1;\mathfrak{a})\big).$$

Proof. The lemma is a result of Lemma 3.11 and the definition of the metric ρ, see Section 3.4.

Lemma 5.2. *For every $\sigma > \frac{1}{2}, h_2 > 0$ and $t \in \mathbb{R}$, the estimate*

$$\sum_{k=0}^{N} |\zeta(\sigma+ikh_2+it,\alpha;\mathfrak{b})|^2 \ll_{\sigma,\alpha,\mathfrak{b}} N(1+|t|)$$

is valid.

Proof. A proof of the lemma is given in [36]. This proof uses the bounds, for $\sigma > \frac{1}{2}$,

$$\int_{-T}^{T} |\zeta(\sigma+it,\alpha;\mathfrak{b})|^2 dt \ll_{\sigma,\alpha,\mathfrak{b}} T,$$

$$\int_{-T}^{T} |\zeta'(\sigma+it,\alpha;\mathfrak{b})|^2 dt \ll_{\sigma,\alpha,\mathfrak{b}} T,$$

and Lemma 3.1.

Lemma 5.3. *Under hypothesis of Lemma 5.1, the equality*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho(\zeta(s+ikh_2,\alpha;\mathfrak{b}), \zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})) = 0$$

is valid.

Proof. In virtue of the definition of the metric ρ, it is sufficient to show that the equality

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+ikh_2,\alpha;\mathfrak{b}) - \zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})| = 0 \quad (5.1)$$

holds for every compact set $K \subset D$. To prove this, we apply the integral representation (4.4), i.e., that, for $\sigma > \frac{1}{2}$,

$$\zeta_{u_N}(s, \alpha; \mathfrak{b}) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z, \alpha; \mathfrak{b}) \widehat{l}_{u_N}(z) dz, \tag{5.2}$$

where

$$\widehat{l}_{u_N}(z) = \frac{1}{\theta} \Gamma\left(\frac{z}{\theta}\right) u_N^z.$$

We note that the lemma is a discrete analogue of the equality (4.3), therefore, we will apply similar arguments as in the proof of Lemma 4.2

Thus we fix a compact set $K \subset D$. Then there exists a number $0 < \varepsilon < \frac{1}{6}$ such that $\frac{1}{2} + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for $s = \sigma + it \in K$. Take $\theta = \frac{1}{2} + \varepsilon$, and denote $\theta_1 = \frac{1}{2} + \varepsilon - \sigma$. Then $-\frac{1}{2} + 2\varepsilon \ll \theta_1 \leqslant -\varepsilon$. Therefore, the integrand of (5.2), in the strip $\theta_1 \leqslant \sigma \leqslant \theta$, has a simple pole at $z = 0$ and a possible simple pole at $z = 1 - s$. Hence, by the residue theorem, we find, for $s \in K$,

$$\zeta_{u_N}(s, \alpha; \mathfrak{b}) - \zeta(s, \alpha; \mathfrak{b}) = \frac{1}{2\pi i} \int_{\theta_1 - \infty}^{\theta_1 + \infty} \zeta(s + z, \alpha; \mathfrak{b}) \widehat{l}_{u_N}(z) dz + \widehat{\widehat{b l}}_{u_N}(1 - s).$$

The latter equality, for $s = \sigma + it \in K$, gives

$$\zeta_{u_N}(s + ikh_2, \alpha; \mathfrak{b}) - \zeta(s + ikh_2, \alpha; \mathfrak{b}) =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{2} + \varepsilon + it + ikh_2 + i\tau, \alpha; \mathfrak{b}\right) \widehat{l}_{u_N}\left(\frac{1}{2} + \varepsilon - \sigma + i\tau\right) d\tau +$$
$$\widehat{b}\, \widehat{l}_{u_N}(s + ikh_2) \ll$$
$$\int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + \varepsilon + ikh_2 + i\tau, \alpha; \mathfrak{b}\right)\right| \sup_{s \in K} \left|\widehat{l}_{u_N}\left(\frac{1}{2} + \delta - s + i\tau\right)\right| d\tau +$$
$$\sup_{s \in K} |\widehat{l}_N(1 - s - ikh_2)|.$$

Here we used the shift $t + \tau \to \tau$. Therefore,

$$\frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} |\zeta_{u_N}(s + ikh_2, \alpha; \mathfrak{b}) - \zeta(s + ikh_2, \alpha; \mathfrak{b})| \ll$$

$$\int_{-\infty}^{\infty} \left(\frac{1}{N+1} \sum_{k=0}^{N} \left|\zeta\left(\frac{1}{2} + \varepsilon + ikh_2 + i\tau, \alpha; \mathfrak{b}\right)\right|\right)$$

$$\sup_{s\in K}\left|\widehat{l}_{u_N}\left(\frac{1}{2}+\varepsilon-s+i\tau\right)\right| d\tau+$$

$$\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\widehat{l}_{u_N}(1-s-ikh_2)| \overset{\text{def}}{=} I_N+S_N. \tag{5.3}$$

Using Lemma 2.2, we find that, for $s\in K$,

$$l_{u_N}\left(\frac{1}{2}+\varepsilon-s+i\tau\right) \ll_\varepsilon u_N^{\frac{1}{2}+\varepsilon-\sigma}\exp\left\{-\frac{c}{\theta}|\tau-t|\right\} \ll_{\varepsilon,K} u_N^{-\varepsilon}\exp\{-c_1|\tau|\}, c_1>0,$$

because of boundedness of t. This, and (5.3) show that

$$I_N \ll_{\varepsilon,h_2,\alpha,\mathfrak{b}} u_N^{-\varepsilon}\int_{-\infty}^{\infty}(1+|\tau|)^{\frac{1}{2}}\,d\tau \ll_{\varepsilon,h_2,\alpha,\mathfrak{b},K} u_N^{-\varepsilon}. \tag{5.4}$$

Using Lemma 2.2 again, for $s\in K$, we have

$$l_{u_N}(1-s-ikh_2) \ll_\varepsilon u_N^{1-\sigma}\exp\{c_2|kh_2-t|\} \ll_{\varepsilon,K} \exp\{-c_3kh_2\}, c_2, c_3>0.$$

Therefore,

$$S_N \ll_{\varepsilon,K,\alpha,\mathfrak{b}} u_N^{\frac{1}{2}-2\varepsilon}\frac{1}{N}\sum_{k=0}^{N}\exp\{-c_3kh_2\} \ll_{\varepsilon,K}$$

$$u_N^{\frac{1}{2}-2\varepsilon}\left(\frac{\log N}{N}+\frac{1}{N}\sum_{k\geqslant\log N}\exp\{-c_3kh_2\}\right) \ll_{\varepsilon,K,\alpha,\mathfrak{b},h_2} u_N^{\frac{1}{2}-2\varepsilon}\frac{\log N}{N}.$$

Thus, in view of (5.4),

$$I_N+S_N \ll_{\varepsilon,K,\alpha,\mathfrak{b},h_2} u_N^{-\varepsilon}+u_N^{\frac{1}{2}-2\varepsilon}\frac{\log N}{N}.$$

Since $u_N\to\infty$ and $u_N\ll N^2$, this shows that

$$\lim_{N\to\infty}(I_N+S_N)=0,$$

and, by (5.3), we have

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+ikh_2,\alpha;\mathfrak{b})-\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})|=0.$$

The lemma is proved.

Now we state the main result on the closeness of $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$ and $\underline{\zeta}_{u_N}(s+ikh_2,\alpha;\mathfrak{a},\mathfrak{b})$.

Lemma 5.4. *Suppose that $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$. Then, for every positive h_1 and h_2,*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho_2 \Big(\underline{\zeta}(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b}), \underline{\zeta}_{u_N}(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b}) \Big) = 0.$$

Proof. By the definition of the metric ρ_2, see Section 4.3, it suffices to prove that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\big(\zeta(s+ikh_1;\mathfrak{a}), \zeta_{u_N}(s+ikh_1;\mathfrak{a})\big) = 0$$

and

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\big(\zeta(s+ikh_2,\alpha;\mathfrak{b}), \zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})\big) = 0.$$

Therefore, the lemma is a consequence of Lemmas 5.1 and 5.3.

5.3 Limit theorems

The proof of Theorem 5.1 relies on a discrete limit theorem for $\underline{\zeta}_{u_N}(s,\alpha;\mathfrak{a},\mathfrak{b})$ in the space $H^2(D)$ on weakly convergent probability measures.

Recall that $P_{\underline{\zeta}}$ is the distribution of the random element $\underline{\zeta}(s,\alpha,\omega_1,\omega_2;\mathfrak{a},\mathfrak{b})$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H\{(\omega_1,\omega_2) \in \Omega : \underline{\zeta}(s,\alpha,\omega_1,\omega_2;\mathfrak{a},\mathfrak{b}) \in A\}, A \in \mathcal{B}(H^2(D)).$$

For $A \in \mathcal{B}(H^2(D))$, define

$$P_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(A) = \frac{1}{N+1} \#\{0 \leqslant k \leqslant N : \underline{\zeta}(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b}) \in A\}.$$

Lemma 5.5. *Suppose that the set $L(\mathbb{P},\alpha,h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,\alpha,\mathfrak{a},\mathfrak{b},h}$ converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.*

Proof. The lemma is proved in [36].

Lemmas 5.4 and 5.3 lead to a limit theorem for $\underline{\zeta}_{u_N}(s,\alpha;\mathfrak{a},\mathfrak{b})$. Let, for $A\in\mathcal{B}(H^2(D))$,

$$P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(A)=\frac{1}{N+1}\#\{0\leqslant k\leqslant N:\underline{\zeta}_{u_N}(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{a})\in A\}.$$

Theorem 5.2. *Suppose that the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$, and $u_N\to\infty$ and $u_N\ll N^2$ as $N\to\infty$. Then $P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}$ converges weakly to $P_{\underline{\zeta}}$ as $N\to\infty$.*

Proof. Let θ_N be a random variable defined on a certain probability space $(\widehat{\Omega},\mathcal{A},\mu)$, and having the distribution

$$\mu\{\theta_N=k\}=\frac{1}{N+1},k=0,1,\ldots,N.$$

We will use Lemma 4.4, i.e., the equivalent of weak convergence in terms of closed sets. Fix a closed set $F\subset H^2(D)$, and, for $\varepsilon>0$, define the set

$$F_\varepsilon=\{\underline{g}\in H^2(D):\inf_{\underline{\widehat{g}}\in F}\{g_2(\underline{g},\underline{\widehat{g}})\leqslant\varepsilon\}\}.$$

Then the set F_ε is closed in the space $H^2(D)$ as well. Define two $H^2(D)-$ valued random elements

$$\underline{X}_N=X_n(s)=\underline{\zeta}(s+i\theta_N\underline{h},\alpha;\mathfrak{a},\mathfrak{b})$$

and

$$\underline{Y}_N=Y_n(s)=\underline{\zeta}_{u_N}(s+i\theta_N\underline{h},\alpha;\mathfrak{a},\mathfrak{b}).$$

By the definition of the random variable θ_N, the random elements $\underline{X}_N$ and $\underline{Y}_N$ have the distributions $P_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}$ and $P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}$, respectively, i.e., for $A\in\mathcal{B}(H^2(D))$,

$$\mu\{\underline{X}_N\in A\}=\frac{1}{N+1}\#\{0\leqslant k\leqslant N:\underline{\zeta}(s+i\theta_N\underline{h},\alpha;\mathfrak{a},\mathfrak{b})\in A\}$$

and

$$\mu\{Y_N\in A\}=\frac{1}{N+1}\#\{0\leqslant k\leqslant N:\underline{\zeta}_{u_N}(s+i\theta_N\underline{h},\alpha;\mathfrak{a},\mathfrak{b})\in A\}.$$

Moreover, the inclusion

$$\{\underline{Y}_N\in F\}\subset\{\underline{X}_N\in F_\varepsilon\}\cup\{\rho_2\left(\underline{X}_N,\underline{Y}_N\right)\geqslant\varepsilon\}$$

is valid. Hence, by properties of probability measures,

$$\mu(F) \leqslant \mu(F_\varepsilon) + \mu\{(\underline{X}_N, Y_N) \geqslant \varepsilon\},$$

and, by the above remark,

$$P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(F) \leqslant P_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(F_\varepsilon) + \mu\{\rho_2\left(\underline{X}_n, Y_n \geqslant \varepsilon\right).\} \tag{5.5}$$

By Lemmas 5.5 and 4.4, we have

$$\limsup_{N\to\infty} P_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(F) \leqslant P_\zeta(F_\varepsilon). \tag{5.6}$$

An application of the Chebyshev type inequality and Lemma 5.4 gives

$$\begin{aligned}
&\limsup_{N\to\infty} \mu\{\rho_2(\underline{X}_n, \underline{Y}_n) \geqslant \varepsilon\} = \\
&\limsup_{N\to\infty} \frac{1}{N+1} \#\{0 \leqslant k \leqslant N : \rho_2(\underline{\zeta}(s+ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}), \\
&\underline{\zeta}_{u_N}(s+ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b})) \geqslant \varepsilon\} \leqslant \\
&\limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{k=0}^{N} \rho_2\left(\underline{\zeta}(s+ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_{u_N}(s+ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b})\right) = 0.
\end{aligned}$$

Thus, in view of (5.5) and (5.6),

$$\limsup_{N\to\infty} P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(F_\xi) \leqslant P_{\underline{\zeta}}(F_\varepsilon). \tag{5.7}$$

If $\varepsilon \to +0$, then F_ε tends to F. Therefore, inequality (5.7) implies

$$\limsup_{N\to\infty} P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(F) \leqslant P_{\underline{\zeta}}(F),$$

and Lemma 4.4 shows that $P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}$ converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.

Theorem 5.2 implies the weak convergence for the corresponding probability measures in the space $\mathbb{R}^2$. Let K_1, K_2, and $f_1(s), f_2(s)$ be the same as in Theorem 5.1. For $A \in \mathcal{B}(\mathbb{R}^2)$, define

$$Q_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(A) = \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \big(\sup_{s\in K_1} |\zeta_{u_N}(s+ikh_1; \mathfrak{a}) - f_1(s)|,$$

$$\sup_{s\in K_2} |\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b}) - f_2(s)|\big) \in A\Big\}.$$

Corollary 5.2. *Suppose that the set $L(\mathbb{P};\frac{1}{\pi},h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$, and $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$. Let K_1, K_2, and $f_1(s), f_2(s)$, be as in Theorem 5.1. Then $Q_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}$ converges weakly to the measure*

$$m_H\{(\omega_1,\omega_2)\in\Omega : \big(\sup_{s\in K_1} |\zeta(s,\omega_1;\mathfrak{a}) - f_1(s)|,$$
$$\sup_{s\in K_2} |\zeta(s,\alpha,\omega_2;\mathfrak{b}) - f_2(s)|\big) \in A\}, A\in\mathcal{B}(\mathbb{R}^2),$$

as $N\to\infty$.

Proof. As in Section 4.4, we use the mapping $v : H^2(D) \to \mathbb{R}^2$ given by

$$v(g_1,g_2) = \left(\sup_{s\in K_1} |g_1(s) - f_1(s)|, \sup_{s\in K_2} |g_2(s) - f_2(s)|\right), g_1,g_2\in H(D).$$

In the proof of Corollary 4.2, the continuity of v was obtained. By the definitions of v, $P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}$ and $Q_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}$, for $A\in\mathcal{B}(\mathbb{R}^2)$, we have

$$Q_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(A) = \frac{1}{N+1}\#\{0\leqslant k\leqslant N : v\left(\underline{\zeta}(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b})\right)\in A\} =$$

$$\frac{1}{N+1}\#\{0\leqslant k\leqslant N : \underline{\zeta}_N\left(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b}\right)\in v^{-1}A\} =$$
$$P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(v^{-1}A) = P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}v^{-1}(A).$$

Since the set A is arbitrary, this shows that $Q_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}} = P_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}v^{-1}(A)$. The latter equality, continuity of v, and Theorem 5.2 imply that the measure $Q_{N,u_N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}$ converges weakly to $P_{\underline{\zeta}}v^{-1}$ as $N\to\infty$, i.e. to

$$P_{\underline{\zeta}}(v^{-1}A) = m_H\{(\omega_1,\omega_2)\in\Omega : \underline{\zeta}(s,\alpha,\omega_1,\omega_2;\mathfrak{a},\mathfrak{b})\in v^{-1}A\} =$$

$$m_H\{(\omega_1,\omega_2)\in\Omega : v(\underline{\zeta}(s,\alpha,\omega_1,\omega_2;\mathfrak{a},\mathfrak{b}))\in A\} =$$

$$m_H\Big\{(\omega_1,\omega_2)\in\Omega : \big(\sup_{s\in K_1} |\zeta(s,\omega_1;\mathfrak{a}) - f_1(s)|,$$
$$\sup_{s\in K_2} |\zeta(s,\alpha,\omega_2;\mathfrak{b}) - f_2(s)|\big) \in A\Big\},\ A\in\mathcal{B}(\mathbb{R}^2)$$

5.4 Proof of Theorem 5.1

As in Chapter 4, for the proof of Theorem 5.1, we will apply two-dimensional distribution functions and their weak convergence.

Proof of Theorem 5.1. Define the functions

$$F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(\varepsilon_1,\varepsilon_2) = \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K_1} |\zeta_{u_N}(s+ikh_1;\mathfrak{a}) - f_1(s)| < \varepsilon_1,$$
$$\sup_{s\in K_2} |\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon_2\Big\}$$

and

$$F(\varepsilon_1,\varepsilon_2) = m_H\Big\{(\omega_1,\omega_2)\in\Omega : \sup_{s\in K_1} |\zeta(s,\omega_1;\mathfrak{a}) - f_1(s)| < \varepsilon_1,$$
$$\sup_{s\in K_2} |\zeta(s,\alpha,\omega_2;\mathfrak{b}) - f_2(s)| < \varepsilon_2\Big\}.$$

Note that $F(\varepsilon_1,\varepsilon_2)$ is the same as in Section 4.4. The function $F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(\varepsilon_1,\varepsilon_2)$ corresponds the measure $Q_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}$, while $F(\varepsilon_1,\varepsilon_2)$ corresponds the limit measure in Corollary 5.2. The above functions are non-decreasing and left continuous with respect to ε_1 and ε_2,
$F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(-\infty,\varepsilon_2) = F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(\varepsilon_1,-\infty) = 0$, $F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(+\infty,+\infty) = 1$, $F(-\infty,\varepsilon_2) = F(\varepsilon_1,-\infty) = 0$ and $F(+\infty,+\infty) = 1$. Moreover, for every rectangle A, say, $a_1 \leqslant \varepsilon_1 < a_2, b_1 \leqslant \varepsilon_2 < b_2$,

$$F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(a_2,b_2) - F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(a_1,b_2) - F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(a_2,b_1) + F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(a_1,b_1) =$$

$$\frac{1}{N+1}\#\{0 \leqslant k \leqslant N : (\sup_{s\in K_1} |\zeta_{u_N}(s+ikh_1;\mathfrak{a}) - f_1(s)|,$$
$$\sup_{s\in K_2} |\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b}) - f_2(s)|) \in A\} \geqslant 0,$$

and

$$F(a_2,b_2) - F(a_1,b_2) - F(a_2,b_1) + F(a_1,b_1) =$$

$$m_H\{(\omega_1,\omega_2)\in\Omega : (\sup_{s\in K_1} |\zeta(s,\omega_1;\mathfrak{a}) - f_1(s)|,$$
$$\sup_{s\in K_2} |\zeta(s,\alpha,\omega_2;\mathfrak{b}) - f_2(s)|) \in A\} \geqslant 0.$$

Thus, the functions $F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}$ and $F(\varepsilon_1, \varepsilon_2)$ are distribution functions.

Let, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be continuity points of the distribution functions $F(\varepsilon_1, +\infty)$ and $F(+\infty, \varepsilon_2)$. Then, by Corollary 5.2,

$$\lim_{N\to\infty} F_{N,\alpha,\mathfrak{a},\mathfrak{b},\underline{h}}(\varepsilon_1, \varepsilon_2) = F(\varepsilon_1, \varepsilon_2). \tag{5.8}$$

Since the set of discontinuity points of a distribution functions is at most countable, equality (5.8) holds for all but at most countably many $\varepsilon > 0$. The positivity of $F(\varepsilon_1, \varepsilon_2)$ is proved in Section 4.5. For this, Lemmas 4.3 and 1.8 are applied. The theorem is proved.

Chapter 6

Conclusions

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be periodic sequences of complex numbers, and, for $\theta > \frac{1}{2}$ and $u > 0$,

$$v_u(m) = \exp\left\{-\left(\frac{m}{u}\right)^\theta\right\}, m \in \mathbb{N}, n \in \mathbb{N},$$

$$v_u(m, \alpha) = \exp\left\{-\left(\frac{m+\alpha}{u}\right)^\theta\right\}, m \in \mathbb{N}_0, n \in \mathbb{N}.$$

In the dissertation, approximation of analytic functions by shifts of absolutely convergent Dirichlet series

$$\zeta_{u_T}(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_{u_T}(m)}{m^s}$$

and

$$\zeta_{u_T}(s, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m v_{u_T}(m, \alpha)}{(m+\alpha)^s}$$

with $s = \sigma + it, s \in \mathbb{C}, t \in \mathbb{R}$ and $u_T \to \infty$ as $T \to \infty$ is considered.
The following results are obtained.

1. Suppose that $\mathfrak{a}$ is a multiplicative sequence, and $u_T \ll T^2$. Then the set of shifts $\zeta_{u_T}(s + i\tau; \mathfrak{a})$, approximating uniformly on compact sets a given analytic function defined on a strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ has a positive density in the interval $[0, T]$.

2. Analogical result is true for the set of discrete shifts $\zeta_{u_N}(s+ikh; \mathfrak{a})$ with $u_N \to \infty$ and $u_N \ll N^2$ as $T \to \infty, h > 0$ and $k \in \{0, 1, \ldots, N\}$.

3. Suppose that $\mathfrak{a}$ is a multiplicative sequence, α is a transcendental parameter, and $u_T \ll T^2$. Then the set of shifts $\big(\zeta_{u_T}(s+i\tau;\mathfrak{a}), \zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b})\big)$ approximating simultaneously uniformly on compact sets a given pair $\big(f_1(s), f_2(s)\big)$ of analytic functions on D, $f_1(s) \neq 0$, has a positive density in the interval $[0, T]$.

4. Suppose that the set $\big\{(h_1 \log p : p \in \mathbb{P}), (h_2 \log(m+\alpha) : m \in \mathbb{N}), 2\pi\big\}$, $h_1 > 0, h_2 > 0$, is linearly independent over field of rational numbers. Then analogical result is true for the set of discrete shifts $\big(\zeta_{u_N}(s+ikh_1;\mathfrak{a}), \zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})\big)$ with $u_N \to \infty$, $u_N \ll N^2$, and $k \in \{0, 1, \ldots, N\}$.

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Santrauka (Summary in Lithuanian)

Tyrimo objektas

Disertacija yra skirta aproksimavimo problemoms kai kuriomis Dirichlė eilutėmis. Primename, kad paprastoji Dirichlė eilutė turi pavidalą

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

čia $\{a_m : m \in \mathbb{N}\} \subset \mathbb{C}$ ir $s = \sigma + it$. Bendromis Dirichlė eilutėmis vadinamos eilutės

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s},$$

čia $\{\lambda_m\} \subset \mathbb{R}$ yra didėjanti į $+\infty$ seka. Dirichlė eilučių tiek konvergavimo, tiek ir absoliutaus konvergavimo sritis yra pusplokštumė.

Dirichlė eilutės yra bendrosios analizės tyrimo objektas, tačiau jos dažnai naudojamos ir analizinėje skaičių teorijoje, kadangi daugelis vadinamųjų dzeta funkcijų yra apibrėžiamos Dirichlė eilutėmis. Pavyzdžiui, garsioji Rymano dzeta funkcija $\zeta(s)$ pusplokštumėje $\sigma > 1$ apibrėžiama Dirichlė eilute

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

t.y., $a_m \equiv 1$. Funkcija $\zeta(s)$ yra analiziškai pratęsiama į visą kompleksinę plokštumą, išskyrus tašką $s = 1$, kuris yra paprastasis polius su reziduumu 1. Pastebime, jog funkcija $\zeta(s)$ yra pagrindinis pirminių skaičių pasiskirtymo aibėje $\mathbb{N}$ tyrimo instrumentas. Naudodami B.Rymano (Riemann) idėjas, Adamaras (Hadamard) ir de la Valė Pusenas (de la Vallée Poussin) nepriklausomai 1896 m. įrodė, kad

$$\sum_{p\leqslant x} 1 = \int_2^\infty \frac{du}{\log u}, x \to \infty.$$

Funkcija $\zeta(s)$ gali būti apibrėžiama srityje $\sigma > 1$ ir begaline sandauga pagal pirminius skaičius

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

kas rodo jos sąryšį su pirminiais skaičiais.

Tarkime, kad funkcija

$$\chi : \mathbb{N} \to \mathbb{C}$$

turi tokias savybes:

1° $\chi(m)$ yra visiškai multiplikatyvi, t.y. $\chi(m_1, m_2) = \chi(m_1)\chi(m_2)$ su visais $m_1, m_2 \in \mathbb{N}$ ir $\chi(1) = 1$;

2° $\chi(m)$ yra periodinė su minimaliu periodu $q \in \mathbb{N}$, t.y., $\chi(m+q) = \chi(m)$ su visais $m \in \mathbb{N}$;

3° $\chi(m) = 0$ su visais $(m, q) > 1$;

4° $\chi(m) \neq 0$ su visais $(m, q) = 1$.

Tuomet $\chi(m)$ sutampa su vienu iš $\phi(q)$ ($\phi(q)$ yra Oilerio funkcija) Dirichlė charakteriu moduliu q. Dirichlė L funkcija $L(s, \chi)$ su charakteriu χ pusplokštumėje $\sigma > 1$ yra apibrėžiama Dirichlė eilute

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}$$

ir yra analiziškai pratęsiama į visą kompleksinę plokštumą su vieninteliu galimu paprastuoju poliu taške $s = 1$, kai charakteris χ yra pagrindinis, t.y. $\chi(m) \equiv 1$ su visais $(m, q) = 1$. Dirichlė L funkcijas apibrėžė L. Dirichlė (Dirichlet) ir panaudojo jas pirminių skaičių aritmetinėse progresijose $\{an+b : n \in \mathbb{N}\}$, $(a, b) = 1$, pasiskirstymo tyrimui. Yra įrodyta, jog

$$\sum_{\substack{p\leqslant x \\ p\equiv b \bmod a}} 1 = \frac{1}{\phi(a)} \int_2^x \frac{du}{\log u}, x \to \infty.$$

Tegul $0 < \alpha \leqslant 1$ yra fiksuotas parametras. Hurvico dzeta funkciją $\zeta(s, \alpha)$ 1882 m. pirmą kartą panaudojo Hurvicas (Hurwitz). Srityje $\sigma > 1$ ji yra apibrėžiama eilute

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s}=\sum_{m=0}^{\infty}e^{-s\log(m+\alpha)},$$

taigi šiuo atveju $\lambda_m = \log(m+\alpha)$. Kai $\alpha = 1$, tada $\zeta(s;\alpha)$ tampa Rymano dzeta funkcija. Funkcija $\zeta(s,\alpha)$, kaip ir $\zeta(s)$, yra meromorfinė, ji turi paprastąjį polių taške $s=1$ su reziduumu 1.

Disertacijoje nagrinėjamos Dirichlė eilutės, susijusios su vadinamosiomis dzeta funkcijomis. Tarkime, kad $\mathfrak{a}=\{a_m : m\in\mathbb{N}\}$ ir $\mathfrak{b}=\{b_m : m\in\mathbb{N}_0\}$ yra dvi periodinės kompleksinių skaičių sekos atitinkamai su periodais $q_1\in\mathbb{N}$ ir $q_2\in\mathbb{N}$.

Periodinė dzeta funkcija $\zeta(s;\mathfrak{a})$ ir periodinė Hurvico dzeta funkcija $\zeta(s,\alpha;\mathfrak{b})$ pusplokštumėje $\sigma>1$ yra apibrėžiamos Dirichlė eilutėmis

$$\zeta(s;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m}{m^s}\text{ ir }\zeta(s,\alpha;\mathfrak{b})=\sum_{m=0}^{\infty}\frac{b_m}{(m+\alpha)^s}.$$

Iš sekų $\mathfrak{a}$ ir $\mathfrak{b}$ periodiškumo srityje $\sigma>1$ išplaukia lygybės

$$\zeta(s;\mathfrak{a})=\frac{1}{q_1^s}\sum_{k=1}^{q_1}a_k\,\zeta\left(s,\frac{k}{q}\right)$$

ir

$$\zeta(s,\alpha;\mathfrak{b})=\frac{1}{q_2^s}\sum_{k=0}^{q_2-1}b_k\,\zeta\left(s,\frac{k+\alpha}{q_2}\right).$$

Iš šių lygybių ir Hurvico dzeta funkcijos savybių turime, kad funkcijos $\zeta(s;\mathfrak{a})$ ir $\zeta(s,\alpha;\mathfrak{b})$ yra analiziškai pratęsiamos į visą kompleksinę plokštumą, išskyrus tašką $s=1$, kuris yra paprastasis polius atitinkamai su reziduumais

$$\frac{1}{q_1}\sum_{k=1}^{q_1}a_k\text{ ir }\frac{1}{q_2}\sum_{k=0}^{q_2-1}b_k.$$

Tarkime, jog $\theta>\frac{1}{2}$ yra fiksuotas skaičius, $u>0$,

$$v_u(m)=\exp\left\{-\left(\frac{m}{u}\right)^{\theta}\right\},m\in\mathbb{N},$$

ir

$$v_u(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{u}\right)^{\theta}\right\}, m \in \mathbb{N}_0.$$

Kadangi $v_u(m)$ ir $v_u(m,\alpha)$ mažėja eksponentiškai m atžvilgiu, eilutės, apibrėžiančios $\zeta_u(s;\mathfrak{a})$ ir $\zeta_u(s,\alpha;\mathfrak{b})$, konverguoja absoliučiai pusplokštumėje $\sigma > \sigma_0$ su bet kuriuo baigtiniu σ_0 ir kiekvienu $u > 0$. Disertacija yra skirta analizinių funkcijų aproksimavimui postūmiais $\zeta_{u_T}(s+i\tau;\mathfrak{a})$ ir $\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b})$, $\tau \in \mathbb{R}$, su tam tikra funkcija $u_T \to \infty$, kai $T \to \infty$.

Tikslas ir uždaviniai

Disertacijos tikslas yra analizinių funkcijų aproksimavimas absoliučiai konverguojančių eilučių $\zeta_{u_T}(s;\mathfrak{a})$ ir $\zeta_{u_T}(s,\alpha;\mathfrak{b})$ postūmiais. Yra sprendžiami šie uždaviniai.

1. Analizinių funkcijų klasės aproksimavimas tolydžiais postūmiais $\zeta_{u_T}(s+i\tau;\mathfrak{a}), \tau \in \mathbb{R}$ su multiplikatyviąja seka $\mathfrak{a}$.

2. Analizinių funkcijų klasės aproksimavimas diskrečiais postūmiais $\zeta_{u_N}(s+ikh;\mathfrak{a}), h > 0, k \in \mathbb{N}_0$, su multiplikatyviąja seka $\mathfrak{a}$.

3. Analizinių funkcijų porų klasės aproksimavimas tolydžiais postūmiais $\left(\zeta_{u_T}(s+i\tau;\mathfrak{a}), \zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b})\right), \tau \in \mathbb{R}$, su multiplikatyviąja seka $\mathfrak{a}$.

4. Analizinių funkcijų porų klasės aproksimavimas diskrečiais postūmiais $\left(\zeta_{u_T}(s+ikh_1;\mathfrak{a}), \zeta_{u_T}(s+ikh_2,\alpha;\mathfrak{b})\right), h_1 > 0, h_2 > 0, k \in \mathbb{N}_0$, su multiplikatyviąja seka $\mathfrak{a}$.

Aktualumas

Analizinių funkcijų aproksimavimas yra vystomas ir taikomas daugelyje matematikos ir kitų mokslo sričių. Sprendžiant įvairius matematinius ir praktinius uždavinius, pasirodo sudėtingos analizinės funkcijos, todėl iškyla problema pakeisti jas paprastesnėmis, kas veda prie aproksimavimo problemų. Gerai žinoma Mergeliano (Mergelyan) teorema tvirtina, kad kiekvieną analizinę funkciją su kai kuriais aproksimavimo srities apribojimais, galima aproksimuoti polinomais. Taigi, kiekvieną analizinę funkciją atitinka tą funkciją aproksimuojantis polinomas. Apie 1970 - 1980 m. skaičių teorijos specialistai S.M. Voroninas (Voronin), S.M. Gonekas (Gonek), B.Bagčis (Bagchi) surado naujus analizinius objektus, kurių kiekvienas aproksimuoja ištisą analizinių funkcijų klasę. Tie objektai yra apibrėžiami Dirichlė eilutėmis ir apima daugelį analizinėje skaičių teorijoje nagrinėjamų dzeta ir L funkcijų. Naujieji aproksimavimo objektai yra universalūs, vieno ir to paties objekto postūmiai aproksimuojantys plačią analizinių funkcijų klasę. Universalių aproksimavimo objektų atradimas iškėlė naujas problemas, susijusias su aproksimavimo efektyvizavimu, jungtiniu universalumu, apibendrintųjų postūmių naudojimu ir kt. Todėl Dirichlė eilučių, įskaitant ir dzeta funkcijas, universalumo teorija yra vystoma toliau. Nemažai dėmesio Dirichlė universalumo problemoms yra skiriama ir Lietuvoje. Yra svarbu išplėsti universalių funkcijų klasę, supaprastinti jų struktūrą. Todėl disertacijoje yra nagrinėjamos analizinių funkcijų aproksimavimas naudojant absoliučiai konverguojančių Dirichlė eilučių postūmius. Tokių eilučių paprastumas leidžia lengviau charakterizuoti aproksimuojamas funkcijas.

Metodai

Mes naudojame tiek analizinius, tiek ir tikimybinius metodus. Analiziniai metodai apima Dirichlė eilučių teoriją, integravimą, reziduumų teoriją, Koši integralinės formulės taikymą bei Mergeliano teoremą. Silpnasis tikimybinių matų konvergavimas, atsitiktinių elementų konvergavimas pagal pasiskirstymą, tikimybinių matų ir jų atramų savybės sudaro tikimybinių metodų taikymo turinį.

Naujumas

Visi disertacijos rezultatai yra nauji. Absoliučiai konverguojančių Dirichlė eilučių, susijusių su periodinėmis dzeta funkcijomis, universalumas disertacijoje yra nagrinėjamas pirmą kartą.

Problemos istorija ir rezultatai

Tarkime, turime kokią nors kompleksinių skaičių seką $\{a(m) : m \in \mathbb{N}\}$ ir mokame apskaičiuoti sekos narius $a(m)$. Tačiau dažnai reikšmės $a(m)$ yra pasiskirstę chaotiškai ir suteikia nedaug informacijos apie visą seką. Todėl vietoje individualių reikšmių $a(m)$ yra nagrinėjamas jų vidurkis

$$M(x) \stackrel{\text{def}}{=} \sum_{m \leqslant x} a(m), x \to \infty.$$

$M(x)$ tyrimui yra naudojamos sekos $\{a(m)\}$, generuojančios Dirichlė eilutes

$$Z(s) \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \sigma > \sigma_0.$$

Yra žinomos formulės, vadinamos Perono formulėmis, surišančios $M(x)$ su $Z(s)$, t.y.,

$$M(x) = Z(s) \text{ integralinis operatorius + liekamasis narys.}$$

Taigi, vidurkio $M(x)$ tyrimo problema pakeičiama funkcijos $Z(s)$ analizinių savybių nagrinėjimu.

Pateikiame daliklių funkcijos

$$d(m) = \sum_{d|m} 1, m \in \mathbb{N},$$

pavyzdį. Aritmetinės funkcijos $d(m)$ generuojanti funkcija yra $\zeta^2(s)$, t.y.,

$$\zeta^2(s) = \sum_{m=1}^{\infty} \frac{d(m)}{m^s}, \sigma > 1.$$

Panaudojus funkcijos $\zeta(s)$ savybes, yra gaunama formulė

$$\sum_{m \leqslant x} d(m) = x \log x + (2\widehat{\gamma} - 1)x + O(x^{\delta}),$$

čia

$$\widehat{\gamma} = \lim_{n \to \infty} \left(\sum_{m \leqslant n} \frac{1}{m} - \log n \right) = 0,57721...$$

yra Oilerio konstanta ir $\frac{1}{4} \leqslant \delta \leqslant \frac{1}{2}$. Didžiausias skaičių δ apatinis rėžis yra vadinamas Dirichlė daliklių problema. Geriausias rezultatas priklauso M.Haksliui (Huxley, 2003) ir yra lygus $\frac{131}{416} = 0.3149...$. Šis pavyzdys rodo, kaip yra svarbu tirti Rymano ir kitas dzeta funkcijas. Iš H.Boro (Bohr) ir R.Kuranto (Courant) straipsnio [6] yra žinoma, kad funkcijos $\zeta(s)$ reikšmių aibė yra labai tanki, aibė $\{\zeta(\sigma+it) : t \in \mathbb{R}\}$ su $\frac{1}{2} < \sigma \leqslant 1$ yra visur tiršta erdvėje $\mathbb{C}$. Tam tikra prasme, šis rezultatas paaiškina, kodėl funkcija $\zeta(s)$ pasirodo ne tik daugelyje aritmetikos, bet ir kitų matematikos šakų, ir netgi kai kurių kitų gamtos mokslų problemose. Pavyzdžiui, funkcijos $\zeta(s)$ nuliai yra susiję su kai kurių operatorių tikrinėmis reikšmėmis, $\zeta(s)$ yra taikoma kosmologijoje [1] ir netgi muzikoje tonų derinimui [50].

Praėjusio amžiaus aštuntame dešimtmetyje buvo atrasta dar viena svarbi funkcijos $\zeta(s)$ ir kitų dzeta funkcijų savybė. Tapo žinoma, kad postūmiai $\zeta(s+i\tau), \tau \in \mathbb{R}$, aproksimuoja plačią analizinių funkcijų klasę. Tiksliau kalbant, S.M. Voroninas įrodė [37] tokią funkcijos $\zeta(s)$ universalumo savybę.

A teorema. *Tegu $0 < r < \frac{1}{4}$ yra fiksuotas skaičius, o funkcija $f(s)$ yra tolydi, nevirstanti nuliu skritulyje $|s| \leqslant r$ ir analizinė srityje $|s| < r$. Tuomet su kiekvienu $\varepsilon > 0$ egzistuoja toks $\tau = \tau(\varepsilon) \in \mathbb{R}$, su kuriuo yra teisinga nelygybė*

$$\max_{|s|\leqslant r} \left|\zeta(s + \frac{3}{4} + i\tau) - f(s)\right| < \varepsilon.$$

Tegul $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$, o $H(D)$ yra analizinių juostoje D funkcijų erdvė su tolygaus konvergavimo kompaktinėse aibėse topologija. Šioje topologijoje seka $g_n(s) \subset H(D)$ konverguoja į funkciją $g(s) \in H(D)$ kai $n \to \infty$, jei su kiekviena kompaktine aibe $K \subset D$,

$$\lim_{n\to\infty} \sup_{s\in K} |g_n(s) - g(s)| = 0.$$

Kadangi erdvė $H(D)$ yra begaliniamatė, tai A teorema yra minėtos Boro-Kuranto teoremos [6] begaliniamatė versija.

Voronino universalumo teorema susilaukė didelio matematikų susidomėjimo. Buvo sukurti nauji jos įrodymai, sustiprintas teoremos tvirtinimas, universalumo savybė išplėsta kitoms dzeta ir L funkcijoms. Minėtus rezultatus galima rasti disertacijose [2], [15], monografijose [28], [52], bei apžvalginiame straipsnyje [42]. Paskutinioji Voronino teoremos versija naudoja tokią terminologiją. Tegul $\mathcal{K}$ yra juostos D kompaktinių aibių su jungiaisiais papildiniais klasė, o $H_0(K), K \in \mathcal{K}$, tolydžių, nevirstančių nuliu aibėje K ir analizinių aibės K viduje klasė. Simbolis $\mathrm{meas}A$ žymi mačios aibės $A \subset \mathbb{R}$ Lebego matą. Tuomet yra teisingas toks tvirtinimas [2], [28], [52].

B teorema. *Tarkime, kad $K \in \mathcal{K}$, o $f(s) \in H_0(K)$. Tuomet su kiekvienu*

$\varepsilon > 0$ *yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\Big\} = 0.$$

Be to, apatinė riba gali būti pakeista riba, išskyrus ne daugiau negu skaičią $\varepsilon > 0$ *reikšmių aibę.*

B teorema rodo, jog yra be galo daug postūmių$\zeta(s+i\tau)$ aproksimuojančių norimu tikslumu ε duotą funkciją $f(s) \in H_0(K)$. Tačiau B teorema nėra efektyvi ta prasme, kad joks konkretus aproksimuojantis postūmis $\zeta(s+i\tau)$ nėra žinomas.

Kai kurie universalumo teoremos efektyvinimo rezultatai buvo gauti [12] ir [14] darbuose. Šiuose darbuose efektyviai surastas intervalas $[T_0, 2T_0]$, kuriam priklauso toks τ, kad $\zeta(s+i\tau)$ yra aproksimuojantis postūmis. Minėtų darbų apžvalga duota [14].

Pagal Liniko (Linnik) - Ibragimovo (Ibragimov) hipotezę, žr., pvz., [52], visos funkcijos, kurioje nors pusplokštumėje apibrėžiamos Dirichlė eilutėmis, analiziškai pratęsiamos į kairę nuo šios pusplokštumės ir tenkinančios kai kurias natūralias augimo sąlygas, yra universalios Voronino prasme. Tačiau iki šiol egzistuoja Dirichlė eilutės, kurių universalumas nėra įrodytas.

Todėl priminsime kai kuriuos rezultatus apie funkciją $\zeta(s,\mathfrak{a})$ su periodu $q_1 \in \mathbb{N}$ su periodine seka $\mathfrak{a}$. Funkcijos $\zeta(s,\mathfrak{a})$ reikšmių pasiskirstymas metodiškai yra nagrinėjamas [46] monografijos 11 skyriuje. Pavyzdžiui, čia su visais $s \in \mathbb{C}$ randame funkcinę lygtį

$$\zeta(1-s,\mathfrak{a}^{\pm}) = \Big(\frac{q_1}{2\pi}\Big)^s \frac{\Gamma(s)}{\sqrt{q_1}}\Big(\exp\Big\{\frac{\pi i s}{2}\Big\}\zeta(s,\mathfrak{a}^{\pm}) + \exp\Big\{-\frac{\pi i s}{2}\Big\}\zeta(s,\mathfrak{a}^{\pm})\Big),$$

čia

$$\mathfrak{a}^{\pm} = \Bigg\{a_m^{\pm} = \frac{1}{\sqrt{q_1}}\sum_{m=1}^{q_1} a_k \exp\Big\{\pm 2\pi i m k\Big\} : m \in \mathbb{N}\Bigg\}.$$

Pirmą kartą ši lygtis buvo gauta darbe [51].

[40] straipsnyje išnagrinėtas ir funkcijos $\zeta(s,\mathfrak{a})$ nulių pasiskirstymas. Buvo gauta, jog egzistuoja tokia konstanta $B(a)$, kad, kai $\sigma < -B(\mathfrak{a})$, funkcija $\zeta(s;\mathfrak{a})$ gali turėti nulius tik arti realiosios ašies, jei $m_{a^+} = m_{a^-}$, $m_{\mathfrak{a}_{\pm}} = \min\Big\{m : 1 \leqslant m \leqslant q_1 : a_m^{\pm} \neq 0\Big\}$, ir arti tiesės

$$\sigma = 1 + \frac{\pi\tau}{\log\frac{m_{\mathfrak{a}^-}}{m_{\mathfrak{a}^+}}}$$

jei $m_{a^+} \neq m_{a^-}$. Funkcijos $\zeta(s,\mathfrak{a})$ nuliai $\rho = \beta + i\gamma$ yra vadinami trivialiaisias, jei $\beta < -B(\mathfrak{a})$. Netrivialūs nuliai yra juostoje $-B(\mathfrak{a}) \leqslant \sigma < 1 + A(\mathfrak{a})$ su tam tikra konstanta $A(\mathfrak{a}) > 0$. Tegul $N(T;\mathfrak{a})$ yra funkcijos $\zeta(s;\mathfrak{a})$ netrivialiųjų nulių su $|\gamma| \leqslant T$ skaičius. Tada monografijoje [52] įrodyta, jog

$$N(T;\mathfrak{a}) = \frac{T}{\pi}\log\frac{q_1 T}{2\pi e m_{\mathfrak{a}}\sqrt{m_{\mathfrak{a}^-}m_{\mathfrak{a}^+}}} + O(\log T),$$

kitaip tariant, funkcijai $\zeta(s,\mathfrak{a})$ yra teisingas von Mangoldto (von Mangoldt) formulės analogas, galiojantis Rymano dzeta funkcijai.

Disertacijos tikslas yra analizinių funkcijų aproksimavimas postūmiais, artimais postūmiams $\zeta(s+i\tau;\mathfrak{a})$ ir $\zeta(s+i\tau,\alpha;\mathfrak{b})$. Todėl priminsime kai kuriuos funkcijų $\zeta(s;\mathfrak{a})$ ir $\zeta(s,\alpha;\mathfrak{b})$ universalumo rezultatus. Pirmasis rezultatas yra randamas [52] monografijoje. Tegul $H(K), K \in \mathcal{K}$, yra tolydžiųjų aibėje K funkcijų, kurios yra analizinės aibės K viduje, klasė. Taigi, $H_0(K) \subset H(K)$.

C teorema. *Tarkime, kad* $q_1 > 2$, $\mathfrak{a}$ *nėra Dirichlė charakterio moduliu* q_1 *kartotinis ir* $a_m = 0$, *kai* $(m, q_1) > 1$. *Tegul* $K \in \mathcal{K}$ *ir* $f(s) \in H(K)$. *Tuomet su visais* $\varepsilon > 0$ *yra teisinga nelygybė*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\} > 0.$$

Pastebime, kad yra įrodyta [40], kad C teoremos sąlygos užtikrina, jog seka$\mathfrak{a}$ nėra multiplikatyvi.

Funkcijos $\zeta(s;\mathfrak{a})$ universalumas su multiplikatyviąja seka $\mathfrak{a}$ įrodytas [31] darbe. Teisinga tokia teorema.

D teorema [31]. *Tarkime, jog periodinė seka* $\mathfrak{a}$ *yra multiplikatyvi. Tegul* $K \in \mathcal{K}$ *ir* $f(s) \in H_0(K)$. *Tuomet su kiekvienu* $\varepsilon > 0$ *yra teisinga lygybė*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau \in [0,T] \sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\} > 0.$$

Disertacijos 2 skyrius yra skirtas absoliučiai konverguojančios Dirichlė eilutės

$$\zeta_u(s;\mathfrak{a}) = \sum_{m=1}^{\infty}\frac{a_m v_u(m)}{m^s}$$

universalumui. Teoremos formulavimui reikalingi kai kurie objektai. Tegul $\mathbb{P}$

yra visų pirminių skaičių aibė, $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ ir

$$\Omega_1 = \prod_{p \in \mathbb{P}} \gamma_p,$$

čia $\gamma_p = \gamma$ su visais $p \in \mathbb{P}$. Toras Ω_1 yra kompaktinių aibių Dekarto sandauga, todėl pagal klasikinę Tichonovo teoremą su sandaugos topologija ir pataškinės daugybos operacija, yra kompaktinė topologinė Abelio grupė. Tarkime, $\mathcal{B}(\mathbb{X})$ yra erdvės $\mathbb{X}$ Borelio σ-kūnas. Tada mačioje erdvėje $(\Omega_1, \mathcal{B}(\Omega_1)$ egzistuoja tikimybinis Haro (Haar) matas ir gauname tikimybinę erdvę $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$. Tegul $\omega_1 = \{\omega_1(p) : p \in \mathbb{P}\}$ yra toro Ω_1 elementai. Tikimybinėje erdvėje $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$ apibrėžiame $H(D)-$ reikšmį atsitiktinį elementą

$$\zeta(s, \omega; \mathfrak{a}) = \prod_{p \in \mathbb{P}} \Big(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} \omega_1^l(p)}{p^{ls}}\Big).$$

Ši begalinė sandauga su beveik visais $\omega_1 \in \Omega_1$ konverguoja tolygiai juostos D kompaktinėse aibėse.

Pagrindinis skyriaus rezultatas yra tokia teorema.

2.1 teorema. *Tarkime, kad seka $\mathfrak{a}$ yra multiplikatyvi, $u_T \to \infty$ ir $u_T \ll T^2$, kai $T \to \infty$. Tegul $K \in \mathcal{K}$ ir $f(s) \in H_0(K)$. Tuomet su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią jų reikšmių aibę, egzistuoja teigiama riba*

$$\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_{u_T}(s + i\tau; \mathfrak{a}) - f(s)| < \varepsilon \right\} =$$

$$m_{1H} \left\{ \omega_1 \in \Omega_1 : \sup_{s \in K} |\zeta(s, \omega_1; \mathfrak{a}) - f(s)| < \varepsilon \right\}.$$

Iš 2.1 teoremos išplaukia, jog egzistuoja toks skaičius $T_0 = T_0(f, K, \varepsilon) > 0$, kad su visais $T \geqslant T_0$ yra be galo daug postūmių $\zeta_{u_T}(s+i\tau; \mathfrak{a})$, aproksimuojančių tikslumu ε duotą funkciją $f(s) \in H_0(K)$.

2.1 teoremos įrodymui yra naudojama ribinė teorema funkcijai $\zeta_{u_T}(s; \mathfrak{a})$ erdvėje $H(D)$.

2 skyriaus rezultatai yra paskelbti [18] straipsnyje.

Iki šiol minėtos universalumo teoremos funkcijoms $\zeta(s; \mathfrak{a})$ ir $\zeta_{u_T}(s; \mathfrak{a})$ yra vadinamos tolydžiomis, nes τ aproksimuojančiuose postūmiuose $\zeta(s + i\tau; \mathfrak{a})$ ir $\zeta_{u_T}(s + i\tau; \mathfrak{a})$ gali įgyti bet kokias teigiamas reikšmes.

Lygiagrečiai su tolydžiomis universalumo teoremomis dzeta funkcijoms yra nagrinėjamos ir diskrečios universalumo teoremos, kai τ postūmiuose $(s+$

$i\tau; \mathfrak{a})$ įgyja reikšmes iš kokios nors diskrečios aibės, pavyzdžiui, iš aritmetinės progresijos $\{kh : k \in \mathbb{N}_0\}, h > 0$. Tokias teoremas pasiūlė A.Reichas (Reich) [49] straipsnyje. Tegul $\#A$ yra aibės $A \subset \mathbb{R}$ elementų skaičius, o N perbėga aibę $\mathbb{N}_0$. Reichas nagrinėjo [49] algebrinių kūnų $\mathbb{K}$ Dedekindo (Dedekind) dzeta funkciją $\zeta(s)_{\mathbb{K}}$, kuri srityje $\sigma > 1$ apibrėžiama eilute

$$\zeta_{\mathbb{K}}(s) = \sum_{I} \frac{1}{(\mathcal{N}(I))^s},$$

čia sumuojama pagal visus nenulinius kūno $\mathbb{K}$ sveikųjų skaičių žiedo idealus I, o $\mathcal{N}(I)$ yra idealo I norma. Kai $\mathbb{K} = \mathbb{Q}$, funkcija $\zeta_{\mathbb{K}}(s)$ tampa Rymano dzeta funkcija. Todėl formuluojame diskrečią universalumo teoremą funkcijai $\zeta(s)$.

E teorema. *Tegul* $K \in \mathcal{K}, f(s) \in H_0(K)$ *ir* $h > 0$. *Tuomet su kiekvienu* $\varepsilon > 0$ *yra teisinga nelygybė*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon\Big\} > 0.$$

Pastebime, kad E teorema yra pranašesnė už B teoremą, nes yra paprasčiau nustatyti diskrečius aproksimuojančius postūmius $\zeta(s + ikh)$ negu tolydžius $\zeta(s + i\tau)$.

Primename vieną diskrečios svertinės universalumo teoremos funkcijai $\zeta(s; \mathfrak{a})$ [55] išvadą. Tegul $L(\mathbb{P}; h, \pi) = \{(\log p : p \in \mathbb{P}), \frac{2\pi}{h}\}$.

F teorema. *Tarkime, kad seka* $\mathfrak{a}$ *yra multiplikatyvi, o aibė* $L(\mathbb{P}; h, \pi), h \geqslant 1$, *yra tiesiškai nepriklausoma virš* $\mathbb{Q}$. *Tegul* $K \in \mathcal{K}$ *ir* $f(s) \in H_0(K)$. *Tuomet su kiekvienu* $\varepsilon > 0$ *yra teisinga nelygybė*

$$\liminf_{N\to\infty} \frac{1}{N} \#\Big\{1 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh; \mathfrak{a}) - f(s)| < \varepsilon\Big\} > 0.$$

Disertacijos 3 skyriuje yra gaunama diskreti 2.1 teoremos versija. Naudojame 2.1 teoremos žymenis.

3.1 teorema. *Tarkime, kad seka* $\mathfrak{a}$ *yra multiplikatyvi, aibė* $L(\mathbb{P}; h, \pi)$ *yra tiesiškai nepriklausoma virš* $\mathbb{Q}$, *ir* $u_{N\to\infty}$ *bei* $u_N \ll N^2$, *kai* $N \to \infty$. *Tegul* $K \in \mathcal{K}$ *ir* $f(s) \in H_0(K)$. *Tuomet su visais* $\varepsilon > 0$, *išskyrus ne daugiau negu skaičią jų* *reikšmių aibę, egzistuoja teigiama riba*

$$\lim_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a})-f(s)|<\varepsilon\Big\}=$$
$$m_H\Big\{\omega\in\Omega:\sup_{s\in K}|\zeta(s,\omega;\mathfrak{a})-f(s)|<\varepsilon\Big\}.$$

3.1 teoremos įrodymas remiasi diskrečia ribine teorema funkcijai $\zeta_{u_N}(s;\mathfrak{a})$ edvėje $H(D)$.

3 skyriaus rezultatai paskelbti [19] straipsnyje.

Kiti disertacijos rezultatai yra skirti jungtiniam poros $\big(\zeta_{u_T}(s,\mathfrak{a}),$ $\zeta_{u_T}(s,\alpha;\mathfrak{b})\big)$ universalumui, įskaitant jo diskrečiąją versiją.

Jungtinis dzeta ir L funkcijų universalumas turi ilgą ir turtingą istoriją. Pirmasis tokio pobūdžio rezultatas, kaip ir dzeta funkcijų universalumo atradimas, priklauso Voroninui. Jis gavo [56] Dirichlė L funkcijų jungtinį universalumą. Teoremos formulavimui reikalingi kai kurie apibrėžimai. Charakteris χ moduliu q yra generuotas charakterio $\chi_1 \bmod q_1, q_1|q$, jei

$$\chi(m)=\begin{cases}\chi_1(m) & ,\text{kai}(m,q)=1,\\ 0 & ,\text{kai}(m,q)>0.\end{cases}$$

Charakteris $\chi \bmod q$ yra vadinamas primityviuoju, jei jis nėra generuotas jokio charakterio moduliu $\bmod q_1, q_1<q$. Du Dirichlė charakteriai yra vadinami ekvivalenčiais, jei jie yra generuoti to paties primityviojo charakterio. Formuluojame patikslintą Voronino teoremą [56], įrodytą [35] straipsnyje.

G teorema. *Tarkime, kad $\chi,\dots,\chi_r$ yra poromis neekvivalentūs Dirichlė charakteriai. Tegul $K_j\in\mathcal{K}$ ir $f_j(s)\in H_0(K_j)$, $j=1,\dots,r$. Tuomet su kiekvienu $\varepsilon>0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau\in[0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|L(s+i\tau,\chi_j)-f_j(s)|<\varepsilon\Big\}>0.$$

Lengva matyti, kad jungtinio universalumo atveju aproksimuojančios funkcijos tam tikra prasme turi būti nepriklausomos. G teoremoje tas nepriklausomumas yra nusakomas charakterių neekvivalentiškumu.

Jungtinio universalumo teoremos žinomos ir kitoms dzeta funkcijoms. Primename tokią teoremą Hurvico dzeta funkcijoms. Tegul

$$L(\alpha_1,\dots,\alpha_r)=\Big\{\log(m+\alpha_j):m\in\mathbb{N}_0,j=1,\dots,r\Big\}.$$

Tuomet yra teisinga tokia teorema [33].

H teorema. *Tarkime, kad aibė $L(\alpha_1, \ldots, \alpha_r)$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$. Tegul $K_j \in \mathcal{K}$ ir $f_j(s) \in H(K_j)$, $j = 1, \ldots, r$. Tuomet su kiekvienu $\varepsilon > 0$, yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

Primename, kad skaičiai $\alpha_1, \ldots, \alpha_r$ yra vadinami algebriškai nepriklausomais virš $\mathbb{Q}$, jei nėra polinomo $p(s_1, \ldots, s_r) \not\equiv 0$ su racionaliais koeficientais, kad $p(\alpha_1, \ldots, \alpha_r) = 0$.

Pastebime, kad aibė $L(\alpha_1, \ldots, \alpha_r)$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$, jei skaičiai $\alpha_1, \ldots, \alpha_r$ yra algebriškai nepriklausomi virš $\mathbb{Q}$.

G ir H teoremos yra jungtinės universalumo teoremos L ir dzeta to paties tipo funkcijoms (Dirichlė L funkcijoms ir Hurvico dzeta funkcijoms). H.Mišu (Mishou) įrodė [46] jungtinę universalumo teoremą dviems skirtingų tipų funkcijoms: Rymano dzeta funkcijai ir Hurvico dzeta funkcijai. Primename, kad skaičius α yra transcendentus, jei nėra polinomo $p(s) \not\equiv 0$ su racionaliais koeficientais, kad $p(\alpha) = 0$.

Formuluojame Mišu teoremą [46].

I teorema [46]. *Tarkime, kad parametras α transcendentus. Tegul $K_1, K_2 \in \mathcal{K}$ ir $f_1(s) = H_0(K_1), f_2(s) = H(K_2)$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \sup_{s\in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

Funkcija $\zeta(s)$ turi Oilerio sandaugą pagal pirminius skaičius, o funkcija $\zeta(s, \alpha)$ su transcendenčiu α tokios sandaugos neturi. Todėl $f_1(s) \in H_0(K_1)$, o $f_2(s) \in H(K_2)$. I teorema yra vadinama funkcijų $\zeta(s)$ ir $\zeta(s, \alpha)$ mišria jungtine universalumo teorema.

Straipsnyje [21] I teorema buvo išplėsta funkcijoms $\zeta(s; \mathfrak{a})$ ir $\zeta(s, \alpha; \mathfrak{b})$.

J teorema. *Tarkime, kad seka $\mathfrak{a}$ yra multiplikatyvi, o parametras α yra transcendentus. Tegul $K_1, K_2 \in \mathcal{K}$ ir $f_1(s) \in H_0(K_1), f_2(s) \in H(K_2)$. Tuomet* *su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon\Big\} > 0.$$

Straipsnyje [21] J teorema yra pritaikyta funkcijų $\zeta(s;\mathfrak{a})$ ir $\zeta(s,\alpha;\mathfrak{b})$ funkcinio neprilausomumo įrodymui.

Aišku, kad iš J teoremos išplaukia D teorema ir funkcijos $\zeta(s,\alpha;\mathfrak{b})$ su transcendenčiu α universalumas, pirmą kartą įrodytas [20] darbe. Be to, yra keletas darbų apie funkcijų $\zeta(s;\mathfrak{a}_j), j = 1,\dots,r$, jungtinį universalumą, pvz., [34], ir funkcijų $\zeta(s,\alpha_j;\mathfrak{b}), j = 1,\dots,r$, jungtinį universalumą, pvz., [32]. Paminėsime R.Kačinskaitės ir K.Macumoto (Matsumoto) seriją darbų [22], [23], [24], [25], į kurių teoremas įeina funkcijos $\zeta(s,\alpha_j;\mathfrak{b}_j)$.

I teorema turi modifikaciją. Apibrėžiame aibę

$$\Omega_2 = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

čia $\gamma_m = \gamma$ su visais $m \in \mathbb{N}_0$. Tada Ω_2, kaip ir Ω_1, yra kompaktiška topologinė Abelio grupė, todėl erdvėje $(\Omega_2, \mathcal{B}(\Omega_2))$ egzistuoja tikimybinis Haro matas m_{2H}. Apibrėžiame dar vieną Dekarto sandaugą

$$\Omega = \Omega_1 \times \Omega_2.$$

Tada Ω vėl yra kompaktinė topologinė Abelio grupė, ir erdvėje $(\Omega, \mathcal{B}(\Omega)$ egzistuoja tikimybinis Haro matas m_H. Pastebime, kad matas m_H yra matų m_{1H} ir m_{2H} sandauga. Tai reiškia, kad jei $A = A_1 \times A_2 \in \mathcal{B}(\Omega)$ su $A_j \in \mathcal{B}(\Omega_j), j = 1,2$, tai tuomet

$$m_H(A) = m_{1H}(A_1) \cdot m_{2H}(A_2).$$

Tikimybinėje erdvėje $(\Omega, \mathcal{B}(\Omega), m_H)$ apibrėžiame $H^2(D) = H(D)\times H(D)-$ reikšmį atsitiktinį elementą

$$\zeta(s,\omega,\alpha) = \Big(\zeta(s,\omega_1), \zeta(s,\omega_2,\alpha)\Big),$$

čia $\omega = (\omega_1, \omega_2) \in \Omega, \omega_1 \in \Omega_1, \omega_2 \in \Omega_2, \omega_2 = \big\{\omega_2(m) : m \in \mathbb{N}_0\big\}$

$$\zeta(s, \omega_1) = \prod_{p \in \mathbb{P}} \Big(1 - \frac{\omega_1(p)}{p^s}\Big)^{-1}$$

ir

$$\zeta(s, \omega_2, \alpha) = \sum_{m \in \mathbb{N}_0} \frac{\omega_2(m)}{(m+\alpha)^s}.$$

Straipsnyje [39] buvo gauta tokia teoremos I versija.

K teorema. *Tarkime, kad aibė* $L(\mathbb{P}, \alpha) \overset{def}{=} \big\{(\log p : p \in \mathbb{P}), (\log (m+\alpha) : m \in \mathbb{N}_0)\big\}$ *yra tiesiškai nepriklausoma virš* $\mathbb{Q}$*. Tegul* $K_1, K_2 \in \mathcal{K}$*, o* $f_1(s) \in H_0(K_1), f_2(s) \in H(K_2)$*. Tuomet su visais* $\varepsilon_1 > 0$ *ir* $\varepsilon_2 > 0$*, išskyrus ne daugiau negu skaičią jų reikšmių aibę, egzistuoja teigiama riba*

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \text{meas}\big\{\tau \in [0, T] : &\sup_{s \in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon_1, \\ &\sup_{s \in K_2} |\zeta(s+i\tau, \alpha) - f(s)| < \varepsilon_2\big\} = \\ m_H\big\{\omega \in \Omega : &\sup_{s \in K_1} |\zeta(s; \omega_1) - f_1(s)| < \varepsilon_1, \\ &\sup_{s \in K_2} |\zeta(s; \omega_2, \alpha) - f_2(s)| < \varepsilon_2\big\}.\end{aligned}$$

K teorema su transcendenčiu α buvo gauta [38] darbe. Straipsnyje [39], K teorema buvo išplėsta ir absoliučiai konverguojančioms Dirichlė eilutėms

$$\zeta_u(s) = \sum_{m=1}^{\infty} \frac{v_u(m)}{m^s} \quad \text{ir} \quad \zeta_u(s, \alpha) = \sum_{m=0}^{\infty} \frac{v_u(m, \alpha)}{(m+\alpha)^s}.$$

L teorema. *Tarkime, kad aibė* $L(\mathbb{P}, \alpha)$ *yra tiesiškai nepriklausoma virš* $\mathbb{Q}$ *ir* $u_T \to \infty$*,* $u_T \ll T^2$*, kai* $T \to \infty$*. Tegul* $K_1, K_2 \in \mathcal{K}$*, o* $f_1(s) \in H_0(K_1), f_2(s) \in H(K_2)$*. Tada su visais* $\varepsilon_1 > 0$ *ir* $\varepsilon_2 > 0$*, išskyrus ne daugiau* *negu skaičią jų reikšmių aibę, egzistuoja teigiama riba*

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau\in[0,T]:\sup_{s\in K_1}|\zeta_{u_T}(s+i\tau)-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta_{u_T}(s+i\tau,\alpha)-f_2(s)|<\varepsilon_2\Big\}=$$
$$m_H\Big\{\omega\in\Omega:\sup_{s\in K_1}|\zeta(s;\omega_1)-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta(s;\omega_2,\alpha)-f_2(s)|<\varepsilon_2\Big\}$$

Disertacijos 4 skyriuje K teorema yra išplėsta funkcijoms $\zeta_{u_T}(s;\mathfrak{a})$ ir $\zeta_{u_T}(s,\alpha;\mathfrak{b})$. Tikimybinėje erdvėje $(\Omega,\mathcal{B}(\Omega),m_H)$ apibrėžiame $H^2(D)-$reikšmį atsitiktinį elementą

$$\zeta(s,\alpha,\omega;\mathfrak{a},\mathfrak{b})=(\zeta(s,\omega_1;\mathfrak{a}),\zeta(s,\alpha,\omega_2;\mathfrak{b})),$$

čia $\zeta(s,\omega_1;\mathfrak{a})$ yra tas pats $H(D)$ reikšmis atsitiktinis elementas kaip ir 2.1 teoremoje, o

$$\zeta(s,\alpha,\omega_2;\mathfrak{b})=\sum_{m=0}^{\infty}\frac{b_m\omega_2(m)}{(m+\alpha)^s}.$$

Disertacijos 4 skyriuje įrodyta tokia teorema.

4.1 teorema. *Tarkime, kad seka* $\mathfrak{a}$ *yra multiplikatyvi, parametras* α *yra transcendentus, ir* $u_T\to\infty$, $u_T\ll T^2$, *kai* $T\to\infty$. *Tegul* $K_1,K_2\in\mathcal{K}$, *o* $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$. *Tuomet su visais* $\varepsilon_1>0$ *ir* $\varepsilon_2>0$, *išskyrus ne daugiau negu skaičią jų reikšmių aibę, egzistuoja teigiama riba*

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau\in[0,T]:\sup_{s\in K_1}|\zeta_{u_T}(s+i\tau;\mathfrak{a})-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}=$$
$$m_H\Big\{\omega\in\Omega:\sup_{s\in K_1}|\zeta(s,\omega_1,\mathfrak{a})-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta(s,\omega_2,\alpha;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}.$$

Disertacijos 4 skyriaus rezultatai paskelbti [3] straipsnyje.

Paskutinysis disertacijos 5 skyrius yra skirtas 4.1 teoremos diskrečiam variantui. Pirma primename Mišu teoremos diskrečias versijas. Tegul $h>0$ ir

$$L(\mathbb{P};\alpha,h,\pi)=\Big\{(\log p:p\in\mathbb{P}),(\log(m+\alpha):m\in\mathbb{N}_0),\frac{2\pi}{h}\Big\}.$$

M teorema [9]. *Tarkime, kad aibė* $L(\mathbb{P};\alpha,h,\pi)$ *yra tiesiškai nepriklausoma virš* $\mathbb{Q}$. *Tegul* $K_1,K_2\in\mathcal{K}$, *o* $f_1(s)\in H_0(K_1), f_2(s)\in H(K_2)$. *Tuomet su kiekvienu* $\varepsilon>0$ *yra teisinga nelygybė*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K_1}|\zeta(s+ikh)-f_1(s)|<\varepsilon,\\ \sup_{s\in K_2}|\zeta(s+ikh,\alpha)-f_2(s)|<\varepsilon\Big\}>0.$$

Darbe [8] M teorema buvo apibrendinta postūmiams $\zeta(s+ikh_1)$ ir $\zeta(s+ikh_2,\alpha)$ su skirtingais $h_1>0$ ir $h_2>0$. Tegul

$$L(\mathbb{P};\alpha,h_1,h_2,\pi)=\Big\{(h_1\log p:p\in\mathbb{P}),(h_2\log(m+\alpha):m\in\mathbb{N}_0),2\pi\Big\}.$$

N teorema [8]. *Tarkime, kad aibė* $L(\mathbb{P};\alpha,h_1,h_2)$ *yra tiesiškai nepriklausoma virš* $\mathbb{Q}$. *Tuomet, su kiekvienu* $\varepsilon>0$ *yra teisinga nelygybė*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K_1}|\zeta(s+ikh_1)-f_1(s)|<\varepsilon,\\ \sup_{s\in K_2}|\zeta(s+ikh_2,\alpha)-f_2(s)|<\varepsilon\Big\}>0.$$

Disertacijos 5 skyrius yra skirtas N teoremos apibendrinimui funkcijoms $\zeta_{u_N}(s;\mathfrak{a})$ ir $\zeta_{u_N}(s,\alpha;\mathfrak{b})$. Apibendrinimas suformuluotas 5.1 teoremoje.

5.1 teorema. *Tegu seka* $\mathfrak{a}$ *yra multiplikatyvi, aibė* $L(\mathbb{P};\alpha,h_1,h_2,\pi)$ *yra tiesiškai nepriklausoma virš* $\mathbb{Q}$, *ir* $u_N\to\infty$ *ir* $u_N\ll N^2$, *kai* $N\to\infty$. *Tegu* $K_1,K_2\in\mathcal{K}$, *o* $f_1(s)\in H_0(K_1), f_2(s)\in H(K_2)$. *Tada su visais* $\varepsilon>0$ *ir* $\varepsilon_2>0$, *išskyrus ne daugiau negu skaičią jų reikšmių aibę, egzistuoja teigiama riba*

$$\lim_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K_1}|\zeta_{u_N}(s+ikh_1;\mathfrak{a})-f_1(s)|<\varepsilon_1,\\ \sup_{s\in K_2}|\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}=\\ m_H\Big\{(\omega_1,\omega_2)\in\Omega:\sup_{s\in K_1}|\zeta(s,\omega_1;\mathfrak{a})-f_1(s)|<\varepsilon_1,\\ \sup_{s\in K_2}|\zeta(s,\alpha,\omega_2;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}$$

Disertacijos 5 skyriaus rezultatai yra pateikiami [4] straipsnyje.

Aprobacija

Pagrindiniai disertacijos rezultatai buvo pristatyti tarptautinėse MMA (Mathematical Modelling and Analysis) konferencijose (MMA2023, Gegužės 30 – Birželio 2, 2023, Jūrmala, Latvija), (MMA2024, Gegužės 28 – 31 d., 2024 Pernu, Estija), tarptautinėje tikimybių teorijos ir skaičių teorijos konfrencijoje (ICPTNT2024, Rugsėjo 16-20, 2024 Palanga) Lietuvos matematikų draugijos konferencijose (LMD2021, Birželio 16 – 17, 2021, Vilnius), (LMD2022, Birželio 16 – 17, 2022, Kaunas), (LMD2023, Birželio 21 – 22, 2023, Vilnius), (LMD2024, Birželio 27 – 28, 2024, Šiauliai).

Publikacijų disertacijos tema sąrašas

Disertacijos rezultatai buvo paskelbti šiuose straipsniuose:

1. M. Jasas, A. Laurinčikas, D. Šiaučiūnas, On the approximation of analytic functions by shifts of an absolutely convergent Dirichlet series, *Math. Notes* **109** (2021), 832–841.

2. M. Jasas, A. Laurinčikas, M.Stoncelis, D. Šiaučiūnas, Discrete universality of absolutely convergent Dirichlet series, *Mathematical Modelling and Analysis*, **27** (2022), 78–87.

3. A. Balčiūnas, M. Jasas, R. Macaitienė, D. Šiaučiūnas, On the Mishou Theorem for Zeta-Functions with Periodic Coefficients, *Mathematics* (2023), https://doi.org/10.3390/math11092042.

4. A. Balčiūnas, M. Jasas, A. Rimkevičienė, A Discrete version of the Mishou theorem related to periodic Zeta-Functions, *Mathematical Modelling and Analysis*, **29** (2024), 331–346.

Konferencijų tezės:

1. M. Jasas, R. Macaitienė. On the Mishou theorem for periodic zeta-functions. Abstracts of MMA2023, May 30 – June 2, 2023, Jurmala, Latvia, pp. 21.

2. M. Jasas. Approximation of analytic functions by shifts of absolutely convergent Dirichlet series related to periodic zeta-functions. Abstracts of MMA2024, May 28 – 31, 2024, Pärnu, Estonia, pp. 30.

Išvados

Iš disertacijos išplaukia tokios išvados:

1. Tegu $\mathfrak{a}$ yra multiplikatyvi skaičių seka ir $u_T \ll T^2$. Tada postūmiai $\zeta_{u_T}(s + i\tau; \mathfrak{a})$ tolygiai ant kompaktinių aibių aproksimuojantys duotas analizines funkcijas juostoje $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, turi teigiamą tankį intervale $[0, T]$.

2. Analogiškas rezultatas teigingas ir diskrečių postūmių aibei $\zeta_{u_N}(s + ikh; \mathfrak{a})$, $u_N \to \infty$ ir $u_N \ll N^2$, kai $T \to \infty$, $h > 0$ ir $k \in \{0, 1, \dots, N\}$.

3. Tegu $\mathfrak{a}$ yra multiplikatyvi skaičių seka, α yra transcendentus parametras ir $u_T \ll T^2$. Tada postūmių aibė $\big(\zeta_{u_T}(s + i\tau; \mathfrak{a}), \zeta_{u_T}(s + i\tau, \alpha; \mathfrak{b})\big)$ tolygiai ant kompaktinių aibių aproksimuojanti duotą analizinių funkcijų porą $\big(f_1(s), f_2(s)\big)$ juostoje D, $f_1(s) \neq 0$, turi teigiamą tankį intervale $[0, T]$.

4. Tegu aibė $\big\{(h_1 \log p : p \in \mathbb{P}), (h_2 \log(m + \alpha) : m \in \mathbb{N}_0), 2\pi\big\}$, $h_1 > 0, h_2 > 0$, yra tiesiškai nepriklausoma virš racionaliųjų skaičių kūno. Tada analogiškas rezultatas yra teisingas ir diskrečių postūmių aibei $\big(\zeta_{u_N}(s + ikh_1; \mathfrak{a}), \zeta_{u_N}(s + ikh_2, \alpha; \mathfrak{b})\big)$, kai $u_N \to \infty$, $u_N \ll N^2$ ir $k \in \{0, 1, \dots, N\}$.

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