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Joint Approximation of Analytic Functions by Certain Classes of Zeta-Functions

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VILNIAUS UNIVERSITETAS

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Jungtinis analizinių funkcijų aproksimavimas tam tikromis dzeta funkcijų klasėmis

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Contents

Notation

p	prime number
$\mathbb{P}$	set of all prime numbers
$\mathbb{N}$	set of all positive integer numbers
$\mathbb{N}_0$	set of all non-negative integer numbers
$\mathbb{Z}$	set of all integer numbers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
$\Re z$, $\Im z$	the real part and imaginary part of z, resp., both are of real numbers
i	imaginary unity, i.e., $i = \sqrt{-1}$
$s = \sigma + it$	s is a complex number with $\Re s = \sigma$, $\Im s = t$
$\overline{s}$	the complex conjugate of s
rank(A)	rank of the matrix A
$\bigoplus\limits_m A_m$	direct sum of sets A_m
$A \times B$	Cartesian product of the sets A and B
$\prod\limits_m A_m$	Cartesian product of sets A_m
A^m	Cartesian product of m copies of the set A
$\mathrm{meas}\{A\}$	Lebesgue measure of the set $A \subset \mathbb{R}$
$\#A$	cardinality of the set A
∂A	the boundary of the set A
$\mathcal{B}(\mathbb{X})$	class of Borel sets of the space $\mathbb{X}$
$\mathbb{E}X$	expectation of the random variable
$\xrightarrow{\mathcal{D}}$	convergence in distribution
$\pi(x)$	the prime counting function

$\Gamma(s)$	Euler gamma-function
$\zeta(s)$	Riemann zeta-function
$\zeta(s,\alpha)$	Hurwitz zeta-function
$\zeta(s;\mathfrak{A})$	periodic zeta-function
$\zeta(s,\alpha;\mathfrak{B})$	periodic Hurwitz zeta-function
$a \ll_\eta b$, $b>0$	there exists a constant $M = M(\eta) > 0$ such that $\|a\|\leqslant Mb$
$D\big(a,b\big)$	the strip $\{s\in\mathbb{C} : a<\sigma<b\}$, $a,b\in\mathbb{R}$, $a<b$
ε	an arbitrarily small positive number, not necessarily the same at each occurrence
$\|\cdot\|_{\mathbb{C}^m}$	the distance in the space $\mathbb{C}^m$
$H(G)$	space of analytic functions on G
$H^c(K)$	the set of continuous functions defined on K and holomorphic in the interior of K for any subset $K\subset\mathbb{C}$
$H_0^c(K)$	set of all elements of $H^c(K)$ that are non-vanishing on K
$\mathcal{K}_{\mathcal{L}}$	class of compact subsets of the strip $D(a,b)$ (or D_L)
$H_{0L}(K)$	class of continuous non-vanishing functions on K that are analytic in the interior of K

Chapter 1

Introduction

1.1 Research topic

Zeta- and L-functions play a central role in analytic number theory. The theory of these functions is constantly and systematically developing in various directions. The starting point was the work by B. Riemann [46], where he obtained a series of remarkable results for the function $\zeta(s)$ as a complex-variable function defined by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1.$$

It is well-known that the function $\zeta(s)$ has analytic continuation to the whole complex plane, except the point $s = 1$, which is a simple pole with residue 1.

On the other hand, there are other zeta-functions which do not have Euler product. A classical example of such function is the Hurwitz zeta-function $\zeta(s, \alpha)$ introduced by A. Hurwitz [12]. The function $\zeta(s, \alpha)$ with the real parameter α, $0 < \alpha \leqslant 1$, is defined by Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}$$

for $\sigma > 1$. Except for the special cases $\alpha = \frac{1}{2}$ and $\alpha = 1$, $\zeta(s, \alpha)$ does not have Euler product. Recall that this function has analytic continuation to the whole complex plane, except for the point $s = 1$ which is a simple pole with residue 1.

One of the most popular topics in modern number theory is investigations of approximation of the analytic functions by more general functions. In 1952, S. Mergelyan has showed [39] that every complex-variable function $f(s)$, continuous on a compact set of a complex plane $\mathbb{C}$ and analytic in the interior of this set, can be approximated uniformly by polynomials in s.

A special interest in approximation became stronger after the famous result by S.M. Voronin [55] in 1975, where he discovered that any analytic function in the complex plane can be approximated by the shifts of the Riemann zeta-function $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, with a given accuracy.

In nowadays analytic number theory, many Riemann zeta-function $\zeta(s)$ generalizations are known, such as the Dirichlet L-function $L(s, \chi)$, the Matsumoto zeta-function $\varphi(s)$, and other. It is also interesting to study certain generalizations of $\zeta(s)$ which do not have Euler product, such as the classical Hurwitz zeta-function $\zeta(s, \alpha)$ mentioned before or other Hurwitz type zeta-functions.

The main subjects of this doctoral dissertation are the zeta- and L-functions, which generalize the Riemann zeta-function, too. More precisely, the subjects are L-functions belonging to the Selberg-Steuding class $\widetilde{S}$, and the two zeta-functions with periodic coefficients, namely, the periodic zeta-function $\zeta(s; \mathfrak{A})$ and the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{B})$.

We start with the definitions of functions and their classes under our interest.

The structure of the class S was studied by various authors (see, for example, [20, 21, 22, 23, 43, 52]), but until now it is not fully described. However, the class includes all main zeta- and L-functions, for example, $\zeta(s)$, $L(s, \chi)$, the zeta-functions of certain cusp forms, etc.

The class $\mathcal{S}$ consists of Dirichlet series

$$L(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \quad a(m) \in \mathbb{C},$$

satisfying the following hypotheses:

(1) *Ramanujan conjecture.* The estimate $a(m) \ll m^{\varepsilon}$ is valid for any $\varepsilon > 0$, where the implicit constant may depend on ε.

(2) *Analytic continuation.* For some $k \in \mathbb{N}_0$, $(s-1)^k L(s)$ in an entire function of finite order.

(3) *Functional equation.* Let

$$\Lambda_L(s) = L(s) Q^s \prod_{j=1}^{j_0} \Gamma(\lambda_j s + \mu_j),$$

where q and λ_j are positive real numbers, and complex number μ_j such that $\Re\mu_j \geqslant 0$. Then the functional equation

$$\Lambda_L(s) = w\overline{\Lambda_L(1-\overline{s})},$$

where $|w| = 1$, is valid.

(4) *Euler product.* Let

$$\log L_p(s) = \sum_{l=1}^{\infty} \frac{b(p^l)}{p^{ls}}$$

with coefficients $b(p^l)$ satisfying the estimate $b(p^l) \ll p^{\alpha l}$, $\alpha < \frac{1}{2}$. Then the representation

$$L(s) = \prod_{p \in \mathbb{P}} L_p(s)$$

holds.

J. Steuding was the first to study the class $\mathcal{S}$ with respect to universality [52]. He introduced the following axioms.

(5) *An analogue of the prime number theorem.* There exists $\kappa > 0$ such that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leqslant x} |a(p)|^2 = \kappa.$$

Moreover, in [52], it was required the Euler product of the type

(6)

$$L(s) = \prod_{p\in\mathbb{P}} \prod_{j=1}^{l} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}$$

with some complex $\alpha_j(p)$.

The Selberg–Steuding class $\widetilde{\mathcal{S}}$ is called the class of zeta- and L-functions belonging to the Selberg class $\mathcal{S}$ and satisfying Axiom (5).

Note that, in the theory of the Selberg class $\mathcal{S}$, the degree of the functions is an important characteristic. Recall that the degree of $L(s) \in \mathcal{S}$ is defined by

$$d_L = 2\sum_{j=1}^{j_0} \lambda_j.$$

For example, if $d_L = 1$, then $L(s)$ coincides with the Riemann zeta-function $\zeta(s)$, if $d_L = 2$, then $L(s)$ are L-functions associated with holomorphic newforms f, and, if $d_L = 4$, then the Rankin–Selberg L-function of any normalized eigenform is an element of the class $\mathcal{S}$.

We are interested in the simultaneous approximation property and its applications for two zeta-functions with periodic coefficients also.

The first periodic zeta-function under our interests was introduced by W. Schnee [48], and now we call it a periodic zeta-function $\zeta(s;\mathfrak{A})$.

Let $\mathfrak{A} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers a_m with minimal positive period $k \in \mathbb{N}$. For $\sigma > 1$, the periodic zeta-function $\zeta(s;\mathfrak{A})$ is defined by Dirichlet series

$$\zeta(s;\mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

The periodicity of the sequence $\mathfrak{A}$ imply that, for $\sigma > 1$,

$$\zeta(s;\mathfrak{A}) = \frac{1}{k^s} \sum_{q=1}^{k} a_q \zeta\left(s, \frac{q}{k}\right),$$

where $\zeta(s,\alpha)$ is a classical Hurwitz zeta-function. From here, combining with the properties of a function $\zeta(s,\alpha)$, we see that the function $\zeta(s;\mathfrak{A})$ can be analytically continued to the whole complex plane, except possibly, for simple pole at the point $s = 1$ with residue

$$a = \frac{1}{k} \sum_{q=1}^{k} a_q.$$

If $a = 0$, then the function $\zeta(s;\mathfrak{A})$ is entire.

The second zeta-function with periodic coefficients under our consideration is the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{B})$. Note that this function was introduced by A. Laurinčikas in [27].

For the periodic sequence $\mathfrak{B} = \{b_m \in \mathbb{C} : m \in \mathbb{N}_0\}$, with a minimal period $k \in \mathbb{N}_0$, and a fixed real α, $0 < \alpha \leqslant 1$, the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{B})$ is defined by the Dirichlet series

$$\zeta(s,\alpha;\mathfrak{B}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}, \quad \sigma > 1.$$

Since the sequence $\mathfrak{B}$ is a periodic sequence, then

$$\zeta(s,\alpha;\mathfrak{B}) = \frac{1}{k^s} \sum_{m=0}^{k-1} b_m \zeta\left(s, \frac{m+\alpha}{k}\right).$$

From this, we deduce that the function $\zeta(s,\alpha;\mathfrak{B})$ has analytic continuation to the whole complex plane, except for a simple pole at the point

$s = 1$ with residue

$$b = \frac{1}{k} \sum_{m=0}^{k-1} b_m.$$

If $b = 0$, then the function $\zeta(s, \alpha; \mathfrak{B})$ is an entire function also.

In this research, we deal with simultaneous approximation of certain target analytic functions by the suitable shifts of zeta- and L-functions mentioned above, and consider some important applications of such approximations.

1.2 Aim and problems

The aim of the doctoral dissertation is to provide the generalizations of the universality property for L-functions from the Selberg-Steuding class $\widetilde{S}$ and certain results related to the mixed simultaneous approximation for the zeta-functions with periodic coefficients, namely, $\zeta(s; \mathfrak{A})$ and $\zeta(s, \alpha; \mathfrak{B})$.

The problems considered are the following:

1. A continuous joint universality theorem for the functions $L(s)$ and its modification.
2. A discrete joint universality theorem for the L-functions and its modification.
3. The joint functional independence and density of periodic zeta-functions.
4. A mixed simultaneous approximation of the tuples of analytic functions by the shifts of collections of periodic zeta-functions and collections of the periodic Hurwitz zeta-functions.
5. The mixed joint functional independence and density of the multicollections of periodic zeta-functions and collections of periodic Hurwitz zeta-functions.

1.3 Relevance

Over the last three decades, an intensive development of the theory of universality in different directions, such as joint mixed universality, linear operators universality, or universality in short intervals, shows its significant place in modern mathematics in a general sense. Considering the fact that zeta-functions and, particularly, discrete universality has more wide applications in physics than continuous (see, for example, [4, 10, 25, 36, 44]), investigations of the approximation of analytic functions by the shifts of zeta- and L-functions became more attractive.

Together with the universality property for single zeta-function it arrises a question on the directions for possible generalizations. One of them is a generalization for the broad classes of zeta-functions, for example, the Selberg class $\mathcal{S}$, the extended Selberg class $\mathcal{S}^{\#}$, the Matsumoto zeta-functions class $\mathcal{M}$, and so on. Another possible direction could be studies of the simultaneous approximation for the classes. The third direction is the investigations of so called mixed simultaneous approximation. This could be done by studying continuous and discrete types of the universality property in one moment.

As it is known (see [19, 26, 37, 52]), many other related problems can be solved using the property of universality, for example, the denseness, functional independence of zeta- and L-functions, question on the number of zeroes, effectivization problem, and so on.

Therefore, it is important to develop the theory on the approximation of analytic functions by the suitable known classes of zeta-functions, and to look for a new one that keeps approximating properties. In addition, the development of the theory of approximation by zeta-functions is one of the most productive fields in the Lithuanian school of analytical number theory. Moreover, it is an obligation to young mathematicians to keep and extend the traditions of Lithuanian mathematicians.

1.4 Methods

The proofs on universality theorems for the classes of zeta-functions use the probabilistic and analytic methods, as well as the theory of Megelyan on approximation of analytic functions by polynomials. The probabilistic methods are based on limit theorems on weakly convergent probability measures in the space of analytic functions with explicitly given limit measure. For this purpose, the elements of Fourier analysis, Dirichlet series, properties of the weak convergence of probability measures, and ergodic theory are applied.

1.5 Novelty

All results presented in the thesis are recent.

The joint simultaneous approximation for the L-functions belonging to the Selberg-Steuding class $\widetilde{S}$ in both, continuous and discrete, cases are studied firstly. Note that in the aforementioned theorems the lower density and the density for all but not most countable many accuracies of approximating shifts of the sets are considered. Such type of approximation theorems for L-functions from Selberg-Steuding class $\widetilde{S}$ were not studied before.

Joint mixed universality theorems for the zeta-functions with periodic coefficients give two results in a sequel: joint functional independence and denseness theorems. In general case, as they are presented in the dissertation, also appear firstly.

1.6 History of the problem and the main results

The 1975's famous paper [55] by S.M. Voronin gave the origin of the universality theory for zeta- and L-functions. In that paper, it has proved that a wide class of analytic functions on the discs of the right side of the critical strip $\left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$ can be approximated uniformly by vertical shifts of the Riemann zeta-function $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, with a given accuracy.

Theorem A. *Suppose that $0 < r < \frac{1}{4}$. Let $f(s)$ be a continuous non-vanishing function on the disc $|s| < r$, and analytic in the interior of this disc. Then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s|\leqslant r}\left|\zeta\left(s + \frac{3}{4} + i\tau\right) - f(s)\right| < \varepsilon.$$

This property of zeta-functions now is known as the universality in the Voronin sense or the Voronin universality. Note that the existence of the Euler product expression for the function $\zeta(s)$ is essentially used in Voronin's proof of his theorem. Later Voronin's result (Theorem A) was improved and extended (see, for example, [1, 9, 26, 52]).

Theorem B. *Suppose that $K \subset D\big(\frac{1}{2}, 1\big)$ is a compact set with connected complement, and $f(s)$ is a continuous non-vanishing function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

The proof of Theorem B is based on the limit theorem on the weak convergence of probability measures in the space of analytic functions. The latter method was proposed by B. Bagchi in his PhD thesis [1], and was developed in the monographs [26], [30] and [52]. Therefore, one natural way is to consider the generalization of universality to zeta- and L-functions which have Euler products.

In 2013, the universality property in terms of density was proposed by J.-L. Mauclaire [38] and independently by A. Laurinčikas and L. Meška [34]. Therefore, the lower density in Theorem B can be replaced by density, and we have the following statement.

Theorem C. *Suppose that K and $f(s)$ are the same as in Theorem B. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The first universality result related to the Selberg class $\mathcal{S}$ was obtained by J. Steuding in [52]. Let, for $L(s) \in \mathcal{S}$,

$$\sigma_L = \max\left(\frac{1}{2}, 1 - \frac{1}{d_L}\right)$$

and $D_L := D\big(\sigma_L, 1\big)$. In addition to the hypothesis 4 of the class $\mathcal{S}$, it required the existence of a polynomial Euler product and analogous of the prime number theorem (Axioms (6) and (5), respectively).

Denote by $\mathcal{K}_L$ the class of compact subset of the strip D_L with connected complements, and by $H_{0L}(K)$, $K \in \mathcal{K}_L$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Denote by $\widehat{\mathcal{S}}$ the class satisfying hypotheses of the class $\mathcal{S}$, and Axioms 5 and 6. Then the following theorem is true (see [52]).

Theorem D. *Suppose that $L(s) \in \mathcal{S} \cap \widehat{\mathcal{S}}$. Let $K \in \mathcal{K}_L$ and $f(s) \in H_{0L}$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |L(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

We note that the class $\mathcal{S} \cap \widehat{\mathcal{S}}$ consists of all functions satisfying the Hypotheses 2 and 3 of class $\mathcal{S}$, and Axioms 5 and 6.

In [42], Theorem D was improved by removing the condition of Axiom 6. More precisely, Theorem D is valid for $L(s) \in \mathcal{S}$ satisfying Axiom 5 only. This is the first result on the universality of L-functions from the Selberg-Steuding class $\widetilde{\mathcal{S}}$.

We discussed the universality of continuous type in the results overviewed above. Now we turn on to one more direction of the Voronin universality, i.e., we focus on the so-called discrete universality property of zeta- and L-functions. Discrete universality deals with the approximation of the analytic functions when the approximating shifts from certain discrete set are taken.

The first result in this direction for the zeta-functions has been obtained by A. Reich [45]. Let $\mathbb{K}$ be an algebraic number field. The Dedekind zeta-function $\zeta_{\mathbb{K}}(s)$, for $\sigma > 1$, is given by

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \frac{1}{\mathrm{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^s}\right)^{-1},$$

where the sum is taken over all non-zero integral ideals, the product is taken over all prime ideals of the ring of integers of $\mathbb{K}$, and $\mathrm{N}(\mathfrak{a})$ denotes the norm of the ideal $\mathfrak{a}$. When $\mathbb{K} = \mathbb{Q}$, the Dedekind zeta-function $\zeta_{\mathbb{Q}}(s)$ is simply the Riemann zeta-function $\zeta(s)$. Therefore, Reich's result is a discrete analogue of Theorem B, i.e., the values of imaginary parts of complex variables are taken from an arithmetic progression $\{kh, k \in \mathbb{N}_0\}$.

Theorem E. *Suppose that K and $f(s)$ are the same as in Theorem B. Then, for any real non-zero number h and every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0.$$

Recall that in all the discrete type universality theorems an essential role goes to an arithmetic nature of the shifting parameter.

The discrete version of Theorem D has been obtained by A. Laurinčikas and R. Macaitienė in [33].

Theorem F. *Suppose that $L(s)$, K and $f(s)$ are the same as in Theorem D. Then, for every $h > 0$ and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |L(s+ikh) - f(s)| < \varepsilon \right\} > 0.$$

The Voronin universality can be examined in a more broad historical research frame of zeta- and L-functions, particularly, as a direction of the studies on the denseness of values and functional independence.

In 1910, H. Bohr initiated the study of value-distribution of the Riemann zeta-function $\zeta(s)$ using diophantine, geometric, and probabilistic methods. He proved [5] the following result related to the frequency of values of the function $\zeta(s)$.

Theorem G. *For every $\delta > 0$, in the strip $1 < \sigma < 1 + \delta$, $\zeta(s)$ takes non-zero values infinitely many times.*

Three years later H. Bohr and R. Courant studied the case for $\sigma \leqslant 1$, and proved [6] the following result.

Theorem H. *For every fixed σ, $\frac{1}{2} < \sigma \leqslant 1$, the set $\{\zeta(\sigma + i\tau) : \tau \in \mathbb{R}\}$ is dense in $\mathbb{C}$.*

In 1972, S.M. Voronin obtained [53] the multidimensional generalization of this denseness result.

Theorem I. *For every fixed distinct numbers* $s_1, \dots, s_n$, $\frac{1}{2} < \Re s_i < 1$, $1 \leqslant i \leqslant n$, *and* $s_k \neq s_l$ *for* $k \neq l$, *the set*

$$\big\{ \big(\zeta(s_1 + i\tau), \dots, \zeta(s_n + i\tau) \big) : \tau \in \mathbb{R} \big\}$$

is dense in $\mathbb{C}^n$, $n \in \mathbb{N}$. *Moreover, for every fixed number* s, $\frac{1}{2} < \sigma < 1$, *the set*

$$\big\{ \big(\zeta(s + i\tau), \zeta'(s + i\tau), \dots, \zeta^{(n-1)}(s + i\tau) \big) : \tau \in \mathbb{R} \big\}$$

is dense in $\mathbb{C}^n$.

As we see, the last result is closely related to the functional independence of the functions given by the Dirichlet series. This problem was proposed by D. Hilbert who raised a list of 23 challenging problems of the XX century during the 2nd International Congress of Mathematicians (see [11]).

In 1973, S.M. Voronin proved [54] that the Riemann zeta-function $\zeta(s)$ is functionally independent. More precisely, it proves the following.

Theorem J. *The function* $\zeta(s)$ *does not satisfy any differential equation having the form*

$$\begin{aligned} s^m F_m\big(\zeta(s), \zeta'(s), \dots, \zeta^{(N-1)}(s)\big) + \dots \\ + F_0\big(\zeta(s), \zeta'(s), \dots, \zeta^{(N-1)}(s)\big) = 0, \end{aligned}$$

where $F_0, \dots, F_m$ *are continuous functions, not all identically zero.*

Later on the functional independence problem for zeta- and L-functions was considered by many mathematicians (we refer to [26, 37, 52]). It is necessary to mention that probabilistic methods can be adopted in order to show the results of functional independence and the density results, as well as the universality property. It has especially become more attractive after the innovative PhD Thesis of B. Bagchi [1], where he proposed an alternative proof for the Voronin universality theorem. The main novelty of the method proposed was involving the functional limit theorem in the sense of weakly convergent probability measures for the study of values of zeta-functions.

Now we turn our focus to the universality property again. We can consider a simultaneous approximation of a tuple of analytic functions by a tuple of shifts of zeta- or L-functions. This type of universality is called the joint universality property of the zeta- and L-functions. This phenomenon of Dirichlet series was also introduced by S.M. Voronin. In [54], he considered the joint functional independence of Dirichlet L-functions $L(s,\chi)$, and for this, he used the joint universality.

We recall that the function $L(s,\chi)$ attached to a character $\chi \pmod d$, $d \in \mathbb{N}$, for $\sigma > 1$, is given by

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}=\prod_{p\in\mathbb{P}}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}.$$

Theorem K. *Let $\chi_1,\dots,\chi_n$ be pairwise non-equivalent Dirichlet characters, and $L(s,\chi_1),\dots,L(s,\chi_n)$ are the corresponding Dirichlet L-functions. For $j=1,\dots,n$, let K_j denote a compact subset of the strip $D\left(\frac{1}{2},1\right)$ with connected complement, and $f_j(s)$ be a continuous non-vanishing function on K_j and analytic in the interior of K_j. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left(\tau\in[0,T]:\sup_{1\leqslant j\leqslant n}\sup_{s\in K_j}|L(s+i\tau,\chi_j)-f_j(s)|<\varepsilon\right)>0.$$

Naturally, the joint universality is more complicated, on the other hand, it is much more interesting for studies. Obviously, in the case of joint universality, the approximating shifts require some independence conditions. In order to obtain joint universality theorems, various matrix conditions are frequently used. After Voronin's result, mentioned above, the joint universality theorems were proved for zeta-functions that are defined by Dirichlet series with periodic coefficients, for Matsumoto zeta-function, for automorphic L-functions (see, for results, a survey paper by K. Matsumoto [37] or monographs by A. Laurinčikas [26], and J. Steuding [52]).

Since the first decade of the XXI century, one more significant branch of joint universality appeared. More precisely, joint simultaneous approximation involves two different types of zeta- and L-functions. One of them has Euler product representations, while the other one does not. This type of research originally is due to J. Steuding and J. Sandres [47], and independently to H. Mishou [40]. They proved that a pair of analytic functions is simultaneously approximated by shifts of a pair $\big(\zeta(s), \zeta(s, \alpha)\big)$ with transcendental α. We state the result of Mishou.

Theorem L. *Suppose that α is a transcendental number such that $0 < \alpha < 1$. Let K_1 and K_2 be compact subsets of the strip $D\big(\frac{1}{2}, 1\big)$ with connected complements. Assume that functions $f_j(s)$ are continuous on K_j and analytic in the interior of K_j for each $j = 1, 2$. In addition, we suppose that $f_1(s)$ does not vanish on K_1. Then, for all positive ε, it holds*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \max_{s\in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \\ \max_{s\in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

The Mishou proof is based on the Bagchi method, and the essential novelty is that the linear independence of the set $\{\log(m + \alpha) : m \in \mathbb{N}_0\} \cup \{\log p : p \in \mathbb{P}\}$ over the field of rational numbers $\mathbb{Q}$, when α is a transcendental number. While J. Sander and J. Steuding proved the same type of result when α is a rational number using a totally different method.

As in the universality for the Riemann zeta-function $\zeta(s)$, there is a list of results for generalizations of mixed simultaneous approximation of both, continuous and discrete, types. We refer to the survey paper by R. Kačinskaitė and K. Matsumoto [19].

The first result of mixed joint functional independence was obtained by H. Mishou in 2007 (see [40]), who proved that the Riemann zeta-function $\zeta(s)$ and the classical Hurwitz zeta-function $\zeta(s, \alpha)$ with transcendental parameter α, $0 < \alpha < 1$, are functionally independent. The result of Mishou's type for periodic zeta-function $\zeta(s;\mathfrak{A})$ and periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{B})$ with transcendental α was proved by R. Kačinskaitė and A. Laurinčikas (see [14]).

Theorem M. *Suppose that α is transcendental, and, for all $p \in \mathbb{P}$,*

$$\sum_{d=1}^{\infty} \frac{|a_{p^d}|}{p^{d/2}} \leqslant c < 1.$$

Let, for each $j = 0, 1, \ldots, n$, $F_j : \mathbb{C}^{2N} \to \mathbb{C}$ be a continuous function and, $N \in \mathbb{N}$,

$$\sum_{j=0}^{n} s^j F_j\Big(\zeta(s;\mathfrak{A}), \zeta'(s;\mathfrak{A}), \ldots, \zeta^{(N-1)}(s;\mathfrak{A}),$$
$$\zeta(s,\alpha;\mathfrak{B}), \zeta'(s,\alpha;\mathfrak{B}), \ldots, \zeta^{(N-1)}(s,\alpha;\mathfrak{B})\Big) \equiv 0.$$

Then $F_j \equiv 0$ for $j = 0, 1, \ldots, n$.

Along with the functional independence problem, the joint denseness for the functions $\zeta(s;\mathfrak{A})$ and $\zeta(s,\alpha;\mathfrak{B})$ have been solved.

The results obtained in the doctoral dissertation cover all the problems overviewed above. Now we will present our main results.

The dissertation's Chapter 2 is devoted to the simultaneous approximations of a collection of analytic functions $\big(f_1(s), \ldots, f_r(s)\big)$ in the strip $D\big(\sigma_L, 1\big)$ by a collection of shifts $\big(L(s+ia_1\tau), \ldots, L(s+ia_r\tau)\big)$, where $a_1, \ldots, a_r$ are real algebraic numbers linearly independent over the field of rational numbers. The result of this chapter is published in [15].

Since we examine a joint simultaneous approximation by the zeta- and L-functions, certain independence of these functions is required. In our case, we use the following A. Baker's result (see [2]).

Lemma 2.1. *Suppose that the logarithm $\log\lambda_1, \ldots, \log\lambda_r$ of algebraic numbers $\lambda_1, \ldots, \lambda_r$ are linearly independent over $\mathbb{Q}$. Then, for any al-*

gebraic numbers $\beta_0, \beta_1, \ldots, \beta_r$ not all simultaneously zero, the inequality

$$|\beta_0 + \beta_1 \log \lambda_1 + \ldots + \beta_r \log \lambda_r| > h^{-C},$$

where h is the maximum of the heights of the numbers $\beta_0, \beta_1, \ldots, \beta_r$, and C is an effective constant depending on r, $\lambda_1, \ldots \lambda_r$ and the maximum of the powers of the numbers $\beta_0, \beta_1, \ldots, \beta_r$, is valid.

Now, we state the main result of Chapter 2 of our dissertation.

Theorem 2.1. *Suppose that $L(s) \in \widetilde{\mathcal{S}}$, and real algebraic numbers $a_1, \ldots, a_r$ are linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}_L$ and $f_j(s) \in H_{0L}(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(s + ia_j\tau) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, lim inf *can be replaced by* lim *for all but at most countably many $\varepsilon > 0$.*

Our Theorem 2.1 can be approached as a multidimensional case of the Theorem D, and a natural extension of the Voronin result stated in Theorem K. The second part of the results generalizes Theorem C in terms of the density.

In Chapter 3, we study another type of simultaneous approximation as in Chapter 2, i.e., we turn to discrete approximation for L- functions from the class $\widetilde{\mathcal{S}}$. There we use the linear independence over $\mathbb{Q}$ of the multiset $A(\mathbb{P}, \underline{h}, 2\pi) := \big\{(h_j \log p : p \in \mathbb{P}), j = 1, \ldots, r; 2\pi\big\}$ for positive h_j.

Then the main result is the following statement, published in [16].

Theorem 3.1. *Suppose that $L(s) \in \widetilde{\mathcal{S}}$, and the set $A(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}(D_L)$ and $f_j(s) \in \mathcal{H}_0(K_j, D_L)$. Then, for every $\underline{h} \in (\mathbb{R}^+)^r$*

and $\varepsilon > 0$*,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s)-L(s+ikh_j)|<\varepsilon\right\}>0.$$

Moreover, for all but at most countably many $\varepsilon > 0$*, the limit*

$$\lim_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s)-L(s+ikh_j)|<\varepsilon\right\}$$

exists and is positive.

As can be seen, the last theorem generalizes Theorem F in terms of lower density for $L(s) \in \widetilde{\mathcal{S}}$, while the second part in terms of density is quite new.

In Chapter 4 of the dissertation, two results for the periodic zeta-function $\zeta(s;\mathfrak{A})$ are obtained. There we discuss functional independence and denseness for the collection consisting of r_1 number of periodic zeta-functions $\big(\zeta(s;\mathfrak{A}_1), ..., \zeta(s;\mathfrak{A}_{r_1})\big)$. This result has fulfilled the existing gap in the studies of periodic zeta-functions. The results of Chapter 4 are published in [13].

Let $\mathfrak{A}_j = \{a_{jm} : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers a_{jm}, with a minimal period $k_j \in \mathbb{N}$, and let $\zeta(s;\mathfrak{A}_j)$ be the corresponding periodic zeta-function for $j = 1, \ldots, r_1$, $r_1 > 1$. Let $k = [k_1, \ldots, k_{r_1}]$ be the least common multiple of $k_1, \ldots, k_{r_1}$. Denote by $\eta_1, \ldots, \eta_{\varphi(k)}$ a reduced system of residues modulo k, where $\varphi(k)$ is the Euler totient function. Define a matrix A consisting of the coefficients of periodic sequences $\mathfrak{A}_j$, i.e.,

$$A =: \begin{pmatrix} a_{1\eta_1} & a_{2\eta_1} & \cdots & a_{r_1\eta_1} \\ a_{1\eta_2} & a_{2\eta_2} & \cdots & a_{r_1\eta_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1\eta_{\varphi(k)}} & a_{2\eta_{\varphi(k)}} & \cdots & a_{r_1\eta_{\varphi(k)}} \end{pmatrix}.$$

Theorem 4.1. *Suppose that the sequences* $\mathfrak{A}_1, \ldots, \mathfrak{A}_{r_1}$ *are multiplica-*

tive, $\mathrm{rank}(A) = r_1$, *and inequalities*

$$\sum_{d=1}^{\infty} \frac{|a_{jp^d}|}{p^{d/2}} \leqslant c_j < 1, \quad j = 1, \dots, r_1.$$

hold for all $p \in \mathbb{P}$. *Let, for each* $g = 0, 1, \dots, n$, $F_g : \mathbb{C}^{Nr_1} \to \mathbb{C}$ *be a continuous function and* $n, N \in \mathbb{N}$,

$$\sum_{g=0}^{n} s^g F_g \Big(\zeta(s; \mathfrak{A}_1), \zeta'(s; \mathfrak{A}_1), \dots, \zeta^{(N-1)}(s; \mathfrak{A}_1), \dots,$$
$$\zeta(s; \mathfrak{A}_{r_1}), \zeta'(s; \mathfrak{A}_{r_1}), \dots, \zeta^{(N-1)}(s; \mathfrak{A}_{r_1}) \Big) \equiv 0.$$

Then $F_g \equiv 0$ *for* $g = 0, 1, \dots, n$.

The second result of this chapter is intended for the denseness of the set described recently.

Define the mapping $\mu : \mathbb{R} \to \mathbb{C}^{Nr_1}$ by the formula

$$\mu(t) = \Big(\zeta(\sigma + it; \mathfrak{A}_1), \zeta'(\sigma + it; \mathfrak{A}_1), \dots, \zeta^{(N-1)}(\sigma + it; \mathfrak{A}_1), \dots,$$
$$\zeta(\sigma + it; \mathfrak{A}_{r_1}), \zeta'(\sigma + it; \mathfrak{A}_{r_1}), \dots, \zeta^{(N-1)}(\sigma + it; \mathfrak{A}_{r_1}) \Big)$$

with $\frac{1}{2} < \sigma < 1$.

Theorem 4.2. *Suppose that all hypotheses on* $\mathfrak{A}_j$, $j = 1, \dots, r_1$, *and* $\mathrm{rank}(A)$ *are as in Theorem 4.1. Then the image* μ *of* $\mathbb{R}$ *is dense in* $\mathbb{C}^{Nr_1}$.

Three results for the mixed type of collection consisting of periodic zeta- and periodic Hurwitz zeta-functions are presented in Chapter 5.

Suppose that l_j is a positive integer, $j = 1, ..., r$, $\mathfrak{B}_{jl} = \{b_{mjl} : m \in \mathbb{N}_0\}$ is a periodic sequence of complex numbers with minimal period $k_{jl} \in \mathbb{N}$, α_j is a real number, $0 < \alpha_j \leqslant 1$, $\zeta(s, \alpha_j; \mathfrak{B}_{jl})$ is the corresponding periodic Hurwitz zeta-function, $j = 1, ..., r$, $l = 1, ..., l_j$, and $\kappa = l_1 + ... + l_r$. Let k be the least common multiple of the numbers

$k_{11}, ..., k_{1_{1l_1}}, ..., k_{r1}, ..., k_{rl_r}$. Define the matrix

$$B := \begin{pmatrix} b_{111} & b_{112} & ... & b_{11l_1} & ... & b_{1r1} & b_{1r2} & ... & b_{1rl_r} \\ b_{211} & b_{212} & ... & b_{21l_1} & ... & b_{2r1} & b_{2r2} & ... & b_{2rl_r} \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ b_{k11} & b_{k12} & ... & b_{k1l_1} & ... & b_{kr1} & b_{kr2} & ... & b_{krl_r} \end{pmatrix}.$$

In this chapter, we study the tuple consisting of r_1 number of periodic zeta-functions and κ number of periodic Hurwitz zeta-functions. In other words, we compose into one more general result Theorems 4.1 and N (for a statement of the last one, see Subsection 5.1).

Theorem 5.1. *Suppose that the sequences $\mathfrak{A}_1, \dots, \mathfrak{A}_{r_1}$ are multiplicative,* $\operatorname{rank}(A) = r_1$*, and inequalities as in Theorem 4.1 hold. Let $1, \alpha_1, ..., \alpha_r$ be numbers algebraically independent over the field $\mathbb{Q}$,* $\operatorname{rank}(B) = \kappa$*. Suppose that the function $F_h : \mathbb{C}^{N(r_1+\kappa)} \to \mathbb{C}$ is continuous for each $h = 0, 1, ..., n$, and the function*

$$\begin{aligned} &G(s) \\ &= \sum_{h=0}^{n} s^h F_h\Big(\zeta(s; \mathfrak{A}_1), \zeta'(s; \mathfrak{A}_1), \dots, \zeta^{(N-1)}(s; \mathfrak{A}_1), \dots, \\ &\qquad \zeta(s; \mathfrak{A}_{r_1}), \zeta'(s; \mathfrak{A}_{r_1}), \dots, \zeta^{(N-1)}(s; \mathfrak{A}_{r_1}), \\ &\qquad \zeta(s, \alpha_1; \mathfrak{B}_{11}), \zeta'(s, \alpha_1; \mathfrak{B}_{11}), ..., \zeta^{(N-1)}(s, \alpha_1; \mathfrak{B}_{11}), ..., \\ &\qquad \zeta(s, \alpha_1; \mathfrak{B}_{1l_1}), \zeta'(s, \alpha_1; \mathfrak{B}_{1l_1}), ..., \zeta^{(N-1)}(s, \alpha_1; \mathfrak{B}_{1l_1}), ..., \\ &\qquad \zeta(s, \alpha_r; \mathfrak{B}_{r1}), \zeta'(s, \alpha_r; \mathfrak{B}_{r1}), ..., \zeta^{(N-1)}(s, \alpha_r; \mathfrak{B}_{r1}), ..., \\ &\qquad \zeta(s, \alpha_r; \mathfrak{B}_{rl_r}), \zeta'(s, \alpha_r; \mathfrak{B}_{rl_r}), ..., \zeta^{(N-1)}(s, \alpha_r; \mathfrak{B}_{rl_r})\Big) \end{aligned}$$

is identically zero. Then $F_h \equiv 0$ for $h = 1, ..., n$.

The mixed joint universality theorem for the functions $\zeta(s; \mathfrak{A}_j)$, $j = 1, ..., r_1$, and $\zeta(s, \alpha_j; \mathfrak{B}_{jl})$, $j = 1, ..., r$, $l = 1, ..., l_j$, is essential in the proof of functional independence and denseness results. The following proposition is the most significant new result of the chapter.

Theorem 5.2. *Suppose that the sequences $\mathfrak{A}_1, \dots, \mathfrak{A}_{r_1}$ are multiplicative,* $\operatorname{rank}(A) = r_1$*, and inequalities* (4.1) *hold. Let $1, \alpha_1, ..., \alpha_r$ be* *numbers algebraically independent over the field* $\mathbb{Q}$, $\operatorname{rank}(B) = \kappa$. *Let* $f_1(s), \dots, f_{r_1}(s)$ *be continuous functions without zeros in* $K_1, \dots, K_{r_1}$, *respectively, and analytic inside* $K_1, \dots, K_{r_1}$. *Suppose that* $f_{jl}(s)$ *is a continuous function in* K_{jl} *and analytic inside* K_{jl} *for each* $j = 1, ..., r$, $l = 1, ..., l_j$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Bigg(\tau \in [0,T] : \sup_{1\leqslant j\leqslant r_1} \sup_{s\in K_j} |\zeta(s+i\tau; \mathfrak{A}_j) - f_j(s)| < \varepsilon, \sup_{1\leqslant j\leqslant r} \sup_{1\leqslant l\leqslant l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{B}_{jl}) - f_{jl}(s)| < \varepsilon \Bigg) > 0.$$

The third result presented in this chapter is the following general denseness lemma.

Theorem 5.3. *Suppose that all hypotheses of Theorem 5.2 are satisfied. Then the image* $h(\mathbb{R})$ *defined by the formula*

$$\begin{aligned} h(t) := \Big(&\zeta(\sigma+it; \mathfrak{A}_1), \zeta'(\sigma+it; \mathfrak{A}_1), \dots, \zeta^{(N-1)}(\sigma+it; \mathfrak{A}_1), \dots, \\ &\zeta(\sigma+it; \mathfrak{A}_{r_1}), \zeta'(\sigma+it; \mathfrak{A}_{r_1}), \dots, \zeta^{(N-1)}(\sigma+it; \mathfrak{A}_{r_1}), \\ &\zeta(\sigma+it, \alpha_1; \mathfrak{B}_{11}), \zeta'(\sigma+it, \alpha_1; \mathfrak{B}_{11}), \dots, \zeta^{(N-1)}(\sigma+it, \alpha_1; \mathfrak{B}_{11}), \dots, \\ &\zeta(\sigma+it, \alpha_1; \mathfrak{B}_{1l_1}), \zeta'(\sigma+it, \alpha_1; \mathfrak{B}_{1l_1}), \dots, \zeta^{(N-1)}(\sigma+it, \alpha_1; \mathfrak{B}_{1l_1}), \dots, \\ &\zeta(\sigma+it, \alpha_1; \mathfrak{B}_{r1}), \zeta'(\sigma+it, \alpha_1; \mathfrak{B}_{r1}), \dots, \zeta^{(N-1)}(\sigma+it, \alpha_1; \mathfrak{B}_{r1}), \dots, \\ &\zeta(\sigma+it, \alpha_1; \mathfrak{B}_{rl_r}), \zeta'(\sigma+it, \alpha_1; \mathfrak{B}_{rl_r}), \dots, \zeta^{(N-1)}(\sigma+it, \alpha_1; \mathfrak{B}_{rl_r}) \Big) \end{aligned}$$

is dense in $\mathbb{C}^{N(r_1+\kappa)}$.

The results of Chapter 5 are published in [13].

1.7 Approbation

The results of the dissertation were presented at the following conferences:

1. B. Žemaitienė. Du rezultatai susiję su dzeta funkcijų universalumu // 11-oji jaunųjų mokslininkų konferencija "Fizinių ir technologijos mokslų tarpdalykiniai tyrimai" (Vilnius, Lithuania). March 23, 2023. Poster presentation.

2. B. Žemaitienė. On joint universality in the Selberg-Steuding class // 26th International Conference on Mathematical Modelling And Analysis (Jurmala, Latvia). May 30 – June 2, 2023. Oral presentation.

3. B. Žemaitienė. Jungtinis universalumas Selbergo-Štoidingo klasėje // Lietuvos Matematikų Draugijos LXIV Konferencija (Vilnius, Lithuania). June 21–22, 2023. Oral presentation.

4. B. Žemaitienė. Joint discrete universality in the Selberg-Steuding class // International Conference on Probability Theory and Number Theory (Palanga, Lithuania). September 10–16, 2023. Oral presentation.

5. B. Žemaitienė. Apie jungtinį aproksimavimą dzeta funkcijų klasėmis. Vytautas Magnus University, Scientific seminar at Faculty of Informatics. May 3, 2024. Oral presentation.

1.8 Principal publications

The results of the dissertation are published in the following papers:

1. R. Kačinskaitė, B. Kazlauskaitė. Two results related to the universality of zeta-functions with periodic coefficients, *Results Math*, **73(3)** (2018), 1-19.

2. R. Kačinskaitė, A. Laurinčikas, B. Žemaitienė. On joint universality in the Selberg–Steuding class, *Mathematics* **11** (2023), 737.

3. R. Kačinskaitė, A. Laurinčikas, B. Žemaitienė. Joint discrete universality in the Selberg–Steuding class, *Axioms* **12(7)** (2023), 674.

Abstracts of conferences are as follow:

1. B. Žemaitienė. On joint universality in the Selberg-Steuding class. Abstracts of 26th International Conference on Mathematical Modelling and Analysis: Jurmala, Latvia, May 30 – June 2. University of Latvia: Riga, 2023, p. 69. ISBN-978-9934-36-01-4.

2. B. Žemaitienė. Jungtinis universalumas Selbergo-Štoidingo kla-

sėje. Abstracts of Lietuvos Matematikų Draugijos LXIV Konferencija: Vilnius, Lithuania, June 21–22. Vilniaus universiteto leidykla: Vilnius, 2023, p. 20. ISBN 978-609-07-0887-3

1.9 Acknowledgment

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Chapter 2

Simultaneous approximation in the Selberg–Steuding class

In this chapter, we will present a theorem on simultaneous approximation of a collection of analytic functions $\big(f_1(s), \dots, f_r(s)\big)$ in the strip $D\big(\sigma_L, 1\big)$ by a collection of shifts $\big(L(s+ia_1\tau), \dots, L(s+ia_r\tau)\big)$, where $a_1, \dots, a_r$ are real algebraic numbers linearly independent over the field of rational numbers, and $\sigma_L > \frac{1}{2}$ is a certain number depending on L. More precisely, we prove that the set of the above shifts has a positive lower density, or even positive lower density for all but at most countable many accuracies of approximation. Thus, the set of approximating shifts is infinite.

The result of this chapter is published in [15].

2.1 Statement of joint universality theorem

The main result of the chapter is the following theorem. Note that it contains both cases, lower density, and density, of the joint universality for $L(s) \in \widetilde{\mathcal{S}}$.

Theorem 2.1. *Suppose that* $L(s) \in \widetilde{S}$, *and real algebraic numbers* $a_1, \dots, a_r$ *are linearly independent over the field of rational numbers* $\mathbb{Q}$. *For* $j = 1, \dots, r$, *let* $K_j \in \mathcal{K}_L$ *and* $f_j(s) \in H_{0L}(K_j)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |L(s+ia_j\tau) - f_j(s)| < \varepsilon\right\} > 0.$$

Moreover, lim inf *can be replaced by* lim *for all but at most countably many* $\varepsilon > 0$.

2.2 Limit theorems on a group

We begin to consider the weak convergence of probability measures with a case of one compact group. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of a topological space $\mathbb{X}$, and define the set

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p, \tag{2.1}$$

and $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. According to the classical Tikhonov theorem, the infinite-dimensional torus Ω, with the product topology and operation of pairwise multiplication, is a compact topological Abelian group. Define one more set

$$\Omega^r = \Omega_1 \times \ldots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then, again, according to the Tikhonov theorem, Ω^r is a compact topological Abelian group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. For $p \in \mathbb{P}$, denote by $\omega_j(p)$ the pth component of an element $\omega_j \in \Omega$, $j = 1, \dots, r$, and by $\omega = (\omega_1, \dots, \omega_r)$ the elements of Ω^r. Let, for brevity, $\underline{a} = (a_1, \dots, a_r)$.

Now, we will consider a limit lemma on weak convergence for

$$
\begin{aligned}
&Q_{T,\underline{a}}(A) \\
&= \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \Big(\big(p^{-a_1\tau} : p \in \mathbb{P}\big), ..., \big(p^{-a_r\tau} : p \in \mathbb{P}\big)\Big) \in A\Big\},
\end{aligned}
$$

$A \in \mathcal{B}(\Omega^r)$, as $T \to \infty$. For its proof, we will apply Lemma 1.6.

Lemma 2.1. *Suppose that $a_1, \dots, a_r$ real algebraic numbers are linearly independent over $\mathbb{Q}$. Then $Q_{T,\underline{a}}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. For the proofs of weak convergence of probability measures on groups, it is convenient to use a method of Fourier transforms. Thus, denote by $F_{T,\underline{a}}(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \dots, r$, the Fourier transform of $Q_{T,\underline{a}}$, i.e.,

$$
F_{T,\underline{a}}(\underline{k}_1, \dots, \underline{k}_r) = \int_\Omega \Bigg(\prod_{j=1}^{r} \sideset{}{^*}\prod_{p \in \mathbb{P}} \omega_j^{k_{jp}}(p) dQ_{T,\underline{a}}\Bigg),
$$

where the star $*$ shows that only a finite number of integers k_{jp} are distinct from zero. By the definition of $Q_{T,\underline{a}}$, we have

$$
\begin{aligned}
&F_{T,\underline{a}}(\underline{k}_1, \dots, \underline{k}_r) \\
&\quad= \frac{1}{T}\int_0^T \Bigg(\prod_{j=1}^{r} \sideset{}{^*}\prod_{p \in \mathbb{P}} p^{-a_j k_{jp}\tau}\Bigg) d\tau \\
&\quad= \frac{1}{T}\int_0^T \exp\Bigg\{-i\tau \sum_{j=1}^{r} a_j \sideset{}{^*}\sum_{p \in \mathbb{P}} k_{jp} \log p\Bigg\} d\tau. \qquad (2.2)
\end{aligned}
$$

Obviously,

$$
F_{T,\underline{a}}(\underline{0}, \dots, \underline{0}) = 1, \qquad (2.3)
$$

where $\underline{0}$ is a collection consisting of zeros. Now, suppose that $(\underline{k}_1, \dots,$

$\underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$. Let, for brevity, $\underline{\underline{k}} = (\underline{k}_1, \dots, \underline{k}_r)$,

$$A_{\underline{a},\underline{\underline{k}}} \stackrel{def}{=} \sum_{j=1}^{r} a_j \sum_{p\in\mathbb{P}}{}^{*} k_{jp} \log p = \sum_{p\in\mathbb{P}}{}^{*} c_p \log p,$$

where

$$c_p = \sum_{j=1}^{r} a_j k_{jp}.$$

In this case, there exists j such that $\underline{k}_j \neq \underline{0}$. Therefore, k_{jp} are not all zero. Since the numbers a_j are linearly independent over $\mathbb{Q}$, the algebraic numbers c_p are not all simultaneously zero. It is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, for $A_{\underline{a},\underline{\underline{k}}}$, Lemma 1.6 is applicable, and we obtain that $A_{\underline{a},\underline{\underline{k}}} \neq 0$. Hence, integrating in (2.2), we find

$$F_{T,\underline{a}}(\underline{k}_1, \dots, \underline{k}_r) = \frac{1 - \exp\left\{ -iTA_{\underline{a},\underline{\underline{k}}}\right\}}{iTA_{\underline{a},\underline{\underline{k}}}}.$$

This together with (2.3) shows that

$$\lim_{T\to\infty} F_{T,\underline{a}}(\underline{k}_1, \dots, \underline{k}_r) = \begin{cases} 1 & \text{if} \quad (\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if} \quad (\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0}), \end{cases}$$

and the lemma is proved because the right-hand side of the last equality is the Fourier transform of the Haar measure m_H. □

We will apply Lemma 2.1 to obtain a joint limit lemma in the space of analytic functions for absolutely convergent Dirichlet series. Denote by $H(D_L)$ the space of analytic on D_L functions equipped with topology of uniform convergence on compacta, and set

$$H^r(D_L) = \underbrace{H(D_L) \times \ldots \times H(D_L)}_{r}.$$

Let $\theta > 0$ be a fixed number,

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^\theta \right\}, \quad m, n \in \mathbb{N},$$

and

$$L_n(s) = \sum_{m=1}^{\infty} \frac{a_L(m)v_n(m)}{m^s}.$$

Since $a_L(m) \ll m^\varepsilon$ and $v_n(m)$ are decreasing exponentially with respect to m, the latter series is absolutely convergent in any half-plane $\sigma > \sigma_0$. Extend the functions $\omega_j(p)$, $p \in \mathbb{P}$, $j = 1, \dots, r$, to the set $\mathbb{N}$ of all positive integers by

$$\omega_j(m) = \prod_{p^l \| m} \omega_j^l(p), \quad m \in \mathbb{N},$$

where $p^l \| m$ means that $p^l | m$ but $p^{l+1} \nmid m$, and define

$$L_n(s, \omega_j) = \sum_{m=1}^{\infty} \frac{a_L(m)\omega_j(m)v_n(m)}{m^s}, \tag{2.4}$$

the series also being absolutely convergent for $\sigma > \sigma_0$. Define

$$\underline{L}_n(s, \omega) = \big(L_n(s, \omega_1), \dots, L_n(s, \omega_r)\big),$$

and $h_n : \Omega^r \to H^r(D_L)$ by $h_n(\omega) = \underline{L}_n(s, \omega)$. Since the series $L_n(s, \omega_j)$, $j = 1, \dots, r$, are absolutely convergent in any half-plane, the mapping h_n is continuous. Therefore, every probability measure P on $(\Omega^r, \mathcal{B}(\Omega^r))$ defines the unique probability measure Ph_n^{-1} on $(H^r(D_L), \mathcal{B}(H^r(D_L)))$, where

$$Ph_n^{-1}(A) = P(h_n^{-1}A), \quad A \in \mathcal{B}(H^r(D_L)).$$

For $A \in \mathcal{B}(H^r(D_L))$, define

$$P_{T,n,\underline{a}}(A) = \frac{1}{T}\text{meas}\big\{\tau \in [0, T] : \underline{L}_n(s + i\underline{a}\tau) \in A\big\},$$

where

$$\underline{L}_n(s+i\underline{a}\tau)=\big(\underline{L}_n(s+ia_1\tau),\dots,\underline{L}_n(s+ia_r\tau)\big).$$

Moreover, a property of preservation of weak convergence under continuous mappings (see, for example, Theorem 5.1 of [3]), leads to the following lemma.

Lemma 2.2. *Suppose that $a_1,\dots,a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then $P_{T,n,\underline{a}}$ converges weakly to the measure $V_n \stackrel{def}{=} m_H h_n^{-1}$ as $T\to\infty$.*

Proof. By the definitions of $P_{T,n,\underline{a}}$ and $Q_{T,\underline{a}}$, and the mapping h_n, for every $A\in\mathcal{B}(H^r(D_L))$, we have

$$\begin{aligned}&P_{T,n,\underline{a}}(A)\\&=\frac{1}{T}\text{meas}\big\{\tau\in[0,T]:\big(\big(p^{-ia_1\tau}:p\in\mathbb{P}\big),...,\big(p^{-ia_r\tau}:p\in\mathbb{P}\big)\big)\in h_n^{-1}A\big\}\\&=Q_{T,\underline{a}}(h_n^{-1}A)=Q_{T,\underline{a}}h_n^{-1}(A).\end{aligned}$$

Thus, $P_{T,n,\underline{a}}=Q_{T,\underline{a}}h_n^{-1}$. This, the continuity of h_n, Lemma 2.1 and Theorem 5.1 of [3] prove the lemma. □

Consider one more measure

$$\widehat{P}_{T,n,\underline{a}}(A)=\frac{1}{T}\text{meas}\big\{\tau\in[0,T]:\underline{L}_n(s+i\underline{a}\tau,\widehat{\omega})\in A\big\}$$

for $A\in\mathcal{B}(H^r(D_L))$.

Lemma 2.3. *Suppose that $a_1,\dots,a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then $\widehat{P}_{T,n,\underline{a}}$ with every $\widehat{\omega}\in\Omega^r$ also converges weakly to the measure V_n as $T\to\infty$.*

Proof. Define the mapping $\widehat{h}_n:\Omega^r\to H^r(D_L)$ by $\widehat{h}_n(\omega)=\underline{L}_n(s,\omega\widehat{\omega})$. Then the mapping $\widehat{h}_n$ remains continuous, and repeating the arguments of the proof of Lemma 2.2, we obtain that $\widehat{P}_{T,n,\underline{a}}$ converges weakly to the measure $\widehat{V}_n \stackrel{def}{=} m_H\widehat{h}_n^{-1}$ as $T\to\infty$. By the definitions of $\widehat{h}_n$ and h_n, we have $\widehat{h}_n(\omega) = h_n(h(\omega))$ with $h(\omega) = \omega\widehat{\omega}$. At this moment, we use the invariance of the Haar measure m_H, i.e., that

$$m_H(\omega A) = m_H(A\omega) = m_H(A)$$

for all $A \in \mathcal{B}(\Omega^r)$ and $\omega \in \Omega^r$. Thus, we find

$$\widehat{V}_n = m_H(h_n h)^{-1} = (m_H h^{-1})h_n^{-1} = m_H h_n^{-1} = V_n.$$

□

2.3 Limit theorems

In this section, we will prove a joint limit theorem for the function $L(s)$ from class $\widetilde{\mathcal{S}}$. More precisely, we will consider the weak convergence for

$$P_{T,\underline{a}}(A) \stackrel{def}{=} \frac{1}{T}\text{meas}\big\{\tau \in [0,T] : \underline{L}(s+i\underline{a}\tau) \in A\big\}, \quad A \in \mathcal{B}(H^r(D_L)),$$

where

$$\underline{L}(s+i\underline{a}\tau) = \big(L(s+ia_1\tau), \dots, L(s+ia_r\tau)\big),$$

as $T \to \infty$. For the proof, we will apply Lemmas 2.2 and 2.3, some ergodicity results and estimates for difference $|L(s+i\underline{a}\tau) - \underline{L}_n(s+i\underline{a}\tau)|$. We start with the latter problem.

Recall the metric in the space $H^r(D_L)$. For $g_1, g_2 \in H(D_L)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|}.$$

Here $\{K_l : l \in \mathbb{N}\} \subset D_L$ is a sequence of compact embedded sets such that

$$\bigcup_{l=1}^{\infty} K_l = D_L,$$

and each compact set $K \subset D_L$ lies in K_l for some l. Then ρ is a metric in $H(D_L)$ inducing the topology of uniform convergence on compacta.

For $\underline{g}_1 = (g_{11}, \ldots, g_{1r})$, $\underline{g}_2 = (g_{21}, \ldots, g_{2r}) \in H^r(D_L)$, taking

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho(g_{1j}, g_{2j}),$$

we have a metric in $H^r(D_L)$ inducing the product topology.

Lemma 2.4. *Suppose that $a_1, \ldots, a_r$ are arbitrary real numbers. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}(\underline{L}(s + i\underline{a}\tau), \underline{L}_n(s + i\underline{a}\tau))d\tau = 0.$$

Proof. Let the number θ come from the definition of $v_n(m)$, and

$$l_n(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)n^s.$$

Then the Mellin formula

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \Gamma(s)c^{-s}ds = e^{-c}, \quad b, c > 0,$$

implies the representation (see, for example, [52])

$$L_n(s) = \frac{1}{2\pi i}\int_{\theta_1-i\infty}^{\theta_1+i\infty} L(s+z)l_n(z)\frac{dz}{z},$$

where $\theta_1 > \frac{1}{2}$. Hence, by the residue theorem,

$$L_n(s) - L(s) = \frac{1}{2\pi i}\int_{-\theta_2-i\infty}^{-\theta_2+i\infty} L(s+z)l_n(z)\frac{dz}{z} + R(s), \qquad (2.5)$$

where $\theta_2 > 0$ and

$$R(s) = \operatorname*{Res}_{z=1-s} L(s+z)\frac{l_n(z)}{z} = \widehat{a}\cdot\frac{l_n(1-s)}{1-s}, \quad \widehat{a} = \operatorname*{Res}_{s=1} L(s).$$

Let $K \subset D_L$ be an arbitrary compact set. We fix $\varepsilon > 0$ such that $\sigma_L + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for all $s = \sigma + it \in K$, and put $\theta_2 = \sigma - \sigma_L - \varepsilon$. Then $\theta_2 > 0$ for $\sigma + it \in K$. This and the Axiom 5, for $s = \sigma + it \in K$

and $a \in \mathbb{R}$, gives

$$\begin{aligned}
&L_n(s+ia\tau) - L(s+ia\tau)\\
&\ll \int_{-\infty}^{\infty} |L(\sigma_L+\varepsilon-\sigma+\sigma+it+ia\tau+iv)| \cdot \left|\frac{l_n(\sigma_L+\varepsilon-\sigma+iv)}{\sigma_L+\varepsilon-\sigma+iv}\right| dv\\
&\quad + |\widehat{a}|\left|\frac{l_n(1-s-ia\tau)}{1-s-ia\tau}\right|.
\end{aligned}$$

Taking v in place of $t+v$, we have, for $s \in K$,

$$\begin{aligned}
&L_n(s+ia\tau) - L(s+ia\tau)\\
&\quad \ll \int_{-\infty}^{\infty} |L(\sigma_L+\varepsilon+ia\tau+iv)| \cdot \sup_{s\in K}\left|\frac{l_n(\sigma_L+\varepsilon-s+iv)}{\sigma_L+\varepsilon-s+iv}\right| dv\\
&\qquad + |\widehat{a}| \sup_{s\in K}\left|\frac{l_n(1-s-ia\tau)}{1-s-ia\tau}\right|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{T}\int_0^T \sup_{s\in K}|L(s+ia\tau) - L_n(s+ia\tau)| d\tau\\
&\ll \int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T |L(\sigma_L+\varepsilon+ia\tau+iv)| d\tau\right) \sup_{s\in K}\left|\frac{l_n(\sigma_L+\varepsilon-s+iv)}{\sigma_L+\varepsilon-s+iv}\right| dv\\
&\quad + |\widehat{a}| \cdot \frac{1}{T}\int_0^T \sup_{s\in K}\left|\frac{l_n(1-s-ia\tau)}{1-s-ia\tau}\right| d\tau\\
&\stackrel{def}{=} I_T^{(1)} + I_T^{(2)}.
\end{aligned} \tag{2.6}$$

It is known [52] that, for fixed $\sigma_L < \sigma < 1$,

$$\int_{-T}^{T} |L(\sigma+it)|^2 dt \ll_{\sigma,L} T.$$

This, for the same σ and $v \in \mathbb{R}$, gives

$$\begin{aligned}
\int_0^T |L(\sigma+ia\tau+iv)|^2 d\tau &= \frac{1}{a}\int_v^{aT+v} |L(\sigma+it)|^2 dt\\
&\ll_{\sigma,a} T(1+|v|).
\end{aligned} \tag{2.7}$$

Using the well-known estimate

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c>0, \tag{2.8}$$

we find that, for all $s \in K$,

$$\begin{aligned}\frac{l_n(\sigma_L+\varepsilon-s+iv)}{\sigma_L+\varepsilon-s+iv} &\ll_\theta n^{\sigma_L+\varepsilon-\sigma}\left|\Gamma\left(\frac{1}{\theta}(\sigma_L+\varepsilon-\sigma+it+iv)\right)\right| \\ &\ll_\theta n^{-\varepsilon}\exp\{-c_1|v-t|\} \\ &\ll_{\theta,K} \exp\{-c_2|v|\}, \quad c_1,c_2>0.\end{aligned}$$

This and (2.7) show that

$$\begin{aligned} I_T^{(1)} &\ll_{\varepsilon,L,\theta,a,K} \ll n^{-\varepsilon}\int_{-\infty}^{\infty}(1+|v|)^{\frac{1}{2}}\exp\{-c_2|v|\}dv \\ &\ll_{\varepsilon,L,\theta,a,K} n^{-\varepsilon}. \end{aligned} \tag{2.9}$$

Similarly, by (2.8), for $s \in K$,

$$\begin{aligned}\frac{l_n(1-s-ia\tau)}{1-s-ia\tau} &\ll_\theta n^{1-\sigma}\exp\{-c_3|t+a\tau|\} \\ &\ll_{\theta,K,a} n^{1-\sigma_L-2\varepsilon}\exp\{-c_4|\tau|\}, \quad c_4>0.\end{aligned}$$

Thus,

$$I_T^{(2)} \ll_{\theta,K,a} n^{-\varepsilon}\frac{1}{T}\int_0^T \exp\{-c_4|\tau|\}d\tau \ll_{\theta,K,a} \frac{\log T}{T}.$$

The latter estimate, (2.9) and (2.6) prove that, for every compact set $K \subset D_L$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}|L(s+ia\tau)-L_n(s+ia\tau)|d\tau = 0.$$

Therefore, the lemma follows from the definitions of the metrics ρ and $\underline{\rho}$. $\square$

Now, for $\omega \in \Omega^r$, let

$$\underline{L}(s, \omega) = \big(L(s, \omega_1), \ldots, L(s, \omega_r)\big),$$

where

$$L(s, \omega_j) = \sum_{m=1}^{\infty} \frac{a_L(m)\omega_j(m)}{m^s}, \quad j = 1, \ldots, r.$$

Then it is known [52] that the latter series, for almost all ω_j, are uniformly convergent on compact subset of the half-plane $\sigma > \sigma_L$. Since the Haar measure m_H is the product of the Haar measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, we have that $\underline{L}(s, \omega)$ is the $H^r(D_L)$-valued random element. Moreover, an analogue of Lemma 2.4 is valid.

Lemma 2.5. *Suppose that $a_1, \ldots, a_r$ are arbitrary real numbers. Then, for almost all $\omega \in \Omega^r$,*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}\big(\underline{L}(s + i\underline{a}\tau, \omega), \underline{L}_n(s + i\underline{a}\tau, \omega)\big) d\tau = 0.$$

Proof. It is known [52] that, for almost all $\omega \in \Omega$,

$$\int_{-T}^{T} |L(\sigma + it, \omega)|^2 dt \ll_{\sigma,L} T.$$

Therefore, repeating the proof of Lemma 2.4, we obtain that, for a compact set $K \in D_L$ and real number a,

$$\frac{1}{T} \int_0^T \sup_{s\in K} |L(s + ia\tau, \omega) - L_n(s + ia\tau, \omega)| d\tau \ll_{\varepsilon,L,\theta,a,K} n^{-\varepsilon} \quad (2.10)$$

with certain $\varepsilon > 0$. In this case, in the analogous of estimate (2.6), we have not the second term on the right-hand side. Since $m_H = m_{1H} \times \ldots \times m_{rH}$, estimate (2.10), and the definitions of the metrics ρ and $\underline{\rho}$ prove the lemma. □

Now we are ready to consider the measure $P_{T,\underline{a}}$.

Theorem 2.2. *Suppose that real algebraic numbers $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Then, on $(H^r(D_L), \mathcal{B}(H^r(D_L)))$, there ex-*

ists a probability measure P such that $P_{T,\underline{a}}$ converges weakly to P as $T \to \infty$.

Proof. Recall that a family of probability measures $\{Q\}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is called tight if, for every $\varepsilon > 0$, there exists a compact set $K = K(r) \subset \mathbb{X}$ such that

$$Q(K) > 1 - \varepsilon$$

for all Q.

Denote by V_{nj} marginal measures of the measure V_n, $j = 1, \ldots, r$. Since the series for $L_n(s)$ is absolutely convergent, we obtain by a standard way that the sequence $\{V_{nj} : n \in \mathbb{N}\}$ is tight, $j = 1, \ldots, r$. Then, for every $\varepsilon > 0$, there exists a compact set $K_j \subset H(D_L)$ such that, for all $n \in \mathbb{N}$,

$$V_{nj}(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \ldots, r. \tag{2.11}$$

Let $K = K_1 \times \ldots \times K_r$. Then K is a compact set in $H^r(D_L)$. Moreover, by (2.11), for all $n \in \mathbb{N}$,

$$V_n(H^r(D_L) \setminus K) \leqslant \sum_{j=1}^{r} V_{nj}(H(D_L) \setminus K_j) < \varepsilon.$$

Thus,

$$V_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Hence, the sequence $\{V_n\}$ is tight. Therefore, by the Prokhorov theorem (see [3]), the sequence $\{V_n\}$ is relatively compact. This means that every sequence of $\{V_n\}$ contains a subsequence $\{V_{n_k}\}$ weakly convergent to a certain probability measure P on $\big(H^r(D_L), \mathcal{B}(H^r(D_L))\big)$ as $k \to \infty$.

Denote by X_n the $H^r(D_L)$-valued random element having the distribution V_n, and by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Then we have

$$X_{n_k} \xrightarrow[k \to \infty]{\mathcal{D}} P. \tag{2.12}$$

On the certain probability space with measure μ, define the random variable ξ_T which is uniformly distributed on $[0, T]$. Moreover, let

$$X_{T,n,\underline{a}} = X_{T,n,\underline{a}}(s) = \underline{L}_n(s + i\underline{a}\xi_T)$$

and

$$Y_{T,\underline{a}} = Y_{T,\underline{a}}(s) = \underline{L}(s + i\underline{a}\xi_T).$$

By Lemma 2.2,

$$X_{T,n,\underline{a}} \xrightarrow[T\to\infty]{\mathcal{D}} X_n, \tag{2.13}$$

and Lemma 2.4 implies, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{T\to\infty} \mu\{\underline{\rho}(Y_{T,\underline{a}}, X_{T,n,\underline{a}}) \geqslant \varepsilon\} \\ &\quad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T \underline{\rho}\big(\underline{L}(s + i\underline{a}\tau), \underline{L}_n(s + i\underline{a}\tau)\big) d\tau = 0. \end{aligned}$$

This, and relations (2.12) and (2.13) show that all hypotheses of Theorem 4.2 from [52] are satisfied. Therefore,

$$Y_{T,\underline{a}} \xrightarrow[T\to\infty]{\mathcal{D}} P, \tag{2.14}$$

and this proves the theorem. □

By (2.14), the measure P is independent on the sequence $\{X_{n_k}\}$. Since the sequence $\{X_n\}$ is relatively compact, it follows that

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{2.15}$$

On $(H^r(D_L), \mathcal{B}(H^r(D_L)))$, define one more measure

$$\widehat{P}_{T,\underline{a}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0, T] : \underline{L}(s + i\underline{a}\tau, \omega) \in A\}$$

for almost all $\omega \in \Omega$. Then, by (2.14), Lemmas 2.3 and 2.5, similarly as above, we obtain the analogue of Theorem 2.2.

Theorem 2.3. *Suppose that real algebraic numbers $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$. Then $\widehat{P}_{T,\underline{a}}$ also converges weakly to the* *measure P as $T \to \infty$.*

2.4 Identification of the measure

For the proof of Theorem 2.1, the explicit form of the limit measure in Theorems 2.2 and 2.3 is needed. For this, some elements of ergodic theory can be applied.

For brevity, we set

$$\underline{a}(\tau) = \Big(\big(p^{-ia_1\tau} : p \in \mathbb{P}\big), ..., \big(p^{-ia_r\tau} : p \in \mathbb{P}\big) \Big), \quad \tau \in \mathbb{R},$$

and define

$$E(\tau, \omega) = \underline{a}(\tau)\omega, \quad \omega \in \Omega^r.$$

Then $E(\tau, \omega)$ is a measurable measure preserving transformation of the group Ω^r, and $\{E(\tau, \omega) : \tau \in \mathbb{R}\}$ form a group of these transformations. For $A \in \mathcal{B}(\Omega^r)$, let $A(\tau) = E(\tau, A)$. If the sets A and $A(\tau)$ differ one from another at most by a set of m_H-measure zero, then the set A is called invariant. All invariant sets form a σ-field. If this field consists only of sets of m_H-measure 1 or 0, then the group $\{E(\tau, \omega)\}$ is called ergodic.

Lemma 2.6. *Suppose that real algebraic numbers $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$. Then the group $\{E(\tau, \omega) : \tau \in \mathbb{R}\}$ is ergodic.*

Proof. The characters χ of the group Ω^r are of the form

$$\chi(\omega) = \prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p), \tag{2.16}$$

where the sign $*$ means that only a finite number of integers k_{jp} are not zero. This already was used in the proof of Lemma 2.1 for the definition of the Fourier transform of the measure $Q_{T,\underline{a}}$. Suppose that A is an invariant set with respect to $\{E(\tau, \omega)\}$, and χ is a nontrivial character of Ω^r, i.e., $\chi(m) \not\equiv 1$. Then, by (2.16), $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$, and thus $A_{\underline{a},\underline{k}} \neq 0$ in the notation used in the proof of Lemma 2.1. Therefore, there exists a real number τ_0 such that

$$\chi(\underline{a}(\tau_0)) = \exp\{-i\tau_0 A_{\underline{a},\underline{k}}\} \neq 1. \tag{2.17}$$

Take the indicator function $\mathbb{I}_A$ of the set A. By virtue of the invariance of the set A, we have

$$\mathbb{I}_A\big(\underline{a}(\tau_0)\omega\big) = \mathbb{I}_A(\omega)$$

for almost all $\omega \in \Omega^r$. Hence, denoting by $\widehat{g}$ the Fourier transform of a function g, we find

$$\begin{aligned} \widehat{\mathbb{I}}_A(\chi) &= \chi\big(\underline{a}(\tau_0)\big) \int_{\Omega^r} \mathbb{I}_A\big(\underline{a}(\tau_0)\omega\big)\chi(\omega)dm_H \\ &= \chi\big(\underline{a}(\tau_0)\big) \int_{\Omega^r} \mathbb{I}_A(\omega)\chi(\omega)dm_H = \chi\big(\underline{a}(\tau_0)\big)\widehat{\mathbb{I}}_A(\chi). \end{aligned}$$

Therefore, in view of (2.17),

$$\widehat{\mathbb{I}}_A(\chi) = 0. \tag{2.18}$$

Now, suppose that χ_0 denotes the trivial character of Ω^r, and $\widehat{\mathbb{I}}_A(\chi_0) = c$. Then, taking into account (2.18), we have

$$\widehat{\mathbb{I}}_A(\chi) = c \int_{\Omega^r} \chi(m)dm_h = \widehat{c}(\chi)$$

for an arbitrary character χ of Ω^r. This shows that $\mathbb{I}_A(\omega) = c$ for almost all $\omega \in \Omega^r$. However, $\mathbb{I}_A$ is the indicator function of the A, thus, $\mathbb{I}_A(\omega) = 1$ or $\mathbb{I}_A(\omega) = 0$ for almost all $\omega \in \Omega^r$. In other words, $m_H(A) = 1$ or $m_H(A) = 0$, and the lemma is proved. □

Denote by $P_{\underline{L}}$ the distribution of the $H^r(D_L)$-valued random element $\underline{L}(s,\omega)$, i.e.,

$$P_{\underline{L}}(A) = m_H\big\{\omega \in \Omega^r : \underline{L}(s,\omega) \in A\big\}, \quad A \in \mathcal{B}(H^r(D_L)).$$

Lemma 2.7. *The measure P in Theorems 2.2 and 2.3 coincide with $P_{\underline{L}}$.*

Proof. Suppose that A is a continuity set of the measure P, i.e., $P(\partial A) = 0$, where ∂A denotes the boundary of A. Then Theorem 2.3 along with the equivalent of weak convergence of probability measure in terms of continuity sets (see, for example, Theorem 2.1 of [3]) yields

$$\lim_{T\to\infty} \widehat{P}_{T,\underline{a}}(A) = P(A). \tag{2.19}$$

On $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$, define the random variable

$$\theta(\omega) = \begin{cases} 1 & \text{if} \quad \underline{L}(s,\omega) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.6 implies the ergodicity of the random process $\theta\big(E(\tau,\omega)\big)$. Therefore, by the Birkhoff–Khintchine ergodic theorem (see, for example, [8]), we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \theta\big(E(\tau,\omega)\big)\, d\tau = \mathbb{E}\theta = P_{\underline{L}}(A), \tag{2.20}$$

where $\mathbb{E}\theta$ is the expectation of θ. However, by the definitions of $E(\tau,\omega)$ and θ,

$$\begin{aligned} \frac{1}{T} \int_0^T \theta\big(E(\tau,\omega)\big)\, d\tau &= \frac{1}{T} \text{meas}\big\{\tau \in [0,T] : \underline{L}(s + i\underline{a}\tau, \omega) \in A\big\} \\ &= \widehat{P}_{T,\underline{a}}(A). \end{aligned}$$

This and (2.20) show that

$$\lim_{T\to\infty} \widehat{P}_{T,\underline{a}}(A) = P_{\underline{L}}(A).$$

Thus, in view of (2.19), $P(A) = P_{\underline{L}}(A)$ for all continuity sets A of P. It is well known that continuity sets constitute the defining class. Thus, $P = P_{\underline{L}}$. □

It remains to find the support of the measure $P_{\underline{L}}$. We recall that the support of $P_{\underline{L}}$ is a minimal closed set S_L such that $P_{\underline{L}}(S_L) = 1$.

Let

$$S_L = \{g \in H(D_L) : g(s) = 0 \text{ or } g(s) \equiv 0\}.$$

Lemma 2.8. *The support of the measure $P_{\underline{L}}$ is the set S_L^r.*

Proof. It is known that the support of the measure

$$P_L(A) \overset{def}{=} m_{jH}\{\omega_j \in \Omega_j : L(s, \omega_j) \in A\}, \quad A \in \mathcal{B}(H(D_L)),$$

$j = 1, ..., r$, is the set S_L (see [52] or [42]). Since the space $H(D_L)$ is separable, we have

$$\mathcal{B}(H^r(D_L)) = \underbrace{\mathcal{B}(H(D_L)) \times \ldots \times \mathcal{B}(H(D_L))}_{r}.$$

Therefore, it suffices to consider the measure $P_{\underline{L}}$ on rectangular sets

$$A = A_1 \times \ldots \times A_r, \quad A_j \in \mathcal{B}(H(D_L)), \quad j = 1, ..., r.$$

Moreover, $m_H = m_{1H} \times \ldots \times m_{rH}$. These remarks show that

$$m_H\big\{\omega \in \Omega^r : \underline{L}(s, \omega) \in A\big\} = \prod_{j=1}^{r} m_{jH}\{\omega_j \in \Omega_j : L(s, \omega_j) \in A_j\}.$$

This and the minimality of the support prove the lemma. □

2.5 Proof of continuous joint universality theorems

Theorem 2.1 follows from Theorem 2.2, Lemmas 2.7 and 2.8, and the Mergelyan's theorem on approximation of analytic functions by polynomials (see [39]). We separate the proof into two pieces: for the case of lower density and the case of density.

Firstly, since the Mergelyan theorem plays an essential role in our studies, we will give it in exact form.

Theorem 2.4. *Let $K \in \mathbb{C}$ be a compact set with connected complements, and the function $f(s)$ be continuous on K and analytic in the* *interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

Proof. The proof of the theorem can be found in [39]. □

Secondly, we recall one more important statement which supports the proof of universality.

We recall that a set $A \in \mathcal{B}(S)$ is said to be a continuity set of the probability measure P if $P(\partial A) = 0$. Note that the set ∂A is closed; therefore, it belongs to the class $\mathcal{B}(S)$.

Theorem 2.5. *Let P_n and P be probability measures on $(S, \mathcal{B}(S))$. Then, the following assertions are equivalent:*

(1) P_n converges weakly to P as $n \to \infty$,

(2) $\lim_{n\to\infty} P_n(A) = P(A)$ for all continuity sets A of P.

3) $\liminf_{n\to\infty} P_n(G) \geqslant P(G)$ for all open sets G.

Proof. For the proof, see Theorem 2.1 of [3]. □

2.5.1 Case of lower density

According to the Mergelyan theorem, there exists polynomials $p_1(s)$, $\ldots, p_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2}. \tag{2.21}$$

Let

$$G_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D_L) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - e^{p_j(s)}| < \varepsilon \right\}.$$

Then, in view of Lemma 2.8, G_ε is an open neighborhood of an element $\left(e^{p_1(s)}, \ldots, e^{p_r(s)}\right)$ of the support S_L^r of the measure $P_{\underline{L}}$. Therefore, by a property of the support,

$$P_{\underline{L}}(G_\varepsilon) > 0. \tag{2.22}$$

Hence, by Theorem 2.2, Lemma 2.7 and the equivalent of weak convergence in terms of open sets (see Theorem 2.5),

$$\liminf_{T\to\infty} P_{T,\underline{a}}(G_\varepsilon) \geqslant P_{\underline{L}}(G_\varepsilon) > 0.$$

Thus, by the definitions of $P_{T,\underline{\varepsilon}}$ and G_ε,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(s+ia_j\tau) - e^{p_j(s)}| < \frac{\varepsilon}{2}\right\} > 0. \tag{2.23}$$

Define one more set

$$A_\varepsilon = \left\{(g_1,\dots,g_r) \in H^r(D_L) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon\right\}.$$

Then, in view of (2.21), we have $G_\varepsilon \subset A_\varepsilon$. This and (2.23) show that

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(s+ia_j\tau) - f_j(s)| < \varepsilon\right\} > 0.$$

2.5.2 Case of density

The boundaries ∂A_ε of the set A_ε do not intersect for different values of ε. Therefore, $P_{\underline{L}}(\partial A_\varepsilon) > 0$ at most for countable many $\varepsilon > 0$, i.e., A_ε is a continuity set of $P_{\underline{L}}$ for all but at most countable many $\varepsilon > 0$. Moreover, since $G_\varepsilon \subset A_\varepsilon$, we have $P_{\underline{L}}(A_\varepsilon) > 0$ by (2.22). Therefore, Theorem 2.2, Lemma 2.7 and the equivalent of weak convergence in terms of continuity sets (see Theorem 2.5) yield

$$\lim_{T\to\infty} P_{T,\underline{a}}(A_\varepsilon) = P_{\underline{L}}(A_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. This and the definitions of $P_{T,\underline{a}}$ and A_ε complete the proof of the theorem.

Chapter 3

Discrete simultaneous approximation in the Selberg–Steuding class

This chapter of the dissertation is devoted to the joint discrete universality theorem for the function $L(s) \in \widetilde{\mathcal{S}}$. Using the linear independence over $\mathbb{Q}$ of the multiset $\big\{(h_j \log p : p \in \mathbb{P}), j = 1, \dots, r; 2\pi\big\}$ for positive h_j, we obtain that there are many infinite shifts $\big(L(s + ikh_1), \dots, L(s + ikh_r)\big)$, $k = 0, 1, \dots$, approximating every collection $\big(f_1(s), \dots, f_r(s)\big)$ of analytic non-vanishing functions defined in the strip $D\big(\sigma_L, 1\big)$, where σ_L is a degree of the function $L(s)$.

3.1 Statement of the discrete joint universality

Let $h_1, ..., h_r$ be fixed positive numbers, and denote $\underline{h} = (h_1, \dots, h_r)$. We define the multiset

$$A(\mathbb{P}, \underline{h}, 2\pi) = \big\{(h_j \log p : p \in \mathbb{P}),\ j = 1, \dots, r;\ 2\pi\big\}.$$

Then the following theorem is the main result of this chapter.

Theorem 3.1. *Suppose that $L(s) \in \widetilde{S}$, and the set $A(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}(D_L)$ and $f_j(s) \in \mathcal{H}_0(K_j, D_L)$. Then, for every $\underline{h} \in (\mathbb{R}^+)^r$ and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - L(s+ikh_j)| < \varepsilon\right\} > 0.$$

Moreover, for all but at most countably many $\varepsilon > 0$, the limit

$$\lim_{N\to\infty} \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - L(s+ikh_j)| < \varepsilon\right\}$$

exists and is positive.

It can be seen, that the latter theorem covers lower density and density cases.

3.2 Case of the torus

In this chapter, we preserve the same meaning for notations as in Chapter 2.

The infinite-dimensional torus Ω^r is the set

$$\Omega^r = \Omega_1 \times \ldots \times \Omega_r$$

with $\Omega_j = \Omega$, $j = 1, \dots, r$. As previously, we denote by $\omega = (\omega_1, .., \omega_r)$, $\omega_j \in \Omega_j$, the elements of Ω^r, where $\omega_j = (\omega_j(p) : p \in \mathbb{P})$, $j = 1, \dots, r$.

For $A \in \mathcal{B}(\Omega^r)$, we set

$$Q_{N,\Omega^r,\underline{h}}(A) = \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \Big(\big(p^{-ikh_1} : p \in \mathbb{P}\big), \dots, \big(p^{-ikh_r} : p \in \mathbb{P}\big)\Big) \in A\right\}.$$

In this section, we consider the weak convergence for $Q_{N,\Omega^r,\underline{h}}$ as $N \to \infty$.

Proposition 3.2. *Suppose that the set* $A(\mathbb{P}, \underline{h}, 2\pi)$ *is linearly independent over* $\mathbb{Q}$*. Then,* $Q_{N,\Omega^r,\underline{h}} \xrightarrow[n\to\infty]{w} m^H$*, where* m^H *is the probability Haar measure on* $(\Omega^r, \mathcal{B}(\Omega^r))$.

Proof. The characters of the Ω^r are of the form

$$\prod_{j=1}^{r} \prod_{p\in\mathbb{P}}^{*} \omega_j^{l_{jp}}(p)$$

with integers l_{jp}, where the symbol star indicates that only a finite number of l_{jp} are not zeroes. Therefore, the Fourier transform $\mathcal{F}_{N,\Omega^r,\underline{h}}(\underline{l}_1, .., \underline{l}_r)$, $\underline{l}_j = (l_{jp} : l_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \ldots, r$, can be represented by

$$\begin{aligned} \mathcal{F}_{N,\Omega^r,\underline{h}}(\underline{l}_1, .., \underline{l}_r) &= \int_{\Omega^r} \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}^{*} \omega_j{}^{l_{jp}}(p) dQ_{N,\Omega^r,\underline{h}} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}^{*} p^{-ikl_{jp}h_j} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\Big\{ - ik \sum_{j=1}^{r} h_j \sum_{p\in\mathbb{P}}^{*} l_{jp} \log p \Big\}. \qquad (3.1) \end{aligned}$$

By the continuity theorem of the compact groups, to prove the Proposition 3.2, it is sufficient to show that the Fourier transform $\mathcal{F}_{N,\Omega^r,\underline{h}}(\underline{l}_1, .., \underline{l}_r)$ converges, as $N \to \infty$, to the Fourier transform

$$\mathcal{F}_{m^H}(\underline{l}_1, \ldots, \underline{l}_r) = \begin{cases} 1 & \text{if} \quad (\underline{l}_1, \ldots, \underline{l}_r) = (\underline{0}, \ldots, \underline{0}), \\ 0 & \text{otherwise} \end{cases}$$

of the Haar measure m^H. Here, $\underline{0} = (0, 0, \ldots)$.

Equality (3.1), obviously, gives

$$\mathcal{F}_{N,\Omega^r,\underline{h}}(\underline{0}, \ldots, \underline{0}) = 1. \qquad (3.2)$$

Thus, it remains to consider only the case $(\underline{l}_1, \ldots, \underline{l}_r) \neq (\underline{0}, \ldots, \underline{0})$. Since the set $A(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent over $\mathbb{Q}$, we have, in this case,

$$\exp\Big\{-i\sum_{j=1}^{r}h_j{\sum_{p\in\mathbb{P}}}^{*}l_{jp}\log p\Big\}\neq 1. \tag{3.3}$$

Actually, if (3.3) is false, then

$$\sum_{j=1}^{r}h_j{\sum_{p\in\mathbb{P}}}^{*}l_{jp}\log p=2\pi m$$

for some $m\in\mathbb{Z}$ and the integers $l_{jp}\neq 0$. However, this contradicts the assumption that the set $A(\mathbb{P},\underline{h},2\pi)$ is linearly independent. Now, using (3.3) and the formula for the sum of geometric progressions, we deduce from (3.1) that, for $(\underline{l}_1,\dots,\underline{l}_r)\neq(\underline{0},\dots,\underline{0})$,

$$\mathcal{F}_{N,\Omega^r,\underline{h}}(\underline{l}_1,\dots,\underline{l}_r)=\frac{1-\exp\big\{-i(N+1)\sum_{j=1}^{r}h_j\sum_{p\in\mathbb{P}}^{*}l_{jp}\log p\big\}}{(N+1)\big(1-\exp\big\{-i\sum_{j=1}^{r}h_j\sum_{p\in\mathbb{P}}^{*}l_{jp}\log p\big\}\big)}.$$

Hence,

$$\lim_{N\to\infty}\mathcal{F}_{N,\Omega^r,\underline{h}}(\underline{l}_1,\dots,\underline{l}_r)=0$$

for $(\underline{l}_1,\dots,\underline{l}_r)\neq(\underline{0},\dots,\underline{0})$. This, together with (3.2), shows that

$$\lim_{N\to\infty}\mathcal{F}_{N,\Omega^r,\underline{h}}(\underline{l}_1,\dots,\underline{l}_r)=\mathcal{F}_{m^H}(\underline{l}_1,\dots,\underline{l}_r),$$

thus proving the Proposition 3.2. □

We apply Proposition 3.2 for the proof of weak convergence for the measures defined by means of certain absolutely convergent Dirichlet series connected to the function $L(s)$. We fix a number $\beta>\frac{1}{2}$, and

$$v_n(m;\beta)=\exp\Big\{-\Big(\frac{m}{n}\Big)^{\beta}\Big\},\quad m,n\in\mathbb{N}.$$

We define the functions

$$L_n(s) = \sum_{m=1}^{\infty} \frac{a(m) v_n(m;\beta)}{m^s}$$

and

$$L_n(s,\omega_j) = \sum_{m=1}^{\infty} \frac{a(m)\omega_j(m) v_n(m;\beta)}{m^s}, \quad j = 1, \dots, r,$$

where, for $m \in \mathbb{N}$,

$$\omega_j(m) = \prod_{p^l \| m} \omega_j^l(p).$$

If $L(s) \in \widetilde{S}$, then $a(m) \ll m_\varepsilon^\varepsilon$ with arbitrary $\varepsilon > 0$. Obviously, $v_n(m;\beta)$ decreases exponentially with respect to m. Therefore, the series for $L_n(s)$ and $L_n(s,\omega_j)$ are absolutely convergent for $\sigma > \sigma_a$ with arbitrary finite σ_a and fixed $n \in \mathbb{N}$. Let

$$\underline{L}_n(s + ik\underline{h}) = \big(L_n(s + ikh_1), \dots, L_n(s + ikh_r)\big)$$

and

$$\underline{L}_n(s,\omega) = \big(L_n(s,\omega_1), \dots, L_n(s,\omega_r)\big).$$

Moreover, let $\mathcal{H}(D_L)$ stand for the space of analytic on D_L functions endowed with the topology of uniform convergence on compact sets, and let

$$\mathcal{H}^r(D_L) = \prod_{j=1}^{r} \mathcal{H}(D_L).$$

For $A \in \mathcal{B}(\mathcal{H}^r(D_L))$, we set

$$P_{N,n,\underline{h}}(A) = \frac{1}{N+1} \#\big\{0 \leqslant k \leqslant N : \underline{L}_n(s + ik\underline{h}) \in A\big\}.$$

Proposition 3.3. *On* $(\mathcal{H}^r(D_L), \mathcal{B}(\mathcal{H}^r(D_L)))$*, a probability measure* P_n

exists such that $P_{N,n,\underline{h}} \xrightarrow[N\to\infty]{w} P_n$.

Proof. Let the mapping $u_n : \Omega^r \to \mathcal{H}^r(D_L)$ be given by $u_n(\omega) = \underline{L}_n(s,\omega)$. The absolute convergence of the series for $L_n(s,\omega_j)$, $j = 1,\dots,r$, implies the continuity of u_n. Hence, u_n is $(\Omega^r, \mathcal{H}^r(D_L))$-measurable. Therefore, every probability measure P on $(\Omega^r, \mathcal{B}(\Omega^r))$ induces the unique probability measure Pu_n^{-1} on $(\mathcal{H}^r(D_L), \mathcal{B}(\mathcal{H}^r(D_L)))$ given by

$$Pu_n^{-1}(A) = P(u_n^{-1}A), \quad A \in \mathcal{B}(\mathcal{H}^r(D_L)).$$

Let $Q_{N,\Omega^r,\underline{h}}$ be from Proposition 3.2. Then

$$\begin{aligned}
&P_{N,n,\underline{h}}(A)\\
&= \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \left(\left(p^{-ikh_j} : p\in\mathbb{P}\right), j=1,\dots,r\right) \in u_n^{-1}A\right\}\\
&= Q_{N,\Omega^r,\underline{h}}(u_n^{-1}A) = Q_{N,\Omega^r,\underline{h}}u_n^{-1}(A)
\end{aligned}$$

for every $A \in \mathcal{B}(\mathcal{H}^r(D_L))$.

Hence, we have $P_{N,n,\underline{h}} = Q_{N,\Omega^r,\underline{h}}u_n^{-1}$. Therefore, Proposition 3.2, the continuity of u_n and Theorem 5.1 in [3] show that $P_{N,n,\underline{h}} \xrightarrow[N\to\infty]{w} P_n$, where $P_n = m^H u_n^{-1}$. □

We see that the measure P_n is independent of $\underline{h}$. This allows us to obtain the weak convergence of P_n as $n \to \infty$, and identify the limit measure. Let

$$L(s,\omega_j) = \sum_{m=1}^{\infty} \frac{a(m)\omega_j(m)}{m^s}, \quad j = 1,\dots,r.$$

It is known [52] that the Dirichlet series for $L(s,\omega_j)$, for almost all ω_j, is uniformly convergent on compact subsets of the strip D_L. Thus, $L(s,\omega_j)$, for $j = 1,\dots,r$, is a $\mathcal{H}(D_L)$-valued random element. The probability Haar measure m^H on $(\Omega, \mathcal{B}(\Omega))$ is the product of the Haar measure m_j^H on $(\Omega_j, \mathcal{B}(\Omega_j))$, i.e., for $A = A_1 \times \dots \times A_r \in \mathcal{B}(\Omega^r)$,

$$m^H(A) = m_1^H(A_1) \cdot \ldots \cdot m_r^H(A_r).$$

The above remarks show that

$$\underline{L}(s,\omega) = \big(L(s,\omega_1),\dots,L(s,\omega_r)\big)$$

is a $\mathcal{H}^r(D_L)$-valued random element defined on the probability space $(\Omega^r, \mathcal{B}(\Omega^r))$. We denote by $P_{\underline{L}}$ the distribution of $\underline{L}(s,\omega)$.

The measure P_n coincides with that studied in the continuous case (see Section 2.4). Therefore, we have the following proposition.

Lemma 3.1. *The relation* $P_n \xrightarrow[n\to\infty]{w} P_{\underline{L}}$ *holds. Moreover, the support of the measure* $P_{\underline{L}}$ *is set as*

$$\Big(\big\{g \in \mathcal{H}(D_L) :\ \textit{either}\ \ g(s) \neq 0\ \ \textit{or}\ \ g(s) \equiv 0\big\}\Big)^r.$$

Proof. The first assertion of the lemma is contained in Lemma 2.6, while the second one is in Lemma 2.8. □

3.3 Limit theorem

We begin this section with a mean value estimate for the collection of L-functions we are interested in.

Let

$$\underline{L}(s+ik\underline{h}) = \big(L(s+ikh_1),\dots,L(s+ikh_r)\big).$$

In this section, we estimate the distance between $\underline{L}(s+ik\underline{h})$ and $L_n(s+ik\underline{h})$ in the mean. Let $\underline{d}$ be the metric on the space $\mathcal{H}^r(D_L)$, i.e., for $\underline{g}_l = (g_{l1},\dots,g_{lr})$, $l = 1,2$,

$$\underline{d}(\underline{g}_1,\underline{g}_2) = \max_{1\leqslant m\leqslant r} d(g_{1m},g_{2m}),$$

and d is the metric in $\mathcal{H}(D_L)$ which induces its uniform convergence topology on compact sets.

Lemma 3.2. *For arbitrary positive fixed numbers* $h_1, \dots, h_r$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \underline{d}\big(\underline{L}(s+ik\underline{h}), \underline{L}_n(s+ik\underline{h})\big) = 0.$$

Proof. Since

$$d(g_1, g_2) = \sum_{j=1}^{\infty} 2^{-j} \frac{\sup_{s\in K_j} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_j} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in \mathcal{H}(D_L),$$

where $\{K_j : j \in \mathbb{N}\} \subset D_L$ is a certain sequence of compact sets, it suffices to show that, for every compact set $K \subset D_L$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |L(s+ikh_j) - L_n(s+ikh_j)| = 0, \tag{3.4}$$

$j = 1, \dots, r$.

We fix a compact set K, a positive number h, and $L(s) \in \widetilde{\mathcal{S}}$. We use the integral representation (see Section 2.3)

$$L_n(s) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} L(s+z) l_n(z;\beta) dz, \tag{3.5}$$

where

$$l_n(s;\beta) = \frac{1}{\beta} \Gamma\left(\frac{s}{\beta}\right) n^s,$$

and the fixed number $\beta > \frac{1}{2}$ is the same as in the definition of $v_n(m;\beta)$. There exists $\delta = \delta(K)$ such that $\sigma_L + 2\delta \leqslant \sigma \leqslant 1 - \delta$ for $\sigma + it \in K$. Thus, $\beta_1 \stackrel{def}{=} \sigma - \sigma_L - \delta > 0$. Let $\beta = \sigma_L + \delta$. The integrand in (3.5) has a simple pole at the point $z = 0$, and a possible simple pole at the point $z = 1 - s$. Therefore, by the residue theorem and (3.1),

$$L_n(s) - L(s) = \frac{1}{2\pi i} \int_{-\beta_1 - i\infty}^{-\beta_1 + i\infty} L(s+z) l_n(z;\beta) dz + r(s),$$

where

$$r(s) = \operatorname*{Res}_{z=1-s} L(s+z) l_n(z;\beta) = \gamma l_n(1-s;\beta),$$

and $\gamma = \mathrm{Res}_{s=1} L(s)$. If $\alpha = 0$ in Axiom 2, then $r(s) = 0$. Hence, for $s = \sigma + it \in K$,

$$\begin{aligned}
&L(s+ikh) - L_n(s+ikh) \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(s+ikh+\sigma_L-\sigma+\delta+i\tau) l_n(\sigma_L-\sigma+\delta+i\tau;\beta) d\tau \\
&\quad + r(s+ikh) \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(\sigma_L+\delta+ikh+i\tau) l_n(\sigma_L+\delta-s+i\tau) d\tau + r(s+ikh) \\
&\ll \int_{-\infty}^{\infty} \big|L(\sigma_L+\delta+ikh+i\tau)\big| \sup_{s\in K} |l_n(\sigma_L+\delta-s+i\tau)| d\tau \\
&\quad + \sup_{s\in K} |r(s+ikh)|.
\end{aligned}$$

From this, we have

$$\begin{aligned}
&\frac{1}{N+1} \sum_{k=2}^{N} \sup_{s\in K} |L(s+ikh) - L_n(s+ikh)| \\
&\qquad \ll \int_{-\infty}^{\infty} \left(\frac{1}{N+1} \sum_{k=2}^{N} |L(\sigma_L+\delta+ikh+i\tau)| \right) \\
&\qquad\quad \times \sup_{s\in K} |l_n(\sigma_L+\delta-s+i\tau)| d\tau \\
&\qquad\qquad + \frac{1}{N+1} \sum_{k=2}^{N} \sup_{s\in K} |r(s+ikh)|. \qquad (3.6)
\end{aligned}$$

By the Cauchy–Schwarz inequality,

$$\frac{1}{N+1} \sum_{k=2}^{N} |L(\sigma_L+\delta+ikh+i\tau)| \ll \left(\frac{1}{N} \sum_{k=2}^{N} |L(\sigma_L+\delta+ikh+i\tau)|^2 \right)^{\frac{1}{2}}. \qquad (3.7)$$

To estimate the last mean square, we apply the Gallagher lemma, see Lemma 1.4 in [41], and the known estimate [52]

$$\int_{-T}^{T} |L(\sigma+it)|^2 dt \ll_\sigma T, \tag{3.8}$$

which is valid for fixed σ, $\sigma_L < \sigma < 1$. Application of the Gallagher lemma gives

$$\begin{aligned} \sum_{k=2}^{N} &|L(\sigma_L+\delta+ikh+i\tau)|^2 \\ &\ll_h \int_{\frac{3}{2}h}^{Nh} |L(\sigma_L+\delta+iv+i\tau)|^2 dv + \\ &\quad + \Bigg(\int_{\frac{3}{2}h}^{Nh} |L(\sigma_L+\delta+iv+i\tau)|^2 dv \\ &\quad \times \int_{\frac{3}{2}h}^{Nh} |L'(\sigma_L+\delta+iv+i\tau)|^2 dv \Bigg)^{\frac{1}{2}}. \end{aligned} \tag{3.9}$$

The Cauchy integral formula along with (3.8) gives, for $\sigma_L < \sigma < 1$, the bound

$$\int_{-T}^{T} |L'(\sigma+it)|^2 dt \ll_\sigma T.$$

This, and (3.8) and (3.9) lead to the estimate

$$\sum_{k=2}^{N} |L(\sigma_L+\delta+ikh+i\tau)|^2 \ll_{h,\delta} N(1+|\tau|). \tag{3.10}$$

To estimate $l_n(\sigma_L+\delta-s+i\tau)$ for $s \in K$, we use the well-known estimate

$$\Gamma(\sigma+it) \ll e^{-c|t|}, \quad c>0,$$

which is valid for large $|t|$ uniformly in any fixed strip. Thus, for $s \in K$, we find

$$l_n(\sigma_L+\delta-s+i\tau) \ll_\beta n^{\sigma_L+\delta-\sigma} e^{-\frac{c}{\beta}|\tau-t|} \ll_{\beta,K} n^{-\delta} c^{-c_1|\tau|}$$

with $c_1 > 0$. Now, the latter estimate, and (3.7) and (3.10) show that

$$\int_{-\infty}^{\infty} \left(\frac{1}{N+1} \sum_{k=2}^{N} |L(\sigma_L + \delta + ikh + i\tau)| \right) \sup_{s \in K} |l_n(\sigma_L + \delta - s + i\tau)| d\tau$$
$$\ll_{\beta,K,h,\delta} n^{-\delta} \int_{-\infty}^{\infty} e^{-c_1|\tau|}(1+|\tau|)^{\frac{1}{2}} d\tau \ll_{\beta,K,h,\delta} n^{-\delta}. \tag{3.11}$$

Similarly, the definition of $r(s)$ yields that, for $s \in K$,

$$r(s+ikh) \ll_\beta n^{1-\sigma} e^{-\frac{c}{\beta}|kh+t|} \ll_{\beta,K} n^{1-\sigma_L-2\delta} e^{-c_2 kh}$$

with $c_2 > 0$. Hence,

$$\frac{1}{N+1} \sum_{k=2}^{N} \sup_{s \in K} |r(s+ikh)|$$
$$\ll_{\beta,K} n^{1-\sigma_L-2\delta} \frac{1}{N} \sum_{k=2}^{N} e^{-c_2 kh}$$
$$\ll_{\beta,K,h} n^{1-\sigma_L-2\delta} \left(\frac{\log N}{N} + \frac{1}{N} \sum_{k \geqslant \log N}^{\infty} e^{-c_2 kh} \right)$$
$$\ll_{\beta,K,h} n^{1-\sigma_L-2\delta} \frac{\log N}{N}.$$

This, and (3.6) and (3.11) lead to the estimate

$$\frac{1}{N+1} \sum_{k=2}^{N} \sup_{s \in K} |L(s+ikh) - L_n(s+ikh)|$$
$$\ll_{\beta,K,h,\delta} \left(n^{-\delta} + n^{1-\sigma_L-2\delta} \frac{\log N}{N} \right).$$

Therefore, taking $N \to \infty$ and then $n \to \infty$, we obtain

$$\lim_{n\to\infty} \liminf_{N\to\infty} \frac{1}{N+1} \sum_{k=2}^{N} \sup_{s \in K} |L(s+ikh) - L_n(s+ikh)| = 0.$$

Since, obviously,

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{1} \sup_{s\in K} |L(s+ikh) - L_n(s+ikh)| = 0,$$

thus proving (3.4). □

Now we are ready to prove the desired joint discrete limit theorem for the collection of L-functions belonging to the class $\widetilde{\mathcal{S}}$. For $A \in \mathcal{B}(\mathcal{H}^r(D_L))$, we set

$$P_{N,\underline{h}}(A) = \frac{1}{N+1} \#\big\{0 \leqslant k \leqslant N : \underline{L}(s+ik\underline{h}) \in A\big\}.$$

Let P_n and $P_{\underline{L}}$ be the same as in Lemma 3.1.

Theorem 3.4. *Suppose that $L(s) \in \widetilde{\mathcal{S}}$, and the set $A(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,\underline{h}} \xrightarrow[N\to\infty]{w} P_{\underline{L}}$.*

Proof. In view of Lemma 3.1, it suffices to show that P_n and $P_{N,\underline{h}}$ have the same limit measure as $n \to \infty$ and $N \to \infty$, respectively.

On some probability space $(\Omega^*, \mathcal{A}, P)$, we define the random variable ξ_N by

$$P\{\xi_N = k\} = \frac{1}{N+1}, \quad k = 0, 1, \ldots, N.$$

Let the $\mathcal{H}^r(D_L)$-valued random elements $X_{N,n,\underline{h}}$ and $X_{N,\underline{h}}$ be defined by

$$X_{N,n,\underline{h}} = X_{N,n,\underline{h}}(s) = \underline{L}_n(s + i\underline{h}\xi_N)$$

and

$$X_{N,\underline{h}} = X_{N,\underline{h}}(s) = \underline{L}(s + i\underline{h}\xi_N).$$

Then the assertion of Proposition 3.3 can be written in the form

$$X_{N,n,\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} P_n. \tag{3.12}$$

Moreover, by Lemma 3.1,

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P_{\underline{L}}, \tag{3.13}$$

where X_n is the $\mathcal{H}^r(D_L)$-valued random element with distribution P_n. Application of Lemma 3.2 and defining the above random elements show that, for $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} P\{\underline{d}(X_{N,\underline{h}}, X_{N,n,\underline{h}}) \geqslant \varepsilon\} \\
&= \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \# \left\{ 0\leqslant k \leqslant N : \underline{d}\big(\underline{L}(s+ik\underline{h}), \underline{L}_n(s+ik\underline{h})\big) \geqslant \varepsilon \right\} \\
&\leqslant \frac{1}{\varepsilon(N+1)} \sum_{k=0}^{N} \underline{d}\big(\underline{L}(s+ik\underline{h}), \underline{L}_n(s+ik\underline{h})\big) = 0.
\end{aligned}$$

Taking into account the separability of the space $(\mathcal{H}^r(D_L), \underline{d})$, the latter equality, and (3.12) and (3.13), we deduce that the hypotheses of Theorem 4.2 in [3] are satisfied. Therefore, we have

$$X_{N,\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\underline{L}}.$$

From the last relation, we obtain an assertion of the theorem. □

3.4 Proof of Theorem 3.1

The proof of Theorem 3.1 we derive from Theorem 3.4, Lemma 3.1 and the Mergelyan theorem (see Theorem 2.4).

Proof of Theorem 3.1. Since $f_j(s) \neq 0$ on K_j, application of the Megelyan theorem for $\log f_j(s)$ implies the existence of polynomials $q_1(s)$, ..., $q_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| f_j(s) - e^{q_j(s)} \right| < \frac{\varepsilon}{2}. \tag{3.14}$$

In view of the second part of Lemma 3.1, the tuple $\big(e^{q_1(s)}, \dots, e^{q_r(s)}\big)$

is an element of the support of the measure $P_{\underline{L}}$. Therefore, the set

$$\mathcal{G}(\varepsilon) = \left\{ (g_1, \ldots, g_r) \in \mathcal{H}^r(D_L) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_j(s) - e^{q_j(s)}| < \frac{\varepsilon}{2} \right\}$$

is an open neighbourhood of the support element, and thus by a property of supports,

$$P_{\underline{L}}(\mathcal{G}(\varepsilon)) > 0. \tag{3.15}$$

Now, Theorem 3.4 and Theorem 2.5 give

$$\liminf_{N \to \infty} P_{N,n,\underline{h}}(\mathcal{G}(\varepsilon)) \geqslant P_{\underline{L}}(\mathcal{G}(\varepsilon)) > 0. \tag{3.16}$$

Inequality (3.14) shows the inclusion of $\mathcal{G}(\varepsilon) \subset \mathcal{G}_1(\varepsilon)$, where

$$\mathcal{G}_1(\varepsilon) = \left\{ (g_1, \ldots, g_r) \in \mathcal{H}^r(D_L) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Therefore, by (3.16),

$$\liminf_{N \to \infty} P_{N,n,\underline{h}}(\mathcal{G}_1(\varepsilon)) > 0,$$

and we have the first assertion of the theorem.

For the proof of second inequality of the theorem, we observe that, for different values of ε, the boundaries of $\mathcal{G}_1(\varepsilon)$ do not intersect. This remark implies that the set $\mathcal{G}_1(\varepsilon)$ is a continuity set of the measure $P_{\underline{L}}$ for all but at most countably many $\varepsilon > 0$. This result, Theorem 3.4 and Theorem 2.5, in virtue of (3.15), imply

$$\liminf_{N \to \infty} P_{N,n,\underline{h}}(\mathcal{G}_1(\varepsilon)) = P_{\underline{L}}(\mathcal{G}_1(\varepsilon)) \geqslant P_{\underline{L}}(\mathcal{G}(\varepsilon)) > 0$$

for all but at most countably many $\varepsilon > 0$.

Theorem 3.1 is therefore proven. □

Chapter 4

Two results related to the joint universality of periodic zeta-function

In this chapter, we focus on two results for the periodic zeta-function $\zeta(s;\mathfrak{A})$ that are derived from simultaneous approximation. More precisely, we discuss on a functional independence and denseness for the collection consisting of r_1 number of periodic zeta-functions $\big(\zeta(s;\mathfrak{A}_1)$, ..., $\zeta(s;\mathfrak{A}_{r_1})\big)$. This result has fulfiled the existing gap in the investigations of periodic zeta-functions.

4.1 Statements of results

First result of the chapter is devoted to the joint functional independence for the periodic zeta-functions $\zeta(s;\mathfrak{A}_j)$, $j = 1, ..., r_1$.

Let $\mathfrak{A}_j = \{a_{jm} : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers a_{jm}, with a minimal period $k_j \in \mathbb{N}$, and let $\zeta(s;\mathfrak{A}_j)$ be the corresponding periodic zeta-function for $j = 1, \ldots, r_1$, $r_1 > 1$. Let $k = [k_1, \ldots, k_{r_1}]$ be the least common multiple of $k_1, \ldots, k_{r_1}$. Denote by $\eta_1, \ldots, \eta_{\varphi(k)}$ a reduced system of residues modulo k, where $\varphi(k)$ is the Euler totient function. Define a matrix A consisting of the coefficients of periodic sequences $\mathfrak{A}_j$, i.e.,

$$A =: \begin{pmatrix} a_{1\eta_1} & a_{2\eta_1} & \dots & a_{r_1\eta_1} \\ a_{1\eta_2} & a_{2\eta_2} & \dots & a_{r_1\eta_2} \\ \dots & \dots & \dots & \dots \\ a_{1\eta_{\varphi(k)}} & a_{2\eta_{\varphi(k)}} & \dots & a_{r_1\eta_{\varphi(k)}} \end{pmatrix}.$$

Moreover, we suppose that, for all $p \in \mathbb{P}$, the following inequality holds

$$\sum_{d=1}^{\infty} \frac{|a_{jp^d}|}{p^{d/2}} \leqslant c_j < 1, \quad j = 1, \dots, r_1. \tag{4.1}$$

Theorem 4.1. *Suppose that the sequences* $\mathfrak{A}_1, \dots, \mathfrak{A}_{r_1}$ *are multiplicative,* $\operatorname{rank}(A) = r_1$ *and inequalities* (4.1) *hold. Let, for each* $g = 0, 1, \dots, n$, $F_g : \mathbb{C}^{Nr_1} \to \mathbb{C}$ *be a continuous function and* $n, N \in \mathbb{N}$,

$$\sum_{g=0}^{n} s^g F_g \Big(\zeta(s; \mathfrak{A}_1), \zeta'(s; \mathfrak{A}_1), \dots, \zeta^{(N-1)}(s; \mathfrak{A}_1), \dots, \zeta(s; \mathfrak{A}_{r_1}), \zeta'(s; \mathfrak{A}_{r_1}), \dots, \zeta^{(N-1)}(s; \mathfrak{A}_{r_1}) \Big) \equiv 0.$$

Then $F_g \equiv 0$ *for* $g = 0, 1, \dots, n$.

The second result deals with the denseness of set under investigation.

Define the mapping $\mu : \mathbb{R} \to \mathbb{C}^{Nr_1}$ by the formula

$$\mu(t) = \Big(\zeta(\sigma + it; \mathfrak{A}_1), \zeta'(\sigma + it; \mathfrak{A}_1), \dots, \zeta^{(N-1)}(\sigma + it; \mathfrak{A}_1), \dots, \zeta(\sigma + it; \mathfrak{A}_{r_1}), \zeta'(\sigma + it; \mathfrak{A}_{r_1}), \dots, \zeta^{(N-1)}(\sigma + it; \mathfrak{A}_{r_1}) \Big)$$

with $\frac{1}{2} < \sigma < 1$.

Theorem 4.2. *Suppose that all hypotheses on* $\mathfrak{A}_j$, $j = 1, \dots, r_1$, *and* $\operatorname{rank}(A)$ *are as in Theorem 4.1. Then the image* μ *of* $\mathbb{R}$ *is dense in* $\mathbb{C}^{Nr_1}$.

4.2 Proof of Theorems 4.1 and 4.2

The powerful tool in the proof of functional independence of zeta-functions is universality theorem in the Voronin sense. For the collection of periodic zeta-functions $\big(\zeta(s;\mathfrak{A}_1),\dots,\zeta(s;\mathfrak{A}_{r_1})\big)$, it has been obtained by A. Laurinčikas and R. Macaitienė in 2009.

Theorem 4.3. *Suppose that the sequences $\mathfrak{A}_1,\dots,\mathfrak{A}_{r_1}$ are multiplicative,* $\operatorname{rank}(A) = r_1$, *and inequalities* (4.1) *hold. Suppose that $f_1(s)$, $\dots, f_{r_1}(s)$ are continuous functions without zeros in a compact subsets $K_1,\dots,K_{r_1}$ of the strip $D\big(\frac{1}{2},1\big)$ and analytic inside $K_1,\dots,K_{r_1}$, respectively. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left(\sup_{1\leqslant j\leqslant r_1}\sup_{s\in K_j}|\zeta(s+i\tau;\mathfrak{A}_j)-f_j(s)|<\varepsilon\right)>0.$$

Proof. The proof of the theorem can be found in [32]. □

Since for the proof of functional independence we use denseness result also, first of all, we will show a denseness of a collection of periodic zeta-functions.

Proof of Theorem 4.2. Let

$$\underline{v}:=\big(v_{10},v_{11},\dots,v_{1N-1},\dots,v_{r_10},v_{r_11},\dots,v_{r_1N-1}\big)$$

be an arbitrary point in the space $\mathbb{C}^{Nr_1}$. We prove that, for every $\varepsilon>0$, there exists a sequence $\{\tau_m:\tau_m\in\mathbb{R}\}$, $\lim_{m\to\infty}\tau_m=+\infty$, such that

$$|\mu(\tau_m)-\underline{v}|_{\mathbb{C}^{Nr_1}}<\varepsilon.$$

This can be shown in a similar way as Lemma 2 from [28], where the denseness of the collection of periodic Hurwitz zeta-functions is considered.

We define the polynomial

$$P_{jN}(s) = \sum_{h=0}^{N-1} \frac{v_{jh}s^h}{h!}, \quad j = 1, \dots, r_1.$$

Then

$$P_{jN}^{(h)}(0) = v_{jh}$$

for $h = 0, 1, \dots, N-1$, $j = 1, \dots, r_1$. Now we fix the number σ_1, $\frac{1}{2} < \sigma_1 < 1$. Let K be a compact subset of the strip $D\big(\frac{1}{2}, 1\big)$ such that σ_1 is an interior point of K. By Theorem 4.3, there exists a sequence $\{\tau_m : \tau_m \in \mathbb{R}\}$, $\lim_{m\to\infty} \tau_m = +\infty$ such that

$$\sup_{1\leqslant j\leqslant r_1} \sup_{s\in K_j} |\zeta(s + i\tau_m; \mathfrak{A}_j) - P_{jN}(s - \sigma_1)| < \frac{\varepsilon\delta_0^N}{2^N N! N r_1},$$

$j = 1, \dots, r_1$, where $\delta_0 = \min_{1\leqslant j\leqslant r_1}\{\delta_{0j}\}$ and δ_{0j} is the distance from the number σ_1 to the boundary of set K_j. Using the integral Cauchy formula, for all $j = 1, \dots, r_1$ and $h = 0, \dots, N-1$, we have

$$\begin{aligned}&|\zeta^{(h)}(\sigma_1 + i\tau_m; \mathfrak{A}_j) - v_{jh}| \\ &= \left| \frac{h!}{2\pi i} \int_{|s-\sigma_1|=\frac{\delta_0}{2}} \frac{\zeta(s + i\tau_m; \mathfrak{A}_j) - P_{jN}(s - \sigma_1)}{(s - \sigma_1)^{h+1}} ds \right| < \frac{\varepsilon}{N r_1}.\end{aligned}$$

This proves the theorem. □

Now we are ready to complete the proof of functional independence for $\big(\zeta(s; \mathfrak{A}_1), \dots, \zeta(s; \mathfrak{A}_{r_1})\big)$.

Proof of Theorem 4.1. Let g from Theorem 4.1 be a fixed number, and let $F_g : \mathbb{C}^{Nr_1} \to \mathbb{C}$ such that

$$\begin{aligned}F_g\Big(&\zeta(s; \mathfrak{A}_1), \zeta'(s; \mathfrak{A}_1), \dots, \zeta^{(N-1)}(s; \mathfrak{A}_1), \dots, \\ &\zeta(s; \mathfrak{A}_{r_1}), \zeta'(s; \mathfrak{A}_{r_1}), \dots, \zeta^{(N-1)}(s; \mathfrak{A}_{r_1})\Big) \equiv 0.\end{aligned}$$

We will prove, that $F_g \equiv 0$.

Assume the contrary. We suppose that $F_g \not\equiv 0$. Then there exists a point $\underline{v} \in \mathbb{C}^{Nr_1}$ such that $F_g(\underline{v}) \neq 0$. The fact that the function F_g is continuous implies that there exists a bounded domain $G \subset \mathbb{C}^{Nr_1}$ such that $\underline{v} \in G$ and that the inequality

$$|F_g(\underline{s})| \geqslant c > 0 \tag{4.2}$$

holds for all $\underline{s} \in G$. Let $\frac{1}{2} < \sigma < 1$. Then, by denseness Theorem 4.2, there exist values of τ_m such that

$$\Big(\zeta(\sigma + i\tau_m; \mathfrak{A}_1), \zeta'(\sigma + i\tau_m; \mathfrak{A}_1), \ldots, \zeta^{(N-1)}(\sigma + i\tau_m; \mathfrak{A}_1), \ldots,$$
$$\zeta(\sigma + i\tau_m; \mathfrak{A}_{r_1}), \zeta'(\sigma + i\tau_m; \mathfrak{A}_{r_1}), \ldots, \zeta^{(N-1)}(\sigma + i\tau_m; \mathfrak{A}_{r_1})\Big) \in G.$$

But this fact and (4.2) contradict the hypothesis that $F_g \not\equiv 0$.

Now, if we allow g to get values from 0 to n, we will have that $F_n \equiv 0, \ldots, F_0 \equiv 0$. The proof of Theorem 4.1 is complete. □

Chapter 5

Generalized mixed joint universality of the zeta-functions with periodic coefficients and two consequences of it

This chapter is devoted to the quite general mixed simultaneous approximation of analytic functions by the shifts of two types zeta-functions with periodic coefficient. Here we also introduce the results closely related to this approximation – the denseness and the functional independence of the collection consisting of the tuples of periodic zeta-functions, and tuples of periodic Hurwitz zeta-functions. In order to prove it, we use certain matrices conditions for coefficients of these zeta-functions and extra requirement on algebraically independence over $\mathbb{Q}$ of parameters appearing in the definitions of periodic Hurwitz zeta-funtions.

5.1 Statements of main results

In Chapter 4, we have shown that the collection of periodic zeta-functions $\big(\zeta(s;\mathfrak{A}_1),\dots,\zeta(s;\mathfrak{A}_{r_1})\big)$ is functionally independent and dense. Note that the functional independence of a collection of periodic Hurwitz zeta-functions with parameters algebraically independent over the field of rational numbers $\mathbb{Q}$, when to each parameter α_j a collection of periodic sequences is attached, was obtained by A. Laurinčikas (see [28]).

Suppose that l_j is a positive integer, $j=1,...,r$, $\mathfrak{B}_{jl}=\{b_{mjl}: m\in\mathbb{N}_0\}$ is a periodic sequence of complex numbers with minimal period $k_{jl}\in\mathbb{N}$, α_j is a real number, $0<\alpha_j\leqslant 1$, $\zeta(s,\alpha_j;\mathfrak{B}_{jl})$ is the corresponding periodic Hurwitz zeta-function, $j=1,...,r$, $l=1,...,l_j$, and $\kappa=l_1+...+l_r$. Let k be the least common multiple of the numbers $k_{11},...,k_{1_{1l_1}},...,k_{r1},...,k_{rl_r}$. Define the matrix

$$B:=\begin{pmatrix} b_{111} & b_{112} & ... & b_{11l_1} & ... & b_{1r1} & b_{1r2} & ... & b_{1rl_r}\\ b_{211} & b_{212} & ... & b_{21l_1} & ... & b_{2r1} & b_{2r2} & ... & b_{2rl_r}\\ ... & ... & ... & ... & ... & ... & ... & ... & ...\\ b_{k11} & b_{k12} & ... & b_{k1l_1} & ... & b_{kr1} & b_{kr2} & ... & b_{krl_r}\end{pmatrix}.$$

Theorem N. *Suppose that* $1,\alpha_1,...,\alpha_r$ *are numbers algebraically independent over the field* $\mathbb{Q}$*,* $\operatorname{rank}(B)=\kappa$*. Suppose that the function* $F_h:\mathbb{C}^{N\kappa}\to\mathbb{C}$ *is continuous for each* $h=0,1,...,n$ *such that the function*

$$G(s)=\sum_{h=0}^{n} s^h F_h\Big(\zeta(s,\alpha_1;\mathfrak{B}_{11}),\zeta'(s,\alpha_1;\mathfrak{B}_{11}),...,\zeta^{(N-1)}(s,\alpha_1;\mathfrak{B}_{11}),...,\\ \zeta(s,\alpha_1;\mathfrak{B}_{1l_1}),\zeta'(s,\alpha_1;\mathfrak{B}_{1l_1}),...,\zeta^{(N-1)}(s,\alpha_1;\mathfrak{B}_{1l_1}),...,\\ \zeta(s,\alpha_r;\mathfrak{B}_{r1}),\zeta'(s,\alpha_r;\mathfrak{B}_{r1}),...,\zeta^{(N-1)}(s,\alpha_r;\mathfrak{B}_{r1}),...,\\ \zeta(s,\alpha_r;\mathfrak{B}_{rl_r}),\zeta'(s,\alpha_r;\mathfrak{B}_{rl_r}),...,\zeta^{(N-1)}(s,\alpha_r;\mathfrak{B}_{rl_r})\Big)$$

is identically zero. Then $F_h\equiv 0$ *for* $h=1,...,n$.

In this chapter, we study the tuple consisting of r_1 number of periodic zeta-functions and κ number of periodic Hurwitz zeta-functions. In other words, we compose into one more general result Theorems 4.1 and N.

Theorem 5.1. *Suppose that the sequences* $\mathfrak{A}_1, \dots, \mathfrak{A}_{r_1}$ *are multiplicative,* $\operatorname{rank}(A) = r_1$, *and inequalities* (4.1) *hold. Let* $1, \alpha_1, ..., \alpha_r$ *be numbers algebraically independent over the field* $\mathbb{Q}$, $\operatorname{rank}(B) = \kappa$. *Suppose that the function* $F_h : \mathbb{C}^{N(r_1+\kappa)} \to \mathbb{C}$ *is continuous for each* $h = 0, 1, ..., n$, *and the function*

$$
\begin{aligned}
&G(s) \\
&\quad = \sum_{h=0}^{n} s^h F_h\Big(\zeta(s;\mathfrak{A}_1), \zeta'(s;\mathfrak{A}_1), \dots, \zeta^{(N-1)}(s;\mathfrak{A}_1), \dots, \\
&\qquad \zeta(s;\mathfrak{A}_{r_1}), \zeta'(s;\mathfrak{A}_{r_1}), \dots, \zeta^{(N-1)}(s;\mathfrak{A}_{r_1}), \\
&\qquad \zeta(s,\alpha_1;\mathfrak{B}_{11}), \zeta'(s,\alpha_1;\mathfrak{B}_{11}), ..., \zeta^{(N-1)}(s,\alpha_1;\mathfrak{B}_{11}), ..., \\
&\qquad \zeta(s,\alpha_1;\mathfrak{B}_{1l_1}), \zeta'(s,\alpha_1;\mathfrak{B}_{1l_1}), ..., \zeta^{(N-1)}(s,\alpha_1;\mathfrak{B}_{1l_1}), ..., \\
&\qquad \zeta(s,\alpha_r;\mathfrak{B}_{r1}), \zeta'(s,\alpha_r;\mathfrak{B}_{r1}), ..., \zeta^{(N-1)}(s,\alpha_r;\mathfrak{B}_{r1}), ..., \\
&\qquad \zeta(s,\alpha_r;\mathfrak{B}_{rl_r}), \zeta'(s,\alpha_r;\mathfrak{B}_{rl_r}), ..., \zeta^{(N-1)}(s,\alpha_r;\mathfrak{B}_{rl_r})\Big)
\end{aligned}
$$

is identically zero. Then $F_h \equiv 0$ *for* $h = 1, ..., n$.

The mixed joint universality theorem for the functions $\zeta(s;\mathfrak{A}_j)$, $j = 1, ..., r_1$, and $\zeta(s,\alpha_j;\mathfrak{B}_{jl})$, $j = 1, ..., r$, $l = 1, ..., l_j$, is essential in the proof of functional independence and denseness results. It is the most significant new result of this chapter.

Theorem 5.2. *Suppose that the sequences* $\mathfrak{A}_1, \dots, \mathfrak{A}_{r_1}$ *are multiplicative,* $\operatorname{rank}(A) = r_1$, *and inequalities* (4.1) *hold. Let* $1, \alpha_1, ..., \alpha_r$ *be numbers algebraically independent over the field* $\mathbb{Q}$, $\operatorname{rank}(B) = \kappa$. *Let* $f_1(s), \dots, f_{r_1}(s)$ *be continuous functions without zeros in* $K_1, \dots, K_{r_1}$, *respectively, and analytic inside* $K_1, \dots, K_{r_1}$. *Suppose that* $f_{jl}(s)$ *is a continuous function in* K_{jl} *and analytic inside* K_{jl} *for each* $j = 1, ..., r$,

$l = 1, ..., l_j$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Bigg(\tau \in [0,T] :$$
$$\sup_{1\leqslant j\leqslant r_1} \sup_{s\in K_j} |\zeta(s+i\tau;\mathfrak{A}_j) - f_j(s)| < \varepsilon,$$
$$\sup_{1\leqslant j\leqslant r} \sup_{1\leqslant l\leqslant l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau,\alpha_j;\mathfrak{B}_{jl}) - f_{jl}(s)| < \varepsilon\Bigg) > 0.$$

Here all $K.$'s are subsets of the strip $D\big(\frac{1}{2}, 1\big)$.

Finally, the third result presented in this chapter is a general denseness lemma (we state it as theorem).

Theorem 5.3. *Suppose that all hypotheses of Theorem 5.2 are satisfied. Then the image* $h(\mathbb{R})$ *defined by the formula*

$$\begin{aligned}
h(t)& \\
:=\Big(&\zeta(\sigma+it;\mathfrak{A}_1), \zeta'(\sigma+it;\mathfrak{A}_1),\ldots,\zeta^{(N-1)}(\sigma+it;\mathfrak{A}_1),\ldots,\\
&\zeta(\sigma+it;\mathfrak{A}_{r_1}), \zeta'(\sigma+it;\mathfrak{A}_{r_1}),\ldots,\zeta^{(N-1)}(\sigma+it;\mathfrak{A}_{r_1}),\\
&\zeta(\sigma+it,\alpha_1;\mathfrak{B}_{11}), \zeta'(\sigma+it,\alpha_1;\mathfrak{B}_{11}),\ldots,\zeta^{(N-1)}(\sigma+it,\alpha_1;\mathfrak{B}_{11}),\ldots,\\
&\zeta(\sigma+it,\alpha_1;\mathfrak{B}_{1l_1}), \zeta'(\sigma+it,\alpha_1;\mathfrak{B}_{1l_1}),\ldots,\zeta^{(N-1)}(\sigma+it,\alpha_1;\mathfrak{B}_{1l_1}),\ldots,\\
&\zeta(\sigma+it,\alpha_1;\mathfrak{B}_{r1}), \zeta'(\sigma+it,\alpha_1;\mathfrak{B}_{r1}),\ldots,\zeta^{(N-1)}(\sigma+it,\alpha_1;\mathfrak{B}_{r1}),\ldots,\\
&\zeta(\sigma+it,\alpha_1;\mathfrak{B}_{rl_r}), \zeta'(\sigma+it,\alpha_1;\mathfrak{B}_{rl_r}),\ldots,\zeta^{(N-1)}(\sigma+it,\alpha_1;\mathfrak{B}_{rl_r})\Big)
\end{aligned}$$

is dense in $\mathbb{C}^{N(r_1+\kappa)}$.

Note that some results on the mixed joint universality for $\big(\zeta(s;\mathfrak{A}_1), \ldots, \zeta(s;\mathfrak{A}_{r_1}), \zeta(s,\alpha_1;\mathfrak{B}_1), \ldots, \zeta(s,\alpha_{r_2};\mathfrak{B}_{r_2})\big)$ were proved by A. Laurinčikas in [29].

5.2 Proof of Theorems 5.1, 5.2 and 5.3

In the proof of Theorem 5.1, the main role plays the universality (Theorem 5.2). For this, following the Bagchi method (see [1]), firstly we need the functional limit theorem in the sense of weakly convergent probability measures for the functions $\zeta(s;\mathfrak{A}_j)$, $j=1,...,r_1$, and $\zeta(s,\alpha_j;\mathfrak{B}_{jl})$, $j=1,...,r$, $l=1,...,l_j$.

5.2.1 Generalized mixed joint limit theorem

Let

$$\underline{H}(D)=\underbrace{H(D)\times ...\times H(D)}_{r_1+\kappa}.$$

In this subsection, we consider the probability measure P_T defined by

$$P_T(A)=:\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\underline{Z}(s+i\tau,\underline{\alpha};\underline{\mathfrak{A}};\underline{\mathfrak{B}})\in A\right\}$$

for $A\in\mathcal{B}(\underline{H}(D))$, where

$$\begin{aligned}\underline{Z}(s+i\tau,\underline{\alpha};\underline{\mathfrak{A}};\underline{\mathfrak{B}})=\Bigg(&\zeta(s;\mathfrak{A}_1),...,\zeta(s;\mathfrak{A}_{r_1}),\\ &\zeta(s,\alpha_1;\mathfrak{B}_{11}),\ldots,\zeta(s,\alpha_1;\mathfrak{B}_{1l_1})\ldots,\\ &\zeta(s,\alpha_r;\mathfrak{B}_{r1}),\ldots,\zeta(s,\alpha_r;\mathfrak{B}_{rl_r})\Bigg)\end{aligned}$$

with $\underline{\alpha}=(\alpha_1,\ldots,\alpha_r)$, $\underline{\mathfrak{A}}=(\mathfrak{A}_1,...,\mathfrak{A}_{r_1})$, $\underline{\mathfrak{B}}=\big(\mathfrak{B}_{11},...,\mathfrak{B}_{1l_1},...,\mathfrak{B}_{r1},...,\mathfrak{B}_{rl_r}\big)$.

For the statement of limit theorems, we need a certain topological structure. To avoid confusion with notations in previous sections, we redefine some of them.

Now by Ω_1 we mean the torus Ω defined by (2.1), i.e.,

$$\Omega_1=\prod_{p\in\mathbb{P}}\gamma_p,$$

where $\gamma_p=\gamma$ for all $p\in\mathbb{P}$. Define one more torus

$$\Omega_2=\prod_{m=0}^{\infty}\gamma_m,$$

with $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. Since, by the Tikhonov theorem, both tori Ω_1 and Ω_2 are compact topological Abelian groups, on $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$, the probability Haar measures m_{H1} and m_{H2} can be defined, which lead to the probability spaces $(\Omega_1, \mathcal{B}(\Omega_1), m_{H1})$ and $(\Omega_2, \mathcal{B}(\Omega_2), m_{H2})$, respectively. We put

$$\underline{\Omega} = \Omega_1 \times \Omega_{21} \times ... \times \Omega_{2r}$$

with $\Omega_{2j} = \Omega_2$ for all $j = 1, ..., r$. Again, by the Tikhonov theorem, $\underline{\Omega}$ is a compact topological Abelian group, and we have the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$ (see [52]). Here m_H is the product of measures m_{H1} and m_{H2j} for each $j = 1, ..., r$.

Denote by $\omega_1(p)$ be the projection of $\omega_1 \in \Omega_1$ to γ_p, $p \in \mathbb{P}$, and by $\omega_{2j}(m)$ the projection of $\omega_{2j} \in \Omega_{2j}$ to γ_m, $m \in \mathbb{N}_0$, $j = 1, ..., r$. Let $\underline{\omega} = (\omega_1, \omega_{21}, ..., \omega_{2r})$, and define on the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$ the $\underline{H}(D)$-valued random element $\underline{Z}(s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{A}}, \underline{\mathfrak{B}})$ by the formula

$$\begin{aligned}\underline{Z}(s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{A}}, \underline{\mathfrak{B}}) = \Big(&\zeta(s, \omega_1; \mathfrak{A}_1), ..., \zeta(s, \omega_1; \mathfrak{A}_{r_1}),\\ &\zeta(s, \alpha_1, \omega_{21}; \mathfrak{B}_{11}), ..., \zeta(s, \alpha_1, \omega_{21}; \mathfrak{B}_{1l_1}), ...,\\ &\zeta(s, \alpha_r, \omega_{2r}; \mathfrak{B}_{r1}), ..., \zeta(s, \alpha_r, \omega_{2r}; \mathfrak{B}_{rl_r})\Big)\end{aligned}$$

with

$$\zeta(s, \omega_1; \mathfrak{A}_j) = \sum_{m=1}^{\infty} \frac{a_{jm}\omega_1(m)}{m^s}, \quad j = 1, ..., r_1,$$

and

$$\zeta(s, \alpha_j, \omega_{2j}; \mathfrak{B}_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl}\omega_{2j}(m)}{(m+\alpha_j)^s}, \quad j = 1, ..., r, \quad l = 1, ..., l_j,$$

for $s \in D$. These series are convergent for almost all $\omega_1 \in \Omega_1$ and $\omega_{2j} \in \Omega_{2j}$, $j = 1, ..., r$. Denote by $P_{\underline{Z}}$ the distribution of the random element $\underline{Z}(s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{A}}, \underline{\mathfrak{B}})$, i.e., that $P_{\underline{Z}}$ is the probability measure on

$(\underline{H}(D), \mathcal{B}(\underline{H}(D)))$ defined as

$$P_{\underline{Z}}(A) = m_H\big(\underline{\omega} \in \underline{\Omega} : \underline{Z}(s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{A}}, \underline{\mathfrak{B}}) \in A\big).$$

Now we state the mixed joint functional limit theorem for the functions $\zeta(s; \mathfrak{A}_j)$, $j = 1, ..., r_1$, and $\zeta(s, \alpha_j; \mathfrak{B}_{jl})$, $j = 1, ..., r$, $l = 1, ..., l_j$.

Theorem 5.4. *Suppose that the sequences $\mathfrak{A}_1, ..., \mathfrak{A}_{r_1}$ are multiplicative, and the numbers $1, \alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then P_T converges weakly to $P_{\underline{Z}}$ as $T \to \infty$.*

The proof of the theorem is similar to that of Theorem 2 from [29]. Therefore, we recall only some essential moments, and will begin with a limit theorem on the torus $\underline{\Omega}$.

On $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$, define

$$\begin{aligned} &Q_T(A) := \\ &\quad \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \\ &\qquad \big((p^{-i\tau} : p \in \mathbb{P}), \big((m + \alpha_j)^{-i\tau} : m \in \mathbb{N}_0, j = 1, ..., r\big)\big) \in A\Big\}. \end{aligned}$$

Lemma 5.1. *Suppose that the numbers $1, \alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the measure Q_T converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. This is Theorem 3 of [29]. It is important to mention that in the proof essential role plays the fact that the system of numbers $\{\log p : p \in \mathbb{P}\} \cup \{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, ..., r\}$ is linear independent over $\mathbb{Q}$. □

Now, for a fixed $\sigma_0 > \frac{1}{2}$, we put

$$v(m, n) = \exp\left\{\left(\frac{m}{n}\right)^{\sigma_0}\right\}, \quad m, n \in \mathbb{N},$$

and

$$v_j(m, n) = \exp\left\{\left(\frac{m + \alpha_j}{n + \alpha_j}\right)^{\sigma_0}\right\}, \quad m, n \in \mathbb{N}_0, \quad j = 1, ..., r.$$

Using the Mellin transform formula and contour integration, we can show that the series defined by

$$\zeta_{jn}(s;\mathfrak{A}_j) = \sum_{m=1}^{\infty} \frac{a_{jm} v(m,n)}{m^s}, \quad j = 1, ..., r_1,$$

and

$$\zeta_{jn}(s,\alpha_j;\mathfrak{B}_{jl}) = \frac{b_{mjl} v_j(m,n)}{(m+\alpha_j)^s}, \quad j = 1, ..., r, \quad l = 1, ..., l_j,$$

converges absolutely for $\sigma > \frac{1}{2}$ (see [35] and [27], respectively).

Let, for $n \in \mathbb{N}$,

$$\zeta_{jn}(s,\omega_1;\mathfrak{A}_j) = \sum_{m=1}^{\infty} \frac{a_{jm}\omega_1(m) v(m,n)}{m^s}, \quad j = 1, ..., r_1,$$

and

$$\zeta_{jn}(s,\omega_{2j},\alpha_j;\mathfrak{B}_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl}\omega_{2j}(m) v_j(m,n)}{(m+\alpha_j)^s}, \ j = 1, ..., r, \ l = 1, ..., l_j,$$

and we extend $\omega_1(p)$ to the set $\mathbb{N}$ using the formula

$$\omega_1(m) = \prod_{p^\gamma || m} \omega_1(p)^\gamma$$

(in view of the factorization of m into primes). Note that the latter series converges absolutely for $\sigma > \frac{1}{2}$ also.

Let $\underline{\omega}_0 = (\widehat{\omega}_1, \widehat{\omega}_{21}, ..., \widehat{\omega}_{2r}) \in \underline{\Omega}$ be fixed. Then, in view of Lemma 5.1, we can prove that, for $A \in \mathcal{B}(\underline{H}(D))$, the measures

$$P_{T,n}(A) := \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \underline{Z}_n(s+i\tau,\underline{\alpha};\underline{\mathfrak{A}},\underline{\mathfrak{B}}) \in A\Big\}$$

and

$$P_{T,n,\underline{\omega}}(A) := \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \underline{Z}_n(s+i\tau,\underline{\omega}_0,\underline{\alpha};\underline{\mathfrak{A}},\underline{\mathfrak{B}}) \in A\Big\}$$

both converge weakly to the same probability measure P_n as $T \to \infty$. Here, for $s \in D$,

$$\underline{Z}_n(s, \underline{\alpha}; \underline{\mathfrak{A}}, \underline{\mathfrak{B}}) := \Big(\zeta_{1n}(s; \mathfrak{A}_1), .., \zeta_{r_1 n}(s; \mathfrak{A}_{r_1}), \\ \zeta_{1n}(s, \alpha_1; \mathfrak{B}_{11}), ..., \zeta_{1n}(s, \alpha_1; \mathfrak{B}_{1l_1}), ..., \\ \zeta_{rn}(s, \alpha_r; \mathfrak{B}_{r1}), ..., \zeta_{rn}(s, \alpha_r; \mathfrak{B}_{rl_r}) \Big)$$

and

$$\underline{Z}_n(s, \underline{\omega}_0 \underline{\alpha}; \underline{\mathfrak{A}}, \underline{\mathfrak{B}}) := \Big(\zeta_{1n}(s, \widehat{\omega}_1; \mathfrak{A}_1), .., \zeta_{r_1 n}(s, \widehat{\omega}_1; \mathfrak{A}_{r1}), \\ \zeta_{1n}(s, \widehat{\omega}_{21}; \alpha_1; \mathfrak{B}_{11}), ..., \zeta_{1n}(s, \widehat{\omega}_{22}, \alpha_1; \mathfrak{B}_{1l_1}), ..., \\ \zeta_{rn}(s, \widehat{\omega}_{2r}, \alpha_r; \mathfrak{B}_{r1}), ..., \zeta_{rn}(s, \widehat{\omega}_{2r}, \alpha_r; \mathfrak{B}_{rl_r}) \Big).$$

In the next step of the proof, we approximate the vectors $\underline{Z}(s, \underline{\alpha}; \underline{\mathfrak{A}}, \underline{\mathfrak{B}})$ and $\underline{Z}(s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{A}}, \underline{\mathfrak{B}})$ by the vectors $\underline{Z}_n(s, \underline{\alpha}; \underline{\mathfrak{A}}, \underline{\mathfrak{B}})$ and $\underline{Z}_n(s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{A}}, \underline{\mathfrak{B}})$, respectively. On $\underline{H}(D)$ we define the metric inducing the topology of uniform convergence on compacta as follows.

It is known (see [26]) that the metric on $H(D)$ inducing its topology of uniform convergence on compacta, for $f, g \in H(D)$, is defined by the formula

$$\varrho(f, g) = \sum_{k=1}^{\infty} 2^{-k} \frac{\sup_{s \in K_k} |f(s) - g(s)|}{1 + \sup_{s \in K_k} |f(s) - g(s)|}.$$

Here $\{K_k : k \in \mathbb{N}\}$ is a sequence of compact subsets of D such that $D = \cup_{k=1}^{\infty} K_k$, $K_k \subset K_{k+1}$ for all $k \in \mathbb{N}$, and, for some k, $K \subset K_k$ if $K \subset D$ is a compact. If, for $\underline{f} = (f_1, ..., f_{r_1}, f_{11}, ..., f_{1l_1}, f_{r1}, ..., f_{rl_r})$ and $\underline{g} = (g_1, ..., g_{r_1}, g_{11}, ..., g_{1l_1}, g_{r1}, ..., g_{rl_r}) \in \underline{H}(D)$,

$$\underline{\varrho}(\underline{f}, \underline{g}) = \max \left(\max_{1 \leqslant j \leqslant r_1} \varrho(f_j, g_j), \max_{1 \leqslant j \leqslant r} \max_{1 \leqslant l \leqslant l_j} \varrho(f_{jl}, g_{jl}) \right),$$

then $\underline{\varrho}$ is a metric on $\underline{H}(D)$ inducing its topology.

Then, using relations from [35] and [27] together with (3) and (4) from [29], we obtain that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \underline{\varrho}\big(\underline{Z}(s+i\tau,\underline{\alpha};\underline{\mathfrak{A}},\underline{\mathfrak{B}}),\underline{Z}_n(s+i\tau,\underline{\alpha};\underline{\mathfrak{A}},\underline{\mathfrak{B}})\big)d\tau=0 \tag{5.1}$$

and, for almost all elements $\underline{\omega}\in\underline{\Omega}$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \underline{\varrho}\big(\underline{Z}(s+i\tau,\underline{\omega},\underline{\alpha};\underline{\mathfrak{A}},\underline{\mathfrak{B}}),\underline{Z}_n(s+i\tau,\underline{\omega},\underline{\alpha};\underline{\mathfrak{A}},\underline{\mathfrak{B}})\big)d\tau=0 \tag{5.2}$$

if $1,\alpha_1,...,\alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$.

Now, on $(\underline{H}(D),\mathcal{B}(\underline{H}(D)))$, we define one more probability measure

$$P_{T,\underline{\omega}}=\frac{1}{T}\text{meas}\big\{\tau\in[0,T]:\underline{Z}(s+i\tau,\omega,\underline{\alpha};\underline{\mathfrak{A}},\underline{\mathfrak{B}})\in A\big\}.$$

Since both probability measures $P_{T,n}$ and $P_{T,n,\underline{\omega}}$ weakly converge to the same limit measure, this fact together with the relations (5.1) and (5.2), shows that both the measures P_T and $P_{T,\underline{\omega}}$ converge to the same probability measure P as $T\to\infty$.

Finally, applying arguments from ergodic theory we can show that $P=P_{\underline{Z}}$.

5.2.2 Support of the measure $P_{\underline{Z}}$

We recall that the support of the measure $P_{\underline{Z}}$ is a minimal closed subset $S_{P_{\underline{Z}}}\subseteq\underline{H}(D)$ such that $P_{\underline{Z}}(S_{P_{\underline{Z}}})=1$. The set $S_{P_{\underline{Z}}}$ consists of all $\underline{f}\in\underline{H}(D)$ such that, for every neighborhood $\mathcal{F}$ of $\underline{f}$, holds the inequality $P_{\underline{Z}}(\mathcal{F})>0$.

Let $S=\{g\in H(D):g(s)\neq 0 \text{ or } g(s)\equiv 0\}$.

Theorem 5.5. *Suppose that the sequences $\mathfrak{A}_1,...,\mathfrak{A}_{r_1}$ are multiplicative,* $\text{rank}(A)=r_1$, *and inequalities* (4.1) *are satisfied. Let* $1,\alpha_1,...,\alpha_r$

be numbers algebraically independent over $\mathbb{Q}$, *and* $\mathrm{rank}(B) = \kappa$. *Then the support of the measure* $P_{\underline{Z}}$ *is the set* $S^{r_1} \times H^{\kappa}(D)$.

Proof. In Lemma 8 of [29], it was proved that the support of the random element $(\zeta(s, \omega_1; \mathfrak{A}_1), ..., \zeta(s, \omega_1; \mathfrak{A}_{r_1}))$ is the set S^{r_1}. In Lemma 4.1 of [31], it was shown that the support of the random element $\zeta(s, \alpha; \omega_2; \mathfrak{B})$ is the whole space $H(D)$.

The space $\underline{H}(D)$ is separable since the space of analytic functions $H(D)$ is separable. The measure m_H is defined as a product of the Haar measures m_{H1} and m_{H2j} on $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_{2j}, \mathcal{B}(\Omega_{2j}))$, $j = 1, ..., r$, respectively. Thus, we find that, for $A \in \mathcal{B}(\underline{H}(D))$,

$$\begin{aligned}
P_{\underline{Z}}(A) =& m_H\Big(\underline{\omega} \in \Omega : \underline{Z}(s, \underline{\omega}, ; \underline{\mathfrak{A}}, \underline{\mathfrak{B}}) \in A\Big) \\
=& m_H\Big(\underline{\omega} \in \Omega : \big((\zeta(s, \omega_1; \mathfrak{A}_1), ..., \zeta(s, \omega_1; \mathfrak{A}_{r_1})) \in A^{r_1}\big), \\
& \big(\zeta(s, \omega_{21}, \alpha_1; \mathfrak{B}_{11}), ..., \zeta(s, \omega_{21}, \alpha_1; \mathfrak{B}_{1l_1}), ..., \\
& \zeta(s, \omega_{2r}, \alpha_r; \mathfrak{B}_{r1}), ..., \zeta(s, \omega_{2r}, \mathfrak{B}_{rl_r})\big) \in A^{\kappa}\Big) \\
=& m_{H1}\Big((\zeta(s, \omega_1; \mathfrak{A}_1), ..., \zeta(s, \omega_1; \mathfrak{A}_{r_1})) \in A^{r_1}\Big) \\
& \times m_{H21}\Big(\omega_{21} \in \Omega_{21} : \zeta(s, \alpha_1, \omega_{21}; \mathfrak{B}_{11}) \in A_{11}\Big) \times ... \\
& \times m_{H21}\Big(\omega_{21} \in \Omega_{21} : \zeta(s, \alpha_1, \omega_{21}; \mathfrak{B}_{1l_1}) \in A_{1l_1}\Big) \times ... \\
& \times m_{H2r}\Big(\omega_{2r} \in \Omega_{2r} : \zeta(s, \alpha_r, \omega_{2r}; \mathfrak{B}_{r1}) \in A_{r1}\Big) \times ... \\
& \times m_{H2r}\Big(\omega_{2r} \in \Omega_{2r} : \zeta(s, \alpha_r, \omega_{2r}; \mathfrak{B}_{rl_r}) \in A_{rl_r}\Big)
\end{aligned}$$

with $A = A^{r_1} \times A^{\kappa}$, $A^{r_1} \in \mathcal{B}(H^{r_1}(D))$ and $A_{\kappa} = A_{11} \times ... \times A_{1l_1} \times ... A_{r1} \times ... \times A_{rl_r} \in \mathcal{B}(H^{\kappa}(D))$ for each $A_{jl} \in \mathcal{B}(H(D))$, $j = 1, ..., r$, $l = 1, ..., l_j$.

This, under the hypotheses of the theorem, together with above remark for supports of random elements $(\zeta(s, \omega_1; \mathfrak{A}_1), ..., \zeta(s, \omega_1; \mathfrak{A}_{r_1}))$ and $\zeta(s, \alpha; \omega_2; \mathfrak{B})$, completes the proof. □

5.2.3 Mixed joint universality theorem

Recall that the main place in the proof of functional independence and denseness results occupies the mixed joint universality theorem, i.e., Theorem 5.2.

Proof. The proof of the theorem follows from Theorems 5.4 and 5.5 using the Mergelyan theorem on the approximation of analytic functions by polynomials (see Theorem 2.4). □

5.2.4 Proof of denseness lemma

Now we will prove the general denseness lemma which is Theorem 5.3 in our dissertation.

Proof. Let

$$\underline{z} := \big(z_{110}, z_{111}, \dots, z_{11N-1}, \dots, z_{1r_10}, z_{1r_11}, \dots, z_{1r_1N-1},\\ z_{2110}, ..., z_{211N-1}, \dots, z_{21l_10}, \dots, z_{21l_1N-1}, ..., z_{2r10}, ...,\\ z_{2r1N-1}, ..., z_{2rl_r0}, ...z_{2rl_rN-1}\big)$$

be an arbitrary point in the space $\mathbb{C}^{N(r_1+\kappa)}$. As in the proof of Lemma 4.2, we will show that, for each $\varepsilon > 0$, there exists a sequence $\{\tau_m\}$ of real numbers, $\lim_{m\to\infty}\tau_m = +\infty$, such that

$$|u(\tau_m) - \underline{z}|_{\mathbb{C}^{N(r_1+\kappa)}} < \varepsilon,$$

i.e., that, for $g = 0, 1, \dots, N-1$ and every $\varepsilon > 0$, the inequalities

$$|\zeta^{(g)}(\sigma + i\tau_m; \mathfrak{A}_j) - z_{1jg}| < \frac{\varepsilon}{Nr_1}, \quad j = 1, \dots, r_1,$$

and

$$|\zeta^{(g)}(\sigma + i\tau_m, \alpha_j; \mathfrak{B}_j) - z_{2jlg}| < \frac{\varepsilon}{Nr_2}, \quad j = 1, \dots, r, \quad l = 1, ..., l_j,$$

hold.

Define the polynomial

$$P_{hdN}(s) = \frac{z_{hdN-1} \cdot s^{N-1}}{(N-1)!} + \frac{z_{hdN-2} \cdot s^{N-2}}{(N-2)!} + \cdots + \frac{z_{hd0}}{0!}.$$

Here $h = 1$ for the function $\zeta(s; \mathfrak{A}_d)$, $d = 1, \ldots, r_1$, and $h = 2$ for the function $\zeta(s, \alpha_j; \mathfrak{B}_d)$ with the meaning of d as double indexes jl, where $j = 1, \ldots, r$, $l = 1, ..., l_j$. Then we have

$$P_{hdN}^{(g)}(0) = z_{hdg}$$

for $g = 0, 1, \ldots, N-1$.

We fix a number σ_0, $\frac{1}{2} < \sigma_0 < 1$, and choose a compact subset K of the strip D, so that the point σ_0 is its internal point. Then, by Theorem 5.2, there exists a sequence $\{\tau_m\}$ of real numbers, $\lim_{m\to\infty} \tau_m = +\infty$, such that

$$\sup_{1\leqslant j\leqslant r_1} \sup_{s\in K_j} |\zeta(s + i\tau_m; \mathfrak{A}_j) - P_{1jN}(s - \sigma_0)| < \frac{\varepsilon\delta^N}{2^N N! N r_1}$$

for $j = 1, \ldots, r_1$, and

$$\sup_{1\leqslant j\leqslant r} \sup_{1\leqslant l\leqslant l_j} \sup_{s\in K_{jl}} |\zeta(s + i\tau_m, \alpha_j; \mathfrak{B}_{jl}) - P_{2jlN}(s - \sigma_0)| < \frac{\varepsilon\delta^N}{2^N N! N\kappa}$$

for $j = 1, \ldots, r$ and $l = 1, ...l_j$, where $\delta = \min(\delta_{1j}, \delta_{2jl})$ and $\delta_{1j}, \delta_{2jl}$ are the distances of σ_0 from the boundaries of the sets K_j, $j = 1, \ldots, r_1$ and K_{jl}, $j = 1, \ldots, r$, $l = 1..., l_j$, respectively. Hence, applying the Cauchy integral theorem, we see that, for all $g = 0, \ldots, N-1$,

$$|\zeta^{(g)}(\sigma_0 + i\tau_m; \mathfrak{A}_j) - z_{1jg}| = \left| \frac{g!}{2\pi i} \int_{|s-\sigma_0|=\frac{\delta}{2}} \frac{\zeta(s + i\tau_m; \mathfrak{A}_j) - P_{1jN}(s - \sigma_0)}{(s - \sigma_0)^{g+1}} ds \right| < \frac{\varepsilon}{N r_1}$$

for all $j = 1, \dots, r_1$, and

$$
\begin{aligned}
&|\zeta^{(g)}(\sigma_0 + i\tau_m, \alpha_j; \mathfrak{B}_{jl}) - z_{2jlg}| \\
&\quad = \left| \frac{g!}{2\pi i} \int_{|s-\sigma_0|=\frac{\delta}{2}} \frac{\zeta(s + i\tau_m, \alpha_j; \mathfrak{B}_{jl}) - P_{2jlN}(s - \sigma_0)}{(s - \sigma_0)^{g+1}} ds \right| < \frac{\varepsilon}{N\kappa}
\end{aligned}
$$

for all $j = 1, \dots, r$, $l = 1, ..., l_j$.

This completes the proof of the theorem. $\square$

5.2.5 Proof of Theorem 5.1

As for the proof of Theorem 4.1, we use the Voronin method. Therefore, using Theorem 5.3, it is sufficient to prove that $F_h \equiv 0$ for a fixed h. Assuming, on the contrary, we suppose that $F_h \not\equiv 0$. Then there exists a bounded region $G \subset \mathbb{C}^{r_1+\kappa}$ such that the inequality

$$
|F_h(\underline{s})| > c > 0 \qquad (5.3)
$$

is true for all points $\underline{s} := (s_1, ..., s_{r_1}, s_{11}, ..., s_{1l_1}, ..., s_{r1}, ..., s_{rl_r}) \in G$. In view of denseness lemma, there exists a sequence $\{\tau_k\}$, $\lim_{k\to\infty} \tau_k = \infty$, such that

$$
\begin{aligned}
&\Big(\zeta(\sigma+i\tau_k;\mathfrak{A}_1),\zeta'(\sigma+i\tau_k;\mathfrak{A}_1),\dots,\zeta^{(N-1)}(\sigma+i\tau_k;\mathfrak{A}_1),\dots,\\
&\zeta(\sigma+i\tau_k;\mathfrak{A}_{r_1}),\zeta'(\sigma+i\tau_k;\mathfrak{A}_{r_1}),\dots,\zeta^{(N-1)}(\sigma+i\tau_k;\mathfrak{A}_{r_1}),\\
&\zeta(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{11}),\zeta'(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{11}),\dots,\zeta^{(N-1)}(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{11}),\dots,\\
&\zeta(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{1l_1}),\zeta'(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{1l_1}),\dots,\zeta^{(N-1)}(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{1l_1}),\dots,\\
&\zeta(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{r1}),\zeta'(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{r1}),\dots,\zeta^{(N-1)}(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{r1}),\dots,\\
&\zeta(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{rl_r}),\zeta'(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{rl_r}),\dots,\zeta^{(N-1)}(\sigma+i\tau_k,\alpha_1;\mathfrak{B}_{rl_r})\Big)\in G.
\end{aligned}
$$

However this and (5.3) contradicts our hypothesis for F_h. Hence, $F_h \equiv 0$.

In the same way we can show that, for $h = 0, ..., n$, in Theorem 5.1 every continuous function F_h is identically equal to zero, i.e., $F_n \equiv 0, \dots, F_0 \equiv 0$.

The proof of Theorem 5.1 is complete.

Chapter 6

Conclusions

In the doctoral dissertation, the main attention is devoted to the simultaneous approximation of wide classes of analytic functions by the shifts of L-functions from the Selberg-Steuding class $\widetilde{\mathcal{S}}$, and the mixed simultaneous approximation by periodic zeta-functions and periodic Hurwitz zeta-functions, as well as some applications of it. Therefore, the dissertation results lead us to the following list of conclusions.

1. Joint continuous universality theorem for the L-functions from the class $\widetilde{\mathcal{S}}$, namely, $\big(L(s+ia_1\tau), ..., L(s+ia_r\tau)\big)$, where the real algebraic numbers $a_1, ..., a_r$ are linearly independent over the field of rational numbers, is valid.

2. The set of shifts of the collection $\big(L(s+ia_1\tau), ..., L(s+ia_r\tau)\big)$ that approximate on compact sets a tuple of given analytic functions with accuracy $\varepsilon > 0$ has a positive density for all but at most countably many $\varepsilon > 0$.

3. The first two conclusions are valid for discrete shifts using the linear independence over $\mathbb{Q}$ of the multiset $\big\{(h_j \log p : p \in \mathbb{P}), j = 1, \dots, r; 2\pi\big\}$ for positive h_j.

4. The collection consisting of r_1 number of periodic zeta-functions $\big(\zeta(s;\mathfrak{A}_1), ..., \zeta(s;\mathfrak{A}_{r_1})\big)$ is functional independent and dense when the rank of the matrix consisting of the coefficients of the functions and these coefficients satisfy certain conditions.

5. Mixed simultaneous approximation for the multi-collections of periodic zeta-functions and periodic Hurwitz zeta-functions is valid in

continuous case. To fulfil this property, the following additional conditions must be satisfied: the ranks for the coefficients matrices of these functions to be $\mathrm{rank}(A) = r_1$ and $\mathrm{rank}(B) = \kappa$, the parameters $1, \alpha_1, ..., \alpha_r$ of periodic Hurwitz zeta-functions to be algebraically independent over the field $\mathbb{Q}$, and certain estimates for the coefficients of periodic zeta-functions.

6. Same, based on the situation in the 5th conclusion, the collection of zeta-functions is functional independent and dense.

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Santrauka (Summary in Lithuanian)

Tyrimo objektas

Analizinėje skaičių teorijoje dzeta ir L funkcijos atlieka pagrindinį vaidmenį. Šių funkcijų teorija pastoviai ir sistemiškai vystosi įvairiomis kryptimis. Atraminiu šios teorijos tašku tapo B. Rymano (Riemann) darbas [46], kuriame gauta eilė svarbių rezultatų dabar taip vadinamajai Rymano dzeta funkcijai $\zeta(s)$. Ši funkcija buvo nagrinėjama kaip kompleksinio kintamojo $s = \sigma + it$ funkcija, kuri pusplokštumėje $\sigma > 1$ yra apibrėžiama Dirichlė eilute arba Oilerio sandauga pagal pirminius skaičius

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Funkcija $\zeta(s)$ yra analiziškai pratęsiama į visą kompleksinę plokštumą, išskyrus tašką $s = 1$, kuris yra paprastasis polius su reziduumu 1.

Tačiau yra dzeta funkcijų neturinčių išraiškos Oilerio sandauga. Klasikinis tokios funkcijos pavyzdys – dzeta funkcija $\zeta(s, \alpha)$, kurią apibrėžė A. Hurvicas (Hurwitz) [12]. Pusplokštumėje $\sigma > 1$ ir realiajam skaičiui α, $0 < \alpha \leqslant 1$, ji yra apibrėžiama Dirichlė eilute

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}.$$

Išskyrus atskirus atvejus, kai $\alpha = \frac{1}{2}$ ir $\alpha = 1$, funkcija $\zeta(s, \alpha)$ nėra išreiškiama Oilerio sandauga. Paminėsime, kad ši funkcija taip pat yra analiziškai pratęsiama į visą kompleksinę plokštumą, išskyrus tašką $s = 1$, kuris yra paprastasis polius su reziduumu 1.

Šiuolaikinėje skaičių teorijoje vienas iš populiariausių sprendžiamų uždavinių – analizinių funkcijų aproksimavimas bendresnėmis funkcijomis. 1952 m. S. Mergelianas (Mergelyan) įrodė [39], kad kiekviena kompleksines reikšmes įgyjanti funkcija $f(s)$, kuri yra tolydi plokštumos $\mathbb{C}$ kompaktiškoje aibėje ir analizinė šios aibės viduje, gali būti tolygiai aproksimuojama s polinomais.

Ypatingai aproksimavimu susidomėta po 1975 m. stipraus S.M. Voronino (Voronin) darbo [55]. Jame buvo parodyta, kad kiekvieną analizinę funkciją kompleksinėje plokštumoje tam tikru tikslumu galima aproksimuoti Rymano dzeta funkcijos postūmiais $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$.

Dabar analizinėje skaičių teorijoje yra žinoma daugelis Rymano dzeta funcijos $\zeta(s)$ apibendrinimų, pavyzdžiui, Dirichlė L funkcijos $L(s, \chi)$, Matsumoto dzeta funkcija $\varphi(s)$ ir kitos. Natūralu, kad taip pat yra įdomūs ir $\zeta(s)$ apibendrinimai, neturintys Oilerio sandaugos. Vienas iš jų – aukščiau minėta Hurvico dzeta funkcija $\zeta(s, \alpha)$, taip pat ir kitos Hurvico tipo dzeta funkcijos.

Šios daktaro disertacijos tyrimo objektai yra dzeta ir L funkcijos, kurios taip pat apibendrina Rymano dzeta funkciją. O tiksliau – L funkcijos, kurios priklauso Selbergo-Štoidingo (Selberg-Steuding) klasei $\widetilde{S}$, bei dvi dzeta funkcijos su periodiniais koeficientais – periodinė $\zeta(s; \mathfrak{A})$ ir periodinė Hurvico $\zeta(s, \alpha; \mathfrak{B})$ dzeta funkcijos.

Pirmiausia pateiksime dzeta funkcijų, kurių elgesį tirsime darbe, apibrėžimus.

A. Selbergas (Selberg) apibrėžė klasę S (žr. [49, 50]), kuri dabar yra žinoma kaip Selbergo klasė. Nors šios klasės struktūrą nagrinėjo daug matematikų (pavyzdžiui žr. [20, 21, 22, 23, 43, 52]), bet iki šiol ji nėra pilnai ištirta. Tačiau žinoma, kad klasei S priklauso visos pagrindinės dzeta ir L funkcijos, pavyzdžiui, $\zeta(s)$, $L(s, \chi)$, tam tikrų parabolinių formų dzeta funkcijos ir pan.

Klasę $\mathcal{S}$ sudaro Dirichlė eilutės

$$L(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \quad a(m) \in \mathbb{C},$$

kurios tenkina šias hipotezes:

(1) *Ramanudžano (Ramanujan) hipotezė.* Įvertis $a(m) \ll m^{\varepsilon}$ galioja su visais $\varepsilon > 0$; čia tam tikra konstanta gali priklausyti nuo ε.

(2) *Analizinis pratęsimas.* Su tam tikru $k \in \mathbb{N}_0$, funkcija $(s-1)^k L(s)$ yra sveikoji baigtinės eilės funkcija.

(3) *Funkcinė lygtis.* Tegul

$$\Lambda_L(s) = L(s) Q^s \prod_{j=1}^{j_0} \Gamma(\lambda_j s + \mu_j);$$

čia q ir λ_j yra teigiami realieji skaičiai, o kompleksinis skaičius μ_j yra toks, kad $\Re \mu_j \geqslant 0$. Tada yra teisinga tokia funkcinė lygtis

$$\Lambda_L(s) = w \overline{\Lambda_L(1 - \overline{s})};$$

čia $|w| = 1$ ir $\overline{s}$ žymi s jungtinį skaičių.

(4) *Oilerio sandauga.* Tegul

$$\log L_p(s) = \sum_{l=1}^{\infty} \frac{b(p^l)}{p^{ls}}$$

su koeficientais $b(p^l)$ tenkinančiais įvertį $b(p^l) \ll p^{\alpha l}$, $\alpha < \frac{1}{2}$. Tada teisinga tokia išraiška

$$L(s) = \prod_{p \in \mathbb{P}} L_p(s).$$

J. Štoidingas (Steuding) pirmasis praėjo nagrinėti klasę $\mathcal{S}$ universalumo aspektu [52]. Jis panaudojo tokias aksiomas:

(5) *Pirminių skaičių teoremos analogas.* Egzistuoja skaičius $\kappa > 0$ toks, kad

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leqslant x} |a(p)|^2 = \kappa.$$

Be to, [52] darbe buvo pareikalauta, kad egzistuotų tokio tipo Oilerio sandauga:

(6)

$$L(s) = \prod_{p\in\mathbb{P}} \prod_{j=1}^{l} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}$$

su tam tikrais kompleksiniais skaičiais $\alpha_j(p)$.

Selbergo-Štoidingo klase $\widetilde{\mathcal{S}}$ vadiname dzeta ir L funkcijas, kurios priklauso Selbergo klasei $\mathcal{S}$ ir tenkina 5-ąją aksiomą.

Pastebėsime, kad Selbergo klasės teorijoje svarbi charakteristika yra funkcijų laipsnis. Priminsime, kad funkcijos $L(s) \in \mathcal{S}$ laipsnis d_L yra apibrėžiamas taip:

$$d_L = 2\sum_{j=1}^{j_0} \lambda_j.$$

Pavyzdžiui, jei $d_L = 1$, tada $L(s)$ sutampa su Rymano dzeta funkcija $\zeta(s)$, jei $d_L = 2$, tada $L(s)$ yra L funkcijos susietos su holomorfine nauja forma f, o jeigu $d_L = 4$, tai kiekvienos normuotos tikrinės reikšmės Rankino-Selbergo L funkcija yra klasės $\mathcal{S}$ elementas.

Mus taip pat domina vienlaikio aproksimavimo savybė ir jos taikymai dviem dzeta funkcijoms su periodiniais koeficientais.

Pirmoji mus dominanti funkcija buvo apibrėžta V. Šni (Schnee) [48], dabar vadinama periodine dzeta funkcija $\zeta(s; \mathfrak{A})$.

Tarkime, kad $\mathfrak{A} = \{a_m : m \in \mathbb{N}\}$ yra komplesinių skaičių a_m seka su minimaliu teigiamu periodu $k \in \mathbb{N}$. Pusplokštumėje $\sigma > 1$ periodinė dzeta funkcija $\zeta(s;\mathfrak{A})$ yra apibrėžta Dirichlė eilute

$$\zeta(s;\mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

Iš sekos $\mathfrak{A}$ periodiškumo seka, kad

$$\zeta(s;\mathfrak{A}) = \frac{1}{k^s} \sum_{q=1}^{k} a_q \zeta\left(s, \frac{q}{k}\right), \quad \sigma > 1;$$

čia $\zeta(s,\alpha)$ yra klasikinė Hurvico dzeta funkcija. Vadinasi, atsižvelgus į funkcijos $\zeta(s,\alpha)$ savybes, matome, kad funkcija $\zeta(s;\mathfrak{A})$ gali būti analiziškai pratęsiama į visą kompleksinę plokštumą, išskyrus, galbūt tašką $s = 1$, kuris yra paprastasis polius su reziduumu

$$a = \frac{1}{k} \sum_{q=1}^{k} a_q.$$

Jei $a = 0$, tada funkcija $\zeta(s;\mathfrak{A})$ yra sveikoji.

Antroji dzeta funkcija su periodiniais koeficientais, kurią nagrinėjame šiame darbe, yra periodinė Hurvico dzeta funkcija $\zeta(s,\alpha;\mathfrak{B})$. Pastebėsime, kad šią funkciją apibrėžė A. Laurinčikas [27].

Dabar tegul $\mathfrak{B} = \{b_m \in \mathbb{C} : m \in \mathbb{N}_0\}$ bus periodinė seka su minimaliu periodu $k \in \mathbb{N}_0$, o α – fiksuotas realusis skaičius, $0 < \alpha \leqslant 1$. Tada periodinė Hurvico dzeta funkcija $\zeta(s,\alpha;\mathfrak{B})$ yra apibrėžta Dirichlė eilute

$$\zeta(s,\alpha;\mathfrak{B}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}, \quad \sigma > 1.$$

Kadangi seka $\mathfrak{B}$ yra periodinė seka, todėl

$$\zeta(s,\alpha;\mathfrak{B}) = \frac{1}{k^s} \sum_{m=0}^{k-1} b_m \zeta\left(s, \frac{m+\alpha}{k}\right).$$

Iš čia matome, kad funkcija $\zeta(s,\alpha;\mathfrak{B})$ yra analiziškai pratęsiama į visą kompleksinę plokštumą, išskyrus, galbūt tašką $s = 1$, kuris yra paprastasis polius su reziduumu

$$b = \frac{1}{k} \sum_{m=0}^{k-1} b_m.$$

Jei $b = 0$, tai $\zeta(s, \alpha; \mathfrak{B})$ taip pat yra sveikoji funkcija.

Šiame darbe mes nagrinėjame vienlaikį tam tikrų funkcijų aproksimavimą tinkamais aukščiau minėtų dzeta ir L funkcijų postūmiais bei gauname svarbias su tuo susijusias išvadas.

Tikslas ir uždaviniai

Daktaro disertacijos tikslas – gauti L funkcijų, priklausančių Selbergo-Štoidingo klasei $\widetilde{S}$, universalumo savybės apibendrinimus ir tam tikrus rezultatus, susijusius su mišriu vienalaikiu aproksimavimu dzeta funkcijų su periodiniais koeficientais, tiksliau $\zeta(s; \mathfrak{A})$ ir $\zeta(s, \alpha; \mathfrak{B})$, postūmiais.

Uždaviniai yra tokie:

1. Tolydi jungtinio universalumo teorema L funkcijoms ir jos modifikacija.

2. Diskreti jungtinio universalumo teorema L fukcijoms ir jos modifikacija.

3. Periodinių dzeta funkcijų jungtinis funkcinis nepriklausomumas ir tirštumas.

4. Analizinių funkcijų rinkinių mišrus vienalaikis aproksimavimas periodinių dzeta funkcijų rinkinių ir periodinių Hurvico dzeta funkcijų rinkinių postūmiais.

5. Periodinių dzeta funkcijų ir periodinių Hurvico dzeta funkcijų multirinkinių mišrus jungtinis funkcinis nepriklausomumas ir tirštumas.

Aktualumas

Pastaruosius tris dešimtmečius universalumo teorija sparčiai vystosi įvairiomis kryptimis, pavyzdžiui, vykdomi jungtinio mišraus universalumo, tiesinių operatorių universalumo, unversalumo trumpuose intervaluose tyrimai ir pan. Tai parodo šios teorijos svarbą bendrai matematikos moksle. Yra žinoma, kad dzeta funkcijos ir jų universalumas, ypač diskretus, plačiai taikomi fizikoje (žr. [4, 10, 25, 36, 44]). Šis faktas analizinių funkcijų aproksimavimo dzeta ir L funkcijų postūmiais tyrimus padaro dar labiau patraukliais.

Kartu su vienos dzeta funkcijos universalumo savybe iškyla klausimas apie galimų apibendrinimų kryptis. Viena iš jų – aproksimavimas plačiomis dzeta funkcijų klasėmis, pavyzdžiui, Selbergo klase $\mathcal{S}$, išplėsta Selbergo klase $\mathcal{S}^{\#}$, Matsumoto dzeta funkcijų klase $\mathcal{M}$ ir pan. Kita galima tyrimų kryptis – vienalaikis aproksimavimas klasėmis. Trečioji kryptis gali būti vadinamasis mišrus vienalaikis aproksimavimas. Tai galima padaryti nagrinėjant tolydų ir diskretų dzeta ir L funkcijų universalumo savybės atvejus.

Kaip yra žinoma (žr. [19, 26, 37, 52]), panaudojant universalumo savybę, galima spręsti eilę kitų problemų: dzeta ir L funkcijų tirštumo ar funkcinio nepriklausomumo, nulių kiekio, efektyvumo problemas ir pan. Todėl svarbu vystyti analizinių funkcijų aproksimavimo tinkamomis žinomomis dzeta funkcijų klasėmis teoriją, taip pat ieškoti naujų klasių, kurios išsaugotų aproksimavimo savybes.

Be to, aproksimavimo dzeta funkcijomis teorijos vystymas yra vienas iš produktyviausių mokslinių laukų Lietuvos analizinės skaičių teorijos mokykloje. Dar daugiau – jaunojo mokslininko pareiga yra išlaikyti ir plėsti Lietuvos matematikų tradicijas.

Metodai

Dzeta funkcijų klasių universalumo savybės įrodymuose taikomi tikimybiniai ir analiziniai metodai bei Mergeliano analizinių funkcijų aproksimavimo polinomais teorija. Tikimybiniai metodai remiasi silpnojo tikimybinio mato konvergavimu ribinėse teoremose analizinių funkcijų erdvėje su išreikštiniu mato pavidalu. Tam panaudojami Furjė analizės, Dirichlė eilučių teorijos elementai, silpnojo tikimybinių matų konvergavimo savybės bei ergodiškumo teorija.

Naujumas

Šioje disertacijoje pateikiami naujai gauti rezultatai.

Jungtinis vienalaikis aproksimavimas L funkcijomis, kurios priklauso Selbergo-Štoidingo klasei $\widetilde{S}$ abiem atvejais, tolydžiu ir diskrečiu, yra nagrinėjami pirmą kartą. Pastebėsime, kad minėtose teoremose yra nagrinėjamas apatinis postūmių tankis ir tankis, išskyrus daugiausiai skaičią aproksimuojančių postūmių tikslumo aibę. Tokio tipo aproksimavimas L funkcijomis, priklausančiomis klasei $\widetilde{S}$, iš viso dar nebuvo tyrinėtas.

Iš jungtinio mišraus universalumo teoremų dzeta funkcijoms su periodiniais koeficientais gauti du rezultatai: jungtinis funkcinis nepriklausomumas ir tirštumas. Bendriausiu atveju – kokie yra disertacijoje – taip pat pateikiami pirmą kartą.

Problemos istorija ir pagrindiniai rezultatai

1975 m. S.M. Voronino publikuotas straipnsnis [55] davė pradžią dzeta ir L funkcijų universalumo teorijai. Jame buvo įrodyta, kad plati analizinių funkcijų klasė skritulyje, priklausančiame kritinės juostos $\left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$ dešiniajai pusei gali būti tam tikru tikslumu tolygiai aproksimuojama vertikaliais Rymano dzeta funkcijos postūmiais $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$.

A teorema. *Tarkime, kad $0 < r < \frac{1}{4}$, o funkcija $f(s)$ yra tolydi ir nevirstanti nuliu skritulyje $|s| \leqslant r$ bei analizinė to skritulio viduje. Tuomet kiekvienam $\varepsilon > 0$ egzistuoja skaičius $\tau = \tau(\varepsilon)$ toks, kad*

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Ši dzeta funkcijų savybė vadinama universalumu Voronino prasme arba tiesiog Voronino universalumu. Pastebėsime, kad funkcijos $\zeta(s)$ išraiškos Oilerio sandauga egzistavimas buvo esminis faktas, Voronino panaudotas šios teoremos įrodyme. Vėliau Voronino gautas rezultatas (A teorema) buvo pagerintas ir praplėstas (pavyzdžiui žr. [1, 9, 26, 52]).

B teorema. *Tarkime, kad* $K \subset D\big(\frac{1}{2},1\big)$ *yra kompaktinė aibė, turinti jungųjį papildinį, o funkcija* $f(s)$ *yra tolydi ir nelygi nuliui aibėje* K *bei analizinė aibės* K *viduje. Tada visiems* $\varepsilon > 0$ *yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

B teoremos įrodymas remiasi ribine teorema silpnojo tikimybinių matų konvergavimo terminais analizinių funkcijų erdvėje. Pastarąjį metodą pirmasis savo disertacijoje panaudojo B. Bagči (Bagchi) [1]. Vėliau jis buvo išplėtotas monografijose [26, 30, 52]. Tai reiškia, kad natūralus šios teorijos vystymo kelias yra nagrinėti dzeta ir L funkcijų, kurios turi Oilerio sandaugą, universalumo apibendrinimus.

2013 m. Ž. L. Mokleras (Mauclaire) [38] ir A. Laurinčikas su L. Meška [34], nepriklausomai vieni nuo kitų, universalumo teoremą suformulavo ir įrodė tankio terminais. Tokiu būdu B teoremoje apatinis tankis gali būti pakeistas tankiu ir teisingas yra toks tvirtinimas, kurį pateikiame toliau.

C teorema. *Tarkime, kad* K *ir* $f(s)$ *yra tokie pat, kaip B teoremoje. Tuomet riba*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0$$

egzistuoja su visais $\varepsilon > 0$*, išskyrus ne daugiau nei skaičią jų aibę.*

Pirmąją universalumo teoremą L funkcijoms iš Selbergo klasės $\mathcal{S}$ įrodė J. Štoidingas (Steuding) [52]. Tegul funkcijoms $L(s) \in \mathcal{S}$

$$\sigma_L = \max\left(\frac{1}{2}, 1 - \frac{1}{d_L}\right)$$

ir $D_L := D\big(\sigma_L, 1\big)$. Kartu su klasės $\mathcal{S}$ 4-ąja hipoteze buvo reikalaujama, kad egzistuotų polinominė Oilerio sandauga ir pirminių skaičių teoremos analogas, t. y. 5 ir 6 aksiomos.

$\mathcal{K}_L$ pažymėkime kompatiškų juostos D_L poaibių klasę su jungiaisiais papildiniais, o $H_{0L}(K)$, $K \in \mathcal{K}_L$, – tolydžių nelygių nuliui srityje K funkcijų, kurios yra analizinės K viduje, klasę. Taip pat $\widehat{\mathcal{S}}$ pažymėkime klasę funkcijų, kurios tenkina Selbergo klasės $\mathcal{S}$ hipotezes bei 5 ir 6 aksiomas. Tada yra teisingas toks teiginys (žr. [52]).

D teorema. *Tarkime, kad $L(s) \in \mathcal{S} \cap \widehat{\mathcal{S}}$. Tegul $K \in \mathcal{K}_L$ ir $f(s) \in H_{0L}$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |L(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Atkreipsime dėmesį, kad klasę $\mathcal{S} \cap \widehat{\mathcal{S}}$ sudaro visos funkcijos, kurios tenkina Selbergo klasės 2 ir 3 hipotezes bei 5 ir 6 aksiomas.

Vėliau D teoremoje buvo atsisakyta 6-osios aksiomos reikalavimo (žr. [42]). Tiksliau tariant, D teoremos tvirtinimas lieka teisingas funkcijoms $L(s) \in \mathcal{S}$, kurios tenkina tik 5-ąją aksiomą. O tai jau – pirmasis universalumo rezultatas L funkcijoms iš Selbergo-Štoidingo klasės $\widetilde{\mathcal{S}}$.

Aukščiau apžvelgtuose rezultatuose buvo pristatytas tolydusis universalumas. Dabar pereisime prie dar vienos Voronino universalumo krypties – dzeta ir L funkcijų diskretaus universalumo savybės.

Diskretus universalumas nagrinėja analizinių funkcijų aproksimavimą funkcijų postūmiais iš tam tikros diskrečios aibės. Pirmąjį šios krypties rezultatą dzeta funkcijoms gavo A. Reichas (Reich) [45]. Tegul $\mathbb{K}$ yra algebrinis skaičių kūnas. Dedekindo (Dedekind) dzeta funkcija $\zeta_{\mathbb{K}}(s)$ pusplokštumėje $\sigma > 1$ yra apibrėžta taip:

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \frac{1}{\text{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\text{N}(\mathfrak{p})^s}\right)^{-1};$$

čia sumuojama pagal visus nenulinius integralumo idealus, o sandauga skaičiuojama pagal visus sveikųjų skaičių žiedo $\mathbb{K}$ pirminius idealus ($\text{N}(\mathfrak{a})$ žymi idealo $\mathfrak{a}$ normą). Kai $\mathbb{K} = \mathbb{Q}$, Dedekindo dzeta funkcija $\zeta_{\mathbb{Q}}(s)$ virsta Rymano dzeya funkcija $\zeta(s)$. Vadinasi, Reicho įrodytas rezultatas yra B teoremos diskretus analogas, t. y. kompleksinio kintamojo menamosios dalys yra imamos iš aritmetinės progresijos $\{kh, k \in \mathbb{N}_0\}$ su $h > 0$.

E teorema. *Tarkime, kad K ir $f(s)$ yra tokie pat, kaip ir B teoremoje.*

Tada kiekvienam realiam skaičiui h, kuris nelygus nuliui, ir visiems $\varepsilon > 0$ yra teisinga nelygybė

$$\liminf_{N\to\infty} \frac{1}{N}\# \left\{0\leqslant k\leqslant N : \sup_{s\in K}|\zeta(s+ikh)-f(s)|<\varepsilon\right\} > 0.$$

Priminsime, kad visose diskretaus tipo universalumo teoremose poslinkio parametro aritmetinė prigimtis atlieka lemiamą vaidmenį.

D teoremos diskretų analogą įrodė A. Laurinčikas ir R. Macaitienė (žr. [33]).

F teorema. *Tegul $L(s)$, K ir $f(s)$ yra tokie pat, kaip D teoremoje. Tuomet su kiekvienu $h > 0$ ir $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \sup_{s\in K}|L(s+ikh)-f(s)|<\varepsilon\right\} > 0.$$

Voronino universalumas gali būti nagrinėjamas platesniame istoriniame dzeta ir L funkcijų tyrimų kontekste. Jis gali būti laikomas reikšmių pasiskirstymo ir funkcinio nepriklausomumo tyrimų kryptimi.

1910 m. H. Boras (Bohr) pasiūlė Rymano dzeta funkcijos reikšmių pasiskirstymo tyrimui panaudoti diofantinius, geometrinius ir tikimybinius metodus. 1911 m. jis gavo [5] tokį rezultatą apie funkcijos $\zeta(s)$ reikšmių dažnį.

G teorema. *Su kiekvienu $\delta > 0$ juostoje $1 < \sigma < 1 + \delta$ funkcija $\zeta(s)$ įgyja bet kurią nenulinę reikšmę be galo daug kartų.*

Vėliau H. Boras ir R. Kurantas (Courant) nagrinėjo $\sigma\leqslant 1$ atvejį ir įrodė [6], kad

H teorema. *Su kiekvienu fiksuotu σ, $\frac{1}{2} < \sigma\leqslant 1$, aibė $\{\zeta(\sigma+i\tau) : \tau \in \mathbb{R}\}$ yra visur tiršta aibėje $\mathbb{C}$.*

1972 m. S.M. Voroninas nagrinėjo daugiamatį atvejį ir pastarąjį rezultatą ženkliai apibendrino (žr. [53]).

I teorema. *Su visais fiksuotais skirtingais* $s_1, \dots, s_n$, $\frac{1}{2} < \Re s_i < 1$, $1 \leqslant i \leqslant n$ *ir* $s_k \neq s_l$ *su* $k \neq l$, *aibė*

$$\left\{ \left(\zeta(s_1 + i\tau), \dots, \zeta(s_n + i\tau) \right) : \tau \in \mathbb{R} \right\}$$

yra visur tiršta aibėje $\mathbb{C}^n$, $n \in \mathbb{N}$. *Kiekvienam fiksuotam* s, $\frac{1}{2} < \sigma < 1$, *aibė*

$$\left\{ \left(\zeta(s + i\tau), \zeta'(s + i\tau), \dots, \zeta^{(n-1)}(s + i\tau) \right) : \tau \in \mathbb{R} \right\}$$

yra visur tiršta aibėje $\mathbb{C}^n$.

Galima pastebėti, kad pastarasis teiginys yra glaudžiai susijęs su dzeta funkcijų, apibrėžiamų Dirichlė eilute, funkciniu nepriklausomumu. Šį klausimą iškėlė D. Hilbertas (Hilbert), 2-ojo Tarptautinio matematikų kongreso metu suformulavęs 23 svarbiausių XX a. problemų sąrašą (žr. [11]). Tačiau tik 1973 m. S. M. Voroninas įrodė [54], kad Rymano dzeta funkcija $\zeta(s)$ yra funkciškai nepriklausoma.

J teorema. *Funkcija* $\zeta(s)$ *netenkina jokios diferencialinės lygties*

$$\begin{aligned} s^m F_m\big(\zeta(s), \zeta'(s), \dots, \zeta^{(N-1)}(s)\big) + \dots \\ + F_0\big(\zeta(s), \zeta'(s), \dots, \zeta^{(N-1)}(s)\big) = 0, \end{aligned}$$

čia $F_0, \dots, F_m$ *– tolydžios funkcijos, iš kurių ne visos yra lygios nuliui,* $N \in \mathbb{N}$.

Vėliau dzeta ir L funkcijų funkcinio nepriklausomumo uždavinį sprendė daugelis matematikų (žr. [26, 37, 52]). Būtina paminėti, jog funkcinio nepriklausomumo ir tirštumo rezultatams bei universalumo savybei įrodyti gali būti taikomi tikimybiniai metodai. Ypač jie tapo patrauklūs po novatoriškos B. Bagči daktaro disertacijos [1], kurioje jis pasiūlė alternatyvų Voronino universalumo teoremos įrodymą. Pagrindinė pasiūlyto metodo naujovė – funkcinės ribinės teoremos silpnai konverguojančių tikimybinių matų terminais panaudojimas dzeta funkcijų tyrimui.

Dabar vėl grįšime prie universalumo savybės. Galima nagrinėti vienalaikį analizinių funkcijų aproksimavimą dzeta arba L funckijų rinkiniais. Tai – jungtinis universalumas. Šią Dirichlė eilučių savybę pirmasis taip pat atrado S.M. Voroninas. Jis, nagrinėdamas Dirichlė L funkcijų $L(s,\chi)$ funkcinį nepriklausomumą, panaudojo jungtinio universalumo savybę (žr. [54]).

Priminsime, kad Dirichlė L funkcijos $L(s,\chi)$, susietos su charakteriu $\chi \pmod d$, $d \in \mathbb{N}$, pusplokštumėje $\sigma > 1$ yra apibrėžiama taip:

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} = \prod_{p\in\mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

K teorema. *Tarkime, kad $\chi_1, \dots, \chi_n$ yra poromis neekvivalentūs Dirichlė charakteriai, o $L(s,\chi_1), \dots, L(s,\chi_n)$ – atitinkamos Dirichlė L funkcijos. Tegul K_j, $j = 1, \dots, n$, yra kompaktinis poaibis juostoje $D\left(\frac{1}{2}, 1\right)$, turintis jungųjį papildinį, o funkcija $f_j(s)$ yra tolydi, nevirstanti nuliu aibėje K_j ir analizinė aibės K_j viduje. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\left(\tau \in [0,T] : \sup_{1\leqslant j\leqslant n} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon\right) > 0.$$

Be abejo, jungtinis universalumas yra sudėtingesnis. Kita vertus, jis ženkliai įdomesnis. Natūralu, kad jungtinio universalumo atveju aproksimuojantiems poslinkiams reikalingos tam tikros nepriklausomumo sąlygos. Norint įrodyti jungtinio universalumo teoremas, dažnai naudojamos įvairios matricų sąlygos. Po aukščiau minėto Voronino rezultato, jungtinės universalumo teoremos buvo įrodytos dzeta funkcijoms, apibrėžtoms Dirichlė eilutėmis su periodiniais koeficientais, Matsumoto dzeta funkcijai, automorfinėms L funkcijoms (žr. apžvalginiame K. Matsumoto straipsnyje [37] arba A. Laurinčiko [26] ir J. Štoidingo [52] monografijose).

XXI a. pirmajame dešimtmetyje atsirado dar viena reikšminga jungtinio universalumo tyrimų šaka – jungtiniam vienalaikiam aproksimavimui naudojama pora, sudaryta iš dviejų skirtingų tipų dzeta ir L funkcijų (viena jų turi Oilerio sandaugą, kita – ne). Pirmuosius šio tipo rezultatus, nepriklausomai vieni nuo kitų, gavo J. Štoidingas su J. Sandersu (Sanders) [47] ir H. Mišu (Mishou) [40]. Jie įrodė, kad analizinių funkcijų pora vienu metu yra aproksimuojama poros $\big(\zeta(s), \zeta(s,\alpha)\big)$ postūmiais. Pateiksime Mišu gautą rezultatą.

L teorema. *Tarkime, kad α yra transcendentusis skaičius toks, kad $0 < \alpha < 1$. Tegul K_1 ir K_2 yra kompaktiniai juostos $D\big(\frac{1}{2}, 1\big)$ poaibiai su jungiaisiais papildiniais. Sakykime, kad funkcija $f_j(s)$ yra tolydi aibėje K_j ir analizinė aibės K_j viduje su kiekvienu $j = 1, 2$. Be to, tegul funkcija $f_1(s)$ yra nelygi nuliui aibėje K_1. Tada visiems teigiamiems ε yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\bigg\{\tau \in [0, T] : \max_{s\in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \\ \max_{s\in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon\bigg\} > 0.$$

Mišu įrodymas remiasi Bagči metodu, o esminis faktas įrodyme – tiesinis aibės $\{\log(m + \alpha) : m \in \mathbb{N}_0\} \cup \{\log p : p \in \mathbb{P}\}$ elementų nepriklausomumas virš $\mathbb{Q}$, kai α yra transcendetusis skaičius. Tuo tarpu Sanderis ir Štoidingas, naudodami kitokį metodą, įrodė analogišką rezultatą, kai α yra racionalusis skaičius.

Kaip ir Rymano dzeta funkcijos $\zeta(s)$ universalumo atveju, yra žinoma eilė abiejų tipų – tolydaus ir diskretaus – mišraus jungtinio aproksimavimo apibendrinimų (žr. R. Kačinskaitės ir K. Matsumoto apžvalginį straipsnį [19]).

Pirmąją mišraus funkcinio nepriklausomumo teoremą 2007 m. įrodė H. Mišu [40]. Jis parodė, kad Rymano dzeta funkcijos $\zeta(s)$ ir klasikinės Hurvico dzeta funkcijos $\zeta(s, \alpha)$ su transcendenčiuoju parametru α, $0 < \alpha < 1$, pora yra funkciškai nepriklausoma. Mišu tipo teoremą periodinei dzeta funkcijai $\zeta(s; \mathfrak{A})$ ir periodinei Hurvico dzeta funkcijai $\zeta(s, \alpha; \mathfrak{B})$ su transcendenčiuoju α įrodė R. Kačinskaitė ir A. Laurinčikas [14].

M teorema. *Tarkime, kad α yra transcendentusis skaičius ir visiems*

$p \in \mathbb{P}$ yra teisinga

$$\sum_{d=1}^{\infty} \frac{|a_{p^d}|}{p^{d/2}} \leqslant c < 1.$$

Tegul funkcijos $F_j : \mathbb{C}^{2N} \to \mathbb{C}$ yra tolydžios su kiekvienu $j = 0, 1, \dots, n$ ir $N \in \mathbb{N}$ bei

$$\sum_{j=0}^{n} s^j F_j \Big(\zeta(s; \mathfrak{A}), \zeta'(s; \mathfrak{A}), \dots, \zeta^{(N-1)}(s; \mathfrak{A}),$$
$$\zeta(s, \alpha; \mathfrak{B}), \zeta'(s, \alpha; \mathfrak{B}), \dots, \zeta^{(N-1)}(s, \alpha; \mathfrak{B}) \Big) \equiv 0.$$

Tada $F_j \equiv 0$ su $j = 0, 1, \dots, n$.

Kartu su funkcinio nepriklausomumo uždaviniu yra sprendžiamas $\zeta(s; \mathfrak{A})$ ir $\zeta(s, \alpha; \mathfrak{B})$ jungtinio tirštumo klausimas.

Disertacijoje gauti rezultatai aprėpia visas anksčiau apžvelgtas problemas. Todėl dabar pateiksime pagrindinius jos rezultatus.

2 skyriuje nagrinėjamas vienalaikis analiznių funkcijų rinkinio $\big(f_1(s), \dots, f_r(s)\big)$ aproksimavimas rinkinio $\big(L(s + ia_1\tau), \dots, L(s + ia_r\tau)\big)$ postūmiais juostoje $D\big(\sigma_L, 1\big)$; čia realieji algebriniai skaičiai $a_1, \dots, a_r$ yra tiesiškai nepriklausomi virš racionaliųjų skaičių kūno. Šio skyriaus rezultatas yra publikuotas [15] straipsnyje.

Kadangi mes nagrinėjame jungtinį vienalaikį aproksimavimą dzeta ir L fukcijomis, reikalingas tam tikras šių funkcijų nepriklausomumas. Mes panaudojame tokį A. Beikerio (Baker) rezutatą (žr. [2]).

2.1 lema. *Tarkime, kad algebrinių skaičių $\lambda_1, \dots, \lambda_r$ logaritmai $\log \lambda_1, \dots, \log \lambda_r$ yra tiesiškai nepriklausomi virš $\mathbb{Q}$. Tada visiems algebriniams skaičiams $\beta_0, \beta_1, \dots, \beta_r$, tuo pačiu metu ne visiems lygiems nuliui, yra teisinga nelygybė*

$$|\beta_0 + \beta_1 \log \lambda_1 + \ldots + \beta_r \log \lambda_r| > h^{-C};$$

čia h yra skaičių $\beta_0, \beta_1, \dots, \beta_r$ aukštis, C – konstanta priklausanti nuo r, $\lambda_1, \dots \lambda_r$ ir skaičių $\beta_0, \beta_1, \dots, \beta_r$ laipsnių maksimumo.

Tuomet pagrindinis 2 skyriaus rezultatas yra toks tvirtinimas.

2.1 teorema. *Tarkime, kad $L(s) \in \widetilde{S}$ ir algebriniai skaičiai $a_1, \ldots, a_r$ yra tiesiškai nepriklausomi virš racionaliųjų skaičių kūno $\mathbb{Q}$. Tegul $K_j \in \mathcal{K}_L$ ir $f_j(s) \in H_{0L}(K_j)$ su $j = 1, \ldots, r$. Tada visiems $\varepsilon > 0$ teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(s+ia_j\tau) - f_j(s)| < \varepsilon\right\} > 0.$$

Be to, $\liminf$ *galima pakeisti* $\lim$ *visiems $\varepsilon > 0$, išskyrus ne daugiau nei skaičią jų aibę.*

Į 2.1 teoremą galima žiūrėti kaip į daugiamatį D teoremos atvejį bei kaip į Voronino rezultato, pateikto K teoremoje, praplėtimą. Antroji teiginio dalis apibendrina C teoremą tankio terminais.

3 skyriuje nagrinėjame kitokį vienalaikio aproksimavimo atvejį nei 2 skyriuje. Čia mes pereiname prie diskretaus L funkcijų iš klasės $\widetilde{S}$ aproksimavimo. Tam naudojame multiaibės $A(\mathbb{P}, \underline{h}, 2\pi) := \big\{(h_j \log p : p \in \mathbb{P}), j = 1, \ldots, r; 2\pi\big\}$ tiesinį nepriklausomumą virš $\mathbb{Q}$; čia visi h_j yra teigiami skaičiai.

Tada teisingas toks tvirtinimas (publikuotas [16] straipsnyje).

3.1 teorema. *Tarkime, kad $L(s) \in \widetilde{S}$ ir aibė $A(\mathbb{P}, \underline{h}, 2\pi)$ yra tiesiškai nepriklausoma virš racionaliųjų skaičių kūno $\mathbb{Q}$. Tegul $K_j \in \mathcal{K}(D_L)$ ir $f_j(s) \in \mathcal{H}_0(K_j, D_L)$ su $j = 1, \ldots, r$. Tada visiems $\underline{h} \in (\mathbb{R}^+)^r$ ir $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - L(s+ikh_j)| < \varepsilon\right\} > 0.$$

Be to, visiems $\varepsilon > 0$, išskyrus ne daugiau nei skaičią jų aibę, egzistuoja riba

$$\lim_{N\to\infty} \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - L(s+ikh_j)| < \varepsilon\right\}$$

ir ji yra teigiama.

Kaip matome, pastarojo tvirtinimo pirmoji dalis apibendrina F teoremą apatinio tankio terminais funkcijai $L(s) \in \widetilde{S}$, o antroji dalis – tankio terminais – visiškai nauja.

4 disertacijos skyriuje gauti du rezultatai periodinei dzeta funkcijai $\zeta(s;\mathfrak{A})$. Mes nagrinėjame rinkinio $\big(\zeta(s;\mathfrak{A}_1), ..., \zeta(s;\mathfrak{A}_{r_1})\big)$, sudaryto iš r_1-os periodinės dzeta funkcijos, funkcinį nepriklausomumą ir tirštumą. Gautieji rezultatai užpildė buvusią spragą periodinių dzeta funkcijų tyrime. Jie yra publikuoti [13] straipsnyje.

Tegul $\mathfrak{A}_j = \{a_{jm} : m \in \mathbb{N}\}$ yra periodinė kompleksinių skaičių a_{jm} seka su mažiausiu periodu $k_j \in \mathbb{N}$, o $\zeta(s;\mathfrak{A}_j)$ – atitinkama periodinė dzeta funkcija su $j = 1, \dots, r_1$, $r_1 > 1$. Sakykime, kad $k = [k_1, \dots, k_{r_1}]$ yra mažiausias skaičių $k_1, \dots, k_{r_1}$ bendrasis kartotinis, o $\eta_1, \dots, \eta_{\varphi(k)}$ – redukuota liekanų moduliu k sistema (čia $\varphi(k)$ yra Oilerio funkcija).

Apibrėžkime matricą A, sudarytą iš periodinių sekų $\mathfrak{A}_j$ koeficientų, t. y.

$$A =: \begin{pmatrix} a_{1\eta_1} & a_{2\eta_1} & \dots & a_{r_1\eta_1} \\ a_{1\eta_2} & a_{2\eta_2} & \dots & a_{r_1\eta_2} \\ \dots & \dots & \dots & \dots \\ a_{1\eta_{\varphi(k)}} & a_{2\eta_{\varphi(k)}} & \dots & a_{r_1\eta_{\varphi(k)}} \end{pmatrix}.$$

4.1 teorema. *Tarkime, kad sekos $\mathfrak{A}_1, \dots, \mathfrak{A}_{r_1}$ yra multiplikatyvios,* $\operatorname{rank}(A) = r_1$ *ir su visais $p \in \mathbb{P}$ galioja nelygybės*

$$\sum_{d=1}^{\infty} \frac{|a_{jp^d}|}{p^{d/2}} \leqslant c_j < 1, \quad j = 1, \dots, r_1.$$

Tegul su kiekvienu $g = 0, 1, \dots, n$ funkcijos $F_g : \mathbb{C}^{Nr_1} \to \mathbb{C}$ yra tolydžios ir $n, N \in \mathbb{N}$, o

$$\sum_{g=0}^{n} s^g F_g\Big(\zeta(s;\mathfrak{A}_1), \zeta'(s;\mathfrak{A}_1), \dots, \zeta^{(N-1)}(s;\mathfrak{A}_1), \dots,$$
$$\zeta(s;\mathfrak{A}_{r_1}), \zeta'(s;\mathfrak{A}_{r_1}), \dots, \zeta^{(N-1)}(s;\mathfrak{A}_{r_1})\Big) \equiv 0.$$

Tada $F_g \equiv 0$ *su* $g = 0, 1, \ldots, n$.

Antrasis šio skyriaus rezultatas skirtas ką tik aprašyto rinkinio tirštumui.

Apibrėžime atvaizdį $\mu : \mathbb{R} \to \mathbb{C}^{Nr_1}$ tokia formule:

$$\mu(t) = \Big(\zeta(\sigma + it; \mathfrak{A}_1), \zeta'(\sigma + it; \mathfrak{A}_1), \ldots, \zeta^{(N-1)}(\sigma + it; \mathfrak{A}_1), \ldots,$$
$$\zeta(\sigma + it; \mathfrak{A}_{r_1}), \zeta'(\sigma + it; \mathfrak{A}_{r_1}), \ldots, \zeta^{(N-1)}(\sigma + it; \mathfrak{A}_{r_1})\Big)$$

su $\frac{1}{2} < \sigma < 1$.

4.2 teorema. *Tarkime, kad visos hipotezės sekų* $\mathfrak{A}_j$ *koeficientams* a_{jm}, $j = 1, \ldots, r_1$, *ir* $\operatorname{rank}(A)$ *yra tokios kaip ir 4.1 teoremoje. Tada aibės* $\mathbb{R}$ *vaizdas* μ *yra tirštas aibėje* $\mathbb{C}^{Nr_1}$.

Paskutiniame, 5 skyriuje, įrodyti trys teiginiai mišraus tipo multirinkiniams sudarytiems iš periodinių dzeta ir periodinių Hurvico dzeta funkcijų rinkinių.

Sakykime, kad l_j yra natūralusis skaičius, $j = 1, ..., r$, $\mathfrak{B}_{jl} = \{b_{mjl} : m \in \mathbb{N}_0\}$ – periodinė kompleksinių skaičių seka su mažiausiu periodu $k_{jl} \in \mathbb{N}$, α_j – realus skaičius, $0 < \alpha_j \leqslant 1$, $\zeta(s, \alpha_j; \mathfrak{B}_{jl})$ yra atitinkama periodinė Hurvico dzeta funkcija, $j = 1, ..., r$, $l = 1, ..., l_j$. Tegul $\kappa = l_1 + ... + l_r$, o k yra periodų $k_{11}, ..., k_{1_{1l_1}}, ..., k_{r1}, ..., k_{rl_r}$ mažiausias bendras kartotinis. Apibrėžkime matricą

$$B := \begin{pmatrix} b_{111} & b_{112} & ... & b_{11l_1} & ... & b_{1r1} & b_{1r2} & ... & b_{1rl_r} \\ b_{211} & b_{212} & ... & b_{21l_1} & ... & b_{2r1} & b_{2r2} & ... & b_{2rl_r} \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ b_{k11} & b_{k12} & ... & b_{k1l_1} & ... & b_{kr1} & b_{kr2} & ... & b_{krl_r} \end{pmatrix}.$$

Šiame skyriuje nagrinėjamas rinkinys, sudarytas iš r_1-os periodinės dzeta funkcijos ir κ skaičiaus periodinių Hurvico dzeta funkcijų. Kitaip tariant, į vieną bendrą teiginį sujungiame 4.1 teoremą ir N teoremą (pastarąją rasite 5.1 skyrelyje).

5.1 teorema. *Tarkime, kad sekos* $\mathfrak{A}_1, \ldots, \mathfrak{A}_{r_1}$ *yra multiplikatyvios,* $\text{rank}(A) = r_1$ *ir galioja 4.1 teoremos nelygybė. Tegul* $1, \alpha_1, ..., \alpha_r$ *yra algebriškai nepriklausomi virš* $\mathbb{Q}$*, o* $\text{rank}(B) = \kappa$*. Tarkime, kad funkcijos* $F_h : \mathbb{C}^{N(r_1+\kappa)} \to \mathbb{C}$ *yra tolydžios su kiekvienu* $h = 0, 1, ..., n$*, o funkcija*

$$\begin{aligned} &G(s) \\ &= \sum_{h=0}^{n} s^h F_h\Big(\zeta(s;\mathfrak{A}_1), \zeta'(s;\mathfrak{A}_1), \ldots, \zeta^{(N-1)}(s;\mathfrak{A}_1), \ldots, \\ &\qquad \zeta(s;\mathfrak{A}_{r_1}), \zeta'(s;\mathfrak{A}_{r_1}), \ldots, \zeta^{(N-1)}(s;\mathfrak{A}_{r_1}), \\ &\qquad \zeta(s,\alpha_1;\mathfrak{B}_{11}), \zeta'(s,\alpha_1;\mathfrak{B}_{11}), ..., \zeta^{(N-1)}(s,\alpha_1;\mathfrak{B}_{11}), ..., \\ &\qquad \zeta(s,\alpha_1;\mathfrak{B}_{1l_1}), \zeta'(s,\alpha_1;\mathfrak{B}_{1l_1}), ..., \zeta^{(N-1)}(s,\alpha_1;\mathfrak{B}_{1l_1}), ..., \\ &\qquad \zeta(s,\alpha_r;\mathfrak{B}_{r1}), \zeta'(s,\alpha_r;\mathfrak{B}_{r1}), ..., \zeta^{(N-1)}(s,\alpha_r;\mathfrak{B}_{r1}), ..., \\ &\qquad \zeta(s,\alpha_r;\mathfrak{B}_{rl_r}), \zeta'(s,\alpha_r;\mathfrak{B}_{rl_r}), ..., \zeta^{(N-1)}(s,\alpha_r;\mathfrak{B}_{rl_r})\Big) \end{aligned}$$

yra tapačiai lygi nuliui. Tada $F_h \equiv 0$ *su* $h = 1, ..., n$.

Mišrioji jungtinė universalumo teorema funkcijoms $\zeta(s;\mathfrak{A}_j)$, $j = 1, ..., r_1$, ir $\zeta(s,\alpha_j;\mathfrak{B}_{jl})$, $j = 1, ..., r$, $l = 1, ..., l_j$, yra pagrindinė įrodant funkcinio nepriklausomumo ir tirštumo rezultatus. Tai yra svarbiausias šio skyriaus teiginys.

5.2 teorema. *Tarkime, kad sekos* $\mathfrak{A}_1, ..., \mathfrak{A}_{r_1}$ *yra multiplikatyvios,* $\text{rank}(A) = r_1$ *ir galioja 4.1 teoremos nelygybė. Tegul* $1, \alpha_1, ..., \alpha_r$ *yra algebriškai nepriklausomi virš kūno* $\mathbb{Q}$*, o* $\text{rank}(B) = \kappa$*. Tarkime, kad* $f_1(s), ..., f_{r_1}(s)$ *yra tolydžios, nelygios nuliui funkcijos atitinkamai aibėse* $K_1, \ldots, K_{r_1}$ *ir analizinės jų viduje. Tegul* $f_{jl}(s)$ *yra tolydi funkcija aibėje* K_{jl} *ir analizinė aibės* K_{jl} *viduje su visais* $j = 1, ..., r$*,*

$l = 1, ..., l_j$. *Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Bigg(\tau \in [0,T] : \sup_{1\leqslant j\leqslant r_1}\sup_{s\in K_j}|\zeta(s+i\tau;\mathfrak{A}_j)-f_j(s)| < \varepsilon, \sup_{1\leqslant j\leqslant r}\sup_{1\leqslant l\leqslant l_j}\sup_{s\in K_{jl}}|\zeta(s+i\tau,\alpha_j;\mathfrak{B}_{jl})-f_{jl}(s)| < \varepsilon\Bigg) > 0.$$

Trečiasis 5 skyriuje įrodytas teiginys yra ši bendra tirštumo lema.

5.3 teorema. *Tarkime, kad galioja visos 5.2 teoremos sąlygos. Tada vaizdas $h(\mathbb{R})$, apibrėžtas formule*

$$\begin{aligned} h(t) &:= \Big(\zeta(\sigma+it;\mathfrak{A}_1), \zeta'(\sigma+it;\mathfrak{A}_1),\ldots, \zeta^{(N-1)}(\sigma+it;\mathfrak{A}_1),\ldots,\\ &\zeta(\sigma+it;\mathfrak{A}_{r_1}), \zeta'(\sigma+it;\mathfrak{A}_{r_1}),\ldots, \zeta^{(N-1)}(\sigma+it;\mathfrak{A}_{r_1}),\\ &\zeta(\sigma+it,\alpha_1;\mathfrak{B}_{11}), \zeta'(\sigma+it,\alpha_1;\mathfrak{B}_{11}),\ldots, \zeta^{(N-1)}(\sigma+it,\alpha_1;\mathfrak{B}_{11}),\ldots,\\ &\zeta(\sigma+it,\alpha_1;\mathfrak{B}_{1l_1}), \zeta'(\sigma+it,\alpha_1;\mathfrak{B}_{1l_1}),\ldots, \zeta^{(N-1)}(\sigma+it,\alpha_1;\mathfrak{B}_{1l_1}),\ldots,\\ &\zeta(\sigma+it,\alpha_1;\mathfrak{B}_{r1}), \zeta'(\sigma+it,\alpha_1;\mathfrak{B}_{r1}),\ldots, \zeta^{(N-1)}(\sigma+it,\alpha_1;\mathfrak{B}_{r1}),\ldots,\\ &\zeta(\sigma+it,\alpha_1;\mathfrak{B}_{rl_r}), \zeta'(\sigma+it,\alpha_1;\mathfrak{B}_{rl_r}),\ldots, \zeta^{(N-1)}(\sigma+it,\alpha_1;\mathfrak{B}_{rl_r})\Big), \end{aligned}$$

yra tirštas aibėje $\mathbb{C}^{N(r_1+\kappa)}$.

5 skyriaus rezultatai yra publikuoti [13] straipsnyje.

Išvados

Disertacijoje pagrindinis dėmesys skiriamas plačių analizinių funkcijų klasių vienalaikiam aproksimavimui L funkcijų, priklausančių Selbergo-Štoidingo klasei $\widetilde{\mathcal{S}}$, postūmiais bei mišriam vienalaikiam aproksimavimui periodinių dzeta ir periodinių Hurvico dzeta funkcijų rinkiniais. Taip pat pastarųjų rezultatų taikymams. Todėl galima daryti tokias išvadas.

1. L funkcijų, priklausančių klasei $\widetilde{\mathcal{S}}$, rinkiniui $\big(L(s + ia_1\tau),$ $..., L(s + ia_r\tau)\big)$ galioja jungtinė tolydi universalumo teorema, kai realieji algebriniai skaičiai $a_1, ..., a_r$ yra tiesiškai nepriklausomi virš racionaliųjų skaičių lauko.

2. Rinkinio $\big(L(s + ia_1\tau), ..., L(s + ia_r\tau)\big)$ postūmių, kuriais kompaktiškose aibėse $\varepsilon > 0$ tikslumu yra aproksimuojamas duotas analizinių funkcijų rinkinys, aibė turi teigiamą tankį visiems ε, išskyrus ne daugiau nei skaičią jų aibę.

3. Pirmosios dvi išvados galioja diskrečių postūmių atveju, kai naudojama multiaibės $\big\{(h_j \log p : p \in \mathbb{P}), j = 1, \dots, r; 2\pi\big\}$ elementų tiesinio nepriklausomumo virš $\mathbb{Q}$ sąlyga; čia h_j yra teigiami skaičiai.

4. Iš r_1 periodinių dzeta funkcijų sudarytas rinkinys $\big(\zeta(s; \mathfrak{A}_1), ...,$ $\zeta(s; \mathfrak{A}_{r_1})\big)$ išlaiko funkcinį nepriklausomumą ir yra tirštas, kai matricos, sudarytos iš periodinių sekų koeficientų, rangas ir patys koeficientai tenkina tam tikras sąlygas.

5. Periodinių dzeta ir periodinių Hurvico dzeta funkcijų multirinkiniams galioja tolydaus mišraus vienalaikio aproksimavimo savybė. Be to, turi būti išpildytos šios sąlygos: iš funkcijų koeficientų sudarytų atitinkamų matricų rangai yra $\operatorname{rank}(A) = r_1$ bei $\operatorname{rank}(B) = \kappa$, periodinių Hurvico dzeta funkcijų parametrai $1, \alpha_1, ..., \alpha_r$ – algebriškai nepriklausomi virš $\mathbb{Q}$ bei tam tikra sąlyga periodinių dzeta funkcijų koeficientams.

6. 5-oje išvadoje aprašytu atveju rinkinį sudarančios dzeta funkcijos yra funkciškai nepriklausomos ir jos reikšmių aibė yra tiršta.

Aprobacija

Disertacijos rezultatai buvo pristatyti šiose konferencijose:

1. B. Žemaitienė. Du rezultatai susiję su dzeta funkcijų universalumu. 11-oji jaunųjų mokslininkų konferencija „Fizinių ir technologijos mokslų tarpdalykiniai tyrimai“ (Vilnius, Lietuva). 2023 m. kovo 23 d. Stendinis pranešimas.

2. B. Žemaitienė. On joint universality in the Selberg-Steuding class. 26th International Conference on Mathematical Modelling and

Analysis (Jurmala, Latvia). May 30 – June 2, 2023. Žodinis pranešimas.

3. B. Žemaitienė. Jungtinis universalumas Selbergo-Štoidingo klasėje. Lietuvos matematikų draugijos LXIV Konferencija (Vilnius, Lietuva). 2023 m. birželio 22–23 d. Žodinis pranešimas.

4. B. Žemaitienė. Joint discrete universality in the Selberg-Steuding class. International Conference on Probability Theory and Number Theory (Palanga, Lithuania). September 10–16, 2023. Žodinis pranešimas.

5. B. Žemaitienė. Apie jungtinį aproksimavimą dzeta funkcijų klasėmis. Vytauto Didžiojo universitetas, Informatikos fakulteto mokslinis seminaras. 2024 m. gegužės 3 d. Žodinis pranešimas.

Publikacijos

Disertacijos rezultatai yra publikuoti šiuose straipsniuose:

1. R. Kačinskaitė, B. Kazlauskaitė, Two results related to the universality of zeta-functions with periodic coefficients, *Results Math.* **73**(3) (2018), 1–19.

2. R. Kačinskaitė, A. Laurinčikas, B. Žemaitienė, On joint universality in the Selberg–Steuding class, *Mathematics* **11** (2023), 737.

3. R. Kačinskaitė, A. Laurinčikas, B. Žemaitienė, Joint discrete universality in the Selberg–Steuding class, *Axioms* **12**(7) (2023), 674.

Taip pat konferencijų pranešimų santraukose:

1. B. Žemaitienė. On joint universality in the Selberg-Steuding class. Abstracts of 26th International Conference on Mathematical Modelling and Analysis: Jurmala, Latvia, May 30 – June 2. University of Latvia: Riga, 2023, p. 69. ISBN-978-9934-36-01-4

2. B. Žemaitienė. Jungtinis universalumas Selbergo-Štoidingo klasėje. Abstracts of Lietuvos matematikų draugijos LXIV konferencija: Vilnius, Lithuania, June 21–22. Vilniaus universiteto leidykla: Vilnius, 2023, p. 20. ISBN 978-609-07-0887-3.

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Main publications by the author

1st publication

Two results related to the universality of zeta-functions with periodic coefficients

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Results in Mathematics. ISSN: 1422-6383, 1420-9012/e. Vol. 73, No. 3, Article no. 95, 21 p. (2018).

2nd publication

On joint universality in the Selberg-Steuding class

R. Kačinskaitė, A. Laurinčikas, and **B. Žemaitienė**

Mathematics. ISSN: 2227-7390. Vol. 11, Issue 3, Article no. 737, 15 p. (2023).

3rd publication

Joint discrete universality in the Selberg-Steuding class

R. Kačinskaitė, A. Laurinčikas, and **B. Žemaitienė**

Axioms. ISSN: 2075-1680. Vol. 12, Article no. 674, 14 p. (2023).

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