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# Asymptotic analysis of heavy tailed distributions

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mathematics N001

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VILNIAUS UNIVERSITETAS

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# Sunkiauodegių skirstinių asimptotinė analizė

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# Contents

# Chapter 1

# Introduction

## 1.1 Scope and relevance of the problem

Investigation of various classes of heavy-tailed distributions attracted intense attention from theoreticians as well as practitioners because of their use in such fields as insurance, communication networks, economics, physics, etc.

The most researched subclass of heavy-tailed distributions is regularly varying distributions. This class was introduced by Karamata [35] in the context of real analysis. The notion of regular variation was introduced in probability theory by Feller [28] when considering limit theorems for sums of i.i.d. r.v.s. Many analytical results on regularly varying functions can be found in the monograph by Bingham et al. [4]. Some applications of regularly varying distributions to finance and insurance are presented by Embrechts et al. [25].

Besides regularly varying distributions, a number of classes, such as subexponential, consistently varying, long-tailed, dominatedly varying distributions, became standard in recent studies of heavy-tailed distributions. For recent books dealing with the classes of heavy-tailed distributions and their properties we mention Foss et al. [29] and Konstantinides [37]. The class of consistently varying distributions was introduced as a generalization of the regularly varying distribution class in [14], and was named there as a class of distributions with "intermediate regular variation". The concept of consistent variation has been used in various papers in the context of applied probability, such as queueing systems, graph theory and ruin theory, see e.g. [2, 5–9, 15, 29, 38, 45, 69].

In particular, an important question studied in many recent papers is closure property of heavy-tailed and related distribution classes. The closure property says that, assuming two or more distributions in some specific class, their transformation (e.g., sum-convolution, product-convolution, mixture, minima, maxima) belongs to the same class of distributions. This property is not only an auxiliary tool in proving various asymptotic results related to heavy-tailed distributions, but also interesting mathematical problem itself. The 'converse' problem to the convolution closure is the so-called convolution-root problem, which raises the question whether inclusion of convolution-power to the certain class of distributions yields the inclusion to the same class of the primary distribution. For important studies of closure properties of heavy-tailed distributions we mention Embrechts et al. [24] and Cline and Samorodnitsky [15].

Other interesting problems related to heavy-tailed distributions include asymptotic behavior of various transformations of r.v.s. Among such types of problems is asymptotic behavior of the left truncated moments of random sums. It was considered in various fields of applied probability, including risk theory and random walks [17, 18, 51]. In addition, this quantity is closely related with the Haezendonck-Goovaerts risk measure, see for instance [32, 39, 61] and [62].

## 1.2 Aim and objectives of research

The aim of this work is to derive asymptotic properties of heavy tailed distribution classes. To do this, the following objectives are:

(i) Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables, and $\eta$ be a counting random variable independent of this sequence. We consider conditions for random variables $\{\xi_1, \xi_2, \ldots\}$ and $\eta$ under which distribution function of random sum $S_\eta = \xi_1 + \xi_2 + \ldots + \xi_\eta$ belongs to the class of consistently varying distributions. In our consideration random variables $\{\xi_1, \xi_2, \ldots\}$ are not necessarily identically distributed.

(ii) Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued, possibly nonidentically distributed, random variables, and let $\eta$ be a non-negative, nondegenerate at 0, and integer-valued random variable, which is independent of $\{\xi_1, \xi_2, \ldots\}$. We consider conditions for $\{\xi_1, \xi_2, \ldots\}$ and $\eta$ under which the distributions of the randomly stopped minimum, maximum, and sum are regularly varying.

(iii) We consider the sum $S_n = \xi_1 + \ldots + \xi_n$ of possibly dependent and non-identically distributed real-valued random variables $\xi_1, \ldots, \xi_n$ with consistently varying distributions. By assuming that collection $\{\xi_1, \ldots, \xi_n\}$ follows the dependence structure, similar to the asymptotic independence, we obtain the asymptotic relations for $\mathbb{E}\left((S_n)^\alpha \mathbb{I}_{\{S_n > x\}}\right)$ and $\mathbb{E}\left(\left(S_n - x\right)^+\right)^\alpha$, where $\alpha$ is an arbitrary nonnegative real number.

## 1.3 Methodology of research

The main results of the thesis are proved using the classical methods of probability theory and mathematical analysis.

## 1.4 Conferences

The results obtained during the preparation of the thesis were presented in following conferences:

- European Actuarial Journal Conference, 22-24 August 2022, Tartu, Estonia.
- The Conference of Lithuanian Mathematical Society, 20-21 June 2019, Vilnius, Lithuania.
- The Conference of Young Scientists, 12 March 2019, Vilnius, Lithuania.

## 1.5 Main publications

The thesis is prepared based on three publications, which are published in journal indexed in Clarivate Analytics Web of Science:

- Edita Kizinevič, Jonas Sprindys, Jonas Šiaulys. Randomly stopped sums with consistently varying distributions. *Modern Stochastics: Theory and Applications* (2016).

- Jonas Sprindys, Jonas Šiaulys. Regularly distributed randomly stopped sum, minimum, and maximum. *Nonlinear Analysis: Modelling and Control* (2020).

- Jonas Sprindys, Jonas Šiaulys. Asymptotic formulas for the left truncated moments of sums with consistently varying distributed increments. *Nonlinear Analysis: Modelling and Control* (2021).

## 1.6 Other publications

During the preparation of the thesis some other publications, which are published in journals indexed in Clarivate Analytics Web of Science, were issued as well:

- Olga Navickienė, Jonas Sprindys, Jonas Šiaulys. Gerber-Shiu discounted penalty function for the bi-seasonal discrete time risk model. *Informatica* (2018).

- Olga Navickienė, Jonas Sprindys, Jonas Šiaulys. Ruin probability for the bi-seasonal discrete time risk model with dependent claims. *Modern Stochastics: Theory and Applications* (2019).

# Chapter 2

# Review on heavy tailed distributions

In this chapter, we recall the definitions of some classes of heavy-tailed distribution functions (d.f.s). We remark only that the tail function $\overline{F}(x) = 1 - F(x)$ for all real $x$ and an arbitrary d.f. $F$.

• *A d.f. $F$ is heavy-tailed ($F \in \mathcal{H}$) if for every fixed $\delta > 0$*

$$\limsup_{x\to\infty} \overline{F}(x)\mathrm{e}^{\delta x} = \infty.$$

• *A d.f. $F$ is long-tailed ($F \in \mathcal{L}$) if for every $y$ (equivalently, for some*

*$y > 0$)*

$$\overline{F}(x+y) \sim \overline{F}(x).$$

• *A d.f. $F$ has dominatingly varying tail ($F \in \mathcal{D}$) if for every fixed $y \in (0,1)$*

*(equivalently, for some $y \in (0,1)$)*

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

• *A d.f. $F$ has consistently varying tail ($F \in \mathcal{C}$) if*

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

• *A d.f. $F$ has regularly varying tail with index $\gamma \geqslant 0$, written as $F \in \mathcal{R}_\gamma$, if, for any $y > 0$, it holds that*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\gamma}.$$

• *A d.f. $F$ supported on the interval $[0,\infty)$ is subexponential ($F \in \mathcal{S}$) if*

$$\lim_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} = 2. \tag{2.0.1}$$

*If d.f. $G$ is supported on $\mathbb{R}$, then we suppose that d.f. $G$ is subexpo-
nential ($G \in \mathcal{S}$) if d.f. $F(x) = G(x)\mathbb{1}_{[0,\infty)}(x)$ satisfies relation (2.0.1).*

It is known (see, e.g., [12], [26], [36], and Chapters 1.4 and A3 in [25]) that these classes satisfy the following inclusions:

$$\mathcal{R} := \bigcup_{\gamma\geqslant 0} \mathcal{R}_\gamma \subset \mathcal{C} \subset \mathcal{L}\cap\mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}, \quad \mathcal{D} \subset \mathcal{H}.$$

These inclusions are depicted in Figure 2.1.

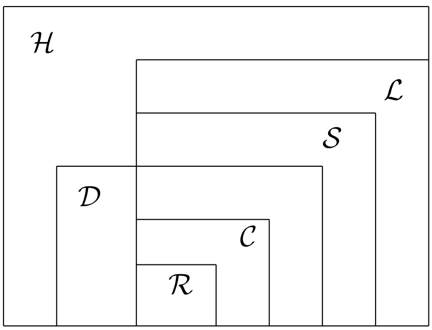


Figure 2.1: Classes of heavy-tailed distributions.

The following two indices are important to the determination whether d.f. $F$ belongs to the aforementioned heavy-tailed distribution classes. The first index is the so-called *upper Matuszewska index* (see, e.g., [4, Section 2], [15, 46]), defined as

$$J_F^+ = \inf_{y>1} \left\{ -\frac{1}{\log y} \log \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right\}.$$

Another index, so-called *L-index*, is defined as

$$L_F = \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

This index was used by [33, 42, 74], among others.

The definitions of the aforementioned heavy-tailed distribution classes imply that

$$\begin{aligned} F \in \mathcal{D} \quad &\Leftrightarrow \quad J_F^+ < \infty \quad \Leftrightarrow \quad L_F > 0, \\ F \in \mathcal{C} \quad &\Leftrightarrow \quad L_F = 1, \\ F \in \mathcal{R}_\gamma \quad &\Rightarrow \quad L_F = 1,\ J_F^+ = \gamma. \end{aligned}$$

In the thesis, we further consider the d.f.s from classes $\mathcal{R}$ and $\mathcal{C}$.

The class $\mathcal{R}$ was introduced by Karamata [35] in the context of real analysis. The notion of regular variation was introduced in probability theory by Feller [28] when considering limit theorems for sums of independent identically distributed (i.i.d.) random variables (r.v.s). Many analytical results on regularly varying functions can be found in the monograph by Bingham et al. [4]. Some applications of regularly varying distributions to finance and insurance are presented by Embrechts et al. [25].

An important property following from the definition of $\mathcal{R}_\gamma$ is that the tail function of an arbitrary regularly varying d.f. can be represented in the form $\overline{F}(x) = x^{-\gamma}L(x)$, where $L$ is a *slowly varying function*, that is,

$$\lim_{x\to\infty} \frac{L(xy)}{L(x)} = 1$$

for any $y > 0$.

The class $\mathcal{C}$ of consistently varying distributions was introduced as a generalization of the class $\mathcal{R}$ in [14], and was named there as a class of distributions with "intermediate regular variation". The concept of consistent variation has been used in various papers in the context of applied probability, such as queueing systems, graph theory and ruin theory, see e.g. [2, 5–9, 15, 29, 38, 45, 69].

# Chapter 3

# Randomly stopped sums with consistently varying distributions

## 3.1 Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent r.v.s with d.f.s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let $\eta$ be a counting r.v., i.e. $\eta$ be integer-valued, nonnegative and not-degenerate at zero. In addition, suppose that r.v. $\eta$ and r.v.s $\{\xi_1, \xi_2, \ldots\}$ are independent. Let $S_0 = 0$, $S_n = \xi_1+\xi_2+\ldots+\xi_n$ for $n \in \mathbb{N}$ and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the randomly stopped sum of r.v.s $\{\xi_1, \xi_2, \ldots\}$.

In this chapter, we are interested in conditions under which d.f. of $S_\eta$

$$F_{S_\eta}(x) = \mathbb{P}(S_\eta \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n \leqslant x) \tag{3.1.1}$$

belongs to the class of consistently varying distributions.

Throughout this chapter, $f(x) = o\,(g(x))$ means that $\lim_{x\to\infty} f(x)/g(x) = 0$ and $f(x) \sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$ for arbitrary two vanishing (at infinity) functions $f$ and $g$. Also, we denote the support of counting r.v. $\eta$ by

$$\operatorname{supp}(\eta) := \{n \in \mathbb{N}_0 : \mathbb{P}(\eta = n) > 0\}.$$

There exist many results about sufficient or necessary and sufficient conditions in order that the d.f. of randomly stopped sum (3.1.1) belongs to some heavy-tailed distribution class. Here we present a few known results concerning the belonging of d.f. $F_{S_\eta}$ to some class. The first result about subexponential distributions was proved by Embrechts and Goldie (see Theorem 4.2 in [23]) and Cline (see Theorem 2.13 in [13]).

**Theorem 3.1.1.** *Let $\{\xi_1, \xi_2, \ldots\}$ be independent copies of a nonnegative r.v. $\xi$ with subexponential d.f. $F_\xi$. Let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbb{E}(1+\delta)^\eta < \infty$ for some $\delta > 0$, then d.f. $F_{S_\eta} \in \mathcal{S}$.*

Similar results for class $\mathcal{D}$ can be found in paper of Leipus and Šiaulys [40]. Below we present the assertion of Theorem 5 from this work.

**Theorem 3.1.2.** *Let $\{\xi_1, \xi_2, \ldots\}$ be i.i.d. nonnegative r.v.s with common d.f. $F_\xi \in \mathcal{D}$ and finite mean $\mathbb{E}\xi$. Let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ with d.f. $F_\eta$ and finite mean $\mathbb{E}\eta$. Then d.f. $F_{S_\eta} \in \mathcal{D}$ iff $\min\{F_\xi, F_\eta\} \in \mathcal{D}$.*

We recall only (see pages 12-13), that d.f. $F$ belongs to the class $\mathcal{D}$ if and only if the upper Matuszewska index $J_F^+ < \infty$, where, by definition,

$$J_F^+ = -\lim_{y\to\infty} \frac{1}{\log y} \log\left(\liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}\right).$$

The random convolution closure for the class $\mathcal{L}$ was considered, for instance, in [1], [40], [70] and [71]. Below we present the particular assertion of Theorem 1.1 from [71].

**Theorem 3.1.3.** *Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s and $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ with d.f. $F_\eta$. Then d.f. $F_{S_\eta} \in \mathcal{L}$ if* *the following five conditions are satisfied:*

$$\begin{aligned}
&\text{(i)} && \mathbb{P}(\eta \geqslant \kappa) > 0 \ for\ some\ \kappa \in \mathbb{N};\\
&\text{(ii)} && for\ all\ k \geqslant \kappa\ d.f.\ F_{S_k}\ of\ sum\ S_k\ is\ long\ tailed;\\
&\text{(iii)} && \sup_{k\geqslant 1}\sup_{x\in\mathbb{R}} \left(F_{S_k}(x) - F_{S_k}(x-1)\right)\sqrt{k} < \infty;\\
&\text{(iv)} && \limsup_{z\to\infty}\sup_{k\geqslant\kappa}\sup_{x\geqslant k(z-1)+z} \frac{\overline{F}_{S_k}(x-1)}{\overline{F}_{S_k}(x)} = 1;\\
&\text{(v)} && \overline{F}_{\eta}(ax) = o\left(\sqrt{x}\,\overline{F}_{S_\kappa}(x)\right)\ for\ each\ a>0.
\end{aligned}$$

We observe that the case of identically distributed r.v.s is considered in Theorems 3.1.1 and 3.1.2. In Theorem 3.1.3 r.v.s $\{\xi_1, \xi_2, \ldots\}$ are independent, but not necessary identically distributed. Similar result, but for r.v.s having d.f.s with dominatingly varying tails can be found in paper [21]. Below we present Theorem 2.1 from this article.

**Theorem 3.1.4.** *Let r.v.s $\{\xi_1, \xi_2, \ldots\}$ be nonnegative independent, but not necessary identically distributed, and $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then d.f $F_{S_\eta}$ belongs to the class $\mathcal{D}$ if the following three conditions are satisfied:*

$$\begin{aligned}
&\text{(i)} && F_{\xi_\kappa} \in \mathcal{D}\ \ for\ some\ \kappa \in \operatorname{supp}(\eta),\\
&\text{(ii)} && \limsup_{x\to\infty}\sup_{n\geqslant\kappa} \frac{1}{n\overline{F}_{\xi_\kappa}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty,\\
&\text{(iii)} && \mathbb{E}\eta^{p+1} < \infty\ \ for\ some\ \ p > J^{+}_{F_{\xi_\kappa}}.
\end{aligned}$$

In this work, we consider randomly stopped sums of independent and not necessarily identically distributed r.v.s. As was noted above, we restrict our consideration on the class $\mathcal{C}$. If r.v.s $\{\xi_1, \xi_2, \ldots\}$ are not identically distributed then different collections of conditions on r.v.s $\{\xi_1, \xi_2, \ldots\}$ and $\eta$ imply that $F_{S_\eta} \in \mathcal{C}$. We suppose that some r.v.s from $\{\xi_1, \xi_2, \ldots\}$ have distributions belonging to the class $\mathcal{C}$, and we find minimal conditions for r.v.s $\{\xi_1, \xi_2, \ldots\}$ and $\eta$ in order that distribution of the randomly stopped sum $S_\eta$ remains in the same class. It should be noted that we use methods developed in papers [21] and [22].

The rest of the chapter is organized as follows. In Section 3.2, we present our main results together with two examples of randomly stopped sums $S_\eta$ with d.f.s having consistently varying tails. Section 3.3 is a collection of auxiliary lemmas, and the proofs of the main results are presented in Section 3.4.

## 3.2 Main results

In this section, we present three assertions in which we describe the belonging of a randomly stopped sum to the class $\mathcal{C}$. In the conditions of Theorem 3.2.1 counting r.v. $\eta$ has a finite support. Theorem 3.2.2 describes the situation when no moment conditions on r.v.s $\{\xi_1, \xi_2, \ldots\}$ are required, but there is strict requirement for $\eta$. Theorem 3.2.3 deals with the opposite case: r.v.s $\{\xi_1, \xi_2, \ldots\}$ should have finite means, while the requirement for $\eta$ is weaker. It should be noted that the case of real-valued r.v.s $\{\xi_1, \xi_2, \ldots\}$ is considered in Theorems 3.2.1 and 3.2.2, while Theorem 3.2.3 deals with nonnegative r.v.s.

**Theorem 3.2.1.** *Let $\{\xi_1, \xi_2, \ldots, \xi_D\}$, $D \in \mathbb{N}$, be independent real-valued r.v.s, and $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots, \xi_D\}$. Then d.f. $F_{S_\eta}$ belongs to the class $\mathcal{C}$ if the following conditions are satisfied:*

(a) $\mathbb{P}(\eta \leqslant D) = 1$,

(b) $F_{\xi_1} \in \mathcal{C}$,

(c) *for each* $k = 2, \ldots, D,$ *either* $F_{\xi_k} \in \mathcal{C}$ *or* $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$.

**Theorem 3.2.2.** *Let $\{\xi_1, \xi_2, \ldots\}$ be independent real-valued r.v.s, and $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then d.f. $F_{S_\eta}$ belongs to the class $\mathcal{C}$ if the following conditions are satisfied:*

(a) $F_{\xi_1} \in \mathcal{C}$,

(b) *for each* $k \geqslant 2,$ *either* $F_{\xi_k} \in \mathcal{C}$ *or* $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$,

(c) $\limsup\limits_{x\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n\overline{F}_{\xi_1}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$,

(d) $\mathbb{E}\eta^{p+1} < \infty$ *for some* $p > J^+_{F_{\xi_1}}$.

When $\{\xi_1, \xi_2, \ldots\}$ are identically distributed with common d.f. $F_\xi \in \mathcal{C}$, conditions (a), (b) and (c) of Theorem 3.2.2 are satisfied obviously. Hence, we have the following corollary.

**Corollary 3.2.1.** *(see also Theorem 3.4 by [9]) Let* $\{\xi_1, \xi_2, \ldots\}$ *be i.i.d. real-valued r.v.s with d.f.* $F_\xi \in \mathcal{C}$, *and* $\eta$ *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *Then d.f.* $F_{S_\eta}$ *belongs to the class* $\mathcal{C}$ *if* $\mathbb{E}\eta^{p+1} < \infty$ *for some* $p > J^+_{F_\xi}$.

**Theorem 3.2.3.** *Let* $\{\xi_1, \xi_2, \ldots\}$ *be independent nonnegative r.v.s, and* $\eta$ *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *Then d.f.* $F_{S_\eta}$ *belongs to the class* $\mathcal{C}$ *if the following conditions are satisfied:*

(a) $F_{\xi_1} \in \mathcal{C}$,

(b) *for each* $k \geqslant 2$, *either* $F_{\xi_k} \in \mathcal{C}$ *or* $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$,

(c) $\mathbb{E}\xi_1 < \infty$,

(d) $\overline{F}_\eta(x) = o\left(\overline{F}_{\xi_1}(x)\right)$,

(e) $\limsup\limits_{x\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n\overline{F}_{\xi_1}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$,

(f) $\limsup\limits_{u\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n} \sum\limits_{\substack{k=1 \\ \mathbb{E}\xi_k \geqslant u}}^{n} \mathbb{E}\xi_k = 0.$

Similarly to Corollary 3.2.1, we could formulate the following assertion. We note that in the i.i.d. case conditions (a), (b), (e) and (f) of Theorem 3.2.3 are satisfied.

**Corollary 3.2.2.** *Let* $\{\xi_1, \xi_2, \ldots\}$ *be i.i.d. nonnegative r.v.s with common d.f.* $F_\xi \in \mathcal{C}$, *and* $\eta$ *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *Then d.f.* $F_{S_\eta}$ *belongs to the class* $\mathcal{C}$ *under the following two conditions:* $\mathbb{E}\xi < \infty$ *and* $\overline{F}_\eta(x) = o\left(\overline{F}_\xi(x)\right)$.

Further in this section, we present two examples of r.v.s $\{\xi_1, \xi_2, \ldots\}$ and $\eta$ for which random sum $F_{S_\eta}$ has a consistently varying tail.

**Example 3.2.1.** *Let* $\{\xi_1, \xi_2, \ldots\}$ *be independent r.v.s such that* $\xi_k$ *are exponentially distributed for all even* $k$, *i.e.*

$$\overline{F}_{\xi_k}(x) = \mathrm{e}^{-x},\ x \geqslant 0,\ k \in \{2, 4, 6, \ldots\},$$

*while for each odd* $k$ *r.v.* $\xi_k$ *is a copy of r.v.*

$$(1 + \mathcal{U})\, 2^{\mathcal{G}},$$

*where* $\mathcal{U}$, $\mathcal{G}$ *are independent,* $\mathcal{U}$ *is uniformly distributed on interval* $[0,1]$*, and* $\mathcal{G}$ *is geometrically distributed with parameter* $q \in (0,1)$*, i.e.*

$$\mathbb{P}(\mathcal{G} = l) = (1-q)\, q^l,\ \ l = 0,1,\ldots.$$

*In addition, let* $\eta$ *be a counting r.v independent of* $\{\xi_1, \xi_2, \ldots\}$ *distributed according to the Poisson law.*

Theorem 3.2.2 implies that d.f. of the randomly stopped sum $S_\eta$ belongs to the class $\mathcal{C}$, because:

(a) $F_{\xi_1} \in \mathcal{C}$ due to considerations in pages 122-123 of [7],

(b) $F_{\xi_k} \in \mathcal{C}$ for $k \in \{3,5,\ldots\}$, and $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$ for $k \in \{2,4,6,\ldots\}$,

(c) $\limsup\limits_{x\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n\overline{F}_{\xi_1}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) \leqslant 1,$

(d) all moments of r.v. $\eta$ are finite.

We should note that $\xi_1$ doesn't satisfy the condition (c) of Theorem 3.2.3 in the case when $q \geqslant 1/2$. Hence, Example 3.2.1 describes the situation where Theorem 3.2.2 should be used instead of Theorem 3.2.3.

**Example 3.2.2.** *Let* $\{\xi_1, \xi_2, \ldots\}$ *be independent r.v.s such that* $\xi_k$ *are distributed according to the Pareto law (with tail index* $\alpha = 2$*) for all odd* $k$*, and* $\xi_k$ *are exponentially distributed (with parameter equal to 1) for all even* $k$*, i.e.*

$$\overline{F}_{\xi_k}(x) = \frac{1}{x^2},\ x \geqslant 1,\ k \in \{1,3,5,\ldots\},$$
$$\overline{F}_{\xi_k}(x) = \mathrm{e}^{-x},\ x \geqslant 0,\ k \in \{2,4,6,\ldots\}.$$

*In addition, let* $\eta$ *be a counting r.v independent of* $\{\xi_1, \xi_2, \ldots\}$ *which has Zeta distribution with parameter equal to 4, i.e.*

$$\mathbb{P}(\eta = m) = \frac{1}{\zeta(4)} \frac{1}{(m+1)^4},\ m \in \mathbb{N}_0,$$

*where* $\zeta$ *denotes the Riemann zeta function.*

Theorem 3.2.3 implies that d.f. of the randomly stopped sum $S_\eta$

belongs to the class $\mathcal{C}$ because:

(a) $F_{\xi_1} \in \mathcal{C}$,

(b) $F_{\xi_k} \in \mathcal{C}$ for $k \in \{3, 5, \ldots\}$, and $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$ for $k \in \{2, 4, 6, \ldots\}$,

(c) $\mathbb{E}\xi_1 = 2$,

(d) $\overline{F}_{\eta}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$,

(e) $\limsup\limits_{x\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n\overline{F}_{\xi_1}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) \leqslant 1$,

(f) $\max\limits_{k\in\mathbb{N}} \mathbb{E}\xi_k = 2$.

Regarding the condition (d), it should be noted, that Zeta distribution with parameter 4 is a discrete version of Pareto distribution with tail index 3.

We should note that $\eta$ doesn't satisfy the condition (d) of Theorem 3.2.2 because $J^{+}_{F_{\xi_1}} = 2$ and $\mathbb{E}\eta^3 = \infty$. Hence, Example 3.2.2 describes the situation, where Theorem 3.2.3 should be used instead of Theorem 3.2.2.

## 3.3 Auxiliary lemmas

This section deals with the row of auxiliary lemmas. The first lemma is Theorem 3.1 by [9] (see also Theorem 2.1 by [64]).

**Lemma 3.3.1.** *Let $\{X_1, X_2, \ldots X_n\}$ be independent real-valued r.v.s. If $F_{X_k} \in \mathcal{C}$ for each $k \in \{1, 2, \ldots, n\}$ then*

$$\mathbb{P}\bigg(\sum_{i=1}^{n} X_i > x\bigg) \sim \sum_{i=1}^{n} \overline{F}_{X_i}(x).$$

The following assertion about subexponential distributions was proved in Proposition 1 of [24], and later generalised to the broader distribution class in Corollary 3.19 of [29].

**Lemma 3.3.2.** *Let $\{X_1, X_2, \ldots X_n\}$ be independent real-valued r.v.s. Assume that $\overline{F}_{X_i}/\overline{F}(x) \underset{x\to\infty}{\to} b_i$ for some subexponential d.f. $F$ and some* *constants* $b_i \geqslant 0,\ i \in \{1, 2, \ldots n\}$. *Then*

$$\frac{\overline{F_{X_1} * F_{X_2} * \ldots * F_{X_n}}(x)}{\overline{F}(x)} \underset{x\to\infty}{\to} \sum_{i=1}^{n} b_i.$$

In the next lemma we show in what cases convolution $F_{X_1} * F_{X_2} * \ldots * F_{X_n}$ belongs to the class $\mathcal{C}$.

**Lemma 3.3.3.** *Let* $\{X_1, X_2, \ldots, X_n\}, n \in \mathbb{N}$, *be independent real-valued r.v.s. Then d.f.* $F_{\Sigma_n}$ *of sum* $\Sigma_n = X_1 + X_2 + \ldots X_n$ *belongs to the class* $\mathcal{C}$ *if the following conditions are satisfied:*

(a) $F_{X_1} \in \mathcal{C}$,

(b) *For each* $k = 2, \ldots, n$, *either* $F_{X_k} \in \mathcal{C}$ *or* $\overline{F}_{X_k}(x) = o\left(\overline{F}_{X_1}(x)\right)$.

*Proof.* Evidently, we can suppose that $n \geqslant 2$, and we can split our proof into two parts.

*First part.* Suppose that $F_{X_k} \in \mathcal{C}$ for all $k \in \{1, 2, \ldots, n\}$. In such a case, the assertion of the lemma follows from Lemma 3.3.1 and inequality

$$\frac{a_1 + a_2 + \ldots + a_m}{b_1 + b_2 + \ldots + b_m} \leqslant \max\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_m}{b_m}\right\} \tag{3.3.1}$$

provided if $a_i \geqslant 0$ and $b_i > 0$ for $i \in \{1, 2, \ldots, m\}$.

Namely, using relation of Lemma 3.3.1 and estimate (3.3.1) we get that

$$\begin{aligned} \limsup_{x\to\infty} \frac{\overline{F}_{\Sigma_n}(xy)}{\overline{F}_{\Sigma_n}(x)} &= \limsup_{x\to\infty} \frac{\sum_{k=1}^{n} \overline{F}_{X_k}(xy)}{\sum_{k=1}^{n} \overline{F}_{X_k}(x)} \\ &\leqslant \max_{1\leqslant k\leqslant n} \limsup_{x\to\infty} \frac{\overline{F}_{X_k}(xy)}{\overline{F}_{X_k}(x)} \end{aligned}$$

for an arbitrary $y \in (0, 1)$.

Since $F_{X_k} \in \mathcal{C}$ for each $k$ the last estimate implies that d.f. $F_{\Sigma_n}$ has consistently varying tail as desired.

*Second part.* Now suppose $F_{X_k} \notin \mathcal{C}$ for some of indexes $k \in \{2, 3, \ldots, n\}$. According to the lemma conditions we have that $\overline{F}_{X_k}(x) = o\big(\overline{F}_{X_1}(x)\big)$ for such $k$. Let $\mathcal{K} \subset \{2, 3, \ldots, n\}$ be the subset of indexes under condition

$$k \in \mathcal{K} \;\Leftrightarrow\; F_{X_k} \notin \mathcal{C} \text{ and } \overline{F}_{X_k}(x) = o\big(\overline{F}_{X_1}(x)\big).$$

Due to Lemma 3.3.2

$$\overline{F}_{\widehat{\Sigma}_n}(x) \sim \overline{F}_{X_1}(x),$$

where

$$\widehat{\Sigma}_n = X_1 + \sum_{k \in \mathcal{K}} X_k.$$

Hence,

$$\limsup_{x\to\infty} \frac{\overline{F}_{\widehat{\Sigma}_n}(xy)}{\overline{F}_{\widehat{\Sigma}_n}(x)} = \limsup_{x\to\infty} \frac{\overline{F}_{X_1}(xy)}{\overline{F}_{X_1}(x)} \tag{3.3.2}$$

for every $y \in (0,1)$.

Equality (3.3.2) implies immediately that d.f. $F_{\widehat{\Sigma}_n}$ belongs to the class $\mathcal{C}$. Therefore d.f. $F_{\Sigma_n}$ also belongs to the class $\mathcal{C}$ according to the first part of the proof because

$$\Sigma_n = \widehat{\Sigma}_n + \sum_{k \notin \mathcal{K}} X_k$$

and $F_{X_k} \in \mathcal{C}$ for each $k \notin \mathcal{K}$. The lemma is proved. □

The following two statements about dominatingly varying distributions are Lemma 3.2 and Lemma 3.3 respectively by [21]. Since any consistently varying distribution is also dominatingly varying, these statements will be useful in the proofs of our main results concerning class $\mathcal{C}$.

**Lemma 3.3.4.** *Let $\{X_1, X_2, \ldots\}$ be independent real-valued r.v.s, and $F_{X_\nu} \in \mathcal{D}$ for some $\nu \geqslant 1$. Suppose, in addition, that the following requirement holds:*

$$\limsup_{x\to\infty} \sup_{n\geqslant\nu} \frac{1}{n\overline{F}_{X_\nu}(x)} \sum_{i=1}^{n} \overline{F}_{X_i}(x) < \infty.$$

*Then for each $p > J^{+}_{F_{X_\nu}}$ there exists a positive constant $c_1$ such that*

$$\overline{F}_{S_n}(x) \leqslant c_1 n^{p+1} \overline{F}_{X_\nu}(x) \tag{3.3.3}$$

*for all* $n \geqslant \nu$ *and* $x \geqslant 0$.

Actually, Lemma 3.3.4 is proved in [21] for nonnegative r.v.s. But the statement of the lemma remains valid for real-valued r.v.s. To see this it is sufficient to observe that $\mathbb{P}(X_1+X_2+\ldots+X_n > x) \leqslant \mathbb{P}(X_1^+ + X_2^+ \ldots + X_n^+ > x)$ and $\mathbb{P}(X_k > x) = \mathbb{P}(X_k^+ > x)$, where $n \in \mathbb{N}$, $k \in \{1, 2, \ldots, n\}$, $x \geqslant 0$, and $a^+$ denotes the positive part of $a$.

**Lemma 3.3.5.** *Let* $\{X_1, X_2, \ldots\}$ *be independent real-valued r.v.s, and* $F_{X_\nu} \in \mathcal{D}$ *for some* $\nu \geqslant 1$. *Let, in addition,*

$$\lim_{u\to\infty} \sup_{n\geqslant\nu} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left(|X_k| \mathbb{1}_{\{X_k \leqslant -u\}}\right) = 0,$$

$$\limsup_{x\to\infty} \sup_{n\geqslant\nu} \frac{1}{n\overline{F}_{X_\nu}(x)} \sum_{i=1}^{n} \overline{F}_{X_i}(x) < \infty,$$

*and* $\mathbb{E}X_k = \mathbb{E}X_k^+ - \mathbb{E}X_k^- = 0$ *for* $k \in \mathbb{N}$. *Then for each* $\gamma > 0$ *there exists a positive constant* $c_2 = c_2(\gamma)$ *such that*

$$\mathbb{P}(S_n > x) \leqslant c_2 n \overline{F}_{X_\nu}(x)$$

*for all* $x \geqslant \gamma n$ *and all* $n \geqslant \nu$.

## 3.4 Proofs of the main results

**Proof of Theorem 3.2.1**. It is sufficient to prove that

$$\limsup_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(xy)}{\overline{F}_{S_\eta}(x)} \leqslant 1. \tag{3.4.1}$$

According to estimate (3.3.1), for $x > 0$ and $y \in (0,1)$ we have

$$\frac{\overline{F}_{S_\eta}(xy)}{\overline{F}_{S_\eta}(x)} = \frac{\sum\limits_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{D} \mathbb{P}\left(S_n > xy\right) \mathbb{P}\left(\eta = n\right)}{\sum\limits_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{D} \mathbb{P}\left(S_n > x\right) \mathbb{P}\left(\eta = n\right)} \leqslant \max_{\substack{1\leqslant n\leqslant D\\ n\in\mathrm{supp}(\eta)}} \frac{\mathbb{P}\left(S_n > xy\right)}{\mathbb{P}\left(S_n > x\right)}.$$

Hence, by Lemma 3.3.3,

$$\limsup_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}_{S_\eta}(xy)}{\overline{F}_{S_\eta}(x)} \leqslant \limsup_{y\uparrow 1}\limsup_{x\to\infty}\max_{\substack{1\leqslant n\leqslant D\\ n\in\mathrm{supp}(\eta)}}\frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)}$$
$$\leqslant \max_{\substack{1\leqslant n\leqslant D\\ n\in\mathrm{supp}(\eta)}}\limsup_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)} = 1,$$

which implies the desired estimate (3.4.1). Theorem is proved. □

**Proof of Theorem 3.2.2.** As in Theorem 3.2.1, it is sufficient to prove inequality (3.4.1). For each $K\in\mathbb{N}$ and $x>0$ we have

$$\mathbb{P}(S_\eta > x) = \left(\sum_{n=1}^{K} + \sum_{n=K+1}^{\infty}\right)\mathbb{P}(S_n > x)\,\mathbb{P}(\eta = n).$$

Therefore for $x>0$ and $y\in(0,1)$ we have

$$\begin{aligned}\frac{\mathbb{P}(S_\eta > xy)}{\mathbb{P}(S_\eta > x)} &= \frac{\sum\limits_{n=1}^{K}\mathbb{P}(S_n > xy)\,\mathbb{P}(\eta = n)}{\mathbb{P}(S_\eta > x)}\\ &\quad + \frac{\sum\limits_{n=K+1}^{\infty}\mathbb{P}(S_n > xy)\,\mathbb{P}(\eta = n)}{\mathbb{P}(S_\eta > x)}\\ &=: \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \tag{3.4.2}$$

Random variable $\eta$ is not degenerate at zero, so there exists $a\in\mathbb{N}$ such that $\mathbb{P}(\eta = a) > 0$. If $K\geqslant a$, then using inequality (3.3.1) we get

$$\mathcal{J}_1 \leqslant \frac{\sum\limits_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{K}\mathbb{P}(S_n > xy)\,\mathbb{P}(\eta = n)}{\sum\limits_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{K}\mathbb{P}(S_n > x)\,\mathbb{P}(\eta = n)} \leqslant \max_{\substack{1\leqslant n\leqslant K\\ n\in\mathrm{supp}(\eta)}}\frac{\mathbb{P}(S_n > xy)}{\mathbb{P}(S_n > x)}.$$

Similarly as in the proof of Theorem 3.2.1, it follows that

$$\limsup_{y\uparrow 1}\limsup_{x\to\infty}\mathcal{J}_1 \leqslant \max_{\substack{1\leqslant n\leqslant K\\ n\in\mathrm{supp}(\eta)}}\limsup_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)} = 1. \tag{3.4.3}$$

Since $\mathcal{C} \subset \mathcal{D}$, we can use Lemma 3.3.4 for the numerator of $\mathcal{J}_2$ to obtain

$$\sum_{n=K+1}^{\infty} \mathbb{P}\left(S_n > xy\right) \mathbb{P}\left(\eta = n\right) \leqslant c_3 \overline{F}_{\xi_1}(xy) \sum_{n=K+1}^{\infty} n^{p+1} \mathbb{P}\left(\eta = n\right)$$

with some positive constant $c_3$. For the denominator of $\mathcal{J}_2$, we have that

$$\begin{aligned} \mathbb{P}\left(S_\eta > x\right) &= \sum_{n=1}^{\infty} \mathbb{P}\left(S_n > x\right) \mathbb{P}(\eta = n) \\ &\geqslant \mathbb{P}(S_a > x) \mathbb{P}\left(\eta = a\right). \end{aligned}$$

Conditions of the theorem imply that

$$S_a = \xi_1 + \sum_{k \in \mathcal{K}_a} \xi_k + \sum_{k \notin \mathcal{K}_a} \xi_k,$$

where $\mathcal{K}_a = \big\{k \in \{2, \ldots, a\} : F_{\xi_k} \notin \mathcal{C}, \overline{F}_{\xi_k}(x) = o\big(\overline{F}_{\xi_1}(x)\big)\big\}$.

According to Lemma 3.3.3 we have that d.f. $F_{\widehat{S}_a}$ of sum

$$\widehat{S}_a = \xi_1 + \sum_{k \in \mathcal{K}_a} \xi_k$$

belongs to the class $\mathcal{C}$. So, due to Lemma 3.3.2,

$$\overline{F}_{\widehat{S}_a}(x) / \overline{F}_{\xi_1}(x) \underset{x \to \infty}{\to} 1.$$

If $k \notin \mathcal{K}_a$ then $F_{\xi_k} \in \mathcal{C}$ by conditions of Theorem. This fact and Lemma 3.3.1 imply that

$$\liminf_{x \to \infty} \frac{\mathbb{P}(S_a > x)}{\overline{F}_{\xi_1}(x)} \geqslant 1 + \sum_{k \notin \mathcal{K}_a} \liminf_{x \to \infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)}.$$

Hence,

$$\mathbb{P}(S_\eta > x) \geqslant \frac{1}{2} \overline{F}_{\xi_1}(x) \mathbb{P}(\eta = a) \tag{3.4.4}$$

if $x$ is sufficiently large. Therefore,

$$\begin{aligned}&\limsup_{y\uparrow 1}\limsup_{x\to\infty}\mathcal{J}_2\\ \leqslant\ &\frac{2\,c_3}{\mathbb{P}\left(\eta=a\right)}\left(\limsup_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_1}(xy)}{\overline{F}_{\xi_1}(x)}\right)\sum_{n=K+1}^{\infty}n^{p+1}\mathbb{P}\left(\eta=n\right)\end{aligned} \tag{3.4.5}$$

Estimates (3.4.2), (3.4.3) and (3.4.5) imply that

$$\limsup_{y\uparrow 1}\limsup_{x\to\infty}\frac{\mathbb{P}\left(S_\eta>xy\right)}{\mathbb{P}\left(S_\eta>x\right)}\leqslant 1+\frac{2\,c_3}{\mathbb{P}\left(\eta=a\right)}\mathbb{E}\eta^{p+1}\mathbb{I}_{\{\eta>K\}}$$

for arbitrary $K\geqslant a$.

Letting $K$ to infinity we get the desired estimate (3.4.1), due to Theorem's condition (d). Theorem is proved. □

**Proof of Theorem 3.2.3.** Once again, it is sufficient to prove inequality (3.4.1).

By Theorem's condition (e), we have that there exist two positive constants $c_4$ and $c_5$ for which

$$\sum_{i=1}^{n}\overline{F}_{\xi_i}(x)\leqslant c_5 n\overline{F}_{\xi_1}(x),\ x\geqslant c_4,\ n\in\mathbb{N}.$$

Therefore,

$$\mathbb{E}S_n=\sum_{j=1}^{n}\mathbb{E}\xi_j=\sum_{j=1}^{n}\left(\int_0^{c_4}+\int_{c_4}^{\infty}\right)\overline{F}_{\xi_j}(u)du\leqslant c_4 n+c_5 n\mathbb{E}\xi_1=:c_6 n, \tag{3.4.6}$$

for a positive constant $c_6$ and all $n\in\mathbb{N}$.
If $K\in\mathbb{N}$ and $x>4Kc_6$, then we have

$$\begin{aligned}\mathbb{P}\left(S_\eta>x\right)\ &=\ \mathbb{P}\left(S_\eta>x,\eta\leqslant K\right)\\ &\quad+\mathbb{P}\Big(S_\eta>x,K<\eta\leqslant\frac{x}{4c_6}\Big)\\ &\quad+\mathbb{P}\Big(S_\eta>x,\eta>\frac{x}{4c_6}\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathbb{P}\left(S_\eta > xy\right)}{\mathbb{P}\left(S_\eta > x\right)} &= \frac{\mathbb{P}\left(S_\eta > xy, \eta \leqslant K\right)}{\mathbb{P}\left(S_\eta > x\right)} \\ &\quad + \frac{\mathbb{P}\big(S_\eta > xy, K < \eta \leqslant \frac{xy}{4c_6}\big)}{\mathbb{P}\left(S_\eta > x\right)} \\ &\quad + \frac{\mathbb{P}\big(S_\eta > xy, \eta > \frac{xy}{4c_6}\big)}{\mathbb{P}\left(S_\eta > x\right)} \\ &=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \end{aligned} \tag{3.4.7}$$

if $xy > 4Kc_6,\ x > 0$ and $y \in (0,1)$.

Random variable $\eta$ is not degenerate at zero, so $\mathbb{P}\left(\eta = a\right) > 0$ for some $a \in \mathbb{N}$. If $K \geqslant a$, then

$$\limsup_{y\uparrow 1} \limsup_{x\to\infty} \mathcal{I}_1 \leqslant 1, \tag{3.4.8}$$

similarly to the estimate (3.4.3) in Theorem 3.2.2.

For the numerator of $\mathcal{I}_2$ we have

$$\begin{aligned}\mathcal{I}_{2,1} &:= \mathbb{P}\Big(S_\eta > xy, K < \eta \leqslant \frac{xy}{4c_6}\Big) \\ &= \sum_{K<n\leqslant \frac{xy}{4c_6}} \mathbb{P}\Big(\sum_{i=1}^{n}(\xi_i - \mathbb{E}\xi_i) > xy - \sum_{j=1}^{n}\mathbb{E}\xi_j\Big)\mathbb{P}(\eta = n) \\ &\leqslant \sum_{K<n\leqslant \frac{xy}{4c_6}} \mathbb{P}\Big(\sum_{i=1}^{n}(\xi_i - \mathbb{E}\xi_i) > \frac{3}{4}xy\Big)\mathbb{P}(\eta = n), \end{aligned} \tag{3.4.9}$$

according to inequality (3.4.6).

R.v.s $\xi_1 - \mathbb{E}\xi_1,\ \xi_2 - \mathbb{E}\xi_2,$ ... satisfy the conditions of Lemma 3.3.5. Namely, $\mathbb{E}(\xi_k - \mathbb{E}\xi_k) = 0$ for $k \in \mathbb{N}$ and $F_{\xi_1 - \mathbb{E}\xi_1} \in \mathcal{C} \subset \mathcal{D}$ obviously. In addition,

$$
\begin{aligned}
& \limsup_{u\to\infty} \sup_{n\geqslant 1} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\left(|\xi_k - \mathbb{E}\xi_k|\mathbb{1}_{\{\xi_k-\mathbb{E}\xi_k\leqslant -u\}}\right)\\
= & \limsup_{u\to\infty} \sup_{n\geqslant 1} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\left((\mathbb{E}\xi_k - \xi_k)\mathbb{1}_{\{\xi_k-\mathbb{E}\xi_k\leqslant -u\}}\right)\\
\leqslant & \limsup_{u\to\infty} \sup_{n\geqslant 1} \frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ \mathbb{E}\xi_k\geqslant u}} \mathbb{E}\xi_k = 0
\end{aligned}
$$

because of Theorem's condition (f). So, using estimation of Lemma 3.3.5 in (3.4.9), we get

$$
\begin{aligned}
\mathcal{I}_{2,1} & \leqslant c_7 \sum_{K<n\leqslant \frac{xy}{4c_6}} n\overline{F}_{\xi_1}\Big(\frac{3}{4}xy + \mathbb{E}\xi_1\Big)\mathbb{P}(\eta = n)\\
& \leqslant c_7\overline{F}_{\xi_1}\Big(\frac{3}{4}xy\Big)\mathbb{E}\eta\mathbb{1}_{\{\eta>K\}}
\end{aligned}
$$

with some positive constant $c_7$. For the denominator of $\mathcal{I}_2$ we can use inequality

$$
\begin{aligned}
\mathbb{P}\left(S_\eta > x\right) & = \sum_{n=1}^{\infty}\mathbb{P}\left(S_n > x\right)\mathbb{P}(\eta = n)\\
& \geqslant \sum_{n=1}^{\infty}\mathbb{P}\left(\xi_1 > x\right)\mathbb{P}(\eta = n)\\
& \geqslant \overline{F}_{\xi_1}(x)\mathbb{P}\left(\eta = a\right) \qquad (3.4.10)
\end{aligned}
$$

because r.v.s $\{\xi_1, \xi_2, \ldots\}$ are nonnegative due to Theorem's conditions. Hence,

$$
\mathcal{I}_2 \leqslant \frac{c_7}{\mathbb{P}(\eta = a)}\mathbb{E}\eta\mathbb{1}_{\{\eta>K\}}\frac{\overline{F}_{\xi_1}\left(\frac{3}{4}xy\right)}{\overline{F}_{\xi_1}\left(x\right)}.
$$

If $y \in (1/2, 1)$ then the last estimate implies that

$$
\limsup_{x\to\infty}\mathcal{I}_2 \leqslant \frac{c_7}{\mathbb{P}(\eta = a)}\mathbb{E}\eta\mathbb{1}_{\{\eta>K\}}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_1}\left(\frac{3}{8}x\right)}{\overline{F}_{\xi_1}\left(x\right)} \leqslant c_8\mathbb{E}\eta\mathbb{1}_{\{\eta>K\}} \qquad (3.4.11)
$$

with some positive constant $c_8$, because $F_{\xi_1} \in \mathcal{C} \subset \mathcal{D}$.

Using inequality (3.4.10) again we obtain

$$\mathcal{I}_3 \leqslant \frac{\mathbb{P}(\eta > \frac{xy}{4c_6})}{\mathbb{P}(S_\eta > x)} \leqslant \frac{1}{\mathbb{P}(\eta = a)} \frac{\overline{F}_\eta\left(\frac{xy}{4c_6}\right)}{\overline{F}_{\xi_1}\left(\frac{xy}{4c_6}\right)} \frac{\overline{F}_{\xi_1}\left(\frac{xy}{4c_6}\right)}{\overline{F}_{\xi_1}(x)}.$$

Therefore, for $y \in (1/2, 1)$ we get

$$\limsup_{x\to\infty} \mathcal{I}_3 \leqslant \frac{1}{\mathbb{P}(\eta = a)} \limsup_{x\to\infty} \frac{\overline{F}_\eta\left(\frac{xy}{4c_6}\right)}{\overline{F}_{\xi_1}\left(\frac{xy}{4c_6}\right)} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_1}\left(\frac{xy}{4c_6}\right)}{\overline{F}_{\xi_1}(x)} = 0, \tag{3.4.12}$$

because of Theorem's condition (d).

Estimates (3.4.7), (3.4.8), (3.4.11) and (3.4.12) imply that

$$\limsup_{y\uparrow 1} \limsup_{x\to\infty} \frac{\mathbb{P}(S_\eta > xy)}{\mathbb{P}(S_\eta > x)} \leqslant 1 + c_8 \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}}$$

for $K \geqslant a$.

Letting $K$ tend to infinity we get the desired estimate (3.4.1), because $\mathbb{E}\eta < \infty$ due to Theorem's conditions (c) and (d). Theorem is proved. □

# Chapter 4

# Regularly distributed randomly stopped structures

## 4.1 Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let $\eta$ be a counting random variable (c.r.v.), that is, a nonnegative, nondegenerate at 0, and integer-valued r.v. In addition, we suppose that the r.v. $\eta$ and the sequence $\{\xi_1, \xi_2, \ldots\}$ are independent.

Let $S_0 := 0$, $S_n := \xi_1 + \ldots + \xi_n$ for $n \in \mathbb{N}$, and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the *randomly stopped sum* of the r.v.s $\xi_1, \xi_2, \ldots$.

Next, let $\xi^{(0)} := 0$, $\xi^{(n)} := \max\{0, \xi_1, \ldots, \xi_n\}$ for $n \in \mathbb{N}$, and let

$$\xi^{(\eta)} := \begin{cases} 0 & \text{if } \eta = 0, \\ \max\{0, \xi_1, \ldots, \xi_\eta\} & \text{if } \eta \geqslant 1, \end{cases}$$

be the *randomly stopped maximum* of the r.v.s $\xi_1, \xi_2, \ldots$.

Similarly, let $\xi_{(0)} := 0$, $\xi_{(n)} := \min\{\xi_1, \dots, \xi_n\}$ for $n \in \mathbb{N}$, and let

$$\xi_{(\eta)} := \begin{cases} 0 & \text{if } \eta = 0, \\ \min\{\xi_1, \dots, \xi_\eta\} & \text{if } \eta \geqslant 1, \end{cases}$$

be the *randomly stopped minimum* of the r.v.s $\xi_1, \xi_2, \dots$.

We denote by $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, and $F_{S_\eta}$ the d.f.s of $\xi_{(\eta)}$, $\xi^{(\eta)}$, and $S_\eta$, respectively. We denote by $\overline{F}$ the tail of a d.f. $F$, that is, $\overline{F}(x) = 1 - F(x)$ for $x \in \mathbb{R}$. It is obvious that the following equalities hold for $x > 0$:

$$\overline{F}_{\xi_{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(\xi_{(n)} > x), \quad \overline{F}_{\xi^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(\xi^{(n)} > x),$$

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n > x).$$

We use the following three notations for the asymptotic relations of arbitrary positive functions $f$ and $g$: $f(x) = o\,(g(x))$ means that $\lim_{x\to\infty} f(x)/g(x) = 0$; $f(x) \sim cg(x)$, $c > 0$, means that $\lim_{x\to\infty} f(x)/g(x) = c$; and $f(x) \asymp g(x)$ means that $0 < \liminf_{x\to\infty} f(x)/g(x) \leqslant \limsup_{x\to\infty} f(x)/g(x) < \infty$.

In this chapter, we consider a sequence $\{\xi_1, \xi_2, \dots\}$ of possibly non-identically distributed r.v.s. We suppose that some of the d.f.s of these r.v.s belong either to the class $\mathcal{R}_\alpha$ with some $\alpha \geqslant 0$ or to the classes $\mathcal{R} = \bigcup_{\alpha\geqslant 0} \mathcal{R}_\alpha$, $\mathcal{R}_+ = \bigcup_{\alpha>0} \mathcal{R}_\alpha$. We find conditions under which the d.f.s $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, and $F_{S_\eta}$ are regularly varying.

We further present a few results on randomly stopped structures related to regularly varying distribution functions.

The following two results present sufficient (Theorem 4.1.1) and necessary (Theorem 4.1.2) conditions for the closure of random sum of regularly varying r.v.s, see [27], Propositions 4.1 and 4.8, respectively.

**Theorem 4.1.1.** *Let $\xi_1, \xi_2, \dots$ be independent and identically distributed (i.i.d.) nonnegative r.v.s, and let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Assume that the d.f. $F_{\xi_1}$ is regularly varying with index $\alpha > 0$, $\mathbb{E}\eta < \infty$, and $\overline{F}_\eta(x) = o\big(\overline{F}_{\xi_1}(x)\big)$. Then the d.f. $F_{S_\eta}$ belongs to the class $\mathcal{R}_\alpha$, and $\overline{F}_{S_\eta}(x) \sim \mathbb{E}\eta\overline{F}_{\xi_1}(x)$.*

**Theorem 4.1.2.** *Let $\xi_1, \xi_2, \ldots$ be i.i.d. nonnegative r.v.s, and let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Assume that $S_\eta$ is regularly varying with index $\alpha > 0$ and $\mathbb{E}\eta^{1\vee p} < \infty$ for some $p > \alpha$. Then the d.f. $F_{\xi_1}$ belongs to the class $\mathcal{R}_\alpha$, and $\overline{F}_{S_\eta}(x) \sim \mathbb{E}\eta\overline{F}_{\xi_1}(x)$.*

The following result on sufficient and necessary conditions for the closure under random maximum of regularly varying r.v.s was obtained in [34] (see Lemma 5.1(i)).

**Theorem 4.1.3.** *Let $\xi_1, \xi_2, \ldots$ be i.i.d. real-valued r.v.s, and let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ such that $\mathbb{E}\eta < \infty$. Then $\overline{F}_{\xi^{(\eta)}}(x) \sim \mathbb{E}\eta\overline{F}_{\xi_1}(x)$, and hence $F_{\xi^{(\eta)}}$ belongs to the class $\mathcal{R}_\alpha$ if and only if $F_{\xi_1}$ belongs to $\mathcal{R}_\alpha$, $\alpha \geqslant 0$.*

Motivated by the presented statements and results obtained in [7, 16, 19, 20, 41, 48–50, 52, 58, 69], we continue to consider conditions under which the d.f.s $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, and $F_{S_\eta}$ belong to either the class $\mathcal{R}_\alpha$ with some $\alpha \geqslant 0$ or the class $\mathcal{R}$. As we mentioned before, we deal with the case where the sequence $\{\xi_1, \xi_2, \ldots\}$ consists of independent but possibly nonidentically distributed r.v.s.

The rest of the chapter is organized as follows. In Section 4.2, we present our main results. Section 4.3 consists of some auxiliary lemmas. The proofs of the main results are given in Section 4.4. Finally, in Section 4.5, we present two examples to expose the usefulness of our results.

## 4.2 Main results

In this section, we present the main results of this chapter. *In all the statements, we suppose that the sequence $\{\xi_1, \xi_2, \ldots\}$ and the c.r.v. $\eta$ are independent.* Our first theorem describes properties of randomly stopped minima.

**Theorem 4.2.1.** *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of real-valued r.v.s. Then $F_{\xi_k} \in \mathcal{R}$ for all $k \in \mathbb{N}$ if and only if $F_{\xi_{(\eta)}} \in \mathcal{R}$ for every c.r.v. $\eta$.*

The second theorem below describes sufficient conditions for the regularity of randomly stopped maxima and sums when the c.r.v. $\eta$ has a finite support.

**Theorem 4.2.2.** *Let* $\xi_1, \ldots, \xi_m, m \in \mathbb{N}$, *be independent real-valued r.v.s, and let* $\eta$ *be a counting r.v. independent of* $\{\xi_1, \ldots, \xi_m\}$ *such that* $\mathbb{P}(\eta \leqslant m) = 1$. *Then the d.f.s* $F_{S_\eta}$ *and* $F_{\xi^{(\eta)}}$ *belong to the class* $\mathcal{R}_\alpha,\ \alpha \geqslant 0$, *if the following two conditions are satisfied:*

(i) $F_{\xi_1} \in \mathcal{R}_\alpha$,

(ii) *for each* $k \geqslant 2$ *either* $F_{\xi_k} \in \mathcal{R}_\alpha$ *or* $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$.

*Furthermore, under conditions* (i)–(ii), *the following tail equivalences hold:*

$$\overline{F}_{\xi^{(\eta)}}(x) \sim \overline{F}_{S_\eta}(x) \sim x^{-\alpha} \sum_{n=1}^{m} \mathbb{P}(\eta = n) \sum_{k \in \mathcal{I}_n} L_k(x), \tag{4.2.1}$$

*where* $\mathcal{I}_n = \{k = 1, \ldots, n : F_{\xi_k} \in \mathcal{R}_\alpha\}$, *and* $L_k$ *are slowly varying functions from the representations* $\overline{F}_{\xi_k}(x) = x^{-\alpha} L_k(x)$.

The following theorem describes properties of randomly stopped sums and maxima when the c.r.v. has a finite support. Here we provide both necessary and sufficient conditions for $\{\xi_1, \xi_2, \ldots\}$, but the initial conditions for the collection of the primary r.v.s are more restrictive than in the previous theorem.

**Theorem 4.2.3.** *Let* $\xi_1, \xi_2, \ldots$ *be a sequence of independent real-valued r.v.s such that* $\overline{F}_{\xi_k}(x) \asymp \overline{F}_{\xi_1}(x)$ *for all* $k \geqslant 2$. *Then the following statements are equivalent:*

(i) $F_{\xi_k} \in \mathcal{R}_+$ *for all* $k \in \mathbb{N}$,

(ii) $F_{S_\eta} \in \mathcal{R}_+$ *for any c.r.v.* $\eta$ *with finite support,*

(iii) $F_{\xi^{(\eta)}} \in \mathcal{R}_+$ *for any c.r.v.* $\eta$ *with finite support.*

In the following theorem, we give sufficient conditions under which the randomly stopped sum is regularly varying in the case of a general c.r.v. $\eta$.

**Theorem 4.2.4.** *Let* $\xi_1, \xi_2, \ldots$ *independent real-valued r.v.s, and let* $\eta$ *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *Assume the following* *conditions are satisfied:*

(i) $F_{\xi_1} \in \mathcal{R}_\alpha, \alpha \geqslant 0,$

(ii) $\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right| = 0$ *for a sequence of nonnegative constants* $\{d_1 = 1, d_2, d_3, \ldots\}$ *such that* $\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d_k < \infty,$

(iii) $\mathbb{E}\eta^{p+1} < \infty$ *for some* $p > \alpha$.

*Then the following tail equivalence holds*

$$\overline{F}_{S_\eta}(x) \sim \overline{F}_{\xi_1}(x) \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \sum_{k=1}^{n} d_k,$$

*and hence* $F_{S_\eta} \in \mathcal{R}_\alpha$.

The following result on sufficient and necessary conditions for the closure under random maximum of regularly varying r.v.s is a direct generalization of Theorem 4.1.3.

**Theorem 4.2.5.** *Let* $\{\xi_1, \xi_2, \ldots\}$ *be real-valued r.v.s such that* $\overline{F}_{\xi_1}(x) > 0$ *for all* $x \in \mathbb{R}$, *and let* $\eta$ *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *In addition, suppose that*

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right| = 0 \quad and \quad \max\left\{\mathbb{E}\eta, \mathbb{E}\Big(\sum_{k=1}^{\eta} d_k\Big)\right\} < \infty$$

*for a sequence of nonnegative constants* $\{d_1 = 1, d_2, \ldots\}$. *Then*

$$\overline{F}_{\xi^{(\eta)}}(x) \sim \overline{F}_{\xi_1}(x)\, \mathbb{E}\Big(\sum_{k=1}^{\eta} d_k\Big),$$

*and hence* $F_{\xi^{(\eta)}}$ *belongs to the class* $\mathcal{R}_\alpha$ *if and only if* $F_{\xi_1}$ *belongs to* $\mathcal{R}_\alpha$, $\alpha \geqslant 0$.

## 4.3 Auxiliary lemmas

In this section, we give several auxiliary lemmas. Some of these lemmas are originally stated for wider heavy-tailed distribution classes, which include the class $\mathcal{R}$ as a subclass. Here we restate these lemmas for regularly varying d.f.s. The first lemma follows from Theorem 3.1 of [9] (see also Theorem 2.1 from [64]). This lemma is a specific case of Lemma 3.3.1.

**Lemma 4.3.1.** *Let $X_1, \ldots X_n$ be independent real-valued r.v.s. If $F_{X_k} \in \mathcal{R}$ for $k \in \{1, 2, \ldots, n\}$, then*

$$\mathbb{P}\left(\sum_{k=1}^{n} X_k > x\right) \sim \sum_{k=1}^{n} \overline{F}_{X_k}(x). \tag{4.3.1}$$

The next lemma is Theorem 4.1 from [56]. This lemma provides necessary and sufficient conditions for the max-sum equivalence of regularly varying distributions.

**Lemma 4.3.2.** *Let $\{X_1, X_2, \ldots X_n\}$ be independent real-valued r.v.s. Then $F_{\Sigma_n} \in \mathcal{R}_\alpha$, $\alpha \geqslant 0$, if and only if $\max\left\{0, 1 - \sum_{k=1}^{n} \overline{F}_{X_k}\right\} \in \mathcal{R}_\alpha$, where $F_{\Sigma_n}$ is d.f. of sum $\Sigma_n = X_1 + \ldots + X_n$. In this case, the asymptotic relation* (4.3.1) *holds.*

The next lemma follows from Theorems 3.10, 3.11, and 4.1 by Shimura [56]. It describes the decomposition property of regularly varying distributions.

**Lemma 4.3.3.** *Let $X$ be a real-valued r.v., and suppose that $F_X \in \mathcal{R}_+$. Furthermore, suppose that $X$ can be decomposed into independent r.v.s $X_1$ and $X_2$, that is, $X = X_1 + X_2$. If $F_{X_1} \in \mathcal{R}_+$ and $\overline{F}_{X_2}(x) \asymp \overline{F}_{X_1}(x)$, then $F_{X_2} \in \mathcal{R}_+$.*

The following statement was proved in Proposition 1 of [24] and later was generalized to a broader distribution class in Corollary 3.19 of [29]. This lemma is a specific case of Lemma 3.3.2.

**Lemma 4.3.4.** *Let $\{X_1, \ldots X_n\}$ be a collection of independent real-valued r.v.s. Assume that $\overline{F}_{X_k}(x)/\overline{F}(x) \underset{x\to\infty}{\to} b_k$ for some regularly varying d.f. $F$ and some constants $b_i \geqslant 0$, $i \in \{1, 2, \ldots n\}$. Then*

$$\frac{\mathbb{P}\left(\sum\limits_{k=1}^{n} X_k > x\right)}{\overline{F}(x)} \underset{x\to\infty}{\to} \sum_{k=1}^{n} b_k.$$

In the next lemma, we show in which cases the d.f. $F_{\Sigma_n}$ of the sum $\Sigma_n = X_1 + \ldots + X_n$ and the d.f. $F_{X^{(n)}}$ of the maximum $X^{(n)} = \max\{X_1, \ldots, X_n\}$ belong to the class $\mathcal{R}_\alpha$.

**Lemma 4.3.5.** *Let $X_1, \ldots, X_n$ be independent real-valued r.v.s. Then the d.f.s $F_{\Sigma_n}$ and $F_{X^{(n)}}$ belong to the class $\mathcal{R}_\alpha$, $\alpha \geqslant 0$, if the following conditions are satisfied:*

(i) $F_{X_1} \in \mathcal{R}_\alpha$,

(ii) *for each $k = 2, \ldots, n$, either $F_{X_k} \in \mathcal{R}_\alpha$ or $\overline{F}_{X_k}(x) = o\big(\overline{F}_{X_1}(x)\big)$.*

*Furthermore, under these conditions,*

$$\overline{F}_{X^{(n)}}(x) \sim \overline{F}_{\Sigma_n}(x) \sim x^{-\alpha} \sum_{k \in \widehat{\mathcal{I}}_n} L_k(x), \tag{4.3.2}$$

*where $L_k$ are slowly varying functions from representations $\overline{F}_{X_k}(x) = x^{-\alpha} L_k(x)$, and $\widehat{\mathcal{I}}_n = \{k = 1, \ldots, n : F_{X_k} \in \mathcal{R}_\alpha\}$.*

**Proof.** We first consider the sum $\Sigma_n$. For $n = 2$, the statement is well known (see, e.g., p. 278 in [28], Lemma 1.3.4 in [47], Proposition 4.2.5 in [55] or the case $n = 2$ of Corollary 3.19 of [29]). We use induction. Suppose the statement of the lemma holds for $n = K$. This means that $F_{\Sigma_K} \in \mathcal{R}_\alpha$ and, due to Lemma 4.3.2,

$$\overline{F}_{\Sigma_K}(x) \sim x^{-\alpha} \sum_{k \in \widehat{\mathcal{I}}_K} L_k(x) \sim \sum_{k=1}^{K} \overline{F}_{X_k}(x).$$

According to the conditions of the lemma, either $F_{X_{K+1}} \in \mathcal{R}_\alpha$ or $\overline{F}_{X_{K+1}}(x) = o\big(\overline{F}_{\Sigma_K}(x)\big)$. Since $\Sigma_{K+1} = \Sigma_K + X_{K+1}$, in both cases, we obtain that $F_{\Sigma_{K+1}} \in \mathcal{R}_\alpha$ and

$$\overline{F}_{\Sigma_{K+1}}(x) \sim \sum_{k=1}^{K+1} \overline{F}_{X_k}(x) \sim x^{-\alpha} \sum_{k \in \widehat{\mathcal{I}}_{K+1}} L_k(x)$$

by Proposition 4.2.5 from [55] and Proposition 1.3.6 from [4] on the properties of slowly varying functions. According to the induction principle, the statement of the lemma holds for all sums $\Sigma_n$.

The statement of the lemma for $X^{(n)}$ follows immediately from the following asymptotic relations:

$$\overline{F}_{X^{(n)}}(x) = \mathbb{P}\Big(\bigcup_{k=1}^{n}\{X_k > x\}\Big) \sim \sum_{k=1}^{n}\overline{F}_{X_k}(x) \sim x^{-\alpha}\sum_{k\in\widehat{\mathcal{I}}_n} L_k(x)$$

for each $n \in \mathbb{N}$ by the classical Bonferoni inequalities and properties of slowly varying functions. The lemma is proved. □

The following statement follows from Lemma 3.2 of [21].

**Lemma 4.3.6.** *Let $X_1, \ldots$ be independent real-valued r.v.s, and let $F_{X_\nu} \in \mathcal{R}_\alpha$ for some $\nu \geqslant 1$ and $\alpha \geqslant 0$. Suppose, in addition, that*

$$\limsup_{x\to\infty}\frac{1}{\overline{F}_{X_\nu}(x)}\sup_{n\geqslant\nu}\frac{1}{n}\sum_{k=1}^{n}\overline{F}_{X_k}(x) < \infty.$$

*Then, for any $p > \alpha$, there exists a positive constant $c = c(p)$ such that*

$$\overline{F}_{S_n}(x) \leqslant c\, n^{p+1}\overline{F}_{X_\nu}(x) \tag{4.3.3}$$

*for all $n \geqslant \nu$ and $x \geqslant 0$.*

In fact, Lemma 3.2 in [21] is proved for nonnegative r.v.s, but the statement remains valid for real-valued r.v.s. To see this, it suffices to observe that $\mathbb{P}(X_1 + \ldots + X_n > x) \leqslant \mathbb{P}(X_1^+ \ldots + X_n^+ > x)$ and $\mathbb{P}(X_k > x) = \mathbb{P}(X_k^+ > x)$ for $n \in \mathbb{N}$, $k \in \{1, 2, \ldots, n\}$, $x \geqslant 0$, where $a^+$ denotes the positive part of $a$.

## 4.4 Proofs of main results

In this section, we give detailed proofs of our main results.

**Proof of Theorem 4.2.1** Let $\eta$ be an arbitrary c.r.v., and set

$$\varkappa := \min\{n \geqslant 1 : \mathbb{P}(\eta = n) > 0\}.$$

Then for any $x > 0$, we have

$$\begin{aligned}
\overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbb{P}(\eta = n) \\
&= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa) + \sum_{n=\varkappa+1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbb{P}(\eta = n) \\
&= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa) \left(1 + \sum_{n=\varkappa+1}^{\infty} \left(\prod_{k=\varkappa+1}^{n} \overline{F}_{\xi_k}(x)\right) \frac{\mathbb{P}(\eta = n)}{\mathbb{P}(\eta = \varkappa)}\right) \\
&\leqslant \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa) \left(1 + \overline{F}_{\xi_{\varkappa+1}}(x) \frac{\mathbb{P}(\eta \geqslant \varkappa + 1)}{\mathbb{P}(\eta = \varkappa)}\right)
\end{aligned}$$

and

$$\overline{F}_{\xi_{(\eta)}}(x) \geqslant \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa).$$

Therefore we obtain

$$\overline{F}_{\xi_{(\eta)}}(x) \underset{x\to\infty}{\sim} \mathbb{P}(\eta = \varkappa)\, \overline{F}_{\xi_{(\varkappa)}}(x). \tag{4.4.1}$$

Necessity. If $F_{\xi_k} \in \mathcal{R}$ for all $k \in \mathbb{N}$, then $F_{\xi_1} \in \mathcal{R}_{\alpha_1}$, $F_{\xi_2} \in \mathcal{R}_{\alpha_2}, \ldots$ for some nonnegative parameters $\alpha_1, \alpha_2, \ldots$ This means that, for each $k \in \mathbb{N}$, $\overline{F}_{\xi_k}(x) = x^{-\alpha_k} L_k(x)$ with a slowly varying function $L_k$. Hence, for a finite nonrandom $\varkappa$,

$$F_{\xi_{(\varkappa)}} \in \mathcal{R}_{\alpha_1+\ldots+\alpha_\varkappa} \tag{4.4.2}$$

by the closure properties of slowly varying functions (see, e.g., Proposition 1.3.6 in Bingham et al. [4]) because

$$\overline{F}_{\xi_{(\varkappa)}}(x) = \prod_{k=1}^{\varkappa} \overline{F}_{\xi_k}(x) = x^{-(\alpha_1+\ldots+\alpha_\varkappa)} \prod_{k=1}^{\varkappa} L_k(x)$$

for $x > 0$. Thus, it follows from (4.4.1) and (4.4.2) that

$$F_{\xi_{(\eta)}} \in \mathcal{R}_{\alpha_1+\ldots+\alpha_\varkappa} \subset \mathcal{R}$$

for any c.r.v. $\eta$.

Sufficiency. If $F_{\xi_{(\eta)}} \in \mathcal{R}$ for an arbitrary c.r.v. $\eta$, then from (4.4.1) it follows that $F_{\xi_{(n)}} \in \mathcal{R}$ for any fixed $n \in \mathbb{N}$. In addition, for all $x > 0$, we have that $\overline{F}_{\xi_1}(x) = \overline{F}_{\xi_{(1)}}(x)$ and

$$\overline{F}_{\xi_k}(x) = \frac{\overline{F}_{\xi_{(k)}}(x)}{\overline{F}_{\xi_{(k-1)}}(x)}, \ k \in \{2, 3, \ldots\}.$$

Therefore, by the closure properties of slowly varying functions (see, e.g., Proposition 1.3.6 in Bingham et al. [4]), we obtain that $F_{\xi_k} \in \mathcal{R}$ for each $k \in \mathbb{N}$. Theorem 4.2.1 is proved. □

**Proof of Theorem 4.2.2** To verify that $F_{S_\eta} \in \mathcal{R}_\alpha$, it suffices to prove that

$$\overline{F}_{S_\eta}(x) \sim x^{-\alpha} L(x) \tag{4.4.3}$$

for some slowly varying function $L$.

For all $x > 0$, we have

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{m} \mathbb{P}(\eta = n)\mathbb{P}(S_n > x).$$

By Lemma 4.3.5 we conclude that for each $n \in \{1, \ldots, m\}$,

$$\overline{F}_{S_n}(x) \sim x^{-\alpha} \sum_{k \in \mathcal{I}_n} L_k(x),$$

where $L_k$ are slowly varying functions. Asymptotic relation (4.2.1) now immediately follows.

By the closure properties of slowly varying functions (see Proposition 1.3.6 in Bingham et al. [4]) we conclude that asymptotic relation (4.4.3) holds with slowly varying function

$$L(x) = \sum_{n=1}^{m} \mathbb{P}(\eta = n) \sum_{k \in \mathcal{I}_n} L_k(x).$$

Consequently, $F_{S_\eta} \in \mathcal{R}_\alpha$.

The proof of the theorem for the d.f. $F_{\xi^{(\eta)}}$ is identical to that for $F_{S_\eta}$, and hence we omit it. The theorem is proved. □

**Proof of Theorem 4.2.3** The implication (i) ⇒ (iii) immediately follows from Theorem 4.2.2.

Suppose now assumption (iii) holds, that is, $F_{\xi^{(\eta)}} \in \mathcal{R}_+$ for any c.r.v. $\eta$ with finite support. From this assumption it follows that $F_{\xi^{(n)}} \in \mathcal{R}_{\alpha_n}$ for each $n \in \mathbb{N}$ with some index $\alpha_n > 0$. Applying the classical Bonferroni inequality, we obtain

$$\mathbb{P}\Big(\xi^{(n)} > x\Big) = \mathbb{P}\left(\bigcup_{k=1}^{n}\{\xi_k > x\}\right) \sim \sum_{k=1}^{n}\mathbb{P}(\xi_k > x).$$

Therefore the d.f. $\max\left\{0, 1 - \sum_{k=1}^{n}\overline{F}_k\right\}$ belongs to the class $\mathcal{R}_{\alpha_n}$ as well. Lemma 4.3.2 and the last asymptotic relation imply that $F_{S_n} \in \mathcal{R}_{\alpha_n}$ and

$$\mathbb{P}(S_n > x) \sim \sum_{k=1}^{n}\mathbb{P}(\xi_k > x) \sim \mathbb{P}\big(\xi^{(n)} > x\big) \tag{4.4.4}$$

for $n \in \mathbb{N}$.

Let us consider a c.r.v. $\eta$ with finite support $\{0, 1, \ldots, m\}$, $m \geqslant 1$. In such a case, by the asymptotic relation (4.4.4) we have

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{m}\mathbb{P}(\eta = n)\mathbb{P}(S_n > x) \sim \sum_{n=1}^{m}\mathbb{P}(\eta = n)\mathbb{P}\big(\xi^{(n)} > x\big) = \overline{F}_{\xi^{(\eta)}}(x).$$

Consequently, $F_{S_\eta} \in \mathcal{R}_+$ for c.r.v. $\eta$. The implication (iii) $\Rightarrow$ (ii) is proved.

Finally, we give a proof of the implication (ii) $\Rightarrow$ (i). Since by assumption (ii) $F_{S_\eta} \in \mathcal{R}_+$ for every c.r.v. $\eta$ with finite support, it follows that

$$F_{S_n} \in \mathcal{R}_+ \tag{4.4.5}$$

for each $n \in \mathbb{N}$. In particular, $F_{\xi_1} \in \mathcal{R}_+$ and $F_{\xi_1+\xi_2} \in \mathcal{R}_+$. Lemma 4.3.3 implies that $F_{\xi_1} \in \mathcal{R}_\alpha$ and $F_{\xi_2} \in \mathcal{R}_\alpha$ for some $\alpha > 0$ because $\overline{F}_{\xi_1}(x) \asymp \overline{F}_{\xi_2}(x)$ by the conditions of the theorem.

Let us continue by induction. Suppose that $F_{\xi_1} \in \mathcal{R}_\alpha$, $F_{\xi_2} \in \mathcal{R}_\alpha$, ...,$F_{\xi_K} \in \mathcal{R}_\alpha$ with $K \geqslant 2$. Lemma 4.3.1 implies that $F_{S_K} \in \mathcal{R}_\alpha \subset \mathcal{R}_+$ and

$$\overline{F}_{S_K}(x) \sim \sum_{k=1}^{K}\overline{F}_{\xi_k}(x). \tag{4.4.6}$$

The distribution function $F_{S_{K+1}} \in \mathcal{R}_+$ because of relation (4.4.5). In addition, the conditions of the theorem imply that $\overline{F}_{\xi_k}(x) \asymp \overline{F}_{\xi_1}(x) \asymp \overline{F}_{\xi_{K+1}}(x)$ for each $k \in \{1, 2, \ldots, K\}$. This, together with asymptotic relation (4.4.6), implies that $\overline{F}_{S_K}(x) \asymp \overline{F}_{\xi_{K+1}}(x)$. Using Lemma 4.3.3, we obtain that $F_{\xi_{K+1}} \in \mathcal{R}_\alpha \subset \mathcal{R}_+$. Now statement (i) of Theorem 4.2.3 follows by the induction principle. This completes the proof. □

**Proof of Theorem 4.2.4** As in Theorem 4.2.2, it suffices to prove the tail equivalence formula (4.4.3) with some slowly varying function $L$.

For any $K \in \mathbb{N}$ and all $x > 0$, define the function

$$L_K^*(x) = L_1(x) \sum_{n=1}^{K} \mathbb{P}(\eta = n) \sum_{k=1}^{n} d_k. \tag{4.4.7}$$

In addition, for all $x > 0$, define

$$L_\infty^*(x) := \lim_{K \to \infty} L_K^*(x).$$

We begin with the existence of this limit. First, for each fixed $x$, the sequence $L_K^*(x)$ is nondecreasing. Second, for each fixed $x$, the sequence $L_K^*(x)$ has an upper bound by conditions (ii) and (iii). Indeed, condition (ii) implies that

$$\sum_{k=1}^{n} d_k \leqslant c_1 n$$

for all $n \in \mathbb{N}$ and some positive constant $c_1$, and condition (iii) implies that

$$\sum_{n=1}^{K} \mathbb{P}(\eta = n) \sum_{k=1}^{n} d_k \leqslant c_1 \sum_{n=1}^{\infty} n \mathbb{P}(\eta = n) = c_1 \mathbb{E}\eta < \infty$$

for all $K \in \mathbb{N}$.

Besides that, the function $L_\infty^*(x)$ is slowly varying. Let us prove the asymptotic relation

$$\overline{F}_{S_\eta}(x) \sim x^{-\alpha} L_\infty^*(x),$$

which is analogous to (4.4.3). For all $K \in \mathbb{N}$ and $x > 0$, denote

$$\begin{aligned} \mathcal{J} := \frac{\mathbb{P}(S_\eta > x)}{L_\infty^*(x) x^{-\alpha}} &= \frac{\sum_{n=1}^{K} \mathbb{P}(S_n > x) \mathbb{P}(\eta = n)}{L_\infty^*(x) x^{-\alpha}} + \frac{\sum_{n=K+1}^{\infty} \mathbb{P}(S_n > x) \mathbb{P}(\eta = n)}{L_\infty^*(x) x^{-\alpha}} \\ &=: \mathcal{J}_1(K) + \mathcal{J}_2(K). \end{aligned} \tag{4.4.8}$$

We have to prove the inequalities

$$\liminf_{x\to\infty} \mathcal{J} \geqslant 1 \quad \text{and} \qquad \limsup_{x\to\infty} \mathcal{J} \leqslant 1. \tag{4.4.9}$$

Condition (ii) implies that

$$\lim_{x\to\infty} \left| \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right| = 0$$

for each fixed $k$. Consequently, either $\overline{F}_{\xi_k}(x) \sim d_k \overline{F}_{\xi_1}(x)$ for positive $d_k$, implying that $F_{\xi_k} \in \mathcal{R}_\alpha$, or $\overline{F}_{\xi_k}(x) = o\big(\overline{F}_{\xi_1}(x)\big)$. By Lemma 4.3.5, for all $n \in \mathbb{N}$, we have $F_{S_n} \in \mathcal{R}_\alpha$ and

$$\overline{F}_{S_n}(x) \sim \sum_{k\in\mathcal{I}_n} \overline{F}_{\xi_k}(x) \sim x^{-\alpha} L_1(x) \sum_{k=1}^{n} d_k.$$

From these asymptotic relations we get that

$$\liminf_{x\to\infty} \mathcal{J}_1(K) = \limsup_{x\to\infty} \mathcal{J}_1(K) = \frac{\sum_{n=1}^{K} \mathbb{P}(\eta = n) \sum_{k=1}^{n} d_k}{\sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \sum_{k=1}^{n} d_k}. \tag{4.4.10}$$

Using the obvious inequality $\liminf_{x\to\infty} \mathcal{J} \geqslant \liminf_{x\to\infty} \mathcal{J}_1(K)$ and letting $K$ tend to infinity, we derive from (4.4.10) the first inequality in (4.4.9).

Since

$$\begin{aligned}
&\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n\overline{F}_{\xi_1}(x)} \sum_{k=1}^{n} \overline{F}_{\xi_k}(x) \\
\leqslant\ &\limsup_{x\to\infty} \sup_{n\geqslant 1} \left( \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right| + \frac{1}{n} \sum_{k=1}^{n} d_k \right) < \infty
\end{aligned}$$

by condition (ii) of the theorem, we can use Lemma 4.3.6 for the numerator of $\mathcal{J}_2(K)$ to obtain

$$\sum_{n=K+1}^{\infty} \mathbb{P}\left(S_n > x\right) \mathbb{P}\left(\eta = n\right) \leqslant c_2 \overline{F}_{\xi_1}(x) \sum_{n=K+1}^{\infty} n^{p+1} \mathbb{P}\left(\eta = n\right)$$

with some positive constant $c_2$. Therefore

$$\begin{aligned}\limsup_{x\to\infty}\mathcal{J}_2(K) &\leqslant c_2\limsup_{x\to\infty}\frac{L_1(x)}{L^*_\infty(x)}\sum_{n=K+1}^{\infty}n^{p+1}\mathbb{P}(\eta=n)\\ &\leqslant c_3\sum_{n=K+1}^{\infty}n^{p+1}\mathbb{P}(\eta=n)\end{aligned}$$

with some positive constant $c_3$.

The last inequality together with (4.4.10) implies that

$$\begin{aligned}\limsup_{x\to\infty}\mathcal{J} &\leqslant \limsup_{x\to\infty}\mathcal{J}_1(K)+\limsup_{x\to\infty}\mathcal{J}_2(K)\\ &\leqslant \frac{\sum_{n=1}^{K}\mathbb{P}(\eta=n)\sum_{k=1}^{n}d_k}{\sum_{n=1}^{\infty}\mathbb{P}(\eta=n)\sum_{k=1}^{n}d_k}+c_3\mathbb{E}\big(\eta^{p+1}\mathbb{1}_{\{\eta\geqslant K+1\}}\big).\end{aligned}$$

Letting $K$ tend to infinity, we get the second desired inequality in (4.4.9) by condition (iii) of the theorem. This completes the proof of Theorem 4.2.4. □

**Proof of Theorem 4.2.5** Note that

$$\overline{F}_{\xi^{(n)}}(x)=\mathbb{P}(\xi^{(n)}>x)=\sum_{k=1}^{n}\overline{F}_{\xi_k}(x)\prod_{j=1}^{k-1}F_{\xi_j}(x)$$

for all $x>0$ and $n\in\mathbb{N}$. Therefore

$$\begin{aligned}\frac{\overline{F}_{\xi^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} &= \sum_{n=1}^{K}\mathbb{P}(\eta=n)\sum_{k=1}^{n}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)}\prod_{j=1}^{k-1}F_{\xi_j}(x)\\ &\quad+\sum_{n=K+1}^{\infty}\mathbb{P}(\eta=n)\sum_{k=1}^{n}\Big(\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)}-d_k\Big)\prod_{j=1}^{k-1}F_{\xi_j}(x)\\ &\quad+\sum_{n=K+1}^{\infty}\mathbb{P}(\eta=n)\sum_{k=1}^{n}d_k\prod_{j=1}^{k-1}F_{\xi_j}(x)\\ &=: \mathcal{L}_1(K)+\mathcal{L}_2(K)+\mathcal{L}_3(K) \qquad (4.4.11)\end{aligned}$$

with an arbitrary $K\geqslant 2$.

For the first term, we have

$$\lim_{x\to\infty} \mathcal{L}_1(K) = \sum_{n=1}^{K} \mathbb{P}(\eta = n) \sum_{k=1}^{n} d_k \tag{4.4.12}$$

because $\lim_{x\to\infty} \overline{F}_{\xi_k}(x)/\overline{F}_{\xi_1}(x) = d_k$ for each fixed $k$.

In addition,

$$|\mathcal{L}_2(K)| \leqslant \sup_{n>K} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right| \mathbb{E}\big(\eta \mathbb{1}_{\{\eta>K\}}\big), \tag{4.4.13}$$

$$\mathcal{L}_3(K) \leqslant \mathbb{E}\Big( \sum_{k=1}^{\eta} d_k \mathbb{1}_{\{\eta>K\}} \Big). \tag{4.4.14}$$

Theorem 4.2.5 now follows from equalities (4.4.11), (4.4.12) and estimates (4.4.13), (4.4.14). □

## 4.5 Examples

In this section, we present two examples, which demonstrate the applicability of Theorem 4.2.4.

**Example 4.5.1.** Consider a counting r.v. $\eta$ and a sequence of i.i.d. real-valued r.v.s $\{\xi_1, \xi_2, \ldots\}$ such that $F_{\xi_1} \in \mathcal{R}_\alpha$.

In this case, conditions (i) and (ii) of Theorem 4.2.4 are satisfied with constants $d_1 = d_2 = \ldots = 1$. Hence the theorem implies that $\overline{F}_{S_\eta}(x) \sim \mathbb{E}\eta\, \overline{F}_{\xi_1}(x)$ if $\mathbb{E}\eta^{1+p} < \infty$ for some $p > \alpha$.

Note that this example deals with the same i.i.d. r.v.s as in Theorem 4.1.1. The difference is that Theorem 4.2.4 imposes stricter conditions on the c.r.v., which are sufficient for the d.f. of the random sum to be regularly varying as well as for real valued summands

**Example 4.5.2.** Consider an example similar to that in [42]. Suppose that $\eta$ is an arbitrary counting r.v. and $\{\xi_1, \xi_2, \ldots\}$ is a sequence of independent r.v.s distributed according to the two-sided Pareto laws

$$F_{\xi_k}(x) = \frac{a_k^-}{|x|^\alpha} \mathbb{1}_{(-\infty,-1)} + (1 - b_k - a_k^+) \mathbb{1}_{[-1,1)}(x) + \Big(1 - \frac{a_k^+}{x^\alpha}\Big) \mathbb{1}_{[1,\infty)}(x),$$

where $\alpha > 0$, and $a_k^-$, $a_k^+$, and $b_k$ are nonnegative constants such that $a_k^+ > 0$ and $a_k^- + b_k + a_k^+ \leqslant 1$ for all $k \in \mathbb{N}$.

In this case, if $\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k^+ < \infty$ and $\mathbb{E}\eta^{1+p} < \infty$ for some $p > \alpha$, then conditions (i)–(iii) of Theorem 4.2.4 are satisfied, and

$$\overline{F}_{S_\eta}(x) \sim \frac{1}{x^\alpha} \sum_{k=1}^{\infty} a_k^+ \mathbb{P}(\eta \geq k).$$

Particularly, if $\eta$ is distributed according to the Poisson law with parameter $\lambda > 0$ and $a_k^+ = 1/(k(k+1)),\ k \geq 1$, then

$$\overline{F}_{S_\eta}(x) \sim \frac{1}{\lambda}\big(\mathrm{e}^{-\lambda} + \lambda - 1\big)x^{-\alpha}.$$

# Chapter 5

# Asymptotic formulas for the left truncated moments

Let $n \in \mathbb{N} := \{1, 2, \dots\}$ and let $\{\xi_1, \dots, \xi_n\}$ be a collection of possibly dependent real-valued random variables (r.v.s) with heavy-tailed distributions. Denote

$$S_n \quad := \quad \xi_1 + \cdots + \xi_n. \tag{5.0.1}$$

Throughout the chapter, we assume that random summands have consistently varying distributions.

We explain some notations which will be used throughout the chapter. For two positive functions $f, g$ we write:

$$\begin{aligned} f(x) \lesssim g(x) \quad & \text{if} \quad \limsup_{x\to\infty} f(x)/g(x) \leqslant 1; \\ f(x) = O(g(x)) \quad & \text{if} \quad \limsup_{x\to\infty} f(x)/g(x) < \infty; \\ f(x) \asymp g(x) \quad & \text{if} \quad f(x) = O(g(x)) \text{ and } g(x) = O(f(x)); \\ f(x) \sim g(x) \quad & \text{if} \quad \lim_{x\to\infty} f(x)/g(x) = 1. \end{aligned}$$

In this chapter, we suppose that the random variables $\xi_1, \dots, \xi_n$ are pairwise quasi-asymptotically independent. This dependence structure was introduced in [9] and considered in [31, 43, 44, 65, 67] and other papers. In the definition below and elsewhere, we use the standard notations: $x^+ := \max\{0, x\}$, $x^- := \max\{0, -x\}$.

**Definition 5.0.1.** *Real-valued random variables $\xi_1,\dots,\xi_n$ with distributions supported on $\mathbb{R}$ are called* pairwise quasi-asymptotically independent *(pQAI), if for all pairs of indices $k,l\in\{1,2,\dots,n\}$, $k\neq l$, it holds that*

$$\lim_{x\to\infty}\frac{\mathbb{P}(\xi_k^+>x,\xi_l^+>x)}{\mathbb{P}(\xi_k^+>x)+\mathbb{P}(\xi_l^+>x)} \quad = \quad \lim_{x\to\infty}\frac{\mathbb{P}(\xi_k^+>x,\xi_l^->x)}{\mathbb{P}(\xi_k^+>x)+\mathbb{P}(\xi_l^+>x)} \quad = 0.$$

The following statement is Theorem 3.1 in [9]. The statement provides the asymptotic results for tail probability of sums of pQAI r.v.s having distributions from class $\mathcal{C}$. This statement is similar to Lemma 3.3.1.

**Theorem 5.0.1.** *Let $\{\xi_1,\dots,\xi_n\}$ be a collection of real-valued pQAI r.v.s., such that $F_{\xi_k}\in\mathcal{C}$ for $k\in\{1,\dots,n\}$. Then*

$$\mathbb{P}\big(S_n>x\big) \underset{x\to\infty}{\sim} \sum_{k=1}^{n}\overline{F}_{\xi_k}(x).$$

The following assertion with slightly narrower dependence structure and r.v.s from a wider class $\mathcal{D}$ is derived in Theorem 2.1 of [39].

**Theorem 5.0.2.** *Let $\{\xi_1,\dots,\xi_n\}$ be a collection of real-valued r.v.s., such that*

$$\begin{aligned}\lim_{x\to\infty}\sup_{u\geqslant x}\mathbb{P}(\xi_k^+>x\,|\,\xi_l^+>u) \quad &= \quad \lim_{x\to\infty}\sup_{u\geqslant x}\mathbb{P}(\xi_k^->x\,|\,\xi_l^+>u)\\ &= \quad \lim_{x\to\infty}\sup_{u\geqslant x}\mathbb{P}(\xi_k^+>x\,|\,\xi_l^->u) \quad = \quad 0\end{aligned}$$

*for all pairs of indices $k,l\in\{1,2,\dots,n\}$. In addition, suppose that $F_{\xi_1}\in\mathcal{D}$, $\overline{F}_{\xi_k}(x)\asymp\overline{F}_{\xi_1}(x)$, $\overline{F}_{\xi_k^-}(x)=O(\overline{F}_{\xi_1}(x))$ for $k\in\{1,\dots,n\}$, and $\mathbb{E}\,|\xi_1|^m<\infty$ for some $m\in\mathbb{N}_0:=\{0,1,\dots,\}$. Then*

$$\sum_{k=1}^{n}L_{F_{\xi_k}}\mathbb{E}\big(\xi_k^m\mathbb{1}_{\{\xi_k>x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\Big(\big(S_n\big)^m\mathbb{1}_{\{S_n>x\}}\Big) \underset{x\to\infty}{\lesssim} \sum_{k=1}^{n}\frac{1}{L_{F_{\xi_k}}}\mathbb{E}\big(\xi_k^m\mathbb{1}_{\{\xi_k>x\}}\big).$$

In this chapter, we obtain asymptotic relationships for

$$\mathbb{E}\Big(\big(S_n\big)^\alpha\mathbb{1}_{\{S_n>x\}}\Big) \tag{5.0.2}$$

and

$$\mathbb{E}\left(\left(S_n - x\right)^+\right)^\alpha \tag{5.0.3}$$

for arbitrary power $\alpha \in [0, \infty)$ and for r.v.s $\xi_1, \ldots, \xi_n$ following wider, pQAI, dependence structure. Asymptotic behavior of the left truncated moments of random sums was considered in various fields of applied probability, including risk theory and random walks [17, 18, 51]. In addition, quantity in (5.0.3) is closely related with the Haezendonck-Goovaerts risk measure, see for instance [32,39,61] and [62]. To get the precise asymptotic equivalence relationship we consider r.v.s with d.f.s from class $\mathcal{C}$. The main results on the asymptotics of (5.0.2) and (5.0.3) are presented in Theorems 5.1.1 and 5.1.2 below.

The rest of the chapter is organized as follows. In Section 5.1 we provide formulations of the main results. In Section 5.2 we present the proofs of the asymptotic formulas for the left truncated moments of $S_n$. The last Section 5.3 deals with the examples illustrating the obtained results.

## 5.1 Main results

The first assertion generalizes results of Theorem 5.0.1 which can be derived from theorem below by supposing $\alpha = 0$. In addition, for class $\mathcal{C}$, theorem below gives an analogous result to Theorem 5.0.2 for r.v.s $\xi_1, \ldots, \xi_n$ following a wider dependence structure and for a real-valued nonnegative moment order $\alpha$.

**Theorem 5.1.1.** *Let* $\{\xi_1, \ldots, \xi_n\}$ *be a collection of real-valued pQAI r.v.s, such that* $F_{\xi_k} \in \mathcal{C}$ *and* $\mathbb{E}\left(\xi_k^+\right)^\alpha < \infty$ *for all* $k \in \{1, \ldots, n\}$ *and for some* $\alpha \geqslant 0$*. Then*

$$\mathbb{E}\Big(\left(S_n\right)^\alpha \mathbb{I}_{\{S_n > x\}}\Big) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \mathbb{E}\left(\xi_k^\alpha \mathbb{I}_{\{\xi_k > x\}}\right). \tag{5.1.1}$$

The second theorem shows that the asymptotic behaviour of the left truncated moments of sums depends on consistently varying distributed increments but does not depend on asymptotically lighter increments.

**Theorem 5.1.2.** *Let $\{\xi_1, \ldots, \xi_n\}$ be a collection of real-valued r.v.s such that, for each $k \in \{1, \ldots, n\}$ it holds that $F_{\xi_k} \in \mathcal{C}$ or $\mathbb{P}\left(|\xi_k| > x\right) = o\left(\overline{F}_{\xi_1}(x)\right)$. Suppose that $F_{\xi_1} \in \mathcal{C}$ and $\mathbb{E}\left(\xi_k^+\right)^\alpha < \infty$ for all $k \in \{1, \ldots, n\}$ and some $\alpha \geqslant 0$. Let $\mathcal{I} \subseteq \{1, \ldots, n\}$ be a subset of indices $k$ such that $F_{\xi_k} \in \mathcal{C}$. If the subcollection $\{\xi_k, k \in \mathcal{I}\}$ consists of pQAI r.v.s, then, for each $\beta \in [0, \alpha]$,*

$$\mathbb{E}\Big(\big(S_n\big)^\beta \mathbb{1}_{\{S_n > x\}}\Big) \quad \sim \quad \sum_{k \in \mathcal{I}} \mathbb{E}\left(\xi_k^\beta \mathbb{1}_{\{\xi_k > x\}}\right), \tag{5.1.2}$$

*and, for $\beta \in (0, \alpha]$, it holds that*

$$\mathbb{E}\left(\left(S_n - x\right)^+\right)^\beta \quad \sim \quad \sum_{k \in \mathcal{I}} \mathbb{E}\left(\left(\xi_k - x\right)^+\right)^\beta. \tag{5.1.3}$$

We notice that the basic index in the formulation of Theorem 5.1.2, which is equal to one, can be replaced by any index $l \in \{1, \ldots, n\}$. In addition, it should be noted that dependence of r.v.s $\xi_k, k \in \mathcal{I}^c$, as well as mutual dependence between the sets $\{\xi_k, k \in \mathcal{I}\}$ and $\{\xi_k, k \in \mathcal{I}^c\}$ can be arbitrary.

## 5.2 Proofs of main results

We present two auxiliary lemmas before providing proofs of the main results,

**Lemma 5.2.1.** *Let $\xi$ be a real-valued r.v. such that $\mathbb{E}(\xi^+)^p < \infty$ for some $p > 0$. Then for any $x \geqslant 0$ we have*

$$\mathbb{E}\big(\xi^p \mathbb{1}_{\{\xi > x\}}\big) \quad = \quad x^p \mathbb{P}(\xi > x) + p \int_x^\infty u^{p-1} \mathbb{P}(\xi > u)\, \mathrm{d}u, \tag{5.2.1}$$

*and*

$$\mathbb{E}\big((\xi - x)^+\big)^p \quad = \quad p \int_x^\infty (u - x)^{p-1} \mathbb{P}(\xi > u)\, \mathrm{d}u. \tag{5.2.2}$$

Proof. Both equalities of the lemma follow directly from the following well known formula

$$\mathbb{E}\eta^p = p\int_0^\infty u^{p-1}\mathbb{P}(\eta > u)\,\mathrm{d}u \tag{5.2.3}$$

provided that $p > 0$ and $\eta$ is a nonnegative r.v. (see, for instance, Corollary 2 on page 208 of [57]).

Namely, by supposing $\eta = \xi\,1\!\!1_{\{\xi > x\}}$, from (5.2.3) we obtain

$$\begin{aligned}\mathbb{E}\left(\xi^p\,1\!\!1_{\{\xi>x\}}\right) &= p\int_0^\infty u^{p-1}\mathbb{P}\left(\xi\,1\!\!1_{\{\xi>x\}} > u\right)\,\mathrm{d}u\\ &= p\,\mathbb{P}(\xi > x)\int_0^x u^{p-1}\,\mathrm{d}u + p\int_x^\infty u^{p-1}\mathbb{P}(\xi > u)\,\mathrm{d}u,\end{aligned}$$

and equality (5.2.1) follows.

Similarly, by supposing $\eta = (\xi - x)^+$, from (5.2.3) equality (5.2.2) holds because

$$\begin{aligned}\mathbb{E}\big((\xi - x)^+\big)^p &= p\int_0^\infty u^{p-1}\mathbb{P}\left((\xi - x)^+ > u\right)\,\mathrm{d}u\\ &= p\int_0^\infty u^{p-1}\Big(\mathbb{P}\left((\xi - x)^+ > u, \xi > x\right) + \mathbb{P}\left((\xi - x)^+ > u, \xi \leqslant x\right)\Big)\,\mathrm{d}u\\ &= p\int_0^\infty u^{p-1}\mathbb{P}\left(\xi > x + u\right)\,\mathrm{d}u.\end{aligned}$$

**Lemma 5.2.2.** *Let $\xi$ and $\eta$ be two arbitrarily dependent r.v.s. If $F_\xi \in \mathcal{C}$ and $P\left(|\eta| > x\right) = o\left(\overline{F}_\xi(x)\right)$, then*

$$\mathbb{P}\left(\xi + \eta > x\right) \underset{x\to\infty}{\sim} \overline{F}_\xi(x). \tag{5.2.4}$$

Proof of the lemma is presented in [75] (see part (i) of Lemma 3.3).
□

Proof of Theorem 5.1.1. In the case $\alpha = 0$, the assertion of Theorem 5.1.1 follows from Theorem 5.0.1 immediately. Hence, further we can suppose that $\alpha$ is positive. By Lemma 5.2.1, for all $x \geqslant 0$ we have

$$\frac{\mathbb{E}\Big(\big(S_n\big)^{\alpha}\mathbb{I}_{\{S_n>x\}}\Big)}{\sum_{k=1}^{n}\mathbb{E}\big(\xi_k^{\alpha}\mathbb{I}_{\{\xi_k>x\}}\big)} = \frac{x^{\alpha}\mathbb{P}(S_n > x) + \alpha\int_x^{\infty} u^{\alpha-1}\mathbb{P}(S_n > u)\,\mathrm{d}u}{\sum_{k=1}^{n} x^{\alpha}\mathbb{P}(\xi_k > x) + \alpha\int_x^{\infty} u^{\alpha-1}\sum_{k=1}^{n}\mathbb{P}(\xi_k > u)\,\mathrm{d}u}$$
$$\leqslant \max\left\{\frac{\mathbb{P}(S_n > x)}{\sum_{k=1}^{n}\mathbb{P}(\xi_k > x)}, \frac{\int_x^{\infty} u^{\alpha-1}\frac{\mathbb{P}(S_n>u)}{\sum_{k=1}^{n}\mathbb{P}(\xi_k>u)}\sum_{k=1}^{n}\mathbb{P}(\xi_k > u)\,\mathrm{d}u}{\int_x^{\infty} u^{\alpha-1}\sum_{k=1}^{n}\mathbb{P}(\xi_k > u)\,\mathrm{d}u}\right\}$$
$$\leqslant \max\left\{\frac{\mathbb{P}(S_n > x)}{\sum_{k=1}^{n}\overline{F}_{\xi_k}(x)}, \sup_{u\geqslant x}\frac{\mathbb{P}(S_n > u)}{\sum_{k=1}^{n}\overline{F}_{\xi_k}(u)}\right\}$$

due to right inequality in min-max inequality

$$\min\left\{\frac{a_1}{b_1},\ldots,\frac{a_r}{b_r}\right\} \leqslant \frac{a_1+\ldots+a_r}{b_1+\ldots+b_r} \leqslant \max\left\{\frac{a_1}{b_1},\ldots,\frac{a_r}{b_r}\right\}, \qquad (5.2.5)$$

provided that $a_i \geqslant 0$ and $b_i > 0$ for $i \in \{1,\ldots,r\}$.

By Theorem 5.0.1, we get

$$\limsup_{x\to\infty}\frac{\mathbb{E}\Big(\big(S_n\big)^{\alpha}\mathbb{I}_{\{S_n>x\}}\Big)}{\sum_{k=1}^{n}\mathbb{E}\big(\xi_k^{\alpha}\mathbb{I}_{\{\xi_k>x\}}\big)} \leqslant \limsup_{x\to\infty}\sup_{u\geqslant x}\frac{\mathbb{P}(S_n > u)}{\sum_{k=1}^{n}\overline{F}_{\xi_k}(u)} = 1. \quad (5.2.6)$$

Similarly, using the left inequality in (5.2.5) we obtain

$$\liminf_{x\to\infty}\frac{\mathbb{E}\Big(\big(S_n\big)^{\alpha}\mathbb{I}_{\{S_n>x\}}\Big)}{\sum_{k=1}^{n}\mathbb{E}\big(\xi_k^{\alpha}\mathbb{I}_{\{\xi_k>x\}}\big)} \geqslant \liminf_{x\to\infty}\inf_{u\geqslant x}\frac{\mathbb{P}(S_n > u)}{\sum_{k=1}^{n}\overline{F}_{\xi_k}(u)} = 1. \quad (5.2.7)$$

The derived estimates (5.2.6) and (5.2.7) complete the proof of Theorem 5.1.1. □

Proof of Theorem 5.1.2. If $\mathcal{I} = \{1,\ldots,n\}$, then relation (5.1.2) follows immediately from Theorem 5.1.1. Hence, let us suppose that $\mathcal{I}^c \neq \emptyset$ and denote

$$S_n^{(1)} = \sum_{k\in\mathcal{I}}\xi_k,\ S_n^{(2)} = \sum_{k\in\mathcal{I}^c}\xi_k.$$

Summands in $S_n^{(1)}$ are pQAI r.v.s with consistently varying d.f.s.

Hence, Theorem 5.0.1 implies that

$$\mathbb{P}(S_n^{(1)} > x) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x). \tag{5.2.8}$$

This asymptotic relation and inequality (5.2.5) imply that d.f. $F_{S_n^{(1)}}(x) = \mathbb{P}\left(S_n^{(1)} \leqslant x\right)$ belongs to the class $\mathcal{C}$ due to the following estimate

$$\begin{aligned}\limsup_{x\to\infty} \frac{\mathbb{P}\left(S_n^{(1)} > yx\right)}{\mathbb{P}\left(S_n^{(1)} > x\right)} &= \limsup_{x\to\infty} \frac{\sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(yx)}{\sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x)} \\ &\leqslant \max_{k\in\mathcal{I}} \left\{ \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)} \right\}\end{aligned}$$

provided that $y \in (0,1)$.

In addition, each r.v. $\xi_k$ with index $k \in \mathcal{I}^c$ satisfies condition $\mathbb{P}(|\xi_k| > x) = o\left(\overline{F}_{\xi_1}(x)\right)$ according to requirements of the theorem. The fact that $F_{\xi_1} \in \mathcal{C} \subset \mathcal{D}$ and asymptotic equality (5.2.8) imply that

$$\mathbb{P}\left(|S_n^{(2)}| > x\right) = o\left(\mathbb{P}\left(S_n^{(1)} > x\right)\right) \tag{5.2.9}$$

because

$$\begin{aligned}\frac{\mathbb{P}\left(|S_n^{(2)}| > x\right)}{\mathbb{P}\left(S_n^{(1)} > x\right)} &\leqslant \frac{\mathbb{P}\left(\bigcup\limits_{k\in\mathcal{I}^c} \left\{|\xi_k| > \frac{x}{r}\right\}\right)}{\sum\limits_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x)} \frac{\sum\limits_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x)}{\mathbb{P}\left(S_n^{(1)} > x\right)} \\ &\leqslant \frac{\sum\limits_{k\in\mathcal{I}^c} \mathbb{P}\left(|\xi_k| > \frac{x}{r}\right)}{\overline{F}_{\xi_1}\left(\frac{x}{r}\right)} \frac{\overline{F}_{\xi_1}\left(\frac{x}{r}\right)}{\overline{F}_{\xi_1}(x)} \frac{\sum\limits_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x)}{\mathbb{P}\left(S_n^{(1)} > x\right)},\end{aligned}$$

where $r = |\mathcal{I}^c| \leqslant n-1$.

Consequently, Lemma 5.2.2 and asymptotic relations (5.2.8), (5.2.9) imply that

$$\mathbb{P}(S_n > x) \underset{x\to\infty}{\sim} \mathbb{P}(S_n^{(1)} > x) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x). \tag{5.2.10}$$

Hence the first relation (5.1.2) of Theorem 5.1.2 holds in the case

$\beta = 0$. If $\beta \in (0, \alpha]$, then using the first equality of Lemma 5.2.1 and estimates of (5.2.5), similarly as in the proof of Theorem 5.1.1, we derive that

$$\limsup_{x\to\infty} \frac{\mathbb{E}\Big(\big(S_n\big)^{\beta}\,\mathbb{I}_{\{S_n>x\}}\Big)}{\sum\limits_{k\in\mathcal{I}} \mathbb{E}\big(\xi_k^{\beta}\,\mathbb{I}_{\{\xi_k>x\}}\big)} \leqslant \limsup_{x\to\infty} \sup_{u\geqslant x} \frac{\mathbb{P}(S_n > u)}{\sum\limits_{k\in\mathcal{I}} \overline{F}_{\xi_k}(u)},$$

$$\liminf_{x\to\infty} \frac{\mathbb{E}\Big(\big(S_n\big)^{\beta}\,\mathbb{I}_{\{S_n>x\}}\Big)}{\sum\limits_{k\in\mathcal{I}} \mathbb{E}\big(\xi_k^{\beta}\,\mathbb{I}_{\{\xi_k>x\}}\big)} \geqslant \liminf_{x\to\infty} \inf_{u\geqslant x} \frac{\mathbb{P}(S_n > u)}{\sum\limits_{k\in\mathcal{I}} \overline{F}_{\xi_k}(u)}.$$

The relation (5.1.2) of Theorem 5.1.2 for $\beta \in (0, \alpha]$ follows now from (5.2.10).

The second asymptotic relation (5.1.3) can be obtained in a similar way by using the second equality of Lemma 5.2.1, relation (5.2.10) and estimate (5.2.5). Theorem 5.1.2 is proved. □

## 5.3 Examples

In this section, we provide two examples illustrating our main results.

**Example 5.3.1.** *Let r.v.s $\xi_1, \ldots, \xi_n$ satisfy the assumptions of Theorem 5.1.1. Suppose that for each $k$ r.v. $\xi_k$ is a copy of r.v. $\xi := (1+\mathcal{U})\,2^{\mathcal{G}}$, where $\mathcal{U}$, $\mathcal{G}$ are independent, $\mathcal{U}$ is uniformly distributed on interval $[0,1]$, and $\mathcal{G}$ is geometrically distributed with parameter $q \in (0,1)$, i.e. $\mathbb{P}(\mathcal{G} = l) = (1-q)\,q^{l},\ l \in \mathbb{N}_0$. We derive the asymptotic formulas for*

$$\mathbb{E}\Big(\big(S_n\big)^{\alpha}\,\mathbb{I}_{\{S_n>x\}}\Big) \ \text{ and } \ \mathbb{E}\left(\big(S_n - x\big)^{+}\right)^{\alpha}$$

*in the case of $0 \leqslant \alpha < \log_2(1/q)$, where $S_n = \xi_1 + \ldots + \xi_n$ as usual.*

Due to considerations on pages 122-123 of [7], $F_{\xi} \in \mathcal{C}\backslash\mathcal{R}$. In addition, for $x \geqslant 1$, we have

$$\begin{aligned}
\overline{F}_{\xi}(x) &= \sum_{l=0}^{\infty} \mathbb{P}\left(\mathcal{U} > \frac{x}{2^l} - 1\right) \mathbb{P}\left(\mathcal{G} = l\right) \\
&= \sum_{\log_2 x - 1 < l \leqslant \log_2 x} \left(2 - \frac{x}{2^l}\right)(1-q)q^l + \sum_{l > \log_2 x} (1-q)q^l \\
&= \left(2 - \frac{x}{2^{\lfloor \log_2 x \rfloor}}\right)(1-q)q^{\lfloor \log_2 x \rfloor} + q^{\lfloor \log_2 x \rfloor + 1} \\
&= q^{\log_2 x}\left(\left(2 - 2^{\langle \log_2 x \rangle}\right)(1-q)q^{-\langle \log_2 x \rangle} + q^{1 - \langle \log_2 x \rangle}\right) \\
&= x^{\log_2 q}\left(q^{-\langle \log_2 x \rangle} + (1-q)q^{-\langle \log_2 x \rangle}\left(1 - 2^{\langle \log_2 x \rangle}\right)\right) \\
&= x^{\log_2 q} f\left(\langle \log_2 x \rangle\right),
\end{aligned}$$

where symbol $\lfloor a \rfloor$ denotes the integer part of a real number $a$, symbol $\langle a \rangle$ denotes the fractional part of $a$ and function $f$ is defined by the following equality

$$f(u) = q^{-u} + (1-q)q^{-u}(1 - 2^u),\ 0 \leqslant u < 1.$$

For the function $f$ we have:

$$\begin{aligned}
&f(0) = f(1-0) = 1; \\
&f(u) \geqslant 1,\ u \in [0,1); \\
&f(u) \leqslant f(u_{\max}) = \left(\frac{2-q}{1 - \log_2 q}\right) q^{\log_2\left(\frac{1-q}{2-q}\left(1 - \frac{1}{\log_2 q}\right)\right)} := \mathrm{C}_q; \\
&u_{\max} = \log_2 \frac{(2-q)\log(1/q)}{(1-q)\log(2/q)} \in (0,1).
\end{aligned}$$

Consequently, for $x \geqslant 1$:

$$
\begin{aligned}
&x^{-\log_2(1/q)} \leqslant \overline{F}_\xi(x) \leqslant \mathrm{C}_q\, x^{-\log_2(1/q)},\\
&\mathbb{E}\left(\xi^\alpha \mathbb{I}_{\{\xi > x\}}\right) \geqslant \frac{\log_2(1/q)}{\log_2(1/q)-\alpha}\, x^{\alpha-\log_2(1/q)}, \alpha \in [0, \log_2(1/q)),\\
&\mathbb{E}\left(\xi^\alpha \mathbb{I}_{\{\xi > x\}}\right) \leqslant \frac{\mathrm{C}_q \log_2(1/q)}{\log_2(1/q)-\alpha}\, x^{\alpha-\log_2(1/q)}, \alpha \in [0, \log_2(1/q)),\\
&\mathbb{E}\left((\xi - x)^+\right)^\alpha \leqslant \alpha\, \mathrm{C}_q \int_x^\infty (u-x)^{\alpha-1} u^{\log_2 q} \mathrm{d}u\\
&\qquad\qquad = \alpha \mathrm{C}_q\, B\big(\alpha, \log_2(1/q)-\alpha\big) x^{\alpha-\log_2(1/q)}, \alpha \in (0, \log_2(1/q)),\\
&\mathbb{E}\left((\xi - x)^+\right)^\alpha \geqslant \alpha\, B\big(\alpha, \log_2(1/q)-\alpha\big) x^{\alpha-\log_2(1/q)}, \alpha \in (0, \log_2(1/q)),
\end{aligned}
$$

where $B$ denotes the Beta function

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t, a > 0, b > 0.$$

These relations and theorems 5.1.1, 5.1.2 imply that

$$\frac{n \log_2(1/q)}{\log_2(1/q)-\alpha}\, x^{\alpha-\log_2(1/q)} \underset{x\to\infty}{\lesssim} \mathbb{E}\left((S_n)^\alpha\, \mathbb{I}_{\{S_n > x\}}\right) \underset{x\to\infty}{\lesssim} \frac{n\mathrm{C}_q \log_2(1/q)}{\log_2(1/q)-\alpha}\, x^{\alpha-\log_2(1/q)}$$

for $n \in \mathbb{N}$, $q \in (0,1)$ and $\alpha \in [0, \log_2(1/q))$ and

$$
\begin{aligned}
&\mathbb{E}\left((S_n - x)^+\right)^\alpha \underset{x\to\infty}{\lesssim} n\alpha \mathrm{C}_q\, B\big(\alpha, \log_2(1/q)-\alpha\big) x^{\alpha-\log_2(1/q)},\\
&\mathbb{E}\left((S_n - x)^+\right)^\alpha \underset{x\to\infty}{\gtrsim} n\alpha\, B\big(\alpha, \log_2(1/q)-\alpha\big) x^{\alpha-\log_2(1/q)}
\end{aligned}
$$

for all $n \in \mathbb{N}$, $q \in (0,1)$ and $\alpha \in (0, \log_2(1/q))$.

The derived asymptotic formulas imply the following particular cases:

$$\frac{n}{\log_2(1/q)-1}\, x^{1-\log_2(1/q)} \underset{x\to\infty}{\lesssim} \mathbb{E}\left((S_n - x)^+\right) \underset{x\to\infty}{\lesssim} \frac{n\mathrm{C}_q}{\log_2(1/q)-1}\, x^{1-\log_2(1/q)}$$

if $q \in (0, 1/2)$;

$$\mathbb{E}\left((S_n - x)^+\right)^2 \underset{x\to\infty}{\gtrsim} \frac{2n}{(\log_2(1/q)-1)(\log_2(1/q)-2)}\, x^{2-\log_2(1/q)},$$
$$\mathbb{E}\left((S_n - x)^+\right)^2 \underset{x\to\infty}{\lesssim} \frac{2n\mathrm{C}_q}{(\log_2(1/q)-1)(\log_2(1/q)-2)}\, x^{2-\log_2(1/q)}$$

if $q \in (0, 1/4)$.

**Example 5.3.2.** *Let r.v.s $\xi_1, \xi_2, \ldots, \xi_n$, $n \geqslant 2$, be pQAI. Suppose that $\xi_1$ is distributed according to the following tail function*

$$\overline{F}_{\xi_1}(x) = \exp\left\{-\lfloor \log(1+x)\rfloor + (\log(1+x) - \lfloor \log(1+x)\rfloor)^{1/2}\right\},\ x \geqslant 0.$$

*For other indices $k \in \{2, \ldots, n\}$ let us suppose that*

$$\overline{F}_{\xi_k}(x) = \mathbb{I}_{\{x<0\}} + \mathrm{e}^{-x/k}\mathbb{I}_{\{x\geqslant 0\}}.$$

*Like in Example 5.3.1 we write asymptotic formulas for the left truncated moments*

$$\mathbb{E}\Big(\big(S_n\big)^{\alpha}\mathbb{I}_{\{S_n>x\}}\Big) \ \text{ and }\ \mathbb{E}\left(\big(S_n - x\big)^{+}\right)^{\alpha}$$

*in the case of suitable $\alpha$.*

It is obvious that $\mathbb{P}(|\xi_k| > x) = o\left(\overline{F}_{\xi_1}(x)\right)$ for $k \in \{2, \ldots, n\}$, and, further, $F_{\xi_1} \in \mathcal{C} \setminus \mathcal{R}$ due to results of [15] (see page 87).

Therefore, Theorem 5.1.2 implies that

$$\mathbb{E}\left((S_n)^{\alpha}\, \mathbb{I}_{\{S_n>x\}}\right) \underset{x\to\infty}{\sim} \mathbb{E}\left(\xi_1^{\alpha}\mathbb{I}_{\{\xi_1>x\}}\right)$$

for $\alpha \in [0, 1)$, and

$$\mathbb{E}\left((S_n - x)^+\right)^{\alpha} \underset{x\to\infty}{\sim} \mathbb{E}\left((\xi_1 - x)^+\right)^{\alpha}$$

for $\alpha \in (0, 1)$.

Consequently,

$$\mathbb{P}(S_n > x) \underset{x\to\infty}{\sim} \exp\left\{-\lfloor \log(1+x)\rfloor + (\log(1+x) - \lfloor \log(1+x)\rfloor)^{1/2}\right\},$$
$$\mathbb{P}(S_n > \mathrm{e}^n - 1) \underset{n\to\infty}{\sim} \frac{1}{\mathrm{e}^n},$$

and, for $\alpha \in (0,1)$

$$\begin{aligned}
\frac{1}{1-\alpha}\, x^{\alpha-1} &\underset{x\to\infty}{\lesssim} \mathbb{E}\left((S_n)^{\alpha}\, \mathbb{I}_{\{S_n>x\}}\right) \underset{x\to\infty}{\lesssim} \frac{\mathrm{e}^2}{1-\alpha}\, x^{\alpha-1}, \\
\frac{\alpha\pi}{\sin(\alpha\pi)}\, x^{\alpha-1} &\underset{x\to\infty}{\lesssim} \mathbb{E}\left(\left((S_n-x)^{+}\right)^{\alpha}\right) \underset{x\to\infty}{\lesssim} \frac{\alpha\pi\mathrm{e}^2}{\sin(\alpha\pi)}\, x^{\alpha-1}.
\end{aligned}$$

# Chapter 6

# Santrauka

## 6.1 Disertacinio darbo aprašymas

### 6.1.1 Disertacijos mokslinė problema ir tyrimo objektas

Sunkiauodegių skirstinių klasių tyrimai sulaukė daug dėmesio tiek iš teorinės, tiek iš praktinės pusės tokiose srityse kaip draudimas, komunikacijos tinklai, ekonomika, fizika.

Labiausiai ištirta sunkiauodegių skirstinių klasė yra reguliariai kintantys skirstiniai. Ši klasė buvo pristatyta Karamata [35] realaus kintamojo funkcijų analizės kontekste. Reguliaraus kintamumo sąvoka tikimybių teorijoje buvo pristatyta Feller [28], kur nagrinėjamos ribinės teoremos nepriklausomų ir vienodai pasiskirsčiusių atsitiktinių dydžių sumoms. Daug analitinių rezultatų apie reguliariai kintančias funkcijas galime rasti Bingham ir kt. [4] monografijoje. Reguliariai kintančių skirstinių taikymai finansų ir draudimo srityse pristatomi Embrechts ir kt. [25].

Šalia reguliariai kintančių skirstinių, pastaruoju metu sunkiauodegių skirstinių tyrimuose taip pat sutinkamos tokios klasės kaip subeksponentiniai, nuosaikiai kintantys, ilgauodegiai ir dominuojamai kintantys skirstiniai. Iš naujausių knygų nagrinėjančių sunkiauodegių skirstinių klasių savybes galima paminėti Foss ir kt. [29] ir Konstantinides [37]. Nuosaikiai kintančių skirstinių klasė buvo pristatyta kaip reguliariai kintančių skirstinių klasės apibendrinimas [14], ir šiame straipsnyje pavadinta ”tarpiniais reguliariai kintančiais skirstiniais”. Nuosaikaus kitimo sąvoka buvo naudota daugelyje straipsnių taikomosios tikimybių teorijos srityse, tokiose kaip eilių teorija, grafų teorija ir rizikos teorija, pvz. [2, 5–9, 15, 29, 38, 45, 69].

Svarbi pastaruoju metu nagrinėjama problema yra sunkiauodegių skirstinių klasių uždarumas. Uždarumo savybė reiškia, kad turint kelis skirstinius iš tam tikros klasės jų transformacija (pvz. sumos sąsūka, daugybos sąsūka, mišinys, minimumas, maksimumas) priklauso tai pačiai skirstinių klasei. Šios savybės tyrimas yra ne vien papildomas įrankis įrodinėjant įvairius sunkiauodegių skirstinių klasių asimptotinius rezultatus, bet taip pat ir savaime įdomi matematinė problema. Atvirkštinė problema sąsūkos uždarumui yra vadinama sąsūkos šaknies uždarumo problema, kuri kelia klausimą ar skirstinių sąsūkos priklausymas tam tikrai klasei lemia pradinio skirstinio priklausymą šiai klasei. Iš svarbių straipsnių apie sunkiauodegių skirstinių klasių uždarumą galima paminėti Embrechts ir kt. [24] bei Cline ir Samorodnitsky [15].

Tarp kitų aktualių uždavinių susijusių su sunkiauodegiais skirstiniais galima paminėti asimptotinį įvairių atsitiktinių dydžių transformacijų elgesį. Tokio tipo problema yra kairiųjų nupjautinių atsitiktinių sumų momentų asimptotinis elgesys. Ši problema buvo nagrinėta taikomosios tikimybių teorijos srityse, tokiose kaip rizikos teorija ir atsitiktiniai klaidžiojimai [17, 18, 51]. Taip pat ši atsitiktinių dydžių transformacija yra susijusi su Haezendonck-Goovaerts rizikos mato skaičiavimu [32, 39, 61].

### 6.1.2 Tyrimo tikslas ir pagrindiniai uždaviniai

Šio darbo tikslas yra sunkiauodegių skirstinių klasių asimptotinių savybių tyrimas. Su šiuo tikslu susiję šie uždaviniai:

(i) Tarkime, kad $\{\xi_1, \xi_2, \ldots\}$ yra nepriklausomų atsitiktinių dydžių seka, ir $\eta$ yra skaičiuojantis atsitiktinis dydis nepriklausantis nuo šios sekos. Randamos sąlygos atsitiktiniams dydžiams $\{\xi_1, \xi_2, \ldots\}$ ir $\eta$, prie kurių atsitiktinai sustabdytos sumos $S_\eta = \xi_1 + \xi_2 + \ldots + \xi_\eta$ pasiskirstymo funkcija priklauso skirstinių su nuosaikiai kintančia uodega klasei. Mūsų rezultatuose atsitiktiniai dydžiai $\{\xi_1, \xi_2, \ldots\}$ yra nebūtinai vienodai pasiskirstę.

(ii) Tarkime, kad $\{\xi_1, \xi_2, \ldots\}$ yra nepriklausomų realias reikšmes įgyjančių ir nebūtinai vienodai pasiskirsčiusių atsitiktinių dydžių seka, ir $\eta$ yra neneigiamas, neišsigimęs nulyje ir sveikareikšmis atsitiktinis dydis nepriklausantis nuo šios sekos. Randamos sąlygos atsitiktiniams dydžiams $\{\xi_1, \xi_2, \ldots\}$ ir $\eta$ prie kurių atsitiktinai sustabdytos sumos, minimumo ir maksimumo pasiskirstymo funkcijos priklauso skirstinių su reguliariai kintančia uodega klasei.

(iii) Nagrinėjame sumą $S_n = \xi_1 + \ldots + \xi_n$, kurią sudaro galimai priklausomi ir nevienodai pasiskirstę atsitiktiniai dydžiai $\xi_1, \ldots, \xi_n$, kurių skirstiniai turi nuosaikiai kintančią uodegą. Darant prielaidą, kad $\{\xi_1, \ldots, \xi_n\}$ turi priklausomybės struktūrą panašią į asimptotinį nepriklausomumą, randamos asimptotinės išraiškos kairiesiems nupjautiniams atsitiktinai sustabdytų sumų momentams.

### 6.1.3 Tyrimų metodika

Pagrindiniai disertacijos teiginiai įrodyti naudojant klasikinius tikimybių teorijos ir matematinės analizės metodus.

### 6.1.4 Rezultatų aprobavimas

Disertacijoje gauti rezultatai buvo pristatyti šiose konferencijose:

- European Actuarial Journal Conference, 22-24 August 2022, Tartu, Estonia.
- The Conference of Lithuanian Mathematical Society, 20-21 June 2019, Vilnius, Lithuania.
- The Conference of Young Scientists, 12 March 2019, Vilnius, Lithuania.

## 6.2 Pagrindiniai disertacijos moksliniai rezultatai

Skyriaus pradžioje pateiksime nagrinėjamų sunkiauodegių skirstinių klasių apibrėžimus. Uodegos funkcija apibrėžiama $\overline{F}(x) = 1 - F(x)$ visiems realiems $x$ ir laisvai pasirinktai pasiskirstymo funkcijai (p.f.) $F$.

- *P.f.* $F$ *yra sunkiauodegė* ($F \in \mathcal{H}$) *jei kiekvienam fiksuotam* $\delta > 0$

$$\limsup_{x\to\infty} \overline{F}(x)\mathrm{e}^{\delta x} = \infty.$$

- *P.f.* $F$ *turi nuosaikiai kintančią uodegą* ($F \in \mathcal{C}$) *jei*

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

- *P.f.* $F$ *turi reguliariai kintančią uodegą su indeksu* $\gamma \geqslant 0$, *žymima* $F \in \mathcal{R}_\gamma$, *jei kiekvienam* $y > 0$ *galioja lygybė*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\gamma}.$$

### 6.2.1 Skirstinių su nuosaikiai kintančia uodega atsitiktinai sustabdytos sumos

Tarkime, kad $\{\xi_1, \xi_2, \ldots\}$ yra seka nepriklausomų atsitiktinių dydžių (a.d.) su pasiskirstymo funkcijomis (p.f.) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, ir $\eta$ yra skaičiuojantis a.d., t.y. $\eta$ yra sveikareikšmis, neneigiamas ir neišsigimęs nulyje. Taip pat tarkime, kad $\eta$ ir $\{\xi_1, \xi_2, \ldots\}$ yra nepriklausomi. Pažymėkime $S_0 = 0$, $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$ visiems $n \in \mathbb{N}$ ir

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

a.d. $\{\xi_1, \xi_2, \ldots\}$ atsitiktinai sustabdytą sumą.

Mus domina sąlygos prie kurių $S_\eta$ pasiskirstymo funkcija

$$F_{S_\eta}(x) = \mathbb{P}(S_\eta \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n \leqslant x)$$

turi nuosaikiai kintančią uodegą.

**Teorema 6.2.1.1.** *Tarkime, kad* $\{\xi_1, \xi_2, \ldots, \xi_D\}, D \in \mathbb{N}$, *yra nepriklausomi a.d., ir* $\eta$ *yra skaičiuojantis a.d. nepriklausomas nuo* $\{\xi_1, \xi_2, \ldots, \xi_D\}$. *P.f.* $F_{S_\eta}$ *priklauso klasei* $\mathcal{C}$ *jei tenkinamos šios sąlygos:*

(a) $\mathbb{P}(\eta \leqslant D) = 1$,

(b) $F_{\xi_1} \in \mathcal{C}$,

(c) *kiekvienam* $k = 2, \ldots, D,\ F_{\xi_k} \in \mathcal{C}$ *arba* $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$.

**Teorema 6.2.1.2.** *Tarkime, kad* $\{\xi_1, \xi_2, \ldots\}$ *yra nepriklausomi a.d., ir* $\eta$ *yra skaičiuojantis a.d. nepriklausomas nuo* $\{\xi_1, \xi_2, \ldots\}$. *P.f.* $F_{S_\eta}$ *priklauso klasei* $\mathcal{C}$ *jei tenkinamos šios sąlygos:*

(a) $F_{\xi_1} \in \mathcal{C}$,

(b) *kiekvienam* $k \geqslant 2,\ F_{\xi_k} \in \mathcal{C}$ *arba* $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$,

(c) $\limsup\limits_{x\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n\overline{F}_{\xi_1}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$,

(d) $\mathbb{E}\eta^{p+1} < \infty$ *kažkuriam* $p > J^+_{F_{\xi_1}}$.

**Teorema 6.2.1.3.** *Tarkime, kad* $\{\xi_1, \xi_2, \ldots\}$ *yra nepriklausomi neneigiami a.d., ir* $\eta$ *yra skaičiuojantis a.d. nepriklausomas nuo* $\{\xi_1, \xi_2, \ldots\}$. *P.f.* $F_{S_\eta}$ *priklauso klasei* $\mathcal{C}$ *jei tenkinamos šios sąlygos:*

(a) $F_{\xi_1} \in \mathcal{C}$,

(b) *kiekvienam* $k \geqslant 2,\ F_{\xi_k} \in \mathcal{C}$ *arba* $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$,

(c) $\mathbb{E}\xi_1 < \infty$,

(d) $\overline{F}_\eta(x) = o\left(\overline{F}_{\xi_1}(x)\right)$,

(e) $\limsup\limits_{x\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n\overline{F}_{\xi_1}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$,

(f) $\limsup\limits_{u\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n} \sum\limits_{\substack{k=1 \\ \mathbb{E}\xi_k \geqslant u}}^{n} \mathbb{E}\xi_k = 0$.

### 6.2.2 Reguliariai pasiskirsčiusi atsitiktinė suma, minimumas ir maksimumas

Tarkime, kad $\{\xi_1, \xi_2, \ldots\}$ yra seka nepriklausomų atsitiktinių dydžių (a.d.) su pasiskirstymo funkcijomis (p.f.) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, ir $\eta$ yra skaičiuojantis a.d., t.y. $\eta$ yra sveikareikšmis, neneigiamas ir neišsigimęs nulyje. Taip pat tarkime, kad $\eta$ ir $\{\xi_1, \xi_2, \ldots\}$ yra nepriklausomi. Pažymėkime $S_0 = 0$, $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$ visiems $n \in \mathbb{N}$ ir

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

a.d. $\{\xi_1, \xi_2, \ldots\}$ atsitiktinai sustabdytą sumą.

Pažymėkime $\xi^{(0)} := 0$, $\xi^{(n)} := \max\{0, \xi_1, \ldots, \xi_n\}$ visiems $n \in \mathbb{N}$, ir

$$\xi^{(\eta)} := \begin{cases} 0 & \text{jei } \eta = 0, \\ \max\{0, \xi_1, \ldots, \xi_\eta\} & \text{jei } \eta \geqslant 1, \end{cases}$$

a.d. $\xi_1, \xi_2, \ldots$ atsitiktinį maksimumą.

Pažymėkime $\xi_{(0)} := 0$, $\xi_{(n)} := \min\{\xi_1, \ldots, \xi_n\}$ visiems $n \in \mathbb{N}$, ir

$$\xi_{(\eta)} := \begin{cases} 0 & \text{jei } \eta = 0, \\ \min\{\xi_1, \ldots, \xi_\eta\} & \text{jei } \eta \geqslant 1, \end{cases}$$

a.d. $\xi_1, \xi_2, \ldots$ atsitiktinį minimumą.

Mus domina sąlygos prie kurių p.f. $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, ir $F_{S_\eta}$ turi reguliariai kintančią uodegą.

**Teorema 6.2.2.1.** *$F_{\xi_k} \in \mathcal{R}$ visiems $k \in \mathbb{N}$ tada ir tik tada, kai $F_{\xi_{(\eta)}} \in \mathcal{R}$ kiekvienam skaičiuojančiam a.d. $\eta$.*

**Teorema 6.2.2.2.** *Tarkime, kad $\mathbb{P}(\eta \leqslant m) = 1$. P.f. $F_{S_\eta}$ and $F_{\xi^{(\eta)}}$ priklauso klasei $\mathcal{R}_\alpha$, $\alpha \geqslant 0$, jei tenkinamos šios sąlygos:*

(i) $F_{\xi_1} \in \mathcal{R}_\alpha$,

(ii) *kiekvienam $k \geqslant 2$ $F_{\xi_k} \in \mathcal{R}_\alpha$ arba $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_1}(x)\right)$.*

*Taip pat prie sąlygų* (i)–(ii) *galioja šie sąryšiai:*

$$\overline{F}_{\xi^{(\eta)}}(x) \sim \overline{F}_{S_\eta}(x) \sim x^{-\alpha} \sum_{n=1}^{m} \mathbb{P}(\eta = n) \sum_{k \in \mathcal{I}_n} L_k(x),$$

*kur* $\mathcal{I}_n = \{k = 1, \ldots, n : F_{\xi_k} \in \mathcal{R}_\alpha\}$, *ir* $L_k$ *yra lėtai kintančios funkcijos iš reprezentacijų* $\overline{F}_{\xi_k}(x) = x^{-\alpha} L_k(x)$.

**Teorema 6.2.2.3.** *Tarkime, kad* $\overline{F}_{\xi_k}(x) \asymp \overline{F}_{\xi_1}(x)$ *visiems* $k \geqslant 2$. *Tuomet šie teiginiai ekvivalentūs:*

(i) $F_{\xi_k} \in \mathcal{R}_+$ *visiems* $k \in \mathbb{N}$,

(ii) $F_{S_\eta} \in \mathcal{R}_+$ *visiems* $\eta$ *su baigtine atrama,*

(iii) $F_{\xi^{(\eta)}} \in \mathcal{R}_+$ *visiems* $\eta$ *su baigtine atrama.*

**Teorema 6.2.2.4.** *Tarkime, kad patenkintos šios sąlygos:*

(i) $F_{\xi_1} \in \mathcal{R}_\alpha, \alpha \geqslant 0$,

(ii) $\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right| = 0$ *sekai neneigiamų konstantų* $\{d_1 = 1, d_2, d_3, \ldots\}$ *su savybe* $\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d_k < \infty$,

(iii) $\mathbb{E}\eta^{p+1} < \infty$ *kažkuriems* $p > \alpha$.

*Tada galioja sąryšis*

$$\overline{F}_{S_\eta}(x) \sim \overline{F}_{\xi_1}(x) \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \sum_{k=1}^{n} d_k,$$

*ir todėl* $F_{S_\eta} \in \mathcal{R}_\alpha$.

**Teorema 6.2.2.5.** *Tarkime, kad* $\overline{F}_{\xi_1}(x) > 0$ *visiems* $x \in \mathbb{R}$. *Taip pat tarkime, kad galioja sąlygos*

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right| = 0 \quad ir \quad \max\left\{ \mathbb{E}\eta, \mathbb{E}\Big( \sum_{k=1}^{\eta} d_k \Big) \right\} < \infty$$

*sekai neneigiamų konstantų* $\{d_1 = 1, d_2, \ldots\}$. *Tada*

$$\overline{F}_{\xi^{(\eta)}}(x) \sim \overline{F}_{\xi_1}(x)\,\mathbb{E}\Big(\sum_{k=1}^{\eta} d_k\Big),$$

*ir todėl* $F_{\xi^{(\eta)}}$ *priklauso klasei* $\mathcal{R}_\alpha$ *tada ir tik tada, kai* $F_{\xi_1}$ *priklauso klasei* $\mathcal{R}_\alpha,\ \alpha \geqslant 0$.

### 6.2.3 Asimptotinės formulės kairiesiems nupjautiniams atsitiktinių dydžių su nuosaikiai kintančiomis uodegomis sumų momentams

Sakykime, kad $n \in \mathbb{N} := \{1, 2, \ldots\}$, ir $\{\xi_1, \ldots, \xi_n\}$ yra seka a.d., kurių skirstiniai turi nuosaikiai kintančias uodegas. Pažymėkime

$$S_n \quad := \quad \xi_1 + \cdots + \xi_n.$$

Laikykime, kad $\xi_1, \ldots, \xi_n$ yra poromis kvazi-asimptotiškai nepriklausomi. Pažymėkime $x^+ := \max\{0, x\}$, $x^- := \max\{0, -x\}$.

**Apibrėžimas 6.2.3.1.** *A.d.* $\xi_1, \ldots, \xi_n$ *vadinami poromis kvazi-asimptotiškai nepriklausomais (pQAI), jei visoms indeksų poroms* $k, l \in \{1, 2, \ldots, n\}$, $k \neq l$, *galioja lygybės*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^+ > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} \;=\; \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^- > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} \;=\; 0.$$

**Teorema 6.2.3.1.** *Tarkime, kad* $\{\xi_1, \ldots, \xi_n\}$ *yra seka pQAI a.d. tokių, kad* $F_{\xi_k} \in \mathcal{C}$ *ir* $\mathbb{E}\left(\xi_k^+\right)^\alpha < \infty$ *visiems* $k \in \{1, \ldots, n\}$ *ir kažkuriems* $\alpha \geqslant 0$. *Tada*

$$\mathbb{E}\Big(\left(S_n\right)^\alpha \mathbb{1}_{\{S_n > x\}}\Big) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \mathbb{E}\left(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\right).$$

**Teorema 6.2.3.2.** *Tarkime, kad* $\{\xi_1, \ldots, \xi_n\}$ *yra a.d. seka tokia kad, kiekvienam* $k \in \{1, \ldots, n\}$ *galioja* $F_{\xi_k} \in \mathcal{C}$ *arba* $\mathbb{P}\left(|\xi_k| > x\right) = o\left(\overline{F}_{\xi_1}(x)\right)$. *Taip pat tarkime, kad* $F_{\xi_1} \in \mathcal{C}$ *ir* $\mathbb{E}\left(\xi_k^+\right)^\alpha < \infty$ *visiems* $k \in \{1, \ldots, n\}$ *ir kažkuriems* $\alpha \geqslant 0$. *Tarkime, kad* $\mathcal{I} \subseteq \{1, \ldots, n\}$ *yra poaibis indeksų* $k$ *tokių, kad* $F_{\xi_k} \in \mathcal{C}$. *Jei rinkinys* $\{\xi_k, k \in \mathcal{I}\}$ *yra* *sudarytas iš pQAI a.d., tuomet kiekvienam* $\beta \in [0, \alpha]$ *galioja*

$$\mathbb{E}\Big(\big(S_n\big)^{\beta}\mathbb{I}_{\{S_n>x\}}\Big) \;\sim\; \sum_{k\in\mathcal{I}}\mathbb{E}\left(\xi_k^{\beta}\mathbb{I}_{\{\xi_k>x\}}\right),$$

*ir kiekvienam* $\beta \in (0, \alpha]$ *galioja*

$$\mathbb{E}\left(\left(S_n - x\right)^{+}\right)^{\beta} \;\sim\; \sum_{k\in\mathcal{I}}\mathbb{E}\left(\left(\xi_k - x\right)^{+}\right)^{\beta}.$$

## 6.3 Trumpos žinios apie disertantą

**Gimimo data ir vieta**

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**Išsilavinimas**

2012 – 2016 m. Vilniaus universitetas, Matematikos ir informatikos fakultetas, finansų ir draudimo matematikos bakalauras

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## 6.4 Publikacijų sąrašas

Pristatomos disertacijos rezultatai publikuojami moksliniuose straipsniuose:

- Edita Kizinevič, Jonas Sprindys, Jonas Šiaulys. Randomly stopped sums with consistently varying distributions. *Modern Stochastics: Theory and Applications* (2016).

- Jonas Sprindys, Jonas Šiaulys. Regularly distributed randomly stopped sum, minimum, and maximum. *Nonlinear Analysis: Modelling and Control* (2020).

- Jonas Sprindys, Jonas Šiaulys. Asymptotic formulas for the left truncated moments of sums with consistently varying distributed increments. *Nonlinear Analysis: Modelling and Control* (2021).

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NOTES

NOTES

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