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Weighted Universality Theorems for the Riemann and Hurwitz Zeta-Functions

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Svertinės universalumo teoremos Rymano ir Hurvico dzeta funkcijoms

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Notation

j, k, l, m, n	natural numbers
p	prime number
$\mathbb{P}$	set of all prime numbers
$\mathbb{N}$	set of all natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	set of all integer numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
i	imaginary unity: $i = \sqrt{-1}$
$s = \sigma + it$, $\sigma, t \in \mathbb{R}$	complex variable
$\bigoplus\limits_{m} A_m$	direct sum of sets A_m
$A \times B$	Cartesian product of the sets A and B
$\prod\limits_{m} A_m$	Cartesian product of sets A_m
A^m	Cartesian product of m copies of the set A
$\mathrm{meas}A$	Lebesgue measure of the set $A \subset \mathbb{R}$
$\#A$	cardinality of the set A
$H(G)$	space of analytic functions on G
$\mathcal{B}(\mathbb{X})$	class of Borel sets of the space $\mathbb{X}$
$\mathbb{E}X$	expectation of the random variable
$\xrightarrow{\mathcal{D}}$	convergence in distribution
$\Gamma(s)$	Euler gamma-function
$\zeta(s)$	Riemann zeta-function
$\zeta(s, \alpha)$	Hurwitz zeta-function
$a \ll_\eta b$, $b > 0$	there exists a constant $C = C(\eta) > 0$ such that $\lvert a \rvert \leqslant Cb$

Chapter 1

Introduction

1.1 Research topic

In the dissertation, we consider some analytic properties of the Riemann zeta-function $\zeta(s)$ and of its generalization the Hurwitz zeta-function $\zeta(s,\alpha)$, $s=\sigma+it$ and $0<\alpha\leqslant 1$ is a fixed parameter. We recall that these zeta-function are defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}\quad \zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},$$

and have analytic continuations to the whole complex plane, except for the point $s=1$ which is their simple pole with residue 1. More precisely, the dissertation is devoted to the weighted universality of the functions $\zeta(s)$ and $\zeta(s,\alpha)$ for some classes of the parameter α, i. e., to the approximation of analytic functions defined in the strip $\{s\in\mathbb{C}:1/2<\sigma<1\}$ by shifts $\zeta(s+i\tau)$ and $\zeta(s+it,\alpha)$, $\tau\in\mathbb{R}$, having a weighted positive lower density.

It is known that the Hurwitz zeta-function $\zeta(s,\alpha)$ is a generalization of the Riemann zeta-function because

$$\zeta(s,1)=\sum_{m=0}^{\infty}\frac{1}{(m+1)^s}=\sum_{m=1}^{\infty}\frac{1}{m^s}=\zeta(s),\quad \sigma>1.$$

On the other hand, the functions $\zeta(s)$ and $\zeta(s,\alpha)$ are quite different. The function $\zeta(s)$ has the Euler product over primes, i. e., for $\sigma > 1$,

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

while the function $\zeta(s, \alpha)$ has such a product only in the case $\alpha = 1/2$, i. e., for $\sigma > 1$,

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s) = (2^s - 1) \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Hence, some differences for the value distribution of the functions $\zeta(s)$ and $\zeta(s, \alpha)$ follow. For example, the function $\zeta(s) \neq 0$ on the half-plane $\sigma > 1$ [69], while the function $\zeta(s, \alpha)$, with $\alpha \neq 1/2$ has infinitely many zeros lying in the latter region [12], [13] and [9].

The function $\zeta(s)$, for all $s \in \mathbb{C}$, satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

From this equation, it follows that $\zeta(-2k) = 0$ for all $k \in \mathbb{N}$, and the numbers $s = -2k$ are called the trivial zeros of $\zeta(s)$. Moreover, the function $\zeta(s)$ has infinitely many complex nontrivial zeros lying in the so-called critical strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$. The Riemann hypothesis asserts that all nontrivial zeros lie on the critical line $\sigma = 1/2$. At the moment, it is known that more than $5/12$ of non-trivial zeros in the sense of density are on the critical line [63].

The function $\zeta(s)$ with a real variable s was already known to L. Euler, however, the importance of this function was opened by B. Riemann. In [65], he began to study $\zeta(s)$ as a function of a complex variable, proposed a way how to use $\zeta(s)$ in the investigation of the distribution of prime numbers, and stated several hypotheses concerning the nontrivial zeros of $\zeta(s)$. Let, for $x \geqslant 2$,

$$\pi(x) = \sum_{p \leqslant x} 1,$$

i. e., $\pi(x)$ is the number of prime numbers not exceeding x. Using Riemann's ideas, C. J. de la Vallée Poussin [70] and J. Hadamard [22] independently proved the asymptotic distribution law of prime numbers, i. e., that

$$\lim_{x\to\infty} \frac{\pi(x)}{\int_2^x \frac{\mathrm{du}}{\log u}} = 1.$$

To prove the above asymptotic equality, the non-vanishing of $\zeta(s)$ in the region $\sigma \geqslant 1$ is applied. Using of the zero-free region

$$\sigma > 1 - \frac{c}{\log t}, \quad t \geqslant t_0,\, c > 0,$$

gives the estimate for the reminder term in the distribution low of prime numbers.

The function $\zeta(s, \alpha)$ was introduced by A. Hurwitz in [23], and is used in the theory of Dirichlet *L*-functions which are the main tool for the investigation of prime numbers in arithmetic progressions, i. e., for the asymptotics of

$$\pi(x, a, q) = \sum_{\substack{p\leqslant x\\ p\equiv a (\mathrm{mod}\, q)}} 1, \quad x \to \infty,$$

where a and q are coprime positive integers.

1.2 Aims and problems

The aim of the dissertation is the weighted universality and joint universality of the Riemann and Hurwitz zeta-functions. The problems considered are the following:

1. Weighted discrete universality of the Riemann zeta-function.
2. Weighted continuous universality of the Hurwitz zeta-function.
3. Weighted joint continuous universality of Hurwitz zeta-functions.
4. Weighted mixed joint continuous universality for the Riemann and Hurwitz zeta-functions.

1.3 Actuality

Approximation problems of analytic functions are ones of the most important chapters of modern mathematics. In the eights decade of the 20th century

[72], it became known that wide classes of analytic functions can be approximated by shifts of zeta-functions which are widely studied in analytic number theory. Thus, approximation of analytic functions was reduced in a certain sense to approximation properties of zeta-functions. This observation opened new mathematical problems. Among them, the definition of classes of zeta-functions having approximation properties, the effectivization problems of approximation, various versions of approximation, etc.

In some fields of mathematics (number theory, probability theory, mathematical statistics) weighted theorems and their applications are often investigated. Weighted universality of zeta-functions is comparatively a new branch of universality, there exist only few papers in that direction. Therefore, in our opinion, it was important to continue investigation of weighted universality of the classical Riemann and Hurwitz zeta-functions. Also, universality of zeta-functions is one of directions of the Lithuanian school of analytic number theory, the study of weighted universality continues traditions of Lithuanian researchers.

1.4 Methods

The proofs of weighted universality theorems for the Riemann and Hurwitz zeta-functions include elements of the theory of Dirichlet series, of weak convergence of probability measures, of Fourier analysis and measure theory.

1.5 Novelty

All results obtained in the dissertation are new. A weighted discrete universality theorem for the Rieman zeta-function is proved for a new class of weight functions. Weighted universality theorems for the Hurwitz zeta-function earlier were not investigated.

1.6 History of the problem and the main results

The Riemann zeta-function $\zeta(s)$ and the Hurwitz zeta-function $\zeta(s, \alpha)$ are the main classical zeta-functions. Therefore, there are many results devoted to value distribution of these functions. The classical theory of the function $\zeta(s)$ can be found in the monographs [69], [15], [24], [32] as well in the books [62], [31] and [25]. The probabilistic theory of $\zeta(s)$ is given in the monographs

[35] and [42]. The main attention in the classical theory of the Riemann zeta-function is devoted to the distribution of non-trivial zeros and allied problems. In applications, the zero-free regions play a crucial role. The best known result in this direction says that there exists an absolute constant $c > 0$ such that $\zeta(s) \neq 0$ for

$$\sigma > 1 - \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}}, \quad t \geqslant t_0.$$

Also, the moments

$$\mathrm{I}_k(\sigma, T) \overset{def}{=} \int_1^T |\zeta(\sigma + it)|^{2k}\mathrm{dt}, \quad \sigma \geqslant \frac{1}{2},\ k > 0,$$

and estimate for $\zeta(s)$ are widely studied. For example, there exists a conjecture that

$$\mathrm{I}_k\left(\frac{1}{2}, T\right) \sim c_k T(\log T)^{k^2}, \quad T \to \infty,$$

with a certain constant c_k depending on k. However, at the moment the above asymptotics are only known for $k = 1$ and $k = 2$. We note that the moments of $\zeta(s)$ are very important objects because, in the investigation of some problems, concrete values of $\zeta(s)$ can be replaced by its moments.

There exists the Lindelöf hypothesis that

$$\zeta\left(\frac{1}{2} + it\right) \ll_\varepsilon t^\varepsilon, \quad t \geqslant t_0,$$

with every $\varepsilon > 0$. This hypothesis is equivalent to the estimate

$$\mathrm{I}_k\left(\frac{1}{2}, T\right) \ll_k T(\log T)^{k^2}$$

for all $k \in \mathbb{N}$.

The classical analytic theory of the Hurwitz zeta-function can be found in the books [69], [24], [32], [62]. The function $\zeta(s, \alpha)$ is a partial case of the Lerch zeta-function $L(\lambda, \alpha, s)$, $\lambda \in \mathbb{R}$, which is defined, for $\sigma > 1$, by the Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m}}{(m + \alpha)^s},$$

and has a meromorphic continuation to the whole complex plane. Clearly, $L(k, \alpha, s) = \zeta(s, \alpha)$ with $k \in \mathbb{Z}$. Therefore, modern investigations of the function $\zeta(s, \alpha)$ can be found in the monograph [42].

The idea of applications of probability methods in the theory of zeta-functions belongs to H. Bohr. He proposed to take certain sets on the complex plane and to consider how often the values of a given zeta-function lie in those sets. It turned out that this idea leads to probabilistic limit theorems. H. Bohr jointly with B. Jessen proved [8] a limit theorem for the Riemann zeta-function. Let $\mathcal{R}$ be a rectangle on the complex plane with edges parallel to the axles, and let m_J denote the Jordan measure. Then they proved that, for $\sigma > 1$, there exists the limit

$$\lim_{T\to\infty} \frac{1}{T} m_J\{t \in [0, T] : \zeta(\sigma + it) \in \mathcal{R}\}.$$

Two years later, they generalized the above result for $\sigma > 1/2$.

A Bohr's idea was developed by A. Selberg (unpublished) and other number theorists. It is convenient to state probabilistic limit theorems for zeta-functions in terms of weakly convergent probability measures (probability distributions). Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, i. e., the σ-field generated by open sets of the space $\mathbb{X}$. Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We say that P_n, as $n \to \infty$, converges weakly to P, if, for every real continuous bounded function g on $\mathbb{X}$,

$$\lim_{n\to\infty} \int_{\mathbb{X}} g \, \mathrm{d}P_n = \int_{\mathbb{X}} g \, \mathrm{d}P.$$

Using the latter terminology, the mentioned limit theorem of Bohr-Jessen can be stated as follows, see, for example, [35]. Suppose that $\sigma > 1/2$ is fixed. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure P such that

$$\frac{1}{T}\text{meas}\{t \in [0, T] : \zeta(\sigma + it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P as $T \to \infty$.

Other limit theorems for the function $\zeta(s)$ and other zeta-functions can be found in [26], [35] and [42]. Also, see a paper [55].

Now we focus on the universality of zeta-functions which is the subject of our investigations. The universality of zeta-functions is a certain new direction on denseness of values of zeta-functions. The first results of such a kind were obtained by H. Bohr. In [6], he observed that the function $\zeta(s)$ takes every non-zero value infinitely many times in the strip $\{s \in \mathbb{C} : 1 < \sigma < 1 + \delta\}$, with every $\delta > 0$. A bit later, H. Bohr and R. Courant proved [7] that, for every fixed σ, $1/2 < \sigma \leqslant 1$, the set

$$\{\zeta(\sigma + i\tau) : \tau \in \mathbb{R}\} \tag{1.1}$$

is dense in $\mathbb{C}$. S. M. Voronin significantly generalized the above results. He obtained [71] that the set

$$\{(\zeta(s_1 + i\tau), \dots, \zeta(s_n + i\tau)) : \tau \in \mathbb{R}\}$$

with every fixed numbers $s_1, \dots, s_n \in \mathbb{C}$, $1/2 < \text{Re}s_k < 1$, $1 \leqslant k \leqslant n$ and $s_k \neq s_m$ for $k \neq m$, and the set

$$\left\{(\zeta(s + i\tau), \zeta'(s + i\tau), \dots, \zeta^{(n-1)}(s + i\tau) : \tau \in \mathbb{R})\right\}$$

with every $s \in \mathbb{C}$, $1/2 < \sigma < 1$, is dense in $\mathbb{C}^n$. However, a much more important merit of Voronin is his so-called universality theorem for the function $\zeta(s)$ [72]. He proved in [72], that if, for every $0 < r < \frac{1}{4}$, the function $f(s)$ is continuous and has no zeros in the disc $|s| \leqslant r$, and is analytic in the open disc $|s| < r$, then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Thus, Voronin proved that a wide class of analytic functions can be approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, of the function $\zeta(s)$. This famous result was observed by mathematical community and improved and extended in various directions. For a modern version of the Voronin universality theorem it is convenient to use the following notation. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, i. e., D is the right-hand side of the critical strip of the function $\zeta(s)$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then the following theorem is true.

Theorem A. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

By Theorem A, the set of shifts $\zeta(s + it)$ approximating a given function $f(s)$ from the class $H_0(K)$ has a positive lower density, thus, it is infinite.

Denote by $H(G)$, where G is a region on the complex plane, the space of analytic functions on G equipped with the topology of uniform convergence on compacta. In this topology, $\{g_n(s)\} \subset H(G)$ converges to $g(s) \in H(G)$ as $n \to \infty$ if and only if, for every compact set $K \in G$,

$$\lim_{n\to\infty} \sup_{s\in K} |g_n(s) - g(s)| = 0.$$

Since the space $H(D)$ is infinite-dimensional, Theorem A can be considered as an infinite-dimensional generalization of the Bohr-Courant theorem on denseness of the set (1.1). The proof of Theorem A is given in [1] (in slightly different form), and in [21], [35] and [68].

Theorem A is of continuous type: τ in $\zeta(s + i\tau)$ can take arbitrary real values. Also, a discrete version of Theorem A is known when τ takes values from a certain discrete set. Let $h > 0$ be a fixed number, and N run over the set $\mathbb{N}_0$.

Theorem B. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s + ikh) - f(s)| < \varepsilon \right\} > 0.$$

The proof of Theorem B can be found in [1] and [64].

In place of shifts $\zeta(s+ikh)$, more general shifts can be used. For example, in [14], the shifts $\zeta(s+ikh)$ with fixed α, $0 < \alpha < 1$, were applied in Theorem B. Universality theorems with more complicated shifts for the function $\zeta(s)$ and Dirichlet *L*-functions were considered in the papers [60], [47], [19], [20], [41], [51]. For example, in [19] and [20], the shifts $\zeta(s + ih\gamma_k)$, where $\{\gamma_k\}$ is a sequence of positive imaginary parts of nontrivial zeros of the function $\zeta(s)$, were considered. We note that discrete universality theorems for zeta-functions sometimes are more convenient as continuous ones for practical applications. The paper [5] is an example of this remark.

Universality theorems of type of Theorems A and B are also known for other zeta-functions. For example, they are valid for zeta-functions of normalized Hecke eigen cusp forms, see [43], [44] and [45].

In Theorems A and B, the positivity of a lower density of the set of shifts $\zeta(s + i\tau)$ approximating a given function $f(s) \in H_0(K)$ is considered. In [56] and [46], it was obtained that sometimes a lower density can be replaced by a density. Thus, the following statement is valid.

Theorem C. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limits*

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\} > 0$$

and

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0$$

exist for all but at most countably many $\varepsilon > 0$.

The first weighted universality theorem was proposed in [34]. Let $w(t)$ be a function of bounded variation on $[T_0, \infty)$ with such $T_0 > 0$ such that the variation $V_b^a w$ on $[a,b]$ satisfies the inequality $V_b^a w \leqslant cw(a)$ with a certain constant $c > 0$ for any subinterval $[a,b] \subset [T_0, \infty)$. Define

$$U_T = U(T,w) = \int_{T_0}^{T} w(t)\mathrm{dt},$$

and suppose that

$$\lim_{T\to\infty} U(T,w) = +\infty.$$

The function $w(t)$ is called a weight function. In [34], one technical additional property for the function $w(t)$ was required. Let $\zeta(\omega,\tau)$ be an ergodic process defined on a certain probability space, $\tau \in \mathbb{R}$, $\mathbb{E}|\zeta(\omega,\tau)| < \infty$, with sample paths integrable almost surely over any finite interval. Then in [34], it was assumed that almost surely, for any $t \in \mathbb{R}$,

$$\frac{1}{U_T} \int_{T_0}^{T} w(\tau)\zeta(t+\tau,\omega)\,\mathrm{d}\tau = \mathbb{E}(\zeta(0,\omega)) + o(1+|t|^{\alpha}) \tag{1.2}$$

with $\alpha > 0$ as $T \to \infty$. The latter condition is a weighted analogue of the Birkhoff-Khinchine ergodic theorem, see, for example [11], for a statement and definitions. Denote by $I(A)$ the indicator function, i. e.,

$$I(A) = I_\tau(A) = \begin{cases} 1 & \text{if } \tau \in A, \\ 0 & \text{otherwise.} \end{cases}$$

We use the latter non-standard notation, because the set A in our case is quite complicated to be written as an index. Then the theorem of [34] asserts that if

$K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I \left(\left\{ \tau \in [T_0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} \right) \mathrm{d}\tau > 0.$$

In [52], the condition (1.2) for the weight function was removed.

In [17], an analogue of the theorem from [34] for the Lerch zeta-function was obtained.

The case of weighted discrete universality theorems is more complicated than that of continuous, and at the moment can be obtained only for differentiable weight functions. The first theorem of such a kind was given for periodic zeta-functions with multiplicative coefficients in [53], where it was required that $w(t)$ is a non-vanishing positive function for $t \geqslant 1$, and having a continuous derivative such that, for $h > 0$, $w(t) \ll_h w(ht)$ and $(w'(t))^2 \ll w(t)$. Let

$$V_N = V(N, w) = \sum_{k=1}^{N} w(k),$$

and

$$\lim_{N \to \infty} V(N, w) = +\infty.$$

Define the set

$$L(\mathbb{P}, h, \pi) = \left\{ (\log p : p \in \mathbb{P}), \frac{2\pi}{h} \right\},$$

and denote the class of the above weight functions by V. Then the theorem of [53] for the case of $\zeta(s)$ is of the form.

Theorem D. *Suppose that $w \in V$ and the set $L(\mathbb{P}, h, \pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I \left(\left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon \right\} \right) > 0.$$

In [54], an analogue of Theorem D with shifts $\zeta(s + ik\alpha h)$, with fixed α, $0 < \alpha < 1$, for a certain class of weight function was obtained. Note that the latter case is easier because the set $\{k^\alpha a\}$ with every $a \in \mathbb{R} \setminus \{0\}$ is uniformly distributed modulo 1, see [33] for definitions.

Chapter 2 of the dissertation is devoted to weighted discrete universality theorems with shifts $\zeta(s+ikh)$ for another class of weight functions. Let $w(t)$ be a real non-negative function having a continuous derivative on $[1/2, \infty\}$ such that

$$\int_1^N u|w'(u)|\,\mathrm{d}u \ll V_N$$

as $N \to \infty$. Denote by W the class of functions $w(t)$ satisfying the above conditions. Now, we state the main results of Chapter 2 of our dissertation.

Theorem 2.1. *Suppose that $w(t) \in W$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I \left(\left\{ k : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} \right) > 0.$$

We recall that h is always a fixed positive number. Theorem 2.1 has the following modification.

Theorem 2.2. *Under hypothesis of Theorem 2.1, the limit*

$$\lim_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I \left(\left\{ k : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} \right) > 0.$$

exists for all but at most countably many $\varepsilon > 0$.

For example, the function

$$w(t) = \frac{\sin(\log t) + 1}{t}$$

belongs to the class W. Moreover, it is not monotonically decreasing.

The proofs of Theorems 2.1 and 2.2 are based on limit theorems in the sense of weak convergence for

$$\frac{1}{V_N} \sum_{k=1}^{N} w(k) I(\{k : \zeta(s+ikh) \in A\})$$

in the space of analytic functions $H(D)$ as $N \to \infty$. For this, two different types of the number $h > 0$ is considered. We say that h is of type 1 if $\exp\{(2\pi m)/h\}$ is an irrational number for all $m \in \mathbb{Z} \setminus \{0\}$, and h is of type 2 if h is not of type 1. The cases of types 1 and 2 are examined separately.

The results of Chapter 2 are published in [48].

The problems related to the universality of the Hurwitz-zeta function are more complicated and interesting than those of the Riemann zeta-function because the function $\zeta(s,\alpha)$ depends on the parameter α, and arithmetic properties of α has an influence for approximating properties of $\zeta(s,\alpha)$. Moreover, in general, the function $\zeta(s,\alpha)$ has no Euler product over primes. This fact extends the class of approximated analytic functions.

For $K \in \mathcal{K}$, denote by $H(K)$ the class of continuous functions on K that are analytic in the interior of K. Thus, we have that $H_0(K) \subset H(K)$.

We recall that the number α is called algebraic if there exists a polynomial $p(s) \not\equiv 0$ with rational coefficients such that $p(\alpha) = 0$. In the opposite case, α is called transcendental.

The universality of the function $\zeta(s,\alpha)$ is contained in the following theorem.

Theorem E. *Suppose that the parameter α is transcendental or rational $\neq 1, 1/2$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\} > 0.$$

Theorem E in the case of rational α was already known to Voronin [74]. In this case, the function $\zeta(s,\alpha)$ can be expressed by a linear combination of Dirichlet *L*-functions, therefore, its universality reduces to joint universality of Dirichlet *L*-functions having the Euler product over primes. The full proof of Theorem E by different methods is given in the theses [21] and [1], and also can be found in the monograph [42]. The cases $\alpha = 1$ and $\alpha = 1/2$ are excluded in Theorem E because, as we have mentioned above,

$$\zeta(s,1) = \zeta(s) \quad \text{and} \quad \zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s).$$

The function $\zeta(s,\alpha)$ remains universal also in the cases $\alpha = 1$ and $\alpha = 1/2$, however, the approximated function must be from the class $H_0(K)$.

It is well known that with transcendental α the set

$$L(\alpha) \overset{def}{=} \{\log(m+\alpha) : m \in \mathbb{N}_0\}$$

is linearly independent over $\mathbb{Q}$. Therefore, the transcendence of α in Theorem E can be replaced by the linear independence over $\mathbb{Q}$ of the set $L(\alpha)$, and this was done in [36].

Clearly, rational α is algebraic. Thus, it remains an open problem related to universality for $\zeta(s,\alpha)$ with algebraic irrational parameter α. In this direction, the following not effective result is known [2].

Theorem F. *Suppose that α is algebraic irrational number. Then there exists a closed non-empty subset $F_\alpha \subset H(D)$ such that, for every compact set $K \subset D$, $f(s) \in F_\alpha$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Unfortunately, the set F_α in Theorem F is not explicitly defined. For example, F_α can consist only from one function.

Chapter 3 of the dissertation is devoted to weighted universality theorems for the function $\zeta(s,\alpha)$ with transcendental parameter α. Denote by W_1 the class of weight functions $w(t)$ satisfying hypotheses of the theorem from [34], except for the condition (1.2). Chapter 3 contains the following statements.

Theorem 3.1. *Suppose that $w(t) \in W_1$ and the parameter α is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \times I\left(\left\{ \tau \in [T_0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon \right\} \right) \mathrm{d}\tau > 0.$$

As Theorem 5.1, Theorem 3.1 has the following modification

Theorem 3.2. *Under hypothesis of Theorem 3.1, the limit*

$$\lim_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \times I\left(\left\{ \tau \in [T_0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon \right\} \right) \mathrm{d}\tau > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For the proofs of Theorem 3.1 and 3.2, the weak convergence for

$$\frac{1}{U_T}\int_{T_0}^{T} w(\tau)I\left(\{\tau\in[T_0,T]:\zeta(s+i\tau,\alpha)\in A\}\right)\,\mathrm{d}\tau, \quad A\in\mathcal{B}(H(D))$$

as $T\to\infty$ is applied.

Limit theorems of such a kind for the Lerch zeta-function were obtained in [16] and [18].

The results of Chapter 3 are published in [3].

In Chapter 4, a joint generalization of the results of Chapter 3 is given. We note that joint universality for zeta-functions was studied by numerous authors and many results are obtained. In the joint case, a given collection of analytic functions is simultaneously approximated by a collection of zeta-functions. The first joint universality result belongs to S. M. Voronin who obtained a joint universality of Dirichlet *L*-functions [73]. Let $q\in\mathbb{N}$. We recall that a Dirichlet character modulo q is called a function $\chi:\mathbb{N}\to\mathbb{C}$ which satisfies:

1. $\chi(m)$ is completely multiplicative $\chi(m_1m_2)=\chi(m_1)\chi(m_2)$ for all $m_1,m_2\in\mathbb{N}$;
2. $\chi(m)$ is periodic with period q, i. e., $\chi(m+q)=\chi(m)$ for all $m\in\mathbb{N}$;
3. $\chi(m)=0$ for all $m\in\mathbb{N}$ that are not coprime with q;
4. $\chi(m)\neq 0$ for all $m\in\mathbb{N}$ that are coprime with q.

The Dirichlet *L*-function $L(s,\chi)$ with Dirichlet character χ is defined in the half-plane $\sigma>1$ by the Dricihlet series

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s},$$

and has a meromorphic continuation to the whole complex plane. If $q=1$, then $L(s,\chi)=\zeta(s)$. A character χ modulo q is called primitive if it is not a character modulo q_1/q. It is well known that every non-primitive Dirichlet character χ is generated by a primitive character, i. e., there exists a primitive character χ_1 modulo q_1 such that

$$\chi(m)=\begin{cases}\chi_1(m) & \text{if } (m,q_1)=1,\\ 0 & \text{otherwise.}\end{cases}$$

Two Dirichlet characters are called equivalent if they are generated by the same positive character, in the opposite case, they are called non-equivalent.

In the joint universality, the approximating zeta-functions must be, in a certain sense, independent. In the Voronin theorem [73], such independence is ensured by the non-equivalence of Dirichlet characters.

Theorem G. *Suppose that $\chi_1, \ldots, \chi_r$ are pairwise non-equivalent Dirichlet characters. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\Bigg\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon \Bigg\} > 0.$$

A different proof of Theorem G from that of [73] was given in [38].

A significant part of joint universality theorems for zeta-functions are of mixed character. This means that a collection of analytic functions are approximated by a collection of shifts of zeta-functions consisting from different zeta-functions.Usually, zeta-functions having Euler product and others with no such a product are investigated. Much attention is devoted to the so-called periodic zeta-functions. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers. The periodic zeta-function $\zeta(s, \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and has a meromorphic continuation to the whole complex plane. Let $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0\}$ be an another periodic sequence of complex numbers, and $0 \leqslant \alpha \leqslant 1$ be a fixed parameter. The periodic Hurwitz zeta-function $\zeta(s, \alpha, \mathfrak{b})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha, \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m + \alpha)^s},$$

and has a meromorphic continuation to the whole complex plane. If $b_m \equiv 1$, the function $\zeta(s, \alpha, \mathfrak{b})$ becomes the classical Hurwitz zeta-function. Joint universality of periodic zeta-functions with multiplicative coefficients and periodic Hurwitz zeta-functions was studied, for example, in [27], [37], [39] and [40]. In a series of works by R. Kačinskaitė and K. Matsumoto, Hurwitz zeta-functions with multiplicative coefficients were replaced by more general Matsumoto zeta-functions which are defined by polynomial Euler products [28], [29], [30].

We already have mentioned the independence of zeta-functions in joint universality theorems. In the case of Hurwitz type zeta-functions, the algebraic independence of the parameters $\alpha_1, \dots, \alpha_r$ often is applied. We recall that the numbers $\alpha_1, \dots, \alpha_r$ are called algebraically dependant if there exists a polynomial $p(s_1, \dots, s_r) \not\equiv 0$ with rational coefficients such that $p(\alpha_1, \dots, \alpha_r) = 0$. Otherwise, the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent. In [36], the algebraic independence of the numbers $\alpha_1, \dots, \alpha_r$ was replaced by a more general requirement. Let

$$L(\alpha_1, \dots, \alpha_r) = \{(\log(m + \alpha_1) : m \in \mathbb{N}_0), \dots, (\log(m + \alpha_r) : m \in \mathbb{N}_0)\}.$$

Then the theorem of [36] is stated as follows.

Theorem H. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0.$$

In Chapter 4 of the dissertation, a weighted generalization of Theorem H is obtained.

Theorem 4.1. *Suppose that $w(t) \in W_1$ and the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(t) \times I\left(\left\{\tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0.$$

Theorem 4.1 can be stated in terms of density.

Theorem 4.2. *Under hypothesis of Theorem 4.1, the limit*

$$\lim_{T\to\infty}\frac{1}{U_T}\int_{T_0}^{T} w(t)$$
$$\times I\left(\left\{\tau\in[T_0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\right\}\right)\mathrm{d}\tau>0$$

exists for all but at most countably many $\varepsilon > 0$.

The results of Chapter 4 are contained in [50].

In the last chapter of the dissertation, Chapter 5, an example of weighted mixed joint universality theorems is given. Here, weighted joint universality theorems for the Riemann and Hurwitz zeta-functions are obtained. The results of the chapter are weighted generalization of the well-known Mishou theorem [58], which is the best mixed joint universality theorem for zeta-functions. The Mishou theorem connects Theorems A and E with transcendental parameter α.

Theorem I. *Suppose that the parameter* α *is transcendental. Let* $K_1, K_2 \in \mathcal{K}$ *and* $f_1(s) \in H_0(K_1)$*,* $f_2(s) \in H(K_2)$*. Then, for every* $\varepsilon > 0$*,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K_1}|\zeta(s+i\tau)-f_1(s)|<\varepsilon,\right.$$
$$\left.\sup_{s\in K_2}|\zeta(s+i\tau,\alpha)-f_2(s)|<\varepsilon\right\}>0.$$

Theorem I by various authors was extended for more general collections of zeta-functions than $\zeta(s), \zeta(s,\alpha)$.

The main results of Chapter 5 are the following weighted universality theorems. We preserve the above notation.

Theorem 5.1. *Suppose that the parameter* α *is transcendental and* $w(t) \in W_1$*. Let* $K_1, K_2 \in \mathcal{K}$ *and* $f_1(s) \in H_0(K_1)$*,* $f_2(s) \in H(K_2)$*. Then for every* $\varepsilon > 0$*,*

$$\liminf_{T\to\infty}\frac{1}{U_T}\int_{T_0}^{T} w(t)I\left(\left\{\tau\in[T_0,T]:\sup_{s\in K_1}|\zeta(s+i\tau)-f_1(s)|<\varepsilon,\right.\right.$$
$$\left.\left.\sup_{s\in K_2}|\zeta(s+i\tau,\alpha)-f_2(s)|<\varepsilon\right\}\right)\mathrm{d}\tau>0.$$

Theorem 5.1 as previous weighted universality theorems, has a density version.

Theorem 5.2. *Under hypothesis of Theorem 5.1, the limit*

$$\lim_{T\to\infty}\frac{1}{U_T}\int_{T_0}^{T} w(t)I\Bigg(\Bigg\{\tau\in[T_0,T]:\sup_{s\in K_1}|\zeta(s+i\tau)-f_1(s)|<\varepsilon,\\ \sup_{s\in K_2}|\zeta(s+i\tau,\alpha)-f_2(s)|<\varepsilon\Bigg\}\Bigg)\,\mathrm{d}\tau>0$$

for all but at most countably many $\varepsilon>0$.

The main ingredient of the proof of Theorems 5.1 and 5.2 is a probabilistic limit theorem in the space $H^2(D)$ with explicitly given limit measure.

The results of Chapter 5 are published in [49].

1.7 Approbation

The results of the dissertation were presented at the International MMA (Mathematical Modelling and Analysis) conferences (MMA2018, May 29 – June 1, 2018, Sigulda, Latvia), (MMA2019, May 28 – 31, 2019, Tallinn, Estonia), at the XVI International Conference“Algebra and Number Theory: Modern Problems and Applications” (May 13 – 18, 2019, Tula, Russia), at the International Conference on Number Theory Dedicated to 70th Anniversary of Professors Antanas Laurinčikas and Eugenijus Manstavičius (September 9 – 15, 2018, Palanga, Lithuania), at the conferences of Lithuanian Mathematical Society (LMS 2018, June 18 – 19, 2018, Kaunas), (LMS 2019, June 19 – 20, 2019, Vilnius), (LMS 2020, December 4, 2020, Šiauliai) , as well as at the Number Theory Seminar of Vilnius University.

1.8 Main publications

The results of the dissertation are published in the following papers:

1. A. Balčiūnas, G. Vadeikis, A weighted universality theorem for the Hurwitz zeta-function, *Šiauliai Math. Seminar* **12(20)** (2017), 5–18.

2. A. Laurinčikas, G. Vadeikis, Weighted universality of the Hurwitz zeta-function, in: *Algebra, Numb. Th. Discr. Geom.: Modern Probl. App. XV International Conference, Tula, TSPU of L. N. Tolstoy*, 2019, 45–47.

3. A. Laurinčikas, D. Šiaučiūnas, G. Vadeikis, Weighted discrete universality of the Riemann zeta-function, *Math. Modell. Anal.* **25** (2020),

no. 1, 21–36.

4. A. Laurinčikas, D. Šiaučiūnas, G. Vadeikis, A weighted version of the Mishou theorem, *Math. Modell. Anal.* **26** (2021), no. 1, 21–33.

5. A. Laurinčikas, G. Vadeikis, Joint weighted universality of the Hurwitz zeta-function, *Algebra i Analiz* **33** (2021), no. 3, 111–128; *St. Petersburg Math. J.* (to appear).

Abstracts for conferences:

1. G. Vadeikis. The weighted universality of the Hurwitz Zeta-Function. Abstracts of MMA2018, May 29 – June 1, 2018, Sigulda, Latvia, pp.

2. A. Laurinčikas, G. Vadeikis, A joint weighted universality theorem for Hurwitz zeta-function, Abstracts of MMA2019, May 28–31, 2019, Tallinn, Estonia, pp. 50.

Chapter 2

Weighted discrete universality theorems for the Riemann zeta-function

Though the first weighted continuous universality theorem for the Riemann zeta-function $\zeta(s)$ was proved probably 25 years ago in [34], its discrete version was not known. In this chapter, we will present theorems of such kind using the shifts $\zeta(s+ikh)$, $k \in \mathbb{N}$, $h > 0$. Also, we use a new class W of weight functions. We say that a real non-negative function $w(t)$ belongs to the class W if $w(t)$ has a continuous derivative $w'(t)$ in $[1/2, \infty)$ and the following conditions are satisfied:

$$\lim_{N\to\infty} V_N = +\infty \quad \text{and} \quad \int_1^N u|w'(u)|\,\mathrm{d}u \ll V_N, \quad N \to \infty,$$

where

$$V_N = \sum_{k=1}^{N} w(k).$$

2.1 Statements of discrete universality theorems

The sets considered in universality theorems are sufficiently complicated. Therefore, we will use the following notation of indicator functions. Let $A =$

$A(\tau)$ be a set of elements τ. Then the indicator function is denoted by

$$I(A) = I_\tau(A) = \begin{cases} 1, & \text{if } \tau \in A \\ 0, & \text{otherwise.} \end{cases}$$

The main results of the chapter are the following two weighted discrete universality theorems for the Riemann zeta-function. We recall that $\mathcal{K}$ is the class of compact subsets of the set $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ and $H_0(K)$, $K \in \mathcal{K}$, is the class of continuous non-vanishing functions on K that are analytic in the interior of K.

Theorem 2.1. *Suppose that $w(t) \in W$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I\left(\left\{k : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon\right\}\right) > 0.$$

Let $A \subset \mathbb{N}$ be an arbitrary set and $w(k)$ be an arbitrary function with $\lim_{N\to\infty} V_N = +\infty$. Then the lower limit

$$d_l(w, A) \stackrel{def}{=} \liminf_{N\to\infty} \frac{1}{V_N} \sum_{\substack{k=1\\k\in A}}^{N} w(k) = \liminf_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I(A)$$

always exists and is called a weighted lower density of the set A. Similarly, the upper limit

$$d_u(w, A) = \limsup_{N\to\infty} \frac{1}{V_N} \sum_{\substack{k=1\\k\in A}}^{N} w(k) = \limsup_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I(A)$$

always exists and is called a weighted upper density of the set A. If $d_l(w, A) = d_u(w, A)$, i. e., if the limit

$$\lim_{N\to\infty} \frac{1}{V_N} \sum_{\substack{k=1\\k\in A}}^{N} w(k) = \lim_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I(A)$$

exists, we say that the set has a weighted density $d(w, A)$. Then density $d(w, A)$ not always exists. In mathematics, usually, it is more important to know that a certain set has density (or weighted density). Therefore, we give a version of Theorem 2.1 in terms of density.

Theorem 2.2. *Under hypothesis of Theorem 2.1, the limit*

$$\lim_{N\to\infty}\frac{1}{V_N}\sum_{k=1}^{N}w(k)I\left(\left\{k:\sup_{s\in K}|\zeta(s+ikh)-f(s)|<\varepsilon\right\}\right)>0$$

for all but at most countably many $\varepsilon>0$.

As it was mentioned in Chapter 1, the proof of Theorems 2.1 and 2.2 depends on the arithmetic properties of the member $h>0$. We recall that h is of type 1 of the number $\exp\{(2\pi m)/h\}$ is irrational for all $m\in\mathbb{Z}\setminus\{0\}$. In the opposite case, h is of type 2.

2.2 Limit theorems with h of type 1

In this section, we will prove a limit theorem in the space $H(D)$ of analytic functions on D endowed with the topology of uniform convergence on compacta. Thus, a sequence $\{g_n(s)\subset H(D)\}$ converges to $g(s)\in H(D)$ if and only if, for every compact set $K\subset D$,

$$\lim_{n\to\infty}\sup_{s\in K}|g_n(s)-g(s)|=0.$$

Note that the space $H(D)$ is metrisable. It is well known, see, for example [10], that there exists a sequence of compact subsets $\{K_l:l\in\mathbb{N}\}\subset D$ such that

$$D=\bigcup_{l=1}^{\infty}K_l,$$

$K_l\subset K_{l+1}$ for all $l\in\mathbb{N}$, and if $K\subset D$ is a compact set, then $K\subset K_l$ for some $l\in\mathbb{N}$. For $g_1,g_2\in H(D)$, define

$$\rho(g_1,g_2)=\sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}.$$

Then ρ is a metric on the space $H(D)$ which induces its topology of uniform convergence on compacta.

For $A \in \mathcal{B}(H(D))$, define

$$P_N(A) = P_{N,w,h}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I(\{k : \zeta(s+ikh) \in A\}).$$

In this section we consider the weak convergence of $P_{N,w,h}$ with h of type 1 as $N \to \infty$.

In the limit theorem for $P_{N,w,h}$, the explicit form of the limit measure is needed. Therefore, we will state the limit theorem later, and we start with one topological structure. Let γ be the unit circle on the complex plane, i. e., $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. Define

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. The set Ω consists of all functions $\omega : \mathbb{P} \to \gamma$. By the classical Tikhonov theorem, see, for example, [61], with the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, see, for example, [66] on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined, and this procedure gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the pth component of an element $\omega \in \Omega$, $p \in \mathbb{P}$.

Before the statement of the limit theorem for $P_{N,w,h}$, we will prove a few lemmas. We start with a limit lemma on the torus Ω. For $A \in \mathcal{B}(\Omega)$, define

$$Q_N(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I\left(\left\{k : (p^{-ikh} : p \in \mathbb{P}) \in A\right\}\right).$$

Lemma 2.1. *Suppose that $w(t) \in W$, and $h > 0$ is of type* 1. *Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We apply the Fourier transform method. It is well known and used in numerous papers that the characters of the group Ω are of the form

$$\prod_{p \in \mathbb{P}}{}' \omega^{k_p}(p),$$

where the sign "'" shows that only a finite number of integers k_p are distinct from zero. Therefore, the Fourier transform $g_N(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, of Q_N is defined by

$$g_N(\underline{k}) = \int_\Omega \left({\prod_{p\in\mathbb{P}}}' \omega^{k_p}(p) \right) \mathrm{d}Q_N.$$

Then, by the definition of Q_N,

$$\begin{aligned} g_N(\underline{k}) &= \frac{1}{V_N} \sum_{k=1}^{N} w(k) {\prod_{p\in\mathbb{P}}}' p^{-ikk_p h} \\ &= \frac{1}{V_N} \sum_{k=1}^{N} w(k) \exp\left\{ -ikh {\sum_{p\in\mathbb{P}}}' k_p \log p \right\}. \qquad (2.1) \end{aligned}$$

Obviously,

$$g_N(\underline{0}) = 1, \qquad (2.2)$$

where $\underline{0} = (0, \ldots, 0)$. If $\underline{k} \neq \underline{0}$, then

$${\sum_{p\in\mathbb{P}}}' k_p \log p \neq 0,$$

since the logarithms of prime number are linearly independent over the field of rational numbers. Thus,

$$\exp\left\{ -ih {\sum_{p\in\mathbb{P}}}' kp \log p \right\} \neq 1. \qquad (2.3)$$

Indeed, if inequality (2.3) is not true, then

$$\exp\left\{ -ih {\sum_{p\in\mathbb{P}}}' k_p \log p \right\} = \mathrm{e}^{2\pi i r}$$

with some $r \in \mathbb{Z} \setminus \{0\}$. Hence, taking the logarithms, we find that

$$-ih {\sum_{p\in\mathbb{P}}}' k_p \log p = 2\pi i r_1$$

with some $r_1 \in \mathbb{Z} \setminus \{0\}$. Therefore,

$$-{\sum_{p\leqslant\mathbb{P}}}' kp \log p = \frac{2\pi r_1}{h}.$$

Hence,

$$\prod_p{}' p^{-kp} = \exp\left\{\frac{2\pi r_1}{h}\right\}$$

with $r_1 \in \mathbb{R} \setminus \{0\}$. However, the left-hand side of this equality is a rational number, and we arrive to the contradiction that h is of type 1. Thus, equality (2.3) is true, and we find using the formula for geometric progression that, for $u \geqslant 1$,

$$\begin{aligned}
&\sum_{k\leqslant u} \exp\left\{-ikh\sum_{p\in\mathbb{P}}{}' k_p \log p\right\} \\
&\quad = \frac{\exp\left\{-ih\sum_{p\in\mathbb{P}}{}' k_p \log p\right\} - \exp\left\{i([u]+1)h\sum_{p\in\mathbb{P}}{}' k_p \log p\right\}}{1-\exp\left\{-ih\sum_{p\in\mathbb{P}}{}' k_p \log p\right\}} \\
&\quad \stackrel{def}{=} \Sigma(u).
\end{aligned}$$

Therefore, summing by parts, in view of (2.1), we find that, for $\underline{k} \neq \underline{0}$,

$$g_N(\underline{k}) = \frac{w(N)\Sigma(N)}{V_N} - \frac{1}{V_N}\int_1^N \Sigma(u)w'(u)\mathrm{d}u. \tag{2.4}$$

Clearly, $\Sigma(u)$ is bounded by a constant not depending of u. Thus, by (2.4), and the definition of the class W

$$\begin{aligned}
g_N(\underline{k}) &\ll \frac{w(N)}{V_N} + \frac{1}{V_N}\int_1^N |w'(u)|\mathrm{d}u \ll \frac{1}{V_N}\int_1^N |w'(u)|\,\mathrm{d}u \\
&\ll \frac{o(1)}{V_N}\int_1^N u|w'(u)|\,\mathrm{d}u = o(1)
\end{aligned}$$

as $N \to \infty$. This together with 2.2 shows that

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Same the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H, the lemma is proved. □

Lemma 2.1 implies a weighted limit theorem for absolutely convergent Dirichlet series. Before that we recall some properties of weak convergence of probability measures.

Suppose that P is a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and we have a mapping $u : \mathbb{X}_1 \to \mathbb{X}_2$. This mapping is $(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2))$-measurable if $u^{-1}\mathcal{B}(\mathbb{X}_2) \subset \mathcal{B}(\mathbb{X}_1)$, i. e., for every $A \in \mathcal{B}(\mathbb{X}_2)$,

$$u^{-1}A \in \mathcal{B}(\mathbb{X}_1),$$

Then [4] the measure P induces on $(\mathbb{X}_2, \mathcal{B}(\mathbb{X}_2))$ the unique probability measure Pu^{-1} defined by

$$Pu^{-1}(A) = P(u^{-1}A), \quad A \in \mathcal{B}(\mathbb{X}_1)$$

where $u^{-1}A$ is the preimage of A. If the mapping u is continuous, then it is $(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2))$-measurable [4]. Moreover, the following lemma is valid [4].

Lemma 2.2. *Suppose that $u : \mathbb{X}_1 \to \mathbb{X}_2$ is a continuous mapping, and P_n, $n \in \mathbb{N}$, and P are probability measures on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$. If P_n converges weakly to P as $n \to \infty$, then also $P_n u^{-1}$ converges weakly to Pu^{-1} as $n \to \infty$.*

Now, we will define an absolutely convergent Dirichlet series connected to the function $\zeta(s)$. Let $\theta > 1/2$ be a fixed number, and

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^\theta\right\}, \quad m, n \in \mathbb{N},$$

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}, \quad \zeta(s, \omega) = \sum_{m=1}^{\infty} \frac{\omega(m) v_n(m)}{m^s},$$

where

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Then it is known [35] that the series for $\zeta_n(s)$ and $\zeta_n(s, \omega)$ are absolutely convergent for $\sigma > 1/2$. For $A \in \mathcal{B}(H(D))$, define

$$P_{N,n}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I(\{k : \zeta_n(s + ikh) \in A\})$$

and

$$R_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1}A),$$

where $u_n : \Omega \to H(D)$ is given by the formula

$$u_n(\omega) = \zeta_n(s, \omega).$$

Lemma 2.3. *Suppose that $w(t) \in W$ and h is of type 1. Then $P_{N,n}$ converges weakly to P_n as $N \to \infty$.*

Proof. Since the series for $\zeta(s, \omega)$ is absolutely convergent for $\sigma > 1/2$, the mapping u_n is continuous. Moreover, the definitions of $P_{N,n}$ and Q_N show that, for all $A \in \mathcal{B}(H(D))$,

$$P_{N,n}(A) = \frac{1}{V_n} \sum_{k=1}^{N} w(k) I \left(\left\{ k : (p^{-ikh} : p \in \mathbb{P}) \in u_n^{-1} A \right\} \right) = Q_N(u_n^{-1} A).$$

Therefore, we have the equality $P_{N,n} = Q_N u_n^{-1}$. Thus, Lemmas 2.1 and 2.2 and the continuity of u_n prove the lemma. □

The weak convergence of $P_{N,n}$ is a starting point for proving the weak convergence for P_N as $N \to \infty$. The investigation of P_N requires an approximation of $\zeta(s)$ by $\zeta_n(s)$. This approximation is based on mean square estimates. Since we consider the discrete case, a very useful result is the Gallagher lemma connecting continuous and discrete mean squares of some functions. We state it as the next lemma.

Lemma 2.4. *Suppose that $T_0, T > \delta > 0$ are real numbers, $\mathcal{T} \neq \varnothing$ is a finite set in the internal $[T_0 + \delta/2, T_0 + T - \delta/2]$, and*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x| < \delta}} 1.$$

Let the complex-valued function $g(s)$ be continuous in $[T_0, T_0 + T]$ and have a continuous derivative in $(T_0, T_0 + T)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t) |g(t)|^2 \leqslant \frac{1}{\delta} \int_{T_0}^{T_0+T} |g(x)|^2 \, \mathrm{d}x + \left(\int_{T_0}^{T_0+T} |g(x)|^2 \, \mathrm{d}x \int_{T_0}^{T_0+T} |g'(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

Proof of the lemma can be found in [59], Lemma 1.4.

It is well known that, for fixed $\sigma > 1/2$,

$$\int_{-T}^{T} |\zeta(\sigma+it)|^2 \ll_\sigma T \quad \text{and} \quad \int_{-T}^{T} |\zeta'(\sigma+it)|^2 \ll_\sigma T. \tag{2.5}$$

The latter estimates imply that, for $1/2 < \sigma < 1$ and $\tau \in \mathbb{R}$,

$$\begin{aligned}\int_0^T |\zeta(\sigma+it+i\tau)|^2 \,\mathrm{d}t &\ll \int_{-|\tau|}^{T+|\tau|} |\zeta(\sigma+it))|^2 \,\mathrm{d}t \\ &\ll_\sigma T+|\tau| \ll_\sigma T(1+|\tau|) \end{aligned} \tag{2.6}$$

and

$$\int_0^T |\zeta'(\sigma+it+i\tau)|^2 \,\mathrm{d}t \ll_\sigma T(1+|\tau|). \tag{2.7}$$

Now, we are able to obtain a weighted discrete version of (2.6).

Lemma 2.5. *Suppose that* $w \in W$. *Then for fixed* $1/2 < \sigma < 1$, $h > 0$ *and* $\tau \in \mathbb{R}$,

$$\sum_{k+1}^{N} w(k)|\zeta(\sigma+ikh+i\tau)|^2 \ll_\sigma V_N(1+|\tau|).$$

Proof. Lemma 2.4 together with (2.5) and (2.6) gives

$$\begin{aligned}&\sum_{k=1}^{N} |\zeta(\sigma+ikh+i\tau)|^2 \ll_h \int_0^{(N+1/2)h} |\zeta(\sigma+it+i\tau)|^2 \,\mathrm{d}t \\ &+\left(\int_0^{(N+1/2)h} |\zeta(\sigma+it+i\tau)|^2 \,\mathrm{d}t \int_0^{(N+1/2)h} |\zeta'(\sigma+it+i\tau)|^2 \,\mathrm{d}t\right)^{1/2} \\ &\ll_{\sigma,h} N(1+|\tau|).\end{aligned}$$

Hence, for the same σ and τ, summing by parts, we find

$$\begin{aligned}\sum_{k=1}^{N} w(k)|\zeta(\sigma+ikh+i\tau)|^2 &\ll w(N)\sum_{k=1}^{N} |\zeta(\sigma+ikh+i\tau)|^2 \\ &\quad + (1+|\tau|)\int_1^N u|w'(u)| \,\mathrm{d}u \\ &\ll_{\sigma,h} Nw(N)(1+|\tau|) + V_N(1+|\tau|) \\ &\ll_{\sigma,h} V_N(1+|\tau|),\end{aligned}$$

because

$$Nw(N) = \sum_{k=1}^{N} w(k) + \int_1^N \left(\sum_{k \leqslant u} 1 \right) w'(u)\,\mathrm{d}u \ll V_N.$$

□

Lemma 2.5 leads to the approximation lemma.

Lemma 2.6. *Suppose that* $w(t) \in W$. *Then the equality*

$$\lim_{n\to\infty} \limsup_{n\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k)\rho\left(\zeta(s+ikh), \zeta_n(s+ikh)\right) = 0$$

is true for every fixed $h > 0$.

Proof. We use the integral representation for $\zeta_n(s)$ which was obtained in [35]: for $\sigma > 1/2$,

$$\zeta_n(s) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s+z) l_n(z) \frac{\mathrm{d}z}{z}, \tag{2.8}$$

where

$$l_n(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) n^s, \quad n \in \mathbb{N},$$

$\Gamma(s)$ is the Euler gamma-function, and θ comes from the definition of $v_n(m)$.

Let $K \subset D$ be a compact set. Then there exists $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for $s = \sigma + it \in K$. It suffices to show that

$$\lim_{n\to\infty} \limsup_{n\to\infty} \frac{1}{V_n} \sum_{k=1}^{N} w(k) \sup_{s\in K} |\zeta(s+ikh) - \zeta_n(s+ikh)| = 0. \tag{2.9}$$

Let $\theta_1 = \sigma - 1/2 - \varepsilon$. Then $\theta_1 > 0$. Using (2.8) and the residue theorem with poles $z = 0$ and $z = 1 - s$, we find that, for $s \in K$,

$$\zeta_n(s) - \zeta(s) = \frac{1}{2\pi i} \int_{-\theta_1 - i\infty}^{\theta_1 + i\infty} \zeta(s+z) l_n(z) \frac{\mathrm{d}z}{z} + \frac{l_n(1-s)}{1-s}.$$

Thus,

$$\begin{aligned}\zeta_n(s+ikh)&-\zeta(s+ikh)\\&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta(\sigma-\theta_1+it+ikh+iv)\frac{l_n(-\theta_1+iv)}{-\theta_1+iv}\,\mathrm{d}v\\&\quad+\frac{l_n(1-s-ikh)}{1-s-ikh},\end{aligned}$$

and shifting $t+v$ to v leads to

$$\begin{aligned}&\zeta_n(s+ikh)-\zeta(s+ikh)\\&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\left(\frac{1}{2}+\varepsilon+ikh+iv\right)\frac{l_n(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\,\mathrm{d}v\\&\quad+\frac{l_n(1-s-ikh)}{1-s-i\tau}\\&\ll\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+iv\right)\right|\sup_{s\in K}\left|\frac{l_n(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\right|\mathrm{d}v\\&\quad+\sup_{s\in K}\left|\frac{l_n(1-s+ikh)}{1-s-ikh}\right|.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{1}{V_N}\sum_{k=1}^{N}w(k)\sup_{s\in K}|\zeta_n(s+ikh)-\zeta(s+ikh)|\\&\ll\int_{-\infty}^{\infty}\left(\frac{1}{V_N}\sum_{k=1}^{N}w(k)\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+iv\right)\right|\right)\\&\qquad\times\sup_{s\in K}\left|\frac{l_n(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\right|\mathrm{d}v\\&\quad+\frac{1}{V_N}\sum_{k=1}^{N}w(k)\sup_{s\in K}|\frac{l_n(1-s-ikh)}{1-s-ikh}|\overset{def}{=}S+Z.\end{aligned}\tag{2.10}$$

In virtue of Lemma 2.5 and the Cauchy inequality

$$\begin{aligned}
&\frac{1}{V_N}\sum_{k=1}^{N} w(k)\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+i\sigma\right)\right| \\
&\qquad \leqslant \frac{1}{V_N}\left(\sum_{k=1}^{N} w(k)\sum_{k=1}^{N} w(k)\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+iv\right)\right|^2\right)^{1/2} \\
&\qquad \ll_{\varepsilon,h} \frac{1}{V_N}(V_N\cdot V_N(1+|v|))^{1/2} \ll_{\varepsilon,h} (1+|v|). \qquad (2.11)
\end{aligned}$$

It is well known that uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ with every $\sigma_1 < \sigma_2$ the estimate

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\} \qquad (2.12)$$

is valid with a certain constant $c > 0$. Therefore, by the definition of $l_n(s)$, we have, for all $s \in K$,

$$\begin{aligned}
\frac{l_n(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv} &\ll n^{1/2+\varepsilon-\sigma}\left|\Gamma\left(\frac{1}{2}+\varepsilon-\sigma-t+v\right)\right| \\
&\ll_\theta n^{-\varepsilon}\exp\left\{-\frac{4}{3}c|v-t|\right\} \\
&\ll_\theta n^{-\varepsilon}\exp\left\{-\frac{4}{3}c(|v|-|t|)\right\} \\
&\ll_{\varepsilon,K} \exp\{-c_1|v|\}
\end{aligned}$$

with some $c_1 > 0$ if we take

$$\theta = \frac{1}{2}+\varepsilon.$$

This together with (2.11) shows that

$$I \ll_{\varepsilon,K,h} n^{-\varepsilon}\int_{1\infty}^{\infty}(1+|v|)\exp\{-c_1|v|\}\,\mathrm{d}v \ll_{\varepsilon,K,h} n^{-\varepsilon}. \qquad (2.13)$$

Similarly, using (2.12), we find that, for all $s \in K$,

$$\begin{aligned}
\frac{l_n(1-s-ikh)}{1-s-ikh} &\ll_{\varepsilon,h} n^{1-\sigma}\exp\left\{-\frac{4}{3}c|kh-t|\right\} \\
&\ll_{\varepsilon,K,h} n^{1/1-2\varepsilon}\exp\{-c_2kh\}
\end{aligned}$$

with some $c_2 > 0$. Therefore,

$$Z \ll_{\varepsilon,K,h} n^{1/2-2\varepsilon} \frac{1}{V_N} \sum_{k=1}^{N} w(k) \exp\{-c_2 kh\}.$$

Let $\widehat{N} = \widehat{N}(N) \to \infty$ as $N \to \infty$ be such that

$$\sum_{k=1}^{\widehat{N}} w(k) = o(V_N), \quad N \to \infty.$$

Then

$$\sum_{k=1}^{N} w(k) \exp\{-c_2 kh\} \ll \sum_{k=1}^{\widehat{N}} w(k) + \sum_{k=\widehat{N}+1}^{N} w(k) \exp\{-c_2 kh\}$$

$$\ll o(V_N) + \exp\{-c_2(\widehat{N}+1)h\} \sum_{k=1}^{N} w(k) = o(V_N)$$

as $N \to \infty$. Therefore

$$Z = o(1)$$

as $N \to \infty$. This and (2.13) show that

$$I + Z \ll_{\varepsilon,k,h} n^{-\varepsilon} + o(1)$$

as $N \to \infty$. Now, taking $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \limsup_{N\to\infty} (I + Z) = 0,$$

and this together with (2.10) proves (2.9). The lemma is proved. □

To derive a limit theorem for the function $\zeta(s)$, we will use essentially the following lemma. Denote by $\xrightarrow{\mathcal{D}}$ the convergence of random elements in distribution, i. e., the weak convergence of their distributions.

Lemma 2.7. *Let the space $(\mathbb{X}, d)$ is separable, and the $\mathbb{X}$ - valued random elements Y_n, X_{kn}, $n \in \mathbb{N}$, $k \in \mathbb{N}$, be defined on the same probability space with the measure μ. Suppose that, for every $k \in \mathbb{N}$,*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k,$$

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X,$$

and, for every $\varepsilon > 0$*,*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\{d(X_{kn}, Y_n) \geqslant \varepsilon\} = 0.$$

Then

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X.$$

Proof of the lemma is given in [4], Theorem 4.2.

Now, we are in position to prove a weighted discrete limit theorem for the function $\zeta(s)$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s,\omega)$ by the Euler product

$$\zeta(s,\omega) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}$$

Note that latter product, for almost all $\omega \in \Omega$, is uniformly convergent on compact subsets of the strip D, see [35]. Denote by P_ζ the distribution of the random element $\zeta(s,\omega)$, i. e.,

$$P_\zeta(A) = m_H\{\omega \in \Omega : \zeta(s,\omega) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Theorem 2.3. *Suppose that* $w(t) \in W$ *and* $h > 0$ *is of type* 1*. Then* P_N *converges weakly to* P_ζ *as* $N \to \infty$*. Moreover, the support of* P_ζ *is the set*

$$S \stackrel{def}{=} \{g \in H(D) : g(s) \neq 0 \quad or \quad g(s) \equiv 0\}.$$

Proof. We will prove that R_n (the limit measure in Lemma 2.3), as $n \to \infty$, and P_N, as $N \to \infty$, converge weakly to a certain measure P.

Let θ_N be a random variable defined on a certain probability space with probability measure μ and having the distribution

$$\mu\{\theta_N = kh\} = \frac{w(k)}{V_N}, \quad k = 1, \dots, N.$$

Moreover, let $Y_{N,n} = Y_{N,n}(s)$ be the $H(D)$-valued random element defined by

$$Y_{N,n}(s) = \zeta_n(s + i\theta_N),$$

and let $Y_n = Y_n(s)$ be the $H(D)$-valued random element with the distribution

R_n. Then, by Lemma 2.3,

$$Y_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} R_n. \tag{2.14}$$

Using the absolute convergence in D of the series for $\zeta_n(s)$, it can be proved by a standard method, see [35], that the family of probability measures $\{R_n : n \in \mathbb{N}\}$ is tight, i. e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D)$ such that

$$R_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Hence, by the well-known Prokhorov theorem, see [4], this family is relatively compact, i. e., each sequence of $\{R_n\}$ contains a subsequence R_{n_r} weakly convergent to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$. In other words, we have

$$R_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{2.15}$$

Define one more $H(D)$-valued random element

$$X_N = X_N(s) = \zeta(s + i\theta_N).$$

Then the application of Lemma 2.6 gives, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu\{\rho(X_N(s), Y_{N,n}(s)) \geqslant \varepsilon\}$$
$$\leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon V_N} \sum_{k=1}^{N} w(k)\rho(\zeta(s+ikh), \zeta_n(s+ikh)) = 0.$$

This and the relations (2.14) and (2.15) together with Lemma 2.7 imply the relation

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P. \tag{2.16}$$

This shows that P_N converges weakly to P as $N \to \infty$. Moreover, (2.16) implies that the measure P is independent of the sequence R_n. Since the family $\{R_n\}$ is relatively compact, from this we obtain that R_n converges weakly to P as $n \to \infty$. Thus, P_N, as $N \to \infty$, converges weakly to the limit measure P of R_n as $n \to \infty$. However, by the proof of a limit theorem for

$$\frac{1}{T}\text{meas}\{\tau \in [0, T] : \zeta(s + i\tau) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

it is known [35], that R_n, as $n \to \infty$, converges weakly to P_ζ, and the support of P_ζ is the set S. Therefore, the same statement is also true for P_N, and the theorem is proved. □

2.3 Limit theorems with $h > 0$ of type 2

The case of type 2 is more complicated. We must construct a new probability space different from $(\Omega, \mathcal{B}(\Omega), m_H)$. We will index by h the notation related to h of type 2.

Thus, now suppose that $h > 0$ is of type 2. Then there exists the smallest $m_0 \in \mathbb{N}$ such that the number

$$\exp\left\{\frac{2\pi m_0}{h}\right\}$$

is rational. We put

$$\exp\left\{\frac{2\pi m_0}{h}\right\} = \frac{a}{b}, \quad a, b \in \mathbb{N}, \quad (a,b) = 1.$$

Define the set

$$\mathbb{P}_0 = \left\{ p \in \mathbb{P} : \frac{a}{b} = \prod_{p \in \mathbb{P}} p^{\alpha_p} \text{ with } \alpha_p \neq 0 \right\}.$$

Denote by Ω_h the closed subgroup of the group Ω generated by the element $\{p^{-ih} : p \in \mathbb{P}\}$. It is known by Lemma 1 of [45], if h is of type 2, then

$$\Omega_h = \{\omega \in \Omega : \omega(a) = \omega(b)\}.$$

On $(\Omega_h, \mathcal{B}(\Omega_h))$, the probability Harr measure m_H^h exists, and we obtain a new probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. Moreover, in [45], it was observed (formula (3.1)) that the characters χ of the group Ω_h are of the form

$$\chi(\omega) = \prod_{p \in \mathbb{P} \setminus \mathbb{P}_0} \omega^{k_p}(p) \prod_{p \in \mathbb{P}_0} \omega^{k_p + l\alpha_p}(p), \quad l \in \mathbb{Z}. \tag{2.17}$$

Now, we are ready to prove an analogue of Lemma 2.1. For $A \in \mathcal{B}(\Omega_h)$, define

$$Q_{N,h}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I\left(\left\{k : (p^{-ikh} : p \in \mathbb{P}) \in A\right\}\right).$$

Lemma 2.8. *Suppose that h is of type 2. Then $Q_{N,h}$ converges weakly the Haar measure m_H^h as $N \to \infty$.*

Proof. In view of (2.17), we have that the Fourier transform $g_{N,h}(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z},\ p \in \mathbb{P})$, of $Q_{N,h}$ is of the form

$$\begin{aligned} g_{N,h}(\underline{k}) &= \int_{\Omega_h} \chi(\omega) \mathrm{d}Q_{N,h} \\ &= \frac{1}{V_N} \sum_{k=1}^{N} w(k) \prod_{p \in \mathbb{P} \setminus \mathbb{P}_0}{}' p^{-ikk_p h} \prod_{p \in \mathbb{P}_0} p^{-ikh(k_p + l\alpha_p)}, \quad l \in \mathbb{Z}. \quad (2.18) \end{aligned}$$

If $k_p = 0$ for all $p \in \mathbb{P} \setminus \mathbb{P}_0$ and $k_p = r\alpha_p$ for all $p \in \mathbb{P}_0$ with some $r \in \mathbb{Z}$ (case 1), then

$$g_{N,h}(\underline{k}) = 1 \quad (2.19)$$

because $\prod_{p \in \mathbb{P}_0} \omega^{d\alpha_p}(p) = 1 \quad \text{with} \quad d \in \mathbb{Z}$.

Now, suppose that $k_p \neq 0$ for some $p \in \mathbb{P} \setminus \mathbb{P}_0$, or there does not exist $r \in \mathbb{Z}$ such that $k_p = r\alpha_p$ for all $p \in \mathbb{P}_0$ (case 2). In [45], it was obtained that

$$\exp\{-ihA_p(k_p, l\alpha_p)\} \neq 1,$$

where

$$A_p(k_p, l\alpha_p) = \sum_{p \in \mathbb{P} \setminus \mathbb{P}_0}{}' k_p \log p + \sum_{p \in \mathbb{P}_0} (k_p + l\alpha_p) \log p, \quad l \in \mathbb{Z}.$$

Hence, we find that, for $u \geqslant 1$,

$$\begin{aligned} &\sum_{k \leqslant u} \exp\{-ikhA_p(k_p, l\alpha_p)\} \\ &\quad = \frac{\exp\{-ihA_p(k_p, l\alpha_p)\} - \exp\{-ih([u]+1)A_p(k_p, l\alpha_p)\}}{1 - \exp\{-ihA_p(k_p, l\alpha_p)\}} \stackrel{def}{=} \Sigma_h(u). \end{aligned}$$

Therefore, in view of (2.18),

$$g_{N,h}(\underline{k}) = \frac{w(N)\Sigma_h(N)}{V_N} - \frac{1}{V_N} \int_1^N \Sigma_h(u) w'(u) \mathrm{d}u.$$

Using the properties of the function w, hence we find that

$$g_{N,h}(\underline{k}) = 0.$$

This together with (2.19) shows that

$$\lim_{N\to\infty} g_{N,h}(\underline{k}) = \begin{cases} 1 & \text{in the case 1,} \\ 0 & \text{in the case 2.} \end{cases}$$

Since the right-hand side of the equality is the Fourier transform of the Haar measure m_H^h, the lemma follows by a continuity theorem for probability measures on compact groups. □

Now, together with $P_{N,n,h}$, consider

$$\hat{P}_{N,n,h}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I\left(\{k : \zeta_{n,h}(s+ikh,\omega) \in A\}\right), \qquad A \in \mathcal{B}(H(D)),$$

with $\omega \in \Omega_h$.

Lemma 2.9. *Suppose that $w(t) \in W$ and h is of type 2. Then $P_{N,n,h}$ and $\hat{P}_{N,n,h}$ both converge weakly to the measure $m_H^h u_{n,h}^{-1}$ as $N \to \infty$, where $u_{n,h} : \Omega_h \to H(D)$ is given by $u_{n,h}(\omega) = \zeta_{n,h}(s,\omega)$, $\omega \in \Omega_h$.*

Proof. By proving Lemma 2.3, in view of Lemma 2.8, we have that the mesure $P_{N,n,h}$ converges weakly to $m_H^h u_{n,h}^{-1}$ as $N \to \infty$. Similarly, we obtain that if $\hat{u}_{n,h}(\hat{\omega}) : \Omega_h \to H(D)$ is given by

$$\hat{u}_{n,h}(\hat{\omega}) = \zeta_n(s, \omega\hat{\omega}), \quad \hat{\omega} \in \Omega_h,$$

then $\hat{P}_{N,n,h}$ converges weakly to $m_H^h \hat{u}_{n,h}^{-1}$. However, $\hat{u}_{n,h} = u_{n,h}(u)$, where $u : \Omega_h \to \Omega_h$ is given by $u(\hat{\omega}) = \omega\hat{\omega}$. This and the invariance of the Haar measure m_H^h show that $m_H^h \hat{u}_{n,h}^{-1} = m_H^h u_{n,h}^{-1}$. □

For further considerations, we need some elements of the ergodic theory. Let $a_h = (p^{-ih} : p \in \mathbb{P})$. Then a_h is an element of Ω_h. Define the transformation $\varphi_h(\omega)$ of Ω_h by

$$\varphi_h(\omega) = a_h \omega, \quad \omega \in \Omega_h.$$

Then we have that φ_h is a measurable measure preserving transformation on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. We recall that a set $A \in \mathcal{B}(\Omega_h)$ is called invariant with respect to φ_h if the sets A and $\varphi_h(A)$ can differ from each other at most by a set of m_H^h-measure zero. The transformation φ_h is called ergodic if the σ-field of invariant sets of Ω_h consists only of the sets having m_H^h-measure 1 or 0.

Lemma 2.10. *Suppose that h is of type 2. Then the transformation φ_h is ergodic.*

Proof of the lemma is given in [45], Lemma 3.

Let, for $\omega \in \Omega_h$,

$$\zeta_h(s, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

The first application of Lemma 2.10 is devoted to the discrete mean square of $\zeta_h(s, \omega)$.

Lemma 2.11. *Suppose that $w(t) \in W$, $h > 0$ is of type 2, σ, $1/2 < \sigma < 1$, is fixed and $t \in \mathbb{R}$. Then, for almost all $\omega \in \Omega_h$,*

$$\sum_{k=1}^{N} w(k)|\zeta_h(\sigma + it + ikh, \omega)|^2 \ll V_N(1 + |t|).$$

Proof. We have that $\zeta_h(s, \omega)$ coincides with the restriction of the random element $\zeta(s, \omega)$ to the space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. First we consider the expectation $\mathbb{E}|\zeta_h(\sigma + it, \omega)|^2$. We write $\zeta_h(s, \omega)$ in the form

$$\begin{aligned}\zeta_h(\sigma + it, \omega) &= \prod_{p \in \mathbb{P}_0} \left(1 - \frac{\omega(p)}{p^{\sigma+it}}\right)^{-1} \prod_{p \in \mathbb{P} \setminus \mathbb{P}_0} \left(1 - \frac{\omega(p)}{p^{\sigma+it}}\right)^{-1} \\ &\overset{def}{=} X_1 X_2. \end{aligned} \tag{2.20}$$

The random elements X_1 and X_2 are independent, moreover, for almost all $\omega \in \Omega_h$,

$$X_2 = {\sum_m}' \frac{\omega(m)}{m^{\sigma+it}},$$

where the sign $'$ means that the summing runs over $m = 1$ and $m \in \mathbb{N}$ with the canonical representation consisting only of primes $p \in \mathbb{P} \setminus \mathbb{P}_0$. In the series for X_2, the random variables are orthogonal, therefore,

$$\mathbb{E}|X_2|^2 = \sum_m{}' \frac{1}{m^{2\sigma}} < \infty.$$

Clearly, $\mathbb{E}|X_1|^2$ is bounded by a constant. Therefore, there exists a finite constant $c > 0$ such that, for $1/2 < \sigma < 1$ and $t \in \mathbb{R}$,

$$\mathbb{E}|\zeta_h(\sigma + it, \omega)|^2 = \mathbb{E}|X_1|^2 \mathbb{E}|X_2|^2 \leqslant c.$$

Then (2.20), Lemma 2.10, the Birkhoff-Khintchine ergodic theorem, see, for example, [67], and the definition of the transformation φ_h show that, for $1/2 < \sigma < 1$ and $|t_0| \leqslant h$,

$$\begin{aligned}\sum_{k=1}^{N} |\zeta_h(\sigma + it_0 + ikh, \omega)|^2 &= \sum_{k=1}^{N} |\zeta_h(\sigma + it_0, \varphi_h^k(\omega))|^2 \\ &= N\mathbb{E}|\zeta_h(\sigma + it_0, \omega)|^2(1 + o(1)) \ll N\end{aligned}$$

for almost all $\omega \in \Omega_h$ as $N \to \infty$. Hence, denoting by $[u]$ the integer part of $u \in \mathbb{R}$, for $1/2 < \sigma < 1$ and $t \in \mathbb{R}$, we find that

$$\sum_{k=1}^{N} |\zeta_h(\sigma + it + ikh, \omega)|^2 = \sum_{k=1+[t/h]}^{N+[t/h]} |\zeta_h(\sigma + it_0 + ikh, \omega)|^2 \ll N(1 + |t|)$$

for almost all $\omega \in \Omega_h$. From this, summing by parts, we obtain the estimate of the lemma. □

Similarly to the proof of Lemma 2.6, we arrive, by using Lemma 2.11, to

Lemma 2.12. *Suppose that $w(t) \in W$ and $h > 0$ is of type 2. Then, for almost all $\omega \in \Omega_h$,*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k)\rho\left(\zeta_h(s + ikh, \omega), \zeta_{n,h}(s + ikh, \omega)\right) = 0.$$

For $\omega \in \Omega_h$, additionally to the measure $P_{N,h}$, define

$$\hat{P}_{N,h}(A) = \frac{1}{V_N} \sum_{k=1}^{N} w(k) I\left(\{k : \zeta_h(s + ikh, \omega) \in A\}\right), \quad A \in \mathcal{B}(H(D)).$$

Then, using Lemmas 2.9 and 2.12, and repeating the first part of the proof of Theorem 2.3, we obtain

Lemma 2.13. *Suppose that $w(t) \in W$ and $h > 0$ is of type 2. Then, on $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure P_h such that $P_{N,h}$ and $\hat{P}_{N,h}$ both converge weakly to P_h as $N \to \infty$.*

Denote by $P_{\zeta,h}$ the distribution of the random element $\zeta_h(s,\omega)$, $\omega \in \Omega_h$. Then we have the following analogue of Theorem 2.3.

Theorem 2.4. *Suppose that $w(t) \in W$ and $h > 0$ is of type 2. Then $P_{N,h}$ converges weakly $P_{\zeta,h}$ as $N \to \infty$. Moreover, the support of the measure $P_{\zeta,h}$ is the set S.*

Proof. In virtue of Lemma 2.13, it suffices to identify the measure P_h in that lemma, and to find the support of the limit measure. For the first problem, we will apply Lemma 2.10, and the classical Birkhoff-Khintchine theorem on ergodic transformations, see [67]. Let A be a continuity set of P_h. On the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$, define the random variable ξ by the formula

$$\xi(\omega) = \begin{cases} 1 & \text{if } \zeta_h(s,\omega) \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that

$$\mathbb{E}\xi = \int_{\Omega_h} \xi(\omega) \mathrm{d} m_H^h = \hat{P}_{\zeta,h}(A). \tag{2.21}$$

Moreover, by Lemma 2.13,

$$\lim_{N\to\infty} \hat{P}_N(A) = P_h(A). \tag{2.22}$$

In view of Lemma 2.10 and the Birkhoff-Khintchine theorem, for almost all $\omega \in \Omega_h$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \xi(\varphi_h^k(\omega)) = \mathbb{E}\xi.$$

Since $w \in W$, from this it follows that, for almost all $\omega \in \Omega_h$,

$$\lim_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) \xi(\varphi_h^k(\omega)) = \mathbb{E}\xi. \tag{2.23}$$

However, by the definition of φ_h,

$$\frac{1}{V_N}\sum_{k=1}^{N} w(k)\xi(\varphi_h^k(\omega)) = \frac{1}{V_N}\sum_{k=1}^{N} w(k) I\left(\{k : \zeta_h(s+ikh,\omega)\in A\}\right)$$
$$= \hat{P}_{N,h}(A).$$

Therefore, by (2.21) and (2.23),

$$\lim_{N\to\infty} \hat{P}_{N,h}(A) = P_{\zeta,h}(A).$$

This and (2.22) show that $P_h = P_{\zeta,h}$.

For finding the support of $P_{\zeta,h}$, we use the representation (2.20). For $p \in \mathbb{P}\setminus\mathbb{P}_0$, the random variables $\omega(p)$ are independent. Thus, by the proof of Lemma 6.5.5 from [35], we find that the support of the random element X_2 is the set S. Since the random elements X_1 and X_2 are independent and X_1 is not degenerated at zero, we obtain that the support of X_1X_2 is the set S, i. e., the support of the measure $P_{\zeta,h}$ is the set S. The theorem is proved. □

2.4 Proof of universality theorems

Theorems 2.1 and 2.2 follow from the limit theorems (Theorems 2.3 and 2.4), for $\zeta(s)$ as well as from the Mergelyan theorem [57] on the approximation of analytic functions by polynomials. We state it as the next lemma.

Lemma 2.14. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and $g(s)$ is a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(s)$ such that*

$$\sup_{s\in K} |g(s) - p_\varepsilon(s)| < \varepsilon$$

Proof of Theorem 2.1. By Lemma 2.14, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} \left| f(s) - \mathrm{e}^{p(s)} \right| < \frac{\varepsilon}{2}. \qquad (2.24)$$

For brevity, denote the limit measure in Theorems 2.3 and 2.4 by $\hat{P}_\zeta$, i. e.,

$$\hat{P}_\zeta = \begin{cases} P_\zeta & \text{if } h \text{ is of type 1},\\ P_{\zeta,h} & \text{if } h \text{ is of type 2}, \end{cases}$$

and

$$\hat{P}_N = \begin{cases} P_N & \text{if } h \text{ is of type 1,} \\ P_{N,h} & \text{if } h \text{ is of type 2.} \end{cases}$$

Then we have that $\hat{P}_N$ converges weakly to $\hat{P}_\zeta$ as $N \to \infty$. Define the set

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} \left| g(s) - \mathrm{e}^{p(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

Since $e^{p(s)} \neq 0$, and, in view of Theorems 2.3 and 2.4, the support of the measure $\hat{P}_\zeta$ is the set S, the set G_ε is an open neighbourhood of an element of the support, therefore,

$$\hat{P}_\zeta(G_\varepsilon) > 0. \tag{2.25}$$

Moreover, by the first parts of Theorems 2.3 and 2.4, and the equivalent of weak convergence of probability measures in terms of open sets [12, Theorem 2.1], we have that

$$\liminf_{N \to \infty} \hat{P}_N(G_\varepsilon) \geqslant \hat{P}_\zeta(G_\varepsilon).$$

This, (2.25) and the definitions of $\hat{P}_N$ and G_ε show that

$$\liminf_{N \to \infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I \left\{ k : \sup_{s \in K} \left| \zeta(s + ikh) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\}(k) > 0. \tag{2.26}$$

It remains to replace $e^{p(s)}$ by $f(s)$ in the latter inequality. Suppose that k satisfies the inequality

$$\sup_{s \in K} \left| \zeta(s + ikh) - e^{p(s)} \right| < \frac{\varepsilon}{2}.$$

Then, in virtue of (2.24), the same k satisfies the inequality

$$\sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon.$$

Therefore,

$$\left\{ k : \sup_{s \in K} \left| \zeta(s + ikh) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\} \subset \left\{ k : \sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon \right\}.$$

This inclusion together with (2.26) proves the theorem. □

Proof of Theorem 2.2. Define the set

$$\hat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then $\partial \hat{G}_\varepsilon = \{g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon\}$ is the boundary of $\hat{G}_\varepsilon$. Hence, $\partial \hat{G}_{\varepsilon_1} \cap \partial \hat{G}_{\varepsilon_2} = \varnothing$ if $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. Therefore, the set $\partial \hat{G}_\varepsilon$ can have a positive $\hat{P}_\zeta$-measure for at most countably many $\varepsilon > 0$. This means that the set $\hat{G}_\varepsilon$ is a continuity set of the measure $\hat{P}_\zeta$ for all but at most countably many $\varepsilon > 0$. Using Theorems 2.3 and 2.4, and the equivalent of weak convergence of probability measures in terms of continuity sets [4] Theorem 2.1, we have that

$$\lim_{N \to \infty} \hat{P}_N(\hat{G}_\varepsilon) = \hat{P}_\zeta(\hat{G}_\varepsilon) \tag{2.27}$$

for all but at most countably many $\varepsilon > 0$. Moreover, (2.24) shows that $G_\varepsilon \subset \hat{G}_\varepsilon$. Therefore, by (2.25), $\hat{P}_\zeta(\hat{G}_\varepsilon) > 0$. This, (2.27) and the definition of the set $\hat{G}_\varepsilon$ prove the theorem. □

Chapter 3

Weighted universality theorems for the Hurwitz zeta-function

This chapter is devoted to continuous universality theorems for the Hurwitz zeta-function $\zeta(s,\alpha)$ with a fixed parameter $0 < \alpha \leqslant 1$. Recall that the function $\zeta(s,\alpha)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and is meromorphically continued to the whole complex plane. Analytic properties of the function $\zeta(s,\alpha)$ depends on the parameter α. Throughout the chapter, we suppose that α is a transcendental number. Let $w(t)$ be a positive function on $[T_0,\infty)$, $T_0 > 0$, such that the variation $V_a^b w$ on $[a,b]$ satisfies the inequality $V_a^b w \leqslant cw(a)$ with certain $c > 0$ for any subinterval $[a,b] \subset [T_0,\infty)$, and

$$U_T = U_T(T,w) = \int_{T_0}^{T} w(t)\,\mathrm{d}t \to \infty$$

as $T \to \infty$. Denote the class of the above functions $w(t)$ by W_1.

3.1 Statements of the theorems

Recall that $H(K)$, $K \in \mathcal{K}$, denotes the class of continuous functions on K that are analytic in the interior of K. Thus, in this chapter, we will approximate a wider class of analytic functions, differently from Chapter 2, where the subclass $H_0(K) \subset H(K)$ of non-vanishing functions was investigated. We will prove the following theorems.

Theorem 3.1. *Suppose that $w(t) \in W_1$ and the parameter α is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \times I\left(\left\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0.$$

Theorem 3.1 has a modification in terms of density.

Theorem 3.2. *Under the hypotheses of Theorem 3.1, the limit*

$$\lim_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \times I\left(\left\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The proof of Theorems 3.1 and 3.2 are probabilistic, based on weighted limit theorems in the space $H(D)$. Weighted mean square estimates for the function $\zeta(s,\alpha)$ also play an important role.

3.2 Weighted mean square estimate

It is well known that the mean square estimates play an important role in the proofs of universality for zeta-functions. For the proof of Theorems 3.1 and 3.2, we need a weighted mean square estimate for the Hurwitz zeta-function. For this, we will apply the approximation of the function $\zeta(s,\alpha)$ by a finite sum.

Lemma 3.1. *Suppose that $\sigma \geqslant \sigma_0 > 0$ and $2\pi \leqslant |t| \leqslant \pi x$. Then*

$$\zeta(s,\alpha) = \sum_{0\leqslant m\leqslant x} \frac{1}{(m+\alpha)^s} + \frac{(x+\alpha)^{1-s}}{s-1} + \mathrm{O}_{\sigma_0}\left(x^{-\sigma}\right).$$

A proof of the lemma can be found, for example, in [42] Theorem 3.1.3.

Lemma 3.2. *Suppose that* σ, $1/2<\sigma<1$, *is fixed and* $\tau\in\mathbb{R}$. *Then*

$$\int_{T_0}^{T} w(t)|\zeta(\sigma+it+i\tau,\alpha)|^2\,\mathrm{d}t \ll U_T\left(1+|\tau|^2\right).$$

Proof. We take $x=t+|\tau|$ in Lemma 3.1. Then we have

$$\int_{T_0}^{T} w(t)|\zeta(\sigma+it+i\tau,\alpha)|^2\,\mathrm{d}t \ll \int_{T_0}^{T} w(t)\left|\sum_{0\leqslant m\leqslant t+|\tau|}\frac{1}{m^{\sigma+it+i\tau}}\right|^2\mathrm{d}t + U_T\left(1+|\tau|^2\right), \tag{3.1}$$

since

$$\begin{aligned}\int_{T_0}^{T} w(t)\frac{(t+|\tau|)^{2-2\sigma}}{(t+\tau)^2+(\sigma-1)^2}\,\mathrm{d}t \ll &\int_{T_0}^{2|\tau|} w(t)(t+|\tau|)^{2-2\sigma}\,\mathrm{d}t\\ &+\int_{2|\tau|}^{T} w(t)t^{-2}(t+|\tau|)^{2-2\sigma}\,\mathrm{d}t\\ \ll &U_T\left(1+|\tau|^2\right)\end{aligned}$$

and

$$\int_{T_0}^{T} w(t)(t+|\tau|)^{-2\sigma}\,\mathrm{d}t \ll U_T.$$

Let $\max(m,k)=T_1+|\tau|$, where $T_1=T_1(m,k)$. Then we have

$$\begin{aligned}&\int_{T_0}^{T} w(t)\left|\sum_{0\leqslant m\leqslant t+|\tau|}\frac{1}{m^{\sigma+it+i\tau}}\right|^2\mathrm{d}t\\ &=\sum\sum_{T_0+|\tau|\leqslant m,k\leqslant T+|\tau|}\frac{1}{(m+\alpha)^{\sigma+it}(k+\alpha)^{\sigma-it}}\int_{T_1}^{T} w(t)\left(\frac{k+\alpha}{m+\alpha}\right)^{it}\mathrm{d}t\\ &\ll \sum_{m\leqslant T+|\tau|}\frac{1}{(m+\alpha)^{2\sigma}}\int_{T_1}^{T} w(t)\,\mathrm{d}t\\ &\quad+\sum\sum_{T_0+|\tau|\leqslant m<k\leqslant T+|\tau|}\frac{w(k-|\tau|)}{(m+\alpha)^{\sigma}(k+\alpha)^{\sigma}\log((k+\alpha)/(m+\alpha))}.\end{aligned} \tag{3.2}$$

Clearly,

$$\sum_{m\leqslant T+|\tau|}\frac{1}{(m+\alpha)^{2\sigma}}\int_{T_1}^{T} w(t)\,\mathrm{d}t \ll U_T. \tag{3.3}$$

If $m+\alpha < (k+\alpha)/2$, then

$$\log \frac{m+\alpha}{k+\alpha} > \log 2,$$

thus,

$$\begin{aligned}
&\sum\sum_{m<k\leqslant T+|\tau|} \frac{w(k-|\tau|)}{(m+\alpha)^\sigma (k+\alpha)^\sigma \log((k+\alpha)/(m+\alpha))} \\
&\ll \sum\sum_{m<k\leqslant T+|\tau|} \frac{w(k-|\tau|)}{m^\sigma k^\sigma} \ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|} \frac{w(k-|\tau|)}{k^{2\sigma-1}} \\
&\ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|} w(k-|\tau|) = \int_{T_0+|\tau|}^{T+|\tau|} w(u-|\tau|)\,\mathrm{d}[u] \\
&= [u]w(u-|\tau|)\Big|_{T_0+|\tau|}^{T+|\tau|} - \int_{T_0+|\tau|}^{T+|\tau|} (u-\{u\})\,\mathrm{d}w(u-|\tau|) \\
&\ll U_T + \int_{T_0+|\tau|}^{T+|\tau|} \mathrm{d}w(u-|\tau|) \ll U_T. \qquad (3.4)
\end{aligned}$$

If $m+\alpha \geqslant (k+\alpha)/2$, then we write $m = k-r$, where $1 \leqslant r \leqslant k/2+\alpha/2$. In this case,

$$\log \frac{k+\alpha}{m+\alpha} = -\log \frac{k-r+\alpha}{k+\alpha} = -\log\left(1-\frac{r}{k+\alpha}\right) > \frac{r}{k+\alpha} > \frac{r}{k},$$

and

$$\begin{aligned}
&\sum\sum_{T_0+|\tau|\leqslant m<k\leqslant T+|\tau|} \frac{w(k-|\tau|)}{(m+\alpha)^\sigma (k+\alpha)^\sigma \log((k+\alpha)/(m+\alpha))} \\
&\ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|} \sum_{r\leqslant k/2+\alpha/2} \frac{kw(k-|\tau|)}{rk^\sigma (k-r)^\sigma} \\
&\ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|} \frac{w(k-|\tau|)\log k}{k^{2\sigma-1}} \ll U_T.
\end{aligned}$$

This and (3.1) – (3.4) prove the lemma. □

3.3 Limit theorems

For the proof of weighted universality theorems for the function $\zeta(s,\alpha)$, a weighted limit theorem on weakly convergent probability measures in the space of analytic functions will be applied, and such a theorem is the aim of this section.

As in Chapter 2 denote by γ the unit circle $\{s \in \mathbb{C} : |s| = 1\}$ on the complex plane, and define

$$\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$, $m \in \mathbb{N}_0$, the mth component of the element $\omega \in \Omega$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \alpha, \omega)$ by the formula

$$\zeta(s, \alpha, \omega) = \sum_{m=0}^{\infty} \frac{\omega(m)}{(m+\alpha)^s}.$$

We note that the latter series is uniformly convergent on compact subsets of the strip D for almost all $\omega \in \Omega$ with respect to the measure m_H [42]. Let P_ζ be the distribution of the random element $\zeta(s, \alpha, \omega)$, i. e.,

$$P_\zeta(A) = m_H\{\omega \in \Omega : \zeta(s, \alpha, \omega) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

For $A \in \mathcal{B}(H(D))$, let

$$P_T(A) = \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\left(\{\tau \in [T_0, T] : \zeta(s + i\tau, \alpha) \in A\}\right) \mathrm{d}\tau.$$

Theorem 3.3. *Suppose that the weight function $w(t)$ and the parameter α are as in Theorem 3.1. Then P_T converges weakly to P_ζ as $T \to \infty$. Moreover, the support of P_ζ is the whole $H(D)$.*

We divide the proof of Theorem 3.3 into lemmas. The first lemma is a weighted limit theorem on the torus Ω. For $A \in \mathcal{B}(\Omega)$, let

$$\begin{aligned} Q_T(A) = &\frac{1}{U_T} \int_{T_0}^{T} w(\tau) \\ &\times I\left(\left\{\tau \in [T_0, T] : \left((m+\alpha)^{-i\tau} : \ m \in \mathbb{N}_0\right) \in A\right\}\right) \mathrm{d}\tau. \end{aligned}$$

Lemma 3.3. *Suppose that the weight function $w(t)$ and the parameter α are* *as in Theorem 3.1. Then Q_T converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. We consider the Fourier transform $g_T(\underline{k})$, $\underline{k} = (k_m : \ k_m \in \mathbb{Z}, \ m \in \mathbb{N}_0)$. The characters χ of the group Ω have the form

$$\chi(\omega) = \prod_{m \in \mathbb{N}_0} \omega^{k_m}(m),$$

where only a finite number of integers k_m are distinct from zero. Therefore,

$$g_T(\underline{k}) = \int_\Omega \prod_{m \in \mathbb{N}_0} \omega^{k_m}(m) \, \mathrm{d}Q_T.$$

Thus, by the definition of Q_T,

$$\begin{aligned} g_T(\underline{k}) &= \frac{1}{U} \int_{T_0}^{T} w(\tau) \prod_{m \in \mathbb{N}_0} (m+\alpha)^{-ik_m\tau} \, \mathrm{d}\tau \\ &= \frac{1}{U} \int_{T_0}^{T} w(\tau) \exp\left\{ -i\tau \sideset{}{'}\sum_{m \in \mathbb{N}_0} k_m \log(m+\alpha) \right\} \mathrm{d}\tau, \end{aligned} \tag{3.5}$$

where the sum $\sum'$ means that only a finite number of integers k_m are distinct from zero. Obviously,

$$g_T(\underline{0}) = 1. \tag{3.6}$$

Suppose that $\underline{k} \neq \underline{0}$. It is well known that the set $\{\log(m+\alpha) : m \in \mathbb{N}_0\}$ with transcendental α is linearly independent over the field of rational numbers $\mathbb{Q}$. Therefore, in this case,

$$\sideset{}{'}\sum_{m \in \mathbb{N}_0} k_m \log(m+\alpha) \neq 0,$$

and, in view of (3.5),

$$
\begin{aligned}
g_{T,}(\underline{k}) &= -\frac{1}{iU \sum'_{m\in\mathbb{N}_0} k_m \log(m+\alpha)} \\
&\quad \times \int_{T_0}^{T} w(\tau)\,\mathrm{d}\exp\left\{-i\tau \sum'_{m\in\mathbb{N}_0} k_m \log(m+\alpha)\right\} \\
&\ll \left(U\left|\sum'_{m\in\mathbb{N}_0} k_m \log(m+\alpha)\right|\right)^{-1} \left(1+\int_{T_0}^{T} |\,\mathrm{d}w(\tau)|\right) \\
&\ll \left(U\left|\sum'_{m\in\mathbb{N}_0} k_m \log(m+\alpha)\right|\right)^{-1}
\end{aligned}
$$

because the variation of $w(t)$ is bounded. This estimate shows that

$$\lim_{T\to\infty} g_T(\underline{k}) = 0.$$

Thus, taking into account (3.1), we have that

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H, the lemma follows by a continuity theorem for probability measures on compact groups. □

The next lemma deals with absolutely convergent Dirichlet series. Let $\theta > 1/2$ be a fixed number, and, for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$,

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\theta}\right\}.$$

Define two functions

$$\zeta_n(s,\alpha) = \sum_{m=0}^{\infty} \frac{v_n(m,\alpha)}{(m+\alpha)^s}$$

and

$$\zeta_n(s,\alpha,\omega) = \sum_{m=0}^{\infty} \frac{\omega(m)v_n(m)}{(m+\alpha)^s}.$$

Then it is known [42] that the above series are absolutely convergent in the half-plane $\sigma > 1/2$. Consider the function $u_n : \Omega \to H(D)$ given by the formula

$$u_n(\omega) = \zeta_n(s,\alpha,\omega), \quad \omega \in \Omega.$$

The absolute convergence of the series $\zeta_n(s,\alpha,\omega)$ implies the continuity of the function u_n. Thus, the function u_n is $(\mathcal{B}(\Omega), \mathcal{B}(H(D)))$-measurable, and the measure m_H induces the probability measure $\hat{P}_n = m_H u_n^{-1}$ given by

$$\hat{P}_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1}A), \quad A \in \mathcal{B}(H(D)).$$

For $A \in \mathcal{B}(H(D))$, define

$$P_{T,n}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I\left(\{\tau \in [T_0,T] :\ \zeta_n(s+i\tau,\alpha) \in A\}\right) \mathrm{d}\tau.$$

Lemma 3.4. *Suppose that the weight function $w(t)$ and the parameter α are as in Theorem 3.1. Then $P_{T,n}$ converges weakly to the measure $\hat{P}_n$ as $T \to \infty$.*

Proof. By the definition of the function u_n,

$$u_n\left((m+\alpha)^{-i\tau} :\ m \in \mathbb{N}_0\right) = \zeta_n(s+i\tau,\alpha).$$

Hence,

$$\begin{aligned} P_{T,n}(A) = \frac{1}{U}\int_{T_0}^{T} & w(\tau) \\ & \times I\left(\left\{\tau \in [T_0,T] :\ \left((m+\alpha)^{-i\tau} :\ m \in \mathbb{N}_0\right) \in u_n^{-1}A\right\}\right) \mathrm{d}\tau \\ = Q_T&(u_n^{-1}A) = Q_T u_n^{-1}(A), \end{aligned}$$

where Q_T is from Lemma 3.3. This equality, the continuity of the function u_n, Lemma 3.3 and Theorem 5.1 of [4] prove the lemma. ☐

Let ρ be the same metric in $H(D)$ that was used in Chapter 2.

Lemma 3.5. *Suppose that the weight function $w(t) \in W_h$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U_T}\int_{T_0}^{T} w(\tau)\rho(\zeta(s+i\tau,\alpha), \zeta_n(s+i\tau,\alpha))\,\mathrm{d}\tau = 0.$$

Proof. Let, as usual, $\Gamma(s)$ denote the Euler gamma-function, and let θ be from the definition of $\zeta_n(s,\alpha)$. Define

$$l_n(s,\alpha) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)(n+\alpha)^s, \quad n \in \mathbb{N}.$$

Then, for $\zeta_n(s,\alpha)$, the integral representation

$$\zeta_n(s,\alpha) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z,\alpha) l_n(z,\alpha)\frac{\mathrm{d}z}{z}$$

for $1/2 < \sigma < 1$ is known [42]. Let $K \subset D$ be a compact set. Using the above representation and a standard contour integration, we find that, as $T \to \infty$,

$$\begin{aligned} &\frac{1}{U_T}\int_{T_0}^{T} w(\tau) \sup_{s\in K} |\zeta(s+i\tau,\alpha) - \zeta_n(s+i\tau,\alpha)| \,\mathrm{d}\tau \\ &\ll \int_{-\infty}^{\infty} |l_n(\sigma_1+it,\alpha)| \left(\frac{1}{U_T}\int_{T_0}^{T} w(\tau) \left|\zeta(\sigma+it+i\tau,\alpha)\right| \,\mathrm{d}\tau\right) \mathrm{d}t \\ &\quad + o(1), \end{aligned} \tag{3.7}$$

where $\sigma_1 < 0$, and $1/2 < \sigma < 1$. Moreover, by Lemma 3.2,

$$\begin{aligned} &\int_{T_0}^{T} w(\tau)|\zeta(\sigma+it+i\tau,\alpha)| \,\mathrm{d}\tau \\ &\qquad \ll \left(\int_{T_0}^{T} w(\tau)\mathrm{d}\tau \int_{T_0}^{T} w(\tau)|\zeta(\sigma+it+i\tau,\alpha)|^2 \,\mathrm{d}\tau\right)^{1/2} \\ &\qquad \ll U_T\left(1+|t|^2\right)^{1/2} \ll U_T(1+|t|). \end{aligned}$$

Therefore, in view of (3.7),

$$\begin{aligned} &\frac{1}{U_T}\int_{T_0}^{T} w(\tau) \sup_{s\in K} |\zeta(s+i\tau,\alpha) - \zeta_n(s+i\tau,\alpha)| \,\mathrm{d}\tau \\ &\qquad \ll \int_{-\infty}^{\infty} |l_n(\sigma_1+it,\alpha)| \,(1+|t|) \,\mathrm{d}t + o(1) \end{aligned}$$

as $T \to \infty$. Since $\sigma_1 < 0$,

$$\lim_{n\to\infty} l_n(\sigma+it,\alpha) = 0,$$

thus,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) \sup_{s\in K} |\zeta(s+i\tau,\alpha) - \zeta_n(s+i\tau,\alpha)| \,\mathrm{d}\tau = 0.$$

This equality and the definition of the metric ρ prove the lemma. □

3.4 Proof of Theorem 3.3

Proof. We will prove that the limit measure $\hat{P}_n$ of Lemma 3.4 converges weakly to a certain probability measure P as $n \to \infty$, and that P_T, as $T \to \infty$, also converges weakly to P.

Let a random variable θ_T is defined on a certain probability space $(\hat{\Omega}, \mathcal{A}, \mu)$ by

$$\mu(\theta_T \in A) = \frac{1}{U_T} \int_{T_0}^{T} w(t) I_A(t) \,\mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{R}),$$

where I_A is the indicator of the set A. Define the $H(D)$-valued random element

$$X_{T,n} = X_{T,n}(s) = \zeta_n(s + i\theta_T, \alpha).$$

Since $P_{T,n}$, by Lemma 3.4, converges weakly to $\hat{P}_n$ as $T \to \infty$, we have that

$$X_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \hat{X}_n, \tag{3.8}$$

where $\hat{X}_n$ is the $H(D)$-valued random element with the distribution $\hat{P}_n$. Further, we will consider the family of probability measures $\{\hat{P}_n : \ n \in \mathbb{N}\}$ and will prove that this family is tight, i. e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D)$ such that

$$\hat{P}_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Let the set K_l be from the definition of the metric ρ, and $M_l > 0$.

Then we have

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{s\in K_l}|X_{T,n}(s)|>M_l\right)\\
&=\frac{1}{U_T}\int_{T_0}^{T}w(\tau)I\left(\left\{\tau\in[T_0,T]:\ \sup_{s\in K_l}|\zeta_n(s+i\tau,\alpha)|>M_l\right\}\right)\mathrm{d}\tau\\
&\ll\frac{1}{M_lU_T}\int_{T_0}^{T}w(\tau)\sup_{s\in K_l}|\zeta_n(s+i\tau,\alpha)|\,\mathrm{d}\tau. \qquad (3.9)
\end{aligned}
$$

Since the series for $\zeta_n(s,\alpha)$ is absolutely convergent for $\sigma>1/2$,

$$
\begin{aligned}
\lim_{T\to\infty}\frac{1}{U_T}\int_{T_0}^{T}w(t)|\zeta_n(s+it,\alpha)|^2\mathrm{d}t&=\sum_{m=0}^{\infty}\frac{v_n^2(m,\alpha)}{(m+\alpha)^{2\sigma}}\\
&\leqslant\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^{2\sigma}}\leqslant C<\infty.
\end{aligned}
$$

This and the Cauchy integral formula show that

$$
\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{U_T}\int_{T_0}^{T}w(\tau)\sup_{s\in K_l}|\zeta_n(s+i\tau,\alpha)|\,\mathrm{d}\tau\leqslant C_l<\infty. \qquad (3.10)
$$

We fix $\varepsilon>0$ and take $M_l=C_l2^l\varepsilon^{-1}$. Then (3.9) and (3.10) give

$$
\limsup_{T\to\infty}\mu\left(\sup_{s\in K_l}|X_{T,n}(s)|>M_l\right)\leqslant\frac{\varepsilon}{2^l}
$$

for all $l\in\mathbb{N}$ and $n\in\mathbb{N}$. Hence, by (3.8),

$$
\mu\left(\sup_{s\in K_l}|\hat{X}_n(s)|>M_l\right)\leqslant\frac{\varepsilon}{2^l} \qquad (3.11)
$$

for all $l\in\mathbb{N}$ and $n\in\mathbb{N}$. Define the set

$$
K=K(\varepsilon)=\left\{g\in H(D):\sup_{s\in K_l}|g(s)|\leqslant M_l,\ l\in\mathbb{N}\right\}.
$$

Then K is uniformly bounded on compact subsets of the strip D, therefore, it is a compact set of the space $H(D)$. Moreover, in view of (3.10),

$$\mu\left(\hat{X}_n(s) \in K\right) \geqslant 1 - \varepsilon \sum_{l=1}^{\infty} 2^{-l} = 1 - \varepsilon,$$

or

$$\hat{P}_n(K) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Thus the family $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight.

By the Prokhorov theorem, Theorem 6.1 [4], the tightness of the family $\{\hat{P}_n : n \in \mathbb{N}\}$ implies its relative compactness. Therefore, there exists a sequence $\{\hat{P}_{n_k}\} \subset \{\hat{P}_n\}$ weakly convergent to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $k \to \infty$. In other words,

$$\hat{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{3.12}$$

Moreover, an application of Lemma 3.4 shows that, for every $\varepsilon > 0$,

$$\begin{aligned}&\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \\ &\quad \times I\left(\{\tau \in [T_0, T] :\ \rho(\zeta(s+i\tau,\alpha), \zeta_n(s+i\tau,\alpha)) \geqslant \varepsilon\}\right) \mathrm{d}\tau \\ &\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon U_T} \int_{T_0}^{T} w(\tau)\rho(\zeta(s+i\tau,\alpha), \zeta_n(s+i\tau,\alpha))\,\mathrm{d}\tau = 0.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left(\rho(X_T, X_{T,n}) \geqslant \varepsilon\right) = 0, \tag{3.13}$$

where $H(D)$-valued random element $X_T = X_T(s)$ is defined by

$$X_T(s) = \zeta(s + i\theta_T, \alpha).$$

Now relations (3.8), (3.12) and (3.13) together with Theorem 4.2 of [4] imply the relation

$$X_T \xrightarrow[T\to\infty]{\mathcal{D}} P, \tag{3.14}$$

and this is equivalent to weak convergence of P_T to P as $T \to \infty$.

For identification of the measure P, we apply the following arguments. Relation (3.14) shows that the measure P is independent of the choice of the subsequence $\hat{P}_{n_k}$. Since the family $\{\hat{P}_n\}$ is relatively compact, from this we find that

$$\hat{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P,$$

or, in other words, $\hat{P}_n$ converges weakly to P. Let

$$R_T(A) = \frac{1}{T}\,\mathrm{meas}\left\{\tau \in [0,T] : \ \zeta(s+i\tau,\alpha) \in A\right\}, \quad A \in \mathcal{B}(H(D)).$$

Then, in [36], it is obtained that if the set $L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$, then R_T, as $T \to \infty$, also converges weakly to the limit measure P of $\hat{P}_n$ as $n \to \infty$, and that $P = P_\zeta$. Thus, since the set $L(\alpha)$, with transcendental α, is linearly independent over $\mathbb{Q}$, we have from above, that P_T also converges weakly to P_ζ as $T \to \infty$. Moreover, the support of P_ζ is the whole $H(D)$. The theorem is proved. □

3.5 Proof of universality

Proof of Theorem 3.1. By the Mergelyan theorem on the approximation of analytic functions by polynomials, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3.15}$$

Define the set

$$G_\varepsilon = \left\{ g \in H(D) : \ \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Since, by Theorem 3.3, the polynomial $p(s)$ is an element of the support of the measure P_ζ, we have that

$$P_\zeta(G_\varepsilon) > 0. \tag{3.16}$$

Moreover, by Theorem 3.3 again and the equivalent of weak convergence of probability measures in terms of open sets,

$$\liminf_{T\to\infty} P_T(G_\varepsilon) \geqslant P_\zeta(G_\varepsilon).$$

Therefore, by the definition of P_T and (3.16),

$$\liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\left(\{\tau \in [T_0, T]:\ \zeta(s+i\tau,\alpha) \in G_\varepsilon\}\right) \mathrm{d}\tau$$
$$= \liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau)$$
$$\times I\left(\left\{\tau \in [T_0, T]:\ \sup_{s\in K} |\zeta(s+i\tau,\alpha) - p(s)| < \frac{\varepsilon}{2}\right\}\right) \mathrm{d}\tau > 0. \quad (3.17)$$

Inequality (3.15) shows that

$$\left\{\tau \in [T_0, T]:\ \sup_{s\in K} |\zeta(s+i\tau,\alpha) - p(s)| < \frac{\varepsilon}{2}\right\}$$
$$\subset \left\{\tau \in [T_0, T]:\ \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\}.$$

Thus,

$$I\left(\left\{\tau \in [T_0, T]:\ \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\}\right)$$
$$\geqslant I\left(\left\{\tau \in [T_0, T]:\ \sup_{s\in K} |\zeta(s+i\tau,\alpha) - p(s)| < \frac{\varepsilon}{2}\right\}\right).$$

This and (3.16) prove the theorem. □

Proof Theorem 3.2. Define the set

$$\hat{G}_\varepsilon = \left\{g \in H(D):\ \sup_{s\in K} |g(s) - f(s)| < \varepsilon\right\},$$

and let ∂ be the boundary operator. Then we have that $\partial\hat{G}_{\varepsilon_1} \cap \partial\hat{G}_{\varepsilon_2} = \varnothing$ for $\varepsilon_1 \neq \varepsilon_2$. Hence, it follows that the set $\hat{G}_\varepsilon$ is a continuity set of the measure P_ζ, i. e., $P_\zeta(\partial\hat{G}_\varepsilon) = 0$ for all but at most countably many $\varepsilon > 0$. Therefore, using Theorem 3.3 and the equivalent of weak convergence in terms of continuity sets, we obtain that the limit

$$\lim_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) I\left(\left\{\tau \in [T_0, T]:\ \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\}\right)$$
$$= P_\zeta(\hat{G}_\varepsilon) \quad (3.18)$$

exists for all but at most countably many $\varepsilon > 0$. On the other hand, in view of

(3.15), we have that $\hat{G}_\varepsilon \supset G_\varepsilon$. Thus, $P_\zeta(\hat{G}_\varepsilon) \geqslant P_\zeta(G_\varepsilon) > 0$ by (3.16), and (3.18) proves the theorem. □

Remark. The transcendence of the parameter α in Theorems 3.1 and 3.2 can be replaced by the weaker hypothesis of the linear independence over $\mathbb{Q}$ for the set $L(\alpha)$.

Chapter 4

Joint weighted universality theorems for Hurwitz zeta-functions

In this chapter, we generalize weighted universality theorems for the Hurwitz zeta-functions (Theorems 3.1 and 3.2) for the collection of Hurwitz zeta-functions. Thus, let $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$ be a collection of Hurwitz zeta-functions. Since, in the joint theorem, the approximating functions must be in a certain sense independent, we introduce the set

$$L(\alpha_1, \dots, \alpha_r) = \{(\log(m + \alpha_1) : m \in \mathbb{N}_0), \dots, (\log(m + \alpha_r) : m \in \mathbb{N})\}$$

and use the independence of the functions $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$ defined by means of the set $L(\alpha_1, \dots, \alpha_r)$

We preserve the notation of Chapter 3.

4.1 Statement of joint theorems

In this chapter, we will prove the following theorems.

Theorem 4.1. *Suppose that $w(t) \in W_1$ and the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, .., r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then* *for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \times I\left(\left\{\tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0.$$

As Theorem 3.1, Theorem 4.1 also has a modified version in terms of density.

Theorem 4.2. *Under hypothesis of Theorem 4.1, the limit*

$$\lim_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \times I\left(\left\{\tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0$$

exists for all but at most countably many ε>0.

Thus, we see that the linear independence over $\mathbb{Q}$ of the set $L(\alpha_1, \ldots, \alpha_r)$ ensures a certain independence of the functions $\zeta(s, \alpha_1), \ldots, \zeta(s, \alpha_r)$ and allows to obtain joint universality theorems for them.

We observe that the set $L(\alpha_1, \ldots, \alpha_r)$ is linearly independent over $\mathbb{Q}$ if the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, i. e., if for every non-trivial polynomial $p(s_1, \ldots, s_r)$ with rational coefficients $p(\alpha_1, \ldots, \alpha_r) \neq 0$. For example, the numbers $\alpha_1 = 1/\pi, \ldots, \alpha_r = 1/(r\pi)$ are algebraically independent over $\mathbb{Q}$ because π is a transcendental number.

For the proof of Theorems 4.1 and 4.2, as for universality theorems of previous chapters we will apply elements of the theory of weak convergence of probability measures.

4.2 Joint weighted limit theorems

We recall that $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, and denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. For $A \in \mathcal{B}(H(D))$, define

$$P_{T,w}(A) = \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] : \underline{\zeta}(s+i\tau, \underline{\alpha}) \in A\right\}\right) \mathrm{d}\tau,$$

where $\underline{\zeta}(s,\underline{\alpha}) = (\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_r))$, $\underline{\alpha} = (\alpha_1,\dots,\alpha_r)$. In this section, we will prove a limit theorem for $P_{T,w}$ as $T\to\infty$. The statement of the latter theorem involves the following topological structure.

Denote by γ the unit circle on the complex plane, and define the set

$$\Omega = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m\in\mathbb{N}_0$. By the Tikhonov theorem, with the product topology and positive multiplication, the infinite-dimensional torus Ω is a compact topological group. Define one more set

$$\underline{\Omega} = \Omega_1\times\cdots\times\Omega_r,$$

where $\Omega_j = \Omega$ for all $j=1,\dots,r$. Then, again by the Tikhonov theorem, $\underline{\Omega}$ is a compact topological group. Therefore, on $(\underline{\Omega},\mathcal{B}(\underline{\Omega}))$, the probability Haar measure $\underline{m}_H$ can be defined, and we obtain the probability space $(\underline{\Omega},\mathcal{B}(\underline{\Omega}),\underline{m}_H)$. For $j=1,\dots,r$, denote by ω_j the elements of Ω_j. Then $\omega = (\omega_1,\dots,\omega_r)$ are elements of $\underline{\Omega}$. Moreover, for $j=1,\dots,r$, let $\omega_j(m)$, $m\in\mathbb{N}_0$, be the mth component of the element ω_j. Now, on the probability space $(\underline{\Omega},\mathcal{B}(\underline{\Omega}),\underline{m}_H)$, define the $H^r(D)$ - valued random element $\underline{\zeta}(s,\omega,\underline{\alpha})$ by the formula

$$\underline{\zeta}(s,\omega,\underline{\alpha}) = (\zeta(s,\omega_1,\alpha_1),\dots,(s,\omega_r,\alpha_r)),$$

where

$$\zeta(s,\omega_j,\alpha_j) = \sum_{m=0}^{\infty}\frac{\omega_j(m)}{(m+\alpha_j)^s}, \quad j=1,\dots,r.$$

Let $P_{\underline{\zeta}}$ be the distribution of the random element $\underline{\zeta}(s,\omega,\underline{\alpha})$, i. e.,

$$P_{\underline{\zeta}}(A) = \underline{m}_H\left\{\omega\in\underline{\Omega} : \underline{\zeta}(s,\omega,\underline{\alpha})\in A\right\}, \quad A\in\mathcal{B}(H^r(D)).$$

Using the above definitions, leads to the following limit theorem for the $P_{T,w}$.

Theorem 4.3. *Suppose that the set $L(\alpha_1,\dots,\alpha_r)$ is linearly independent over $\mathbb{Q}$. Then $P_{T,w}$ converges weakly to the measure $P_{\underline{\zeta}}$ as $T\to\infty$.*

We divide the proof of Theorem 4.3 into lemmas on weakly convergent measures in certain spaces. The first of them is the space $(\underline{\Omega},\mathcal{B}(\underline{\Omega}))$. For

$A \in \mathcal{B}(\underline{\Omega})$, define

$$Q_{T,w}(A) = \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I \left(\left\{ \tau \in [T_0, T] : \big(((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \ldots, \right.\right.$$
$$\left.\left. ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0) \big) \in A \right\} \right) \mathrm{d}\tau.$$

Lemma 4.1. *Suppose that the set $L(\alpha_1, \ldots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then $Q_{T,w}$ converges weakly to the Haar measure $\underline{m}_H$ as $T \to \infty$.*

Proof. The dual group of $\underline{\Omega}$ is isomorphic to

$$G \overset{def}{=} \bigoplus_{j=1}^{r} \bigoplus_{m \in \mathbb{N}_0} \mathbb{Z}_{mj},$$

where $\mathbb{Z}_{mj} = \mathbb{Z}$ for all $m \in \mathbb{N}_0$ and $j = 1, \ldots, r$. The element $\underline{k} = \{k_{mj} : k_{mj} \in \mathbb{Z}, m \in \mathbb{N}_0\}, \quad j = 1, \ldots, r$, acts on $\underline{\Omega}$ by

$$\omega \to \omega^{\underline{k}} = \prod_{j=1}^{r} {\prod_{m \in \mathbb{N}_0}}' \omega_j^{k_{mj}}(m),$$

where the sign "$'$" means that only a finite number of integers k_{mj} are distinct from zero. Then the Fourier transform $g_{T,w}(\underline{k})$ of $Q_{T,w}$ is defined by

$$g_{T,w}(\underline{k}) = \int_{\underline{\Omega}} \prod_{j=1}^{r} {\prod_{m \in \mathbb{N}_0}}' \omega_j^{k_{mj}}(m) \, \mathrm{d}Q_{T,w}.$$

Thus, by the definition of $Q_{T,w}$, we have that

$$\begin{aligned} g_{T,w}(\underline{k}) &= \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \prod_{j=1}^{r} {\prod_{m \in \mathbb{N}_0}}' (m+\alpha_j)^{-i\tau k_{mj}}(m) \mathrm{d}\tau \\ &= \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \exp\left\{ -i\tau \sum_{j=1}^{r} {\sum_{m \in \mathbb{N}_0}}' k_{mj} \log(m+\alpha_j) \right\} \mathrm{d}\tau. \quad (4.1) \end{aligned}$$

Obviously,

$$g_{T,w}(\underline{0}) = 1. \quad (4.2)$$

Now, suppose that $\underline{k} \neq \underline{0}$. Since the set $L(\alpha_1, \ldots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, then

$$A \stackrel{def}{=} \sum_{j=1}^{r} \sum_{m\in\mathbb{N}_0}{}' k_{mj} \log(m+\alpha_j) \neq 0.$$

Then, in view of (4.1),

$$\begin{aligned} g_{T,w}(\underline{k}) &= O\left(\frac{1}{U_T A}\right) + \frac{1}{iU_T A}\int_{T_0}^{T} \exp\{-i\tau A\}\,\mathrm{d}w(\tau) \\ &= O\left(\frac{1}{U_T A}\right) + O\left(\frac{V_{T_0}^{T} w}{U_T A}\right). \end{aligned}$$

Thus, for $\underline{k} \neq \underline{0}$,

$$\lim_{T\to\infty} g_{T,w}(\underline{k}) = 0.$$

This together with (4.2) shows that

$$\lim_{T\to\infty} g_{T,w}(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure $\underline{m}_H$, the lemma follows by the continuity theorem for probability measures on compact groups. □

Lemma 4.1 is the main ingredient of the proof of a joint weighted limit theorem for absolutely convergent Dirichlet series connected to Hurwitz zeta-functions.

Let $\theta > 1/2$ be a fixed number. For $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, define

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\theta}\right\}, \quad j = 1,\dots,r,$$

and consider the collections

$$\underline{\zeta}_n(s,\underline{\alpha}) = (\zeta_n(s,\alpha_1),\dots,\zeta_n(s,\alpha_r))$$

and

$$\underline{\zeta}_n(s,\omega,\underline{\alpha}) = (\zeta_n(s,\omega_1,\alpha_1),\dots,\zeta_n(s,\omega_r,\alpha_r)),$$

where, for $j = 1, \ldots, r$,

$$\zeta_n(s, \alpha_j) = \sum_{m=0}^{\infty} \frac{v_n(m, \alpha_j)}{(m + \alpha_j)^s}$$

and

$$\zeta_n(s, \omega_j, \alpha_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m) v_n(m, \alpha_j)}{(m + \alpha_j)^s}.$$

It is known [36] that the series for $\zeta_n(s, \alpha_j)$ and $\zeta_n(s, \omega_j, \alpha_j)$ are absolutely convergent for $\sigma > 1/2$.

For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,n,w}(A) = \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] : \underline{\zeta}_n(s + i\tau, \underline{\alpha}) \in A\right\}\right) \mathrm{d}\tau.$$

Lemma 4.2. *Suppose that the set $L(\alpha_1, \ldots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then, on $(H^r(D), \mathcal{B}(H^r(D)))$, there exists a probability measure P_n such that $P_{T,n,w}$ converges weakly to P_n as $T \to \infty$.*

Proof. Define the mapping $u_n : \underline{\Omega} \to H^r(D)$ by the formula

$$u_n(\omega) = \underline{\zeta}_n(s, \omega, \underline{\alpha}).$$

Since the series for $\zeta_n(s, \omega_j, \alpha_j)$ are absolutely convergent for $\sigma > 1/2$, the mapping u_n is continuous. Moreover,

$$u_n(((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \ldots, ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0))) = \underline{\zeta}_n(s+i\tau, \underline{\alpha}).$$

Thus, by the definitions of $P_{T,n,w}$ and $Q_{T,w}$, we have that, for every $A \in \mathcal{B}(H^r(D))$,

$$P_{T,n,w}(A) = \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] : \left(\left((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0\right), \ldots, \left((m + \alpha_r)^{-i\tau} : m \in \mathbb{N}_0\right)\right) \in u^{-1}A\right\}\right) \mathrm{d}\tau.$$

Hence, $P_{T,n,w} = Q_{T,w} u_n^{-1}$. Therefore, the lemma is consequence of Lemma 2.2, continuity of u_n and Lemma 4.1. We have that $P_n = \underline{m}_H u_n^{-1}$, i. e., the limit measure of $P_{T,n,w}$ is independent of the weight function w. □

The next step of the proof of Theorem 4.3 is devoted to the approximation in the mean of $\underline{\zeta}(s, \underline{\alpha})$ by $\underline{\zeta}_n(s, \underline{\alpha})$. For this, we recall the metric in the space

$H^r(D)$. Let, as above, ρ is the metric in the space $H(D)$ inducing its topology of uniform convergence on compacta, and let $\underline{g}_1 = (g_{11}, \dots, g_{1r}), \underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$. Then taking

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho(g_{1j}, g_{2j})$$

gives a metric in the space $H^r(D)$ inducing its product topology.

Lemma 4.3. *The equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \underline{\rho} \left(\underline{\zeta}(s + i\tau, \underline{\alpha}), \zeta_n(s + i\tau, \underline{\alpha}) \right) \mathrm{d}\tau = 0$$

holds.

Proof. Obviously, by the definition of the metric ρ, the lemma follows from the equality

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \rho \left(\zeta(s + i\tau, \alpha), \zeta_n(s + i\tau, \alpha) \right) \mathrm{d}\tau = 0$$

with $\alpha = \alpha_j$, $j = 1, \dots, r$. Therefore, it suffices to show the above equality with arbitrary fixed α, $0 < \alpha < 1$.

Denote by $\Gamma(s)$ the Euler gamma-function, and define

$$l_n(s, \alpha) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) (n + \alpha)^s, \quad n \in \mathbb{N},$$

where the number θ is the same as in the definition of $v_n(s, \alpha_j)$. Using the Mellin formula

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(s) a^{-s} \mathrm{d}s = \mathrm{e}^{-a}, \quad a, b > 0,$$

we find that, for $\sigma > 1/2$,

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z,\alpha) l_n(z,\alpha)\frac{\mathrm{d}z}{z} \\
&\quad = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}\left(\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \Gamma\left(\frac{z}{\theta}\right)\left(\left(\frac{m+\alpha}{n+\alpha}\right)^{-\frac{z}{\theta}}\right)^{\theta z} \mathrm{d}\left(\frac{z}{\theta}\right)\right) \\
&\quad = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}\left(\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma(z)\left(\left(\frac{m+\alpha}{n+\alpha}\right)^{\theta}\right)^{-z} \mathrm{d}z\right) \\
&\quad = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}\exp\left\{-\left(-\frac{m+\alpha}{n+\alpha}\right)^{\theta}\right\} = \zeta_n(s,\alpha). \qquad (4.3)
\end{aligned}
$$

Let $K \subset D$ be an arbitrary compact set. We fix a positive ε such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1-\varepsilon$ for points $s \in K$, and take $\hat{\theta} > 0$. Then the equality (4.3) together with residue theorem yields

$$
\zeta_n(s,\alpha) - \zeta(s,\alpha) = \frac{1}{2\pi i}\int_{-\hat{\theta}-i\infty}^{-\hat{\theta}+i\infty} \zeta(s+z,\alpha) l_n(z,\alpha)\frac{\mathrm{d}z}{z} + R_n(s,\alpha), \qquad (4.4)
$$

where

$$
R_n(s,\alpha) = \frac{l_n(1-s,\alpha)}{1-s}.
$$

Denote by $s = \sigma + iv$ the points of the set K, and take $\hat{\theta} = \sigma - \varepsilon - 1/2$. Then, for $s \in K$, we derive from (4.4)

$$
\begin{aligned}
&|\zeta(s+i\tau,\alpha) - \zeta_n(s+i\tau,\alpha)| \\
&\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} |s+i\tau-\hat{\theta}+it,\alpha| \frac{|l_n(-\hat{\theta}+it,\alpha)|}{|-\hat{\theta}+it|}\,\mathrm{d}t + |R_n(s+i\tau,\alpha)|.
\end{aligned}
$$

Thus, shifting $v+t$ to t gives

$$
\frac{1}{U_T}\int_{T_0}^{T} w(\tau)\sup_{s\in K}|\zeta(s+i\tau,\alpha) - \zeta_n(s+i\tau,\alpha)|\,\mathrm{d}\tau \ll I_1 + I_2, \qquad (4.5)
$$

where

$$
\begin{aligned}
I_1 = &\frac{1}{U_T}\int_{-\infty}^{\infty}\left(\int_{T_0}^{T} w(\tau)\left|\zeta\left(\frac{1}{2}+\varepsilon+i(t+\tau),\alpha\right)\right|\mathrm{d}\tau\right) \\
&+ \sup_{s\in K}\frac{|l_n(1/2+\varepsilon-s+it,\alpha)|}{|1/2+\varepsilon-s+it|}\,\mathrm{d}t
\end{aligned}
$$

and

$$I_2 = \frac{1}{U_T} \int_0^T w(\tau) \sup_{s \in K} |R_n(s + i\tau, \alpha)| \mathrm{d}\tau.$$

For estimation of the quantities I_1 and I_2, we apply the properties of the gamma-function. It is well known that, for every fixed $\sigma_1 < \sigma_2$, uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}$$

with certain absolute constant $c > 0$. Therefore, taking $\theta = 1/2 + \varepsilon$ and using the definition of the function $l_n(s, \alpha)$, we obtain, for $s \in K$, the bound

$$\begin{aligned}
&\frac{|l_n(1/2 + \varepsilon - s + it, \alpha)|}{|1/2 + \varepsilon - s + it|} \\
&= \frac{(n+\alpha)^{1/2+\varepsilon-\sigma}}{\theta} \left| \Gamma\left(\frac{1/2 + \varepsilon - \sigma}{\theta} + \frac{i(t-v)}{\theta} \right) \right| \\
&\ll \frac{(n+\alpha)^{-\varepsilon}}{\theta} \exp\left\{ -\frac{c|t-v|}{\theta} \right\} \ll_K (n+\alpha)^{-\varepsilon} \exp\{-c_1|t|\} \qquad (4.6)
\end{aligned}$$

with $c_1 > 0$. Similarly, for $s \in K$, we obtain

$$\begin{aligned}
|R_n(s + i\tau, \alpha)| &\ll (n+\alpha)^{1-\sigma} \exp\left\{ -\frac{c|\tau - v|}{\theta} \right\} \\
&\ll_K (n+\alpha)^{1-\sigma} \exp\{-c_2|\tau|\} \qquad (4.7)
\end{aligned}$$

with $c_2 > 0$. In Lemma 3.1, it was obtained that, for fixed σ, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,

$$\int_{T_0}^T w(\tau) |\zeta(\sigma + i(t+\tau), \alpha)|^2 \, \mathrm{d}\tau \ll U_T(1 + |t|^2). \qquad (4.8)$$

Therefore,

$$\begin{aligned}
&\int_{T_0}^T w(\tau) \left| \zeta\left(\frac{1}{2} + \varepsilon + i(t+\tau), \alpha \right) \right| \mathrm{d}\tau \\
&\quad \ll \left(\int_{T_0}^T w(\tau) \, \mathrm{d}\tau \int_{T_0}^T w(\tau) \left| \zeta\left(\frac{1}{2} + \varepsilon + i(t+\tau), \alpha \right) \right|^2 \mathrm{d}\tau \right)^{1/2} \\
&\quad \ll U_T(1 + |t|).
\end{aligned}$$

This estimate together with (4.6) shows that

$$I_1 \ll_K (n+\alpha)^{-\varepsilon} \int_{\infty}^{\infty} (1+|t|) \exp\{-c_1|t|\} \, \mathrm{d}t \ll_K (n+\alpha)^{-\varepsilon}. \qquad (4.9)$$

The estimate (4.7) implies the bound

$$I_2 \ll_K \frac{(n+\alpha)^{1-\sigma}}{U_T} \int_{T_0}^{T} w(\tau) \exp\{-c_2|\tau|\} \, \mathrm{d}\tau \ll_K \frac{(n+\alpha)^{1/2-2\varepsilon}}{U_T}. \qquad (4.10)$$

Now, letting $T \to \infty$ and then $n \to \infty$, we obtain by (4.5), (4.9) and (4.10) that

$$\lim_{n\to\infty} \liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \sup_{s\in K} |\zeta(s+i\tau,\alpha) - \zeta_n(s+i\tau,\alpha)| \, \mathrm{d}\tau = 0.$$

This and the definition of the metric ρ gives the equality

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau)\rho\left(\zeta(s+i\tau,\alpha), \zeta_n(s+i\tau,\alpha)\right) \, \mathrm{d}\tau = 0,$$

and the lemma is proved. □

Now, we will deal with the limit measure P_n in Lemma 4.2.

Lemma 4.4. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then the sequence $\{P_n : n \in \mathbb{N}\}$ is relatively compact.*

Proof. By the mentioned above Prokhorov theorem, it suffices to prove the tightness of the sequence $\{P_n\}$.

Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and define

$$P_{T,n,\alpha}(A) = \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I(\{\tau \in [0,T] : \zeta_n(s+i\tau,\alpha) \in A\},$$
$$A \in \mathcal{B}(H(D))),$$

and denote by $P_{n,\alpha}$ the limit measure of $P_{T,n,\alpha}$ as $T \to \infty$. Then in the proof of Theorem 3.3, it was proved that the sequence $\{P_{n,\alpha} : n \in \mathbb{N}\}$ is tight. Since the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, we have that the sets $L(\alpha_1), \dots, L(\alpha_r)$ are linearly independent over $\mathbb{Q}$ as well. Therefore, by the above remark, the sequences $\{P_{n,\alpha_j} : n \in \mathbb{N}\}$, $j = 1, \dots, r$, are tight. On the other hand, P_{n,α_j}, $j = 1, \dots, r$, are the marginal measures of the measure P_n,

i. e.,

$$P_{n,\alpha_j}(A) = P_n(\underbrace{H(D) \times \cdots \times H(D)}_{\text{j-1}} \times A \times H(D) \times \cdots \times H(D)),$$

$$A \in \mathcal{B}(H(D)), \quad (4.11)$$

$j = 1, \ldots, r$. Since the sequence $\{P_{n,\alpha_j}\}$ is tight, for every $\varepsilon > 0$, there exists a compact set $H_j = H_j(\varepsilon) \subset H(D)$ such that

$$P_{n,\alpha_j}(H_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \ldots, r, \tag{4.12}$$

for all $n \in \mathbb{N}$. Then the set $H = H(\varepsilon) = H_1 \times \cdots \times H_r$ is compact in the space $H^r(D)$. Moreover, by (4.11) and (4.12),

$$\begin{aligned}
&P_n(H^r(D) \setminus H) \\
&= P_n\bigg(\bigcup_{j=1}^{r} \big(\underbrace{H(D) \times \cdots \times H(D)}_{j-1} \times (H(D) \setminus K_j) \times \cdots \\
&\quad \times H(D) \times \cdots \times H(D)\big)\bigg) \\
&\leqslant \sum_{j=1}^{n} P_n(\underbrace{H(D) \times \cdots \times H(D)}_{j-1} \times (H(D) \setminus K_j) \times \cdots \\
&\quad \times H(D) \times \cdots \times H(D)) \\
&= \sum_{j=1}^{n} P_{n,\alpha_j}(H(D) \setminus K_j) \leqslant \sum_{j=1}^{r} \frac{\varepsilon}{r} = \varepsilon
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$P_n(H) \geqslant 1 - \varepsilon,$$

for all $n \in \mathbb{N}$. □

Proof of Theorem 4.3. On a certain probability space with the measure ν, define the random variable η_T by the formula

$$\nu(\eta_T \in A) = \frac{1}{U_T} \int_{T_0}^{T} w(t) I(A) \, \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{R}),$$

Denote by X_n the $H^r(D)$-valued random element having the distribution P_n, and define one more $H^r(D)$-valued random element

$$X_{T,n,w} = X_{T,n,w}(s) = \underline{\zeta}(s + i\eta_T, \underline{\alpha}).$$

Then, in view of Lemma 4.2,

$$X_{T,n,w} \xrightarrow{\mathcal{D}} X_n. \tag{4.13}$$

By Lemma 4.4, there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to a certain probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$. Thus,

$$X_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{4.14}$$

We will use one more $H^r(D)$-valued random element

$$X_{T,w} = \underline{\zeta}(s + i\eta_T, \underline{\alpha}).$$

Then using Lemma 4.3 gives, for $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \nu(\underline{\rho}(X_{T,w}, X_{T,n,w}) \geqslant \varepsilon) \\
&= \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \\
&\quad \times I\left\{\tau \in [T_0, T] : \underline{\rho}\left(\underline{\zeta}(s + i\tau, \underline{\alpha}), \underline{\zeta}_n(s + i\tau, \underline{\alpha})\right) \geqslant \varepsilon\right\} \mathrm{d}\tau \\
&\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon U_T} \int_{T_0}^{T} w(\tau)\underline{\rho}\left(\underline{\zeta}(s + i\tau, \underline{\alpha}), \underline{\zeta}_n(s + i\tau, \underline{\alpha})\right) \mathrm{d}\tau = 0.
\end{aligned} \tag{4.15}$$

The space $H^r(D)$ is separable. Therefore, the relations (4.13) and (4.14), and the equality (4.15) show that all hypotheses of Lemma 2.7 are satisfied by the random elements $X_{T,w}$, $X_{T,n,w}$ and X_{n_k}. Therefore,

$$X_{T,w} \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{4.16}$$

The latter relation shows that the measure P is independent of the sequence $\{P_{n_k}\}$. Since the sequence $\{P_{n_k}\}$ is relatively compact, hence, it follows that

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{4.17}$$

Thus, by the relations (4.16) and (4.17), $P_{T,w}$, as $T \to \infty$, converges weakly to the limit measure P of P_n as $n \to \infty$. In Theorem H [36], under the hypothesis that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, it was obtained that

$$\frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \underline{\zeta}(s + i\tau, \underline{\alpha}) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)),$$

also, as $T \to \infty$ converges weakly to the limit measure P of P_n, as $n \to \infty$, and that P coincides with $P_{\underline{\zeta}}$. Thus, in view of the above remarks, $P_{T,w}$ converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$. The theorem is proved. □

4.3 Proof of universality

Theorem 4.3 is the main auxiliary result for the proofs of Theorems 4.1 and 4.2, however, additionally, the explicit form of the support of the measure $P_{\underline{\zeta}}$ is necessary. We recall that a minimal closed set $S_{\underline{\zeta}} \subset H^r(D)$ is called the support of the measure $P_{\underline{\zeta}}$ if $P_{\underline{\zeta}}(S_{\underline{\zeta}}) = 1$. The set $S_{\underline{\zeta}}$ consists of all points $\underline{g}$ such that, for every open neighbourhood G of $\underline{g}$, the inequality $P_{\underline{\zeta}}(G)$ is satisfied.

Lemma 4.5. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then the support of the measure $P_{\underline{\zeta}}$ is the whole of $H^r(D)$.*

The lemma is Theorem 11 of [36].

Proof of Theorem 4.1. By Lemma 2.14, there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{2}. \tag{4.18}$$

Define the set

$$G_\varepsilon = \left\{(g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_j(s) - p_j(s)| < \frac{\varepsilon}{2}\right\}.$$

By Lemma 4.5, we have that G is an open neighbourhood of an element of the support of $P_{\underline{\zeta}}$. Therefore,

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{4.19}$$

This, Theorem 4.3 and the equivalent of weak convergence in terms of open sets show that

$$\liminf_{T \to \infty} P_{T,w}(G_\varepsilon) \geqslant P_{\underline{\zeta}}(G_\varepsilon) > 0.$$

Hence, by the definitions of $P_{T,w}$ and G_ε, we find that

$$\liminf_{T\to\infty}\frac{1}{U(T,w)}\int_{T_0}^{T} w(\tau) \\ I\left(\left\{\tau\in[T_0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-p_j(s)|<\frac{\varepsilon}{2}\right\}\right)\mathrm{d}\tau>0. \tag{4.20}$$

It is easily seen by (4.18) that

$$\left\{\tau\in[T_0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-p_j(s)|<\frac{\varepsilon}{2}\right\} \\ \subset\left\{\tau\in[T_0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\right\}.$$

This remark and (4.20) prove the theorem. □

Proof of Theorem 4.2. Define the set

$$\mathcal{G}_\varepsilon=\left\{(g_1,\ldots,g_r)\in H^r(D):\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|g_j(s)-p_j(s)|<\varepsilon\right\}.$$

Then $\partial\mathcal{G}_{\varepsilon_1}\bigcap\partial\mathcal{G}_{\varepsilon_2}=\varnothing$ for different positive ε_1 and ε_2. This shows that the set $\mathcal{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many values of ε. Therefore, by Theorem 4.3 and the equivalent of weak convergence in terms of continuity sets, the limit

$$\lim_{T\to\infty}P_{T,w}(\mathcal{G}_\varepsilon)=P_{\underline{\zeta}}(\mathcal{G}_\varepsilon) \tag{4.21}$$

exists for all but at most countably many $\varepsilon>0$. In view of (4.18), the inclusion $G_\varepsilon\subset\mathcal{G}_\varepsilon$ is true. Therefore, (4.19) shows that $P_{\underline{\zeta}}(\mathcal{G}_\varepsilon)>0$, and the theorem follows from (4.21) by using the definitions of $P_{T,w}$ and $\mathcal{G}_\varepsilon$. □

Chapter 5

Weighted Mishou universality theorems

In this chapter of the dissertation, we give a weighted generalization of the joint universality theorem for the Riemann and Hurwitz zeta-functions. Such a theorem was obtained by H. Mishou [58], see Theorem I in Introduction. The Mishou theorem is the first so-called mixed universality theorem for zeta-functions. The term "mixed" is used because the Riemann zeta-function $\zeta(s)$ has Euler's product over primes, while the Hurwitz zeta-function $\zeta(s,\alpha)$ with transcendental parameter α has no such a product. Thus, the functions $\zeta(s)$ and $\zeta(s,\alpha)$ are quite different and have not the same properties. On the other hand, their shifts $\zeta(s+i\tau)$ and $\zeta(s+i\tau,\alpha)$ for some classes of α have a property of approximation of analytic functions. By the Mishou theorem, this approximation property is even valid in the joint case.

5.1 Statements of the theorems

For the class $H_0(K)$, $K \in \mathcal{K}$, we preserve the notation of Chapter 2, i. e., $H_0(K)$ is the class of continuous non-vanishing functions on K that are analytic in the interior of K. The class $H(K)$, $K \in \mathcal{K}$, is the same as in Chapter 3 and 4, and differs from $H_0(K)$ that the requirement of non-vanishing is removed. Also, we preserve the notation of the class W_1. Now, we state the main results of the chapter.

Theorem 5.1. *Suppose, that the parameter α is transcendental and $w(t) \in W_1$. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then for every*

$\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\Bigg(\Bigg\{\tau \in [T_0, T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon$$
$$\sup_{s\in K_2} |\zeta(s+i\tau, \alpha) - f_2(s)| < \varepsilon\Bigg\}\Bigg) \mathrm{d}\tau > 0.$$

Theorem 5.1 has its analogue in terms of density.

Theorem 5.2. *Under hypothesis of Theorem 5.1, the limit*

$$\lim_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\Bigg(\Bigg\{\tau \in [T_0, T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon$$
$$\sup_{s\in K_2} |\zeta(s+i\tau, \alpha) - f_2(s)| < \varepsilon\Bigg\}\Bigg) \mathrm{d}\tau > 0$$

exists for all but at most countably many $\varepsilon > 0$

For the proof of Theorems 5.1 and 5.2, we apply a limit probability theorem for probability measures in the space $H^2(D)$.

5.2 A weighted limit theorem on the product of two tori

Let, as in previous chapters, $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. Define two tori

$$\Omega_1 = \prod_{p\in\mathbb{P}} \gamma_p \qquad \text{and} \qquad \Omega_2 = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$ and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. With product topology and pointwise multiplication, the infinite-dimensional tori Ω_1 and Ω_2 are compact topological Abelian groups. Therefore, $\Omega = \Omega_1 \times \Omega_2$ is again a compact topological Abelian group. Hence, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega_1(p)$ the pth component of an element $\omega_1 \in \Omega_1$, $p \in \mathbb{P}$, and by $\omega_2(m)$ the mth component of an element $\omega_2 \in \Omega_2$, $m \in \mathbb{N}_0$. The elements of Ω are of the form $\omega = (\omega_1, \omega_2)$.

In this section, we will consider the weak convergence for

$$Q_{T,w}(A) = \frac{1}{U_T}\int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] : \left(\left(p^{-i\tau} : p \in \mathbb{P}\right),\right.\right.\right.$$
$$\left.\left.\left.\left((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\right)\right) \in A\right\}\right) \mathrm{d}\tau, \quad A \in \mathcal{B}(\Omega).$$

Theorem 5.3. *Suppose that α is transcendental and $w \in W_1$. Then $Q_{T,w}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. The characters of the group Ω are of the form

$$\prod_{p\in\mathbb{P}}{}' \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0}{}' \omega_2^{l_m}(m),$$

where the sign "$'$" means that only a finite number of integers k_p and l_m are distinct from zero. Therefore, the Fourier transform $g_{T,w}(\underline{k}, \underline{l})$, $\underline{k} = (k_p : k_p \in \mathbb{Z},\ p \in \mathbb{P})$, $\underline{l} = (k_p : l_m \in \mathbb{Z},\ m \in \mathbb{N}_0)$, of $Q_{T,w}$ is defined by

$$g_{T,w}(\underline{k}, \underline{l}) = \int_\Omega \prod_{p\in\mathbb{P}}{}' \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0}{}' \omega_2^{l_m}(m)\, \mathrm{d}Q_{T,w}.$$

Therefore, by the definition of $Q_{T,w}$,

$$g_{T,w}(\underline{k}, \underline{l}) = \frac{1}{U_T}\int_{T_0}^{T} w(\tau) \prod_{p\in\mathbb{P}}{}' p^{-ik_p\tau} \prod_{m\in\mathbb{N}_0}{}' (m+\alpha)^{-il_m\tau}\, \mathrm{d}\tau$$
$$= \frac{1}{U_T}\int_{T_0}^{T} w(\tau) \exp\left\{-i\tau\left(\sum_{p\in\mathbb{P}}{}' k_p \log p + \sum_{m\in\mathbb{N}_0}{}' l_m \log(m+\alpha)\right)\right\} \mathrm{d}\tau. \tag{5.1}$$

Clearly,

$$g_{T,w}(\underline{0}, \underline{0}) = \frac{1}{U_T}\int_{T_0}^{T} w(\tau)\, \mathrm{d}\tau = 1. \tag{5.2}$$

Suppose that $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$. Then

$$A(\underline{k}, \underline{l}) \stackrel{def}{=} \sum_{p\in\mathbb{P}}{}' k_p \log p + \sum_{m\in\mathbb{N}_0}{}' l_m \log(m+\alpha) \neq 0. \tag{5.3}$$

Actually, if the latter inequality is not true, then

$$\prod_{p\in\mathbb{P}}{}' p^{k_p} \prod_{m\in\mathbb{N}_0}{}' (m+\alpha)^{l_m} = 1.$$

From this, it follows that

$$\prod_{m\in\mathbb{N}_0}{}' (m+\alpha)^{l_m}$$

is a rational number. However, this contradicts the transcendence of α. If all $l_m = 0$, then $\sum'_{p\in\mathbb{P}} k_p \log p \neq 0$ because the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers. Thus, (5.3) is true. Now, by (5.1), we find

$$\begin{aligned} g_{T,w}(\underline{k},\underline{l}) &= \frac{1}{-iU_T A(\underline{k},\underline{l})} \int_{T_0}^{T} w(\tau)\, \mathrm{d}\exp\{-i\tau A(\underline{k},\underline{l})\} \\ &\ll (U_T|A(\underline{k},\underline{l})|)^{-1} \left(1 + \int_{T_0}^{T} |\mathrm{d}w(\tau)|\right) \ll (U_T|A(\underline{k},\underline{l})|)^{-1} \end{aligned}$$

in view of a property of the variation of $w(\tau)$. Since $\lim_{T\to\infty} U_T = \infty$, this shows that

$$\lim_{T\to\infty} g_{T,w}(\underline{k},\underline{l}) = 0.$$

Therefore, by (5.2),

$$\lim_{T\to\infty} g_{T,w}(\underline{k},\underline{l}) = \begin{cases} 1 & \text{if } (\underline{k},\underline{l}) = (\underline{0},\underline{0}), \\ 0 & \text{if } (\underline{k},\underline{l}) \neq (\underline{0},\underline{0}), \end{cases}$$

and the theorem is proved because the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H. □

5.3 Case of absolute convergence

Theorem 5.3 implies a weighted joint limit theorem in the space $H^2(D) = H(D) \times H(D)$, where $H(D)$ is the space of analytic functions on D endowed with the topology of uniform convergence on compacta. Thus, let $\theta > 1/2$ be a fixed number, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\theta}\right\},$$

and, for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$,

$$v_n(m, \alpha) = \exp\left\{ -\left(\frac{m+\alpha}{n+\alpha}\right)^\theta \right\}.$$

Define the series

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s} \qquad \text{and} \qquad \zeta_n(s, \alpha) = \sum_{m=0}^{\infty} \frac{v_n(m, \alpha)}{(m+\alpha)^s}.$$

Then the latter series are absolutely convergent for $\sigma > 1/2$. For brevity, let

$$\underline{\zeta}_n(s, \alpha) = (\zeta_n(s), \zeta_n(s, \alpha)).$$

Extend the functions $\omega_1(p)$, to the set $\mathbb{N}$ by the formula

$$\omega_1(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_1^l(p), \quad m \in \mathbb{N},$$

and, additionally to $\zeta_n(s)$ and $\zeta_n(s, \alpha)$, define

$$\zeta_n(s, \omega_1) = \sum_{m=1}^{\infty} \frac{\omega_1(m) v_n(m)}{m^s} \quad \text{and} \quad \zeta_n(s, \alpha, \omega_2) = \sum_{m=0}^{\infty} \frac{\omega_2(m) v_n(m, \alpha)}{(m+\alpha)^s},$$

and put

$$\underline{\zeta}_n(s, \omega, \alpha) = (\zeta_n(s, \omega_1), \zeta_n(s, \omega_2, \alpha)).$$

Obviously, the series $\zeta_n(s, \omega_1)$ and $\zeta_n(s, \omega_2, \alpha)$ are absolutely convergent for $\sigma > 1/2$ as well.

Consider the function $u_n : \Omega \to H^2(D)$ given by $u_n(\omega) = \underline{\zeta}_n(s, \omega, \alpha)$. Since the above seeries are absolutely convergent for $\sigma > 1/2$, the function $u_n(\omega)$ is continuous. For $A \in \mathcal{B}(H^2(D))$, define

$$P_{T,n,w}(A) = \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] : \underline{\zeta}_n(s + i\tau, \alpha) \in A\right\}\right) \mathrm{d}\tau.$$

Then we have $P_{T,n,w}(A) = Q_{T,w}(u^{-1}A)$. Thus, the equality $P_{T,n,w} = Q_{T,w}u^{-1}$ is true. This, the continuity of u_n, Theorem 5.3 together with Lemma 2.2 lead to the following theorem.

Theorem 5.4. *Suppose that α is transcendental and $w \in W$. Then $P_{T,n,w}$*

converges weakly to the measure $V_n \stackrel{def}{=} m_H u_n^{-1}$ *as* $T \to \infty$.

The measure V_n plays an important role in the proof of the limit theorem for

$$P_{T,w}(A) = \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I \left(\left\{\tau \in [T_0, T] : \underline{\zeta}(s + i\tau, \alpha) \in A\right\}\right) \, \mathrm{d}\tau, \qquad A \in \mathcal{B}(H^2(D)),$$

where

$$\underline{\zeta}(s, \alpha) = (\zeta(s), \zeta(s, \alpha)) \, .$$

From the proof of the Mishou theorem [58], the following properties of V_n follows. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D)$-valued random element

$$\underline{\zeta}(s, \omega, \alpha) = \left(\prod_{p \in \mathbb{P}} \left(1 - \frac{\omega_1(p)}{p^s}\right)^{-1}, \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m + \alpha)^s} \right),$$

and let $P_{\underline{\zeta}}$ be the distribution of $\underline{\zeta}(s, \omega, \alpha)$, i. e.,

$$P_{\underline{\zeta}}(A) = m_H \left\{\omega \in \Omega : \underline{\zeta}(s, \omega, \alpha) \in A\right\}, \quad A \in \mathcal{B}(H^2(D)).$$

Moreover, let $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$. Under the above notation, we have

Lemma 5.1. *Suppose that* α *is transcendental. Then* V_n *converges weakly to* $P_{\underline{\zeta}}$ *as* $n \to \infty$. *Moreover, the support of* $P_{\underline{\zeta}}$ *is the set* $S \times H(D)$.

To prove that $P_{T,w}$, as $T \to \infty$, also converges weakly to the measure $P_{\underline{\zeta}}$, some approximation of $\underline{\zeta}(s, \alpha)$ by $\underline{\zeta}_n(s, \alpha)$ is needed.

5.4 Approximation in the mean

Let $\underline{g}_1 = (g_{11}, g_{12})$, $\underline{g}_2 = (g_{21}, g_{22}) \in H^2(D)$. Then putting

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant 2} \rho(g_{1j}, g_{2j})$$

gives a metric on $H^2(D)$ inducing the product topology.

The following statement is true.

Theorem 5.5. *Suppose that* $w \in W_1$. *Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau)\underline{\rho}\left(\underline{\zeta}(s+i\tau,\alpha), \underline{\zeta}_n(s+i\tau,\alpha)\right) \mathrm{d}\tau = 0$$

for all $0 < \alpha \leqslant 1$.

Proof. By the definition of the metric ρ, it suffices to prove the equalities

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau)\rho\left(\zeta(s+i\tau), \zeta_n(s+i\tau)\right) \mathrm{d}\tau = 0 \qquad (5.4)$$

and

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau)\rho\left(\zeta(s+i\tau,\alpha), \zeta_n(s+i\tau,\alpha)\right) \mathrm{d}\tau = 0. \qquad (5.5)$$

Obviously, (5.4) is a corollary of (5.5) with $\alpha = 1$. However, (5.3) is proved in Chapter 3, Lemma 3.5. □

5.5 A limit theorem for $\underline{\zeta}(s,\alpha)$

Now we are ready to prove the weak convergence for $P_{T,w}$ as $T \to \infty$.

Theorem 5.6. *Suppose that* α *is transcendental and* $w \in W$. *Then* $P_{T,w}$ *converges weakly to the measure* $P_{\underline{\zeta}}$ *as* $T \to \infty$.

Proof. On a certain probability space with measure μ, define a random variable $\theta_{T,w}$ by

$$\mu\{\theta_{T,w} \in A\} = \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I(A)\, \mathrm{d}\tau, \quad A \in \mathcal{B}(\mathbb{R}).$$

Consider the $H^2(D)$-valued random element

$$\underline{X}_{T,n,w} = \underline{X}_{T,n,w}(s) = \underline{\zeta}_n(s + i\theta_{T,w}, \alpha).$$

Then, in view of Theorem 5.4,

$$\underline{X}_{T,n,w} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{Y}_n, \qquad (5.6)$$

where Y_n is the $H^2(D)$-valued random element with the distribution V_n. Lemma 5.1 implies the relation

$$\underline{Y}_n \xrightarrow[n\to\infty]{\mathcal{D}} P_{\underline{\zeta}}. \tag{5.7}$$

Moreover, an application of Theorem 5.6 shows that, for every $\varepsilon > 0$,

$$\begin{aligned}&\lim_{n\to\infty}\limsup_{T\to\infty}\mu\left(\underline{\rho}\left(\underline{X}_{T,w}(s), \underline{X}_{T,n,w}(s)\right) \geqslant \varepsilon\right)\\ &\ll \lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{\varepsilon U_T}\int_{T_0}^{T} w(\tau)\underline{\rho}\left(\underline{\zeta}(s+i\tau,\alpha), \underline{\zeta}_n(s+i\tau,\alpha)\right)\,\mathrm{d}\tau = 0,\end{aligned} \tag{5.8}$$

where the $H^2(D)$-valued random element $\underline{X}_{T,w} = \underline{X}_{T,w}(s)$ is defined by

$$\underline{X}_{T,w}(s) = \underline{\zeta}(s + i\theta_{T,w}, \alpha).$$

Now, relations (5.6) – (5.8) show that all hypotheses of Lemma 2.7 are satisfied. Therefore, we obtain that

$$\underline{X}_{T,w} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\underline{\zeta}},$$

and this is equivalent to the assertion of the theorem. □

5.6 Proof of universality

Theorem 5.1 follows easily from Theorem 5.6 and the Mergelyan theorem on the approximation of analytic functions by polynomials.

Proof of Theorem 5.1. By the Mergelyan theorem, there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1}\left|f_1(s) - \mathrm{e}^{p_1(s)}\right| < \frac{\varepsilon}{2} \tag{5.9}$$

and

$$\sup_{s\in K_2}|f_2(s) - p_2(s)| < \frac{\varepsilon}{2}. \tag{5.10}$$

Define the set

$$G_\varepsilon = \Big\{ g_1, g_2 \in H(D) : \sup_{s \in K_1} \left| g_1(s) - \mathrm{e}^{p_1(s)} \right| < \frac{\varepsilon}{2},$$
$$\sup_{s \in K_2} |g_2(s) - p_2(s)| < \frac{\varepsilon}{2} \Big\}.$$

We observe that, in virtue of Lemma 5.1, $(\mathrm{e}^{p_1(s)}, p_2(s))$ is an element of the support of the measure $P_{\underline{\zeta}}$. Since G_ε is an open neighbourhood of an element of the support of $P_{\underline{\zeta}}$, the inequality

$$P_{\underline{\zeta}}(G_\varepsilon) > 0 \tag{5.11}$$

is true. Therefore, using the equivalent of the weak convergence of probability measures in terms of open sets and taking into account Theorem 5.6, we have

$$\liminf_{T \to \infty} P_{T,w}(G_\varepsilon) \geqslant P_{\underline{\zeta}}(G_\varepsilon) > 0.$$

Hence, by the definitions of $P_{T,w}$ and G_ε,

$$\liminf_{T \to \infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I \Bigg(\Big\{ \tau \in [T_0, T] : \sup_{s \in K_1} \left| \zeta(s + i\tau) - \mathrm{e}^{p_1(s)} \right| < \frac{\varepsilon}{2},$$
$$\sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - p_2(s)| < \frac{\varepsilon}{2} \Big\} \Bigg) \, \mathrm{d}\tau > 0. \tag{5.12}$$

It remains to replace $\mathrm{e}^{p_1(s)}$ and $p_2(s)$ by $f_1(s)$ and $f_2(s)$, respectively. Suppose that τ satisfy inequalities

$$\sup_{s \in K_1} \left| \zeta(s + i\tau) - \mathrm{e}^{p_1(s)} \right| < \frac{\varepsilon}{2}$$

and

$$\sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - p_2(s)| < \frac{\varepsilon}{2}.$$

Then inequalities (5.9) and (5.10) imply

$$\sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon$$

and

$$\sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon.$$

Consequently,

$$\Big\{\tau \in [T_0, T] : \sup_{s\in K_1} \Big|\zeta(s+i\tau) - \mathrm{e}^{p_1(s)}\Big| < \frac{\varepsilon}{2},$$
$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - p_2(s)| < \frac{\varepsilon}{2}\Big\}$$
$$\subset \Big\{\tau \in [T_0, T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Big\}.$$

This and (5.12) prove the first assertion of the theorem. □

Proof of Theorem 5.2. Define the set

$$\hat{G}_\varepsilon = \Big\{g_1, g_2 \in H(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |g_2(s) - f_2(s)| < \varepsilon\Big\}.$$

Then the boundaries $\partial\hat{G}_{\varepsilon_1}$ and $\partial\hat{G}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. This shows that the set $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Therefore, using the equivalent of weak convergence of probability measures in terms of continuity sets, we obtain by Theorem 5.6 that

$$\lim_{T\to\infty} P_{T,w}(\hat{G}_\varepsilon) = P_{\underline{\zeta}}(\hat{G}_\varepsilon) \tag{5.13}$$

for all but at most countably many $\varepsilon > 0$. Moreover, inequalities (5.9) and (5.10) imply the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$. Thus, by (5.11), the inequality $P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0$ holds. This, the definitions of $P_{T,w}$ and $\hat{G}_\varepsilon$, and (5.13) prove the second assertion of the theorem. □

Chapter 6

Conclusions

The results of t he dissertation lead to the following conclusions.

1. For some class of weight functions, weighted discrete universality theorems for the Riemann zeta-function $\zeta(s)$ on the approximation of non-vanishing analytic functions by shifts $\zeta(s + ikh)$, $k \in \mathbb{N}_0$, $h > 0$, are valid.

2. For some class of weight functions, weighted continuous universality theorems for the Hurwitz zeta-function $\zeta(s, \alpha)$ with transcendental α on the approximation of analytic functions by shifts $\zeta(s + i\tau, \alpha)$, $\tau \in \mathbb{R}$, are valid.

3. For some class of weight functions, weighted joint continuous universality theorems for Hurwitz zeta-functions $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$ such that the set $\{(\log(m + \alpha_1) : m \in \mathbb{N}_0, \dots, (\log(m + \alpha_r) : m \in \mathbb{N}_0)\}$ is linearly independent over $\mathbb{Q}$ on the simultaneous approximation of a collection of analytic functions by shifts $(\zeta(s + i\tau, \alpha_1), \dots, \zeta(s + i\tau, \alpha_r))$ are valid.

4. For some class of weight functions, weighted mixed joint continuous universality theorems for the Riemann and Hurwitz zeta-functions $\zeta(s)$ and $\zeta(s, \alpha)$ with transcendental α on the simultaneous approximation of a pair of analytic functions by shifts $(\zeta(s + i\tau), \zeta(s + i\tau, \alpha))$ are valid.

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Santrauka
(Summary in Lithuanian)

Tyrimo objektas

Disertacijos tyrimo objektas - dvi klasikinės analizinės skaičių teorijos funcijos: Rymano dzeta funkcija ir Hurvico dzeta funkcija. Tegul $s = \sigma + it$, $\sigma, t \in \mathbb{R}$, $i = \sqrt{-1}$, yra kompleksinis kintamasis. Rymano dzeta funkcija $\zeta(s)$ pusplokštumėje $\sigma > 1$ yra apibrėžiama Dirichlė eilute

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

o Hurvico dzeta funckija $\zeta(s, \alpha)$ su parametru $0 < \alpha \leqslant 1$ toje pusplokštumėje apibrėžiama eilute

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}.$$

Abi šios funckijos yra analiziškai pratęsiamos į visą kompleksinę plokštumą, išskyrus tašką $s = 1$, kuris yra paprastasis polius su reziduumu 1. Disertacijoje yra nagrinėjamas funkcijų $\zeta(s)$ ir $\zeta(s, \alpha)$ svertinis universalumas, t.y. plačios analizinių funkcijų klasės aproksimavimas postūmiais $\zeta(s+i\tau)$ ir $\zeta(s+i\tau, \alpha)$, $\tau \in \mathbb{R}$, kurių aibė turi teigiamą svertinį apatinį tankį.

Iš apibrėžimų išplaukia, kad $\zeta(s, 1) = \zeta(s)$ ir

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\,\zeta(s).$$

Taigi, funckija $\zeta(s, \alpha)$ yra funckijos $\zeta(s)$ apibendrinimas. Apskritai, funkcijos $\zeta(s)$ ir $\zeta(s, \alpha)$ yra pakankamai skirtingos, nes funcija $\zeta(s)$, kai $\sigma > 1$, yra užrašoma Oilerio sandauga pagal pirminius skaičius

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

o funkcija $\zeta(s, \alpha)$ tokią sandaugą turi tik atveju $\alpha = 1$ ir $\alpha = \frac{1}{2}$. Iš čia išplaukia kiti skirtumai tarp funkcijų $\zeta(s)$ ir $\zeta(s, \alpha)$. Pirmiausia, $\zeta(s) \neq 0$ pusplokštumėje $\sigma > 1$, o funkcija $\zeta(s, \alpha)$, $\alpha \neq \frac{1}{2}$, šioje pusplokštumėje turi be galo daug nulių.

Funkcija $\zeta(s)$ su visais $s \in \mathbb{C}$ tenkina funkcinę lygtį

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

čia $\Gamma(\cdot)$ yra Oilerio gamma funcija. Iš šios lygties turime, kad $\zeta(-2k) = 0$, $k \in \mathbb{N}$, o skaičiai $s = -2k$ yra vadinami funkcijos trivialiaisiais nuliais. Be to, funkcija $\zeta(s)$ turi be galo daug kompleksinių netrivialiųjų nulių, gulinčių kritinėje juostoje $\{s \in \mathbb{C} : 1/2 \leqslant \sigma < 1\}$. Rymano hipotezė tvirtina, jog visi funkcijos $\zeta(s)$ netrivialieji nuliai yra kritinėje tiesėje $\sigma = 1/2$. Šiuo metu žinoma, kad bent $5/12$ netrivialiųjų nulių tankio prasme yra kritinėje tiesėje.

Funkciją $\zeta(s)$ jau žinojo L. Oileris 18 a. viduryje, tačiau jos svarbą atskleidė B. Rymanas (Riemann) 1859 metais. Oileris nagrinėjo $\zeta(s)$ su $s \in \mathbb{R}$, o Rymanas jau su $s \in \mathbb{C}$ ir pritaikė ją pirminių skaičių pasiskirstymui tirti, t. y., funkcijos

$$\pi(x) = \sum_{p \leqslant x} 1$$

asimptotiniam dėsniui, kai $x \to \infty$, nagrinėti. Remdamasis Rymano idėjomis, 1896 m. C. J. de la Valė Pusenas (de la Vallée Poussin) ir J. Adamaras (Hadamard) neprilausomai įrodė, kad

$$\lim_{x \to \infty} \pi(x) \Big/ \int_2^x \frac{\mathrm{d}u}{\log u} = 1.$$

Pastarosios lygybės įrodyme pagrindinį vaidmenį atlieka faktas, kad srityje $\sigma \geqslant 1$ nėra funkcijos $\zeta(s)$ nulių.

Funkciją $\zeta(s, \alpha)$ 1882 m. apibrėžė A. Hurvicas (Hurwitz) bei panaudojo ją Dirichlė L funkcijų teorijoje. Šios funkcijos yra pagrindinis įrankis nagrinėjant pirminių skaičių pasiskirstymą aritmetinėse progresijose, t. y., funkcijos

$$\pi(x,a,q) = \sum_{\substack{p\leqslant x \\ p\equiv a(\bmod q)}} 1, \quad (a,q)=1,$$

asimptotiką, kai $x \to \infty$.

Tikslas ir uždaviniai

Disertacijos tikslas yra svertinės universalumo bei jungtinio universalumo teoremos Rymano ir Hurvico dzeta funkcijoms. Uždaviniai yra šie:

1. Svertinis diskretusis universalumas Rymano dzeta funkcijai.
2. Svertinis tolydusis universalumas Hurvico dzeta funkcijai.
3. Svertinis jungtinis tolydusis universalumas Hurvico dzeta funkcijoms.
4. Svertinis mišrusis tolydusis universalumas Rymano ir Hurvico dzeta funkcijoms.

Aktualumas

Analizinių funkcijų aproksimavimo problemos yra vieni svarbiausių šiuolaikinės matematikos skyrių. Dvidešimtojo amžiaus aštuntajame dešimtmetyje tapo žinoma, kad plati analizinių funkcijų klasė gali būti aproksimuojama dzeta funkcijų, kurios plačiai nagrinėjamos analizinėje skaičių teorijoje, postūmiais. Taigi, analizinių funkcijų aproksimavimas, tam tikra prasme, buvo pakeistas dzeta funkcijų aproksimacinėmis savybėmis. Tai pagimdė eilę naujų problemų. Tarp jų – dzeta funkcijų su aproksimacinėmis savybėmis klasių aprašymas, aproksimacijos efektyvizacijos problemos, įvairių aproksimavimų tipai ir kt.

Įvairiose matematikos srityse (skaičių teorija, tikimybių teorija, matematinė statistika ir kt.) yra nagrinėjamos svertinės teoremos ir jų taikymai. Svertinis dzeta funkcijų universalumas yra nauja universalumo šaka, žinomi tik keli šios srities straipsniai. Todėl, mūsų nuomone, yra svarbu tęsti Rymano ir Hurvico dzeta funkcijų svertinio universalumo tyrimus. Be to, dzeta funkcijų universalumas yra viena iš Lietuvos analizinės skaičių teorijos mokyklos krypčių, todėl svertinis dzeta funkcijų universalumas tęsia Lietuvos tyrėjų tradicijas.

Metodai

Svertinių universalumo teoremų įrodymai remiasi Dirichlė eilučių silpnojo tikimybinių matų konvergavimo, Furjė analizės bei mato teorijos elementais.

Naujumas

Disertacijoje visi gauti rezultatai yra nauji. Svertinė diskrečioji universalumo teorema Rymano dzeta funkcijai gauta naujai svorio funcijų klasei. Svertinės universalumo teoremos Hurvico dzeta funkcijai ankščiau nebuvo nagrinėjamos.

Problemos istorija ir rezultatai

Rymano dzeta funkcija $\zeta(s)$ ir Hurvico dzeta funkcija $\zeta(s,\alpha)$ yra svarbiausios klasikinės dzeta funkcijos, todėl yra daug rezultatų, skirtų šių funcijų reikšmių pasiskirstymui. Yra net kelios monografijos skirtos vien funkcijai $\zeta(s)$. Funkcija $\zeta(s,\alpha)$ yra sulaukusi mažiau dėmesio. Tai suprantama, nes funkcija $\zeta(s)$ turi platesnį taikymo lauką. Daug dėmesio yra skiriama sritims, kuriose $\zeta(s) \neq 0$. Geriausias šio tipo rezultatas sako, kad egzistuoja tokia absoliuti konstanta $c > 0$, kad $\zeta(s) \neq 0$ srityje

$$\sigma > 1 - \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}}, \quad t \geqslant t_0.$$

Plačiai yra nagrinėjami ir momentai

$$\mathrm{I}_k(\sigma, T) \overset{def}{=} \int_1^T |\zeta(\sigma+it)|^{2k}\,\mathrm{dt}, \quad \sigma \geqslant \frac{1}{2},\ k > 0,$$

Egzistuoja hipotezė, kad

$$\mathrm{I}_k\left(\frac{1}{2}, T\right) \sim c_k T(\log T)^{k^2}, \quad T \to \infty,$$

su konstanta $c_k > 0$, kuri įrodyta tik su $k = 1$ ir $k = 2$.

Svarbūs yra ir funkcijos $\zeta(s)$ įverčiai. Lindelofo (Lindelöf) hipotezė tvirtina, kad su visais $\varepsilon > 0$

$$\zeta\left(\frac{1}{2}+it\right) \ll_\varepsilon t^\varepsilon, \quad t \geqslant t_0,$$

Ši hipotezė yra ekvivalenti įverčiui

$$\mathrm{I}_k\left(\frac{1}{2}, T\right) \ll_k T(\log T)^{k^2}, \quad k \in \mathbb{N}.$$

Tikimybinių metodų taikymas dzeta funkcijų teorijoje priklauso H. Borui (Bohr). Jis pasiūlė imti tam tikrą kompleksinės plokšumos aibių klasę ir nagrinėti, kaip dažnai duotos dzeta funkcijų reikšmės patenka į tas aibes. Ši idėja veda prie tikimybinių ribinių teoremų. Jis kartu su B. Jesenu (Jessen) įrodė tokią teoremą funkcijai $\zeta(s)$. Tegul $\mathcal{R}$ yra stačiakampis kompleksinėje plokštumoje su kraštinėmis lygiagrečiomis koordinatinėms ašims, o m_J yra Žordano matas. Tuomet jie įrodė [8], kad su $\sigma > 1$ egzistuoja riba

$$\lim_{T\to\infty} \frac{1}{T} m_J\{t \in [0, T] : \zeta(\sigma + it) \in \mathcal{R}\}$$

Vėliau jie šį rezultatą išplėtė į sritį $\sigma > \frac{1}{2}$

Toliau Boro idėjas vystė A. Selbergas (Selberg).

Yra patogu ribines tikimybinio pobūdžio teoremas dzeta funkcijoms formuluoti silpnojo tikimybių matų arba tikimybinių skirstinių konvergavimo terminais. Tegul $\mathcal{B}(\mathbb{X})$ yra Borelio σ kūnas, t. y., σ kūnas, generuotas erdvės $\mathbb{X}$ atvirųjų aibių sistemos. Tegul P_n, $n \in \mathbb{N}$, ir P yra tikimybiniai matai aibėje $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Sakoma, kad P_n, kai $n \to \infty$, silpnai konverguoja į P, jei su realia, tolydžia, aprėžta funkcija g aibėje $\mathbb{X}$

$$\lim_{n\to\infty} \int_{\mathbb{X}} g \, \mathrm{d}P_n = \int_{\mathbb{X}} g \, \mathrm{d}P.$$

Naudojant šią terminologiją, minėtą Boro-Jeseno teoremą galima formuluoti taip [35]. Tegul $\sigma > 1/2$ yra fiksuotas. Tuomet aibėje $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ egzistuoja toks tikimybinis matas P, kad

$$\frac{1}{T}\mathrm{meas}\{t \in [0, T] : \zeta(\sigma + it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

kai $T \to \infty$, silpnai konverguoja į P. Čia $\mathrm{meas}B$ yra mačios aibės B Lebego matas.

Dabar koncentruojamės į dzeta funkcijų universalumą, kuris yra mūsų tyrimo objektas. Dzeta funkcijų universalumas yra tam tikra nauja šių funkcijų reikšmių tankumo aprašymo kryptis. Pirmieji dzeta funkcijų reikšmių tankumo rezultatai priklauso taip pat H. Borui. Jis 1914 m. pastebėjo, kad funckija

$\zeta(s)$ juostoje $\{s \in \mathbb{C} : 1 < \sigma < 1+\delta\}$, $\delta > 0$, įgyja bet kurią nenulinę reikšmę be galo daug kartų. Vėliau H. Boras kartu su R. Kurantu (Courant) įrodė, jog su kiekvienu σ, $1/2 < \sigma \leqslant 1$, aibė $\{\zeta(\sigma + it) : t \in \mathbb{R}\}$ yra visur tiršta aibėje $\mathbb{C}$. S. M. Voroninas šį rezultatą ženkliai apibendrino [71] įrodydamas, kad aibė

$$\{(\zeta(s_1 + i\tau), \ldots, \zeta(s_n + i\tau)) : \tau \in \mathbb{R}\}$$

su visais $s_1, \ldots, s_n \in \mathbb{C}$, $1/2 < \mathrm{Re} s_k < 1$, $1 \leqslant k \leqslant n$, $s_k \neq s_m$ su $k \neq m$, ir aibė

$$\{(\zeta(s + i\tau), \zeta'(s + i\tau), ..., \zeta^{(n-1)}(s + i\tau) : \tau \in \mathbb{R})\}$$

su $s \in \mathbb{C}$, $1/2 < \sigma < 1$, yra visur tiršta aibėje $\mathbb{C}^n$. Tačiau daug didesnis Voronino nuopelnas yra universalumo teorema funkcijai $\zeta(s)$. Jis įrodė [72], kad, jei $0 < r < \frac{1}{4}$, o funkcija $f(s)$ yra tolydi ir nelygi nuliui skritulyje $|s| \leqslant r$ bei analizinė jo viduje, tuomet su kiekvienu $\varepsilon > 0$ egzistuoja toks realusis skaičius $\tau = \tau(\varepsilon)$, su kuriuo

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Taigi, Voroninas įrodė, jog plati analizinių funkcijų klasė gali būti aproksimuojama vienos funkcijos $\zeta(s)$ postūmiais $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. Šį nuostabų rezultatą pastebėjo matematikų bendrija ir eilė autorių apibendrino bei išplėtojo įvairiomis kryptimis. Šiuolaikinis Voronino teoremos variantas naudoja tokius žymenis. Tegul $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, $\mathcal{K}$ yra juostos D kompleksinių poaibių su jungiaisiais papildiniais klasė, o $H_0(K)$, $K \in \mathcal{K}$, yra funkcijų, tolydžių ir nelygių nuliui aibėje K bei analizinių K viduje, klasė. Tuomet teisingas toks tvirtinimas, kurį galima rasti [35] monografijoje.

A teorema. *Tegul $K \in \mathcal{K}$ ir $f(s) \in H_0(K)$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Pagal A teoremą postūmių $\zeta(s + i\tau)$, aproksimuojančių duotąją klasės $H_0(K)$ funkciją $f(s)$ aibė, turi teigiamą apatinį tankį, taigi, ji yra begalinė.

Tegul $H(G)$ yra analizinių funkcijų kompleksinės plokštumos srityje aibė su tolygaus konvergavimo kompaktinėse aibėse topologija. Šioje topologijoje seka $\{g_n(s)\} \subset H(G)$ konverguoja į funkciją $g(s) \in H(G)$, kai $n \to \infty$, tada ir tik tada, kai su kiekviena kompaktine aibe $K \in G$, galioja lygybė

$$\lim_{n\to\infty} \sup_{s\in K} |g_n(s) - g(s)| = 0.$$

Erdvė $H(D)$ yra begaliniamatė, todėl į A teoremą galima žiūrėti kaip į Boro-Kuranto teoremos begaliniamatį apibendrinimą.

A teorema yra tolydaus tipo, nes τ postūmyje $\zeta(s+i\tau)$ gali įgyti bet kurią realiąją reikšmę. A teorema turi diskretųjį variantą, kai τ įgyja reikšmes iš kurios nors diskrečiosios aibės. Tegul $h > 0$ yra fiksuotas skaičius, o N perbėga aibę $\mathbb{N}_0$. Tuomet teisingas toks tvirtinimas [1], [64].

B teorema. *Tegul $K \in \mathcal{K}$ ir $f(s) \in H_0(K)$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0.$$

Vietoje postūmių $\zeta(s+ikh)$ gali būti naudojami sudėtingesni postūmiai, pavydžiui, $\zeta(s+ik^\alpha h)$ su fiksuotu $0 < \alpha < 1$ arba net ir $\zeta(s+ih\gamma_k)$, čia $\{\gamma_k\}$ yra Rymano dzeta funkcijos netrivialiųjų nulių menamųjų dalių seka.

Universalumo teroemos, analogiškos A ir B teoremoms, yra žinomos ir kitoms dzeta funkcijoms, pavyzdžiui, parabolinių formų dzeta funkcijoms [43].

Pastebime, kad A ir B teoremose apatinis tankis gali būti pakeistas tankiu (vietoje "$\liminf$" imant "$\lim$") tačiau ne visiems $\varepsilon > 0$: tenka išskirti ne daugiau negu skaičiąją ε reikšmių aibę.

Pirmąją svertinę universalumo teoremą įrodė [34] A. Laurinčikas. Tegul $w(t)$ yra tokia aprėžtos variacijos funkcija intervale $[T_0, \infty)$, $T_0 > 0$, kad jos variacijai $V_b^a w$ intervale $[a, b]$ galioja nelygybė $V_b^a w \leqslant cw(a)$ su $c > 0$. Tarkime, kad

$$U_T = U(T, w) = \int_{T_0}^{T} w(t)\,\mathrm{d}t,$$

ir

$$\lim_{T\to\infty} U(T, w) = +\infty.$$

Simboliu $I(A)$ žymėsime aibės A indikatorių, t. y.

$$I(A) = \begin{cases} 1, & \text{jei } \tau \in A, \\ 0, & \text{jei } \tau \notin A. \end{cases}$$

Naudojame nestandartinį indikatoriaus žymenį, nes aibė A yra per daug sudėtinga ją rašyti indekso vietoje.

Tuomet teorema iš [34] su viena papildoma sąlyga, primenančia Birkhofo-Chinčino teremos svertinį variantą (šios sąlygos galima lengvai atsisakyti) tvirtina, kad su $K \in \mathcal{K}$, $f(s) \in H_0(K)$ ir $\varepsilon > 0$

$$\liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0.$$

Svertinę universalumo teoremą Lercho dzeta funkcijai įrodė R. Garunkštis 1997 m.

Svertinės diskrečiosios universalumo teoremos yra sudėtingesnės ir yra žinomos tik diferencijuojamų svorio funkcijų klasėms. Pavydzdžiui, [53] yra nagrinėjamas svertinis diskretusis universalumas su svorio funkcija $w(t)$, turinčia tolydžiąją išvestinę, kuri tenkina įverčius su $h > 0 : w(t) \ll_h w(ht)$ ir $(w'(t))^2 \ll w(t)$.

Disertacijos 2 skyriuje konstruojama nauja svorio funkcijų klasė W. Jai priklauso realiosios neneigiamos funkcijos $w(t)$, turinčios tolydžiąją išvestinę intervale $[1/2, \infty)$ ir tenkinančios įvertį

$$\int_1^N u|w'(u)| \,\mathrm{d}u \ll V_N.$$

Čia

$$V_N = V(N, w) = \sum_{k=1}^{N} w(k) \xrightarrow[N\to\infty]{} \infty.$$

Pagrindinis 2 skyriaus rezultatas yra tokios svertinės teoremos

2.1 teorema. *Tarkime, kad $w(t) \in W$. Tegul $K \in \mathcal{K}$ ir $f(s) \in H_0(K)$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I\left(\left\{k : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon\right\}\right) > 0.$$

2.1 teorema turi tokią modifikaciją.

2.2 teorema. *Tegul galioja 2.1 teoremos sąlygos. Tuomet riba*

$$\lim_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I \left(\left\{ k : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} \right) > 0$$

egzistuoja su visais $\varepsilon > 0$*, išskyrus ne daugiau negu skaičią reikšmių aibę.*

Pavyzdžiui, funkcija

$$w(t) = \frac{\sin(\log t) + 1}{t}$$

prilauso klasei W.

2,1 ir 2.2 teoremų įrodymai remiasi ribinėmis teoremomis apie silpnąjį

$$\frac{1}{V_N} \sum_{k=1}^{N} w(k) I(\{k : \zeta(s+ikh) \in A\}), \quad A \in \mathcal{B}(H(D))$$

konvergavimą, kai $N \to \infty$. Atskirai yra nagrinėjami du skaičiaus h tipai. Sakome, kad h yra tipo 1, jei $\exp\{(2\pi m)/h\}$ yra iracionalusis skaičius su visais $m \in \mathbb{R} \setminus \{0\}$, ir tipo 2, jei nėra tipo 1.

Universalumo problemos susijusios su Hurvico dzeta funkcija yra sudėtingesnės ir įdomesnės negu Rymano dzeta funkcijos, nes funkcija $\zeta(s, \alpha)$ priklauso nuo parametro α ir jo aritmetinės sąvybės turi įtakos funkcijos aproksimavimo sąvybėms. Be to, funkcija $\zeta(s, \alpha)$ neturi Oilerio sandaugos. Ši aplinkybė išplečia aproksimuojamų analizinių funkcijų klasę.

Tegul $H(K)$, $K \in \mathcal{K}$, yra funkcijų, tolydžių aibėje K ir analizinių jos viduje, klasė. Taigi $H_0(K) \subset H(K)$.

Primename, kad skaičius α yra vadinamas algebriniu, jei egzistuoja toks polinomas $p(s) \not\equiv 0$ su racionaliaisiais koeficientais, kad $p(\alpha) = 0$. Priešingu atveju α yra vadinamas transcendenčiuoju.

Funkcijos $\zeta(s, \alpha)$ universalumas nusakomas tokia teorema.

C teorema. *Tarkime, kad parametras* α *yra transcendentus arba racionalus* $\neq 1/2, 1$*. Tegul* $K \in \mathcal{K}$ *ir* $f(s) \in H(K)$*. Tuomet su kiekvienu* $\varepsilon > 0$ *yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s+i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

Skirtingais metodais C teorema yra įrodyta B. Bagčio (Bagči) ir S. M. Goneko (Gonek) disertacijose. Jos įrodymą galima rasti ir monografijoje [42].

Kai $\alpha = 1$ arba $\alpha = 1/2$, C teorema lieka teisinga, tik funkcija $f(s)$ turi būti iš klasės $H_0(K)$.

Tegul

$$L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\}.$$

Tuomet parametro α transcendentumą galima pakeisti platesne sąlyga, kad aibė $L(\alpha)$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$ [36].

Algebrinio iracionalaus parametro atvejis kol kas yra atvira problema. Kol kas yra žinomas toks rezultatas [2].

D teorema. *Tarkime, kad α yra algebrinis iracionalusis skaičius. Tuomet egzistuoja tokia netuščia uždara aibė $F_\alpha \subset H(D)$, kad su kiekviena kompaktine aibe $K \subset D$, $f(s) \in F_\alpha$ ir $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon\right\} > 0.$$

Be to, riba

$$\lim_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon\right\} > 0$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

Disertacijos 3 skyrius yra skirtas funkcijos $\zeta(s, \alpha)$ su transcendenčiu α svertinėms universalumo teoremoms. Tegul W_1 yra svorio funkcijų klasė iš [34]. Tuomet teisingi tokie tvirtinimai.

3.1 teorema. *Tarkime, kad $w(t) \in W_1$ ir parametras α yra transcendentusis. Tegul $K \in \mathcal{K}$ ir $f(s) \in H(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \times I\left(\left\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0.$$

3.2 teorema. *Tegul galioja 3.1 teoremos sąlygos. Tuomet riba*

$$\lim_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) \times I\left(\left\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon\right\}\right) \mathrm{d}\tau > 0$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

3.1 ir 3.2 teoremų įrodymai naudoja

$$\frac{1}{U_T}\int_{T_0}^{T} w(\tau) I(\{\tau \in [T_0, T] : \zeta(s + i\tau, \alpha) \in A\})\,\mathrm{d}\tau, \quad A \in \mathcal{B}(H(D)),$$

silpnąjį konvergavimą, kai $T \to \infty$.

Disertacijos 4 skyriuje yra gautas 3 skyriaus rezultatų jungtinis apibendrinimas. Jungtiniu atveju analizinių funkcijų rinkinys yra aproksimuojamas dzeta funkcijų postūmių rinkiniu.

Pirmąją jungtinę universalumo teoremą įrodė Voroninas 1975 m. Dirichlė L funkcijoms. Primename, kad Dirichlė charakteris χ moduliu $q \in \mathbb{N}$ yra funkcija $\chi : \mathbb{N} \to \mathbb{C}$, kuri yra visiškai multiplikatyvi ($\chi(mn) = \chi(m)\chi(n)$, $m, n \in \mathbb{N}$), periodinė su periodu q ($\chi(m + q) = \chi(m)$, $m \in \mathbb{N}$), $\chi(m) = 0$ su $(m, q) > 1$ ir $\chi(m) \neq 0$ su $(m, q) = 1$. Dirichlė L funkcija $L(s, \chi)$ su charakteriu χ pusplokštumėje $\sigma > 1$ yra apibrėžiama eilute

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}$$

ir yra meromorfiškai pratęsiama į visą kompleksinę plokštumą. Charakteris χ moduliu q yra vadinamas primityviuoju, jei jis nėra charakteris moduliu $q_1 \mid q$. Sakome, kad primityvusis charakteris χ_1 generuoja charakterį χ, jei

$$\chi(m) = \begin{cases} \chi_1(m), & \text{jei } (m, q_1) = 1, \\ 0, & \text{jei } (m, q) > 1. \end{cases}$$

Du Dirichlė charakteriai vadinami ekvivalenčiais, jei juos generuoja tas pats primityvusis charakteris.

Voronino jungtinė universalumo teorema turi tokį pavidalą.

E teorema. *Tarkime, kad $\chi_1, \dots, \chi_r$ yra poromis neekvivalentūs Dirichlė charakteriai. Su $j = 1, \dots, r$, tegul $K_j \in \mathcal{K}$ ir $f_j(s) \in H_0(K_j)$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Vėliau buvo įrodyta eilė jungtinių universalumo teoremų įvairioms dzeta funkcijoms. Paminėsime jungtinę universalumo teoremą Hurvico dzeta funkcijoms [36]. Tegul

$$L(\alpha_1, \dots, \alpha_r) = \{(\log(m+\alpha_1) : m \in \mathbb{N}_0), \dots, (\log(m+\alpha_r) : m \in \mathbb{N}_0)\}.$$

Tuomet turime tokį tvirtinimą

F teorema. *Tarkime, kad aibė $L(\alpha_1, \dots, \alpha_r)$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$. Su $j = 1, \dots, r$, tegul $K_j \in \mathcal{K}$ ir $f_j(s) \in H(K_j)$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Disertacijos 4 skyriuje yra gautas F teoremos svertinis apibendrinimas. Teisingi tokie tvirtinimai.

4.1 teorema. *Tarkime, kad $w(t) \in W_1$, o aibė $L(\alpha_1, \dots, \alpha_r)$ yra tiesiškai neprilausoma virš $\mathbb{Q}$. Su $j = 1, \dots, r$, tegul $K_j \in \mathcal{K}$ ir $f_j(s) \in H(K_j)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\Bigg(\Bigg\{ \tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon \Bigg\} \Bigg) \mathrm{d}\tau > 0.$$

4.2 teorema. *Tegul galioja 4.1 teoremos sąlygos. Tuomet riba*

$$\lim_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(\tau) I\Bigg(\Bigg\{ \tau \in [T_0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon \Bigg\} \Bigg) \mathrm{d}\tau > 0$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

Pavyzdžiui, 4.1 ir 4.2 teoremose galime imti $w(t) = 1/t$. Kai $w(t) \equiv 1$, iš 4.1 teoremos išplaukia F teorema.

Primename, kad skaičiai $\alpha_1, \dots, \alpha_r$ yra vadinami algebriškai priklausomi virš $\mathbb{Q}$, jei egizstuoja toks polinomas $p(s_1, \dots, s_r) \not\equiv 0$ su racionaliaisiais koeficientais, kad $p(\alpha_1, \dots, \alpha_r) = 0$. Priešingu atveju, skaičia $\alpha_1, \dots, \alpha_r$ yra vadinami algebriškai nepriklausomais virš $\mathbb{Q}$. Nesunku matyti, kad aibė

$L(\alpha_1, \dots, \alpha_r)$ yra tiesiškai neprilausoma virš $\mathbb{Q}$ su algebriškai nepriklausomais $\alpha_1, \dots, \alpha_r$. Taigi, aibės $L(\alpha_1, \dots, \alpha_r)$ panaudojimas universalumo teoremose išplečia jų galiojimo sritį.

Yra nemažai jungtinių universalumo teoremų dzeta funkcijoms, kuriose naudojamos skirtingo tipo dzeta funkcijos. Tarkime, dalis dzeta funkcijų turi Oilerio sandaugą, o kitos - neturi. Tokio tipo jungtinės universalumo teoremos vadinamos mišriomis. Pirmąją mišrią universalumo teoremą Rymano ir Hurvico dzeta funkcijoms įrodė H. Mišu (Mishou) [58].

G teorema. *Tarkime, kad parametras α yra transcendentus. Tegul $K_1, K_2 \in \mathcal{K}$ ir $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\Big\{ \tau \in [0, T] : \sup_{s\in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon \Big\} > 0.$$

Dažnai vietoje funkcijų $\zeta(s)$ ir $\zeta(s, \alpha)$ yra naudojami jų apibendrinimai – periodinė ir periodinės Hurvico dzeta funkcijos. Tegul $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ ir $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0\}$ yra dvi periodinės kompleksinių skaičių sekos. Tuomet pusplokštumėje $\sigma > 1$ periodinė dzeta funkcija $\zeta(s; \mathfrak{a})$ yra apibrėžiama eilute

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

o periodinė Hurvico dzeta funkcija $\zeta(s, \alpha; \mathfrak{b})$ - eilute

$$\zeta(s, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m + \alpha)^s}.$$

Abi šios funkcijos yra meromorfiškai pratęsiamos į visą kompleksinę plokštumą. Kai koeficientai $\mathfrak{a}_m$ yra multiplikatyvūs, tada funkcija $\zeta(s; \mathfrak{a})$ turi Oilerio sandaugą.

Keletą mišrių universalumo teoremų periodinėms dzeta funkcijoms įrodė A. Laurinčikas ir jo mokiniai. R. Kačinskaitė ir K. Matsumotas (Matsumoto) vietoje periodinių dzeta funkcijų nagrinėjo bendrines Matsumoto dzeta funkcijas, kurios yra apibrėžiamos polinomine Oilerio sandauga.

Disertacijos 5 skyriuje gautos svertinės Mišu teoremos (G teoremos) versijos.

5.1 teorema. *Tarkime, kad parametras α yra transcendentus ir $w(t) \in W_1$. Tegul $K_1, K_2 \in \mathcal{K}$ ir $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(t) I\Bigg(\Bigg\{\tau \in [T_0, T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Bigg\}\Bigg) \mathrm{d}\tau > 0.$$

Kaip ir kitų teoremų atveju, 5.1 teorema turi versiją svertinio tankio terminais.

5.2 teorema. *Tegul galioja 5.1 teoremos sąlygos. Tuomet riba*

$$\lim_{T\to\infty} \frac{1}{U_T} \int_{T_0}^{T} w(t) I\Bigg(\Bigg\{\tau \in [T_0, T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Bigg\}\Bigg) \mathrm{d}\tau > 0$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

5.1 ir 5.2 teoremų įrodymai remiasi svertinėmis tikimybinėmis ribinėmis teoremomis erdvėje $H^2(D)$.

Svertines ribines teoremas Lercho dzeta funkcijai nagrinėjo R. Garunkštis

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Disertacijos rezultatai buvo paskelbti šiuose straipsniuose:

1. A. Balčiūnas, G. Vadeikis, A weighted universality theorem for the Hurwitz zeta-function, *Šiauliai Math. Seminar* **12(20)** (2017), 5–18.
2. A. Laurinčikas, G. Vadeikis, Weighted universality of the Hurwitz zeta-function, in: *Algebra, Numb. Th. Discr. Geom.: Modern Probl. App. XV International Conference, Tula, TSPU of L. N. Tolstoy*, 2019, 45–47.
3. A. Laurinčikas, D. Šiaučiūnas, G. Vadeikis, Weighted discrete universality of the Riemann zeta-function, *Math. Modell. Anal.* **25** (2020), no. 1, 21–36.
4. A. Laurinčikas, D. Šiaučiūnas, G. Vadeikis, A weighted version of the Mishou theorem, *Math. Modell. Anal.* **26** (2021), no. 1, 21–33.
5. A. Laurinčikas, G. Vadeikis, Joint weighted universality of the Hurwitz zeta-function, *Algebra i Analiz* **33** (2021), no. 3, 111–128; *St. Petersburg Math. J.* (to appear).

Konferencijų tezės:

1. G. Vadeikis. The weighted universality of the Hurwitz Zeta-Function. Abstracts of MMA2018, May 29 – June 1, 2018, Sigulda, Latvia, pp. 80.
2. A. Laurinčikas, G. Vadeikis, A joint weighted universality theorem for Hurwitz zeta-function, Abstracts of MMA2019, May 28–31, 2019, Tallinn, Estonia, pp. 50.

Išvados

Iš disertacijos išplaukia tokios išvados:

1. Klasei svorio funkcijų galioja svertinės diskrečiosios universalumo teoremos Rymano dzeta funkcijai $\zeta(s)$ apie neturinčių nulių analizinių funkcijų aproksimavimą postūmiais $\zeta(s + ikh)$, $k \in \mathbb{N}_0$, $h > 0$.
2. Klasei svorio funkcijų galoja svertinės tolydžiosios universalumo teoremos Hurvico dzeta funkcijai $\zeta(s, \alpha)$ su transcendenčiuoju α apie analizinių funkcijų aproksimavimą postūmiais $\zeta(s + i\tau, \alpha)$, $\tau \in \mathbb{R}$.

3. Klasei svorio funkcijų galioja svertinės jungtinės tolydžiosios universalumo teoremos Hurvico dzeta funkcijoms $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$, kai aibė $\{(\log(m + \alpha_1) : m \in \mathbb{N}_0), \dots, (\log(m + \alpha_r) : m \in \mathbb{N}_0)\}$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$, apie analizinių funkcijų rinkinio aproksimavimą postūmiais $(\zeta(s + i\tau, \alpha_1), \dots, \zeta(s + i\tau, \alpha_r))$.

4. Klasei svorio funkcijų galioja svertinės mišrios jungtinės universalumo teoremos Rymano ir Hurvico dzeta funkcijoms $\zeta(s)$ ir $\zeta(s, \alpha)$ su transcendenčiuoju α apie poros analizinių funkcijų aproksimavimą postūmiais $(\zeta(s + i\tau), \zeta(s + i\tau, \alpha))$.

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Publications by the Author

1st publication

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A WEIGHTED UNIVERSALITY THEOREM FOR THE HURWITZ ZETA-FUNCTION

Aidas Balčiūnas, Gediminas Vadeikis

Abstract. In the paper, under hypothesis that the weight function $w(t)$ is of bounded variation on $[T_0, \infty)$, $T_0 > 0$, such that the variation $V_a^b w$ on $[a, b]$ satisfies the inequality $V_a^b w \leqslant cw(a)$ with certain $c > 0$ for any subinterval $[a, b] \subset [T_0, \infty)$, and

$$\int_{T_0}^{T} w(t)\mathrm{d}t \to \infty$$

as $T \to \infty$, a weighted universality theorem for the Hurwitz zeta-function $\zeta(s, \alpha)$ with transcendental α is obtained.

Key words and phrases: bounded variation, Hurwitz zeta-function, transcendental number, universality.

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1. Introduction

Let $s = \sigma + it$ be a complex variable and α, $0 < \alpha \leqslant 1$, be a fixed parameter. The Hurwitz zeta function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The function $\zeta(s, \alpha)$ was introduced and considered in [4]. Analytic properties of $\zeta(s, \alpha)$ also can be found in [6].

The function $\zeta(s, \alpha)$ with transcendental or rational parameter α is universal in the sense that its shifts $\zeta(s + i\tau, \alpha)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions. More precisely, let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$, $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and let $H(K)$ with $K \in \mathcal{K}$ denote the class of continuous functions on K that are analytic in the interior of K. Then the universality of $\zeta(s, \alpha)$ is contained in the following theorem.

THEOREM 1.1. *Suppose that the parameter α is transcendental or rational $\neq 1$, $\frac{1}{2}$, $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

Here meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Theorem 1.1 was obtained by S.M. Gonek in his thesis [3], and, by an another method, in [1].

The cases $\alpha = 1$ and $\alpha = \frac{1}{2}$ are excluded in Theorem 1.1 because

$$\zeta(s, 1) = \zeta(s), \quad \text{and} \quad \zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\,\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta-function. The function $\zeta(s, \alpha)$ remains universal also in the cases $\alpha = 1$ and $\alpha = \frac{1}{2}$, however, the approximated function $f(s)$ must be non-vanishing on the set K.

The aim of this paper is to obtain a weighted version of Theorem 1.1 in the case of transcendental α.

Let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$, $T_0 > 0$, such that the variation $V_a^b w$ on $[a, b]$ satisfies the inequality $V_a^b w \leqslant c w(a)$ with certain $c > 0$ for any subinterval $[a, b] \subset [T_0, \infty)$, and

$$U = U(T, w) = \int_{T_0}^{T} w(t) \mathrm{d}t \to \infty$$

as $T \to \infty$. Denote by $I(A)$ the indicator function of the set A.

THEOREM 1.2. *Suppose that the function $w(t)$ satisfies the above conditions, and the parameter α is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I\left(\left\{ \tau \in [T_0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\}\right) \mathrm{d}\tau > 0.$$

Theorem 1.2 has the following modification.

THEOREM 1.3. *Suppose that the function $w(t)$, parameter α, $K \in \mathcal{K}$ and $f(s)$ are as in Theorem 1.2. Then the limit*

$$\lim_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I \left(\left\{ \tau \in [T_0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} \right) \mathrm{d}\tau > 0$$

exists for all but at most countably many $\varepsilon > 0$.

2. Mean square estimate

It is well known that the mean square estimates play an important role in the proofs of universality for zeta-functions. For the proof of Theorems 1.2 and 1.3, we need a weighted mean square estimate for the Hurwitz zeta-function. For this, we will apply the approximation of the function $\zeta(s, \alpha)$ by a finite sum.

LEMMA 2.1. *Suppose that $\sigma \geqslant \sigma_0 > 0$ and $2\pi \leqslant |t| \leqslant \pi x$. Then*

$$\zeta(s, \alpha) = \sum_{0\leqslant m\leqslant x} \frac{1}{(m+\alpha)^s} + \frac{(x+\alpha)^{1-s}}{s-1} + \mathrm{O}_{\sigma_0}\left(x^{-\sigma}\right).$$

A proof of the lemma can be found, for example, in [6, Theorem 3.1.3].

LEMMA 2.2. *Suppose that σ, $\frac{1}{2} < \sigma < 1$, is fixed and $\tau \in \mathbb{R}$. Then*

$$\int_{T_0}^{T} w(t) |\zeta(\sigma + it + i\tau, \alpha)|^2 \mathrm{d}t \ll U\left(1 + |\tau|^2\right).$$

Proof. We take $x = t + |\tau|$ in Lemma 2.1. Then we have

$$\int_{T_0}^{T} w(t) |\zeta(\sigma + it + i\tau, \alpha)|^2 \mathrm{d}t \ll \int_{T_0}^{T} w(t) \left| \sum_{0\leqslant m\leqslant t+|\tau|} \frac{1}{m^{\sigma+it+i\tau}} \right|^2 \mathrm{d}t + U\left(1 + |\tau|^2\right), \tag{2.1}$$

since

$$\int_{T_0}^{T} w(t) \frac{(t+|\tau|)^{2-2\sigma}}{(t+\tau)^2 + (\sigma-1)^2} \mathrm{d}t \ll \int_{T_0}^{2|\tau|} w(t)(t+|\tau|)^{2-2\sigma} \mathrm{d}t + \int_{2|\tau|}^{T} w(t) t^{-2} (t+|\tau|)^{2-2\sigma} \mathrm{d}t \ll U\left(1 + |\tau|^2\right)$$

and

$$\int_{T_0}^{T} w(t)(t+|\tau|)^{-2\sigma}\mathrm{d}t \ll U.$$

Let $\max(m,k) = T_1 + |\tau|$, where $T_1 = T_1(m,k)$. Then we have

$$\begin{aligned}
&\int_{T_0}^{T} w(t)\left|\sum_{0\leqslant m\leqslant t+|\tau|}\frac{1}{m^{\sigma+it+i\tau}}\right|^2 \mathrm{d}t \\
&\quad = \sum\sum_{T_0+|\tau|\leqslant m,k\leqslant T+|\tau|}\frac{1}{(m+\alpha)^{\sigma+it}(k+\alpha)^{\sigma-it}}\int_{T_1}^{T} w(t)\left(\frac{k+\alpha}{m+\alpha}\right)^{it}\mathrm{d}t \\
&\quad \ll \sum_{m\leqslant T+|\tau|}\frac{1}{(m+\alpha)^{2\sigma}}\int_{T_1}^{T} w(t)\mathrm{d}t \\
&\quad\quad + \sum\sum_{T_0+|\tau|\leqslant m<k\leqslant T+|\tau|}\frac{w(k-|\tau|)}{(m+\alpha)^{\sigma}(k+\alpha)^{\sigma}\log((k+\alpha)/(m+\alpha))}. \qquad (2.2)
\end{aligned}$$

Clearly,

$$\sum_{m\leqslant T+|\tau|}\frac{1}{(m+\alpha)^{2\sigma}}\int_{T_1}^{T} w(t)\mathrm{d}t \ll U. \qquad (2.3)$$

If $m+\alpha < \frac{k+\alpha}{2}$, then

$$\log\frac{m+\alpha}{k+\alpha} > \log 2,$$

thus,

$$\begin{aligned}
&\sum\sum_{m<k\leqslant T+|\tau|}\frac{w(k-|\tau|)}{(m+\alpha)^{\sigma}(k+\alpha)^{\sigma}\log((k+\alpha)/(m+\alpha))} \\
&\quad \ll \sum\sum_{m<k\leqslant T+|\tau|}\frac{w(k-|\tau|)}{m^{\sigma}k^{\sigma}} \ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|}\frac{w(k-|\tau|)}{k^{2\sigma-1}} \\
&\quad \ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|} w(k-|\tau|) = \int_{T_0+|\tau|}^{T+|\tau|} w(u-|\tau|)\mathrm{d}[u] \\
&\quad = [u]w(u-|\tau|)\Big|_{T_0+|\tau|}^{T+|\tau|} - \int_{T_0+|\tau|}^{T+|\tau|}(u-\{u\})\mathrm{d}w(u-|\tau|) \\
&\quad \ll U + \int_{T_0+|\tau|}^{T+|\tau|}\mathrm{d}w(u-|\tau|) \ll U. \qquad (2.4)
\end{aligned}$$

If $m+\alpha \geqslant \frac{k+\alpha}{2}$, then we write $m=k-r$, where $1 \leqslant r \leqslant \frac{k}{2}+\frac{\alpha}{2}$. In this case,

$$\log \frac{k+\alpha}{m+\alpha} = -\log \frac{k-r+\alpha}{k+\alpha} = -\log\left(1-\frac{r}{k+\alpha}\right) > \frac{r}{k+\alpha} > \frac{r}{k},$$

and

$$\sum\sum_{T_0+|\tau|\leqslant m<k\leqslant T+|\tau|} \frac{w(k-|\tau|)}{(m+\alpha)^\sigma (k+\alpha)^\sigma \log((k+\alpha)/(m+\alpha))}$$
$$\ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|} \sum_{r\leqslant k/2+\alpha/2} \frac{kw(k-|\tau|)}{rk^\sigma (k-r)^\sigma} \ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|} \frac{w(k-|\tau|)\log k}{k^{2\sigma-1}} \ll U.$$

This and (2.1) – (2.4) prove the lemma.

3. Limit theorems

For the proof of weighted universality theorems for the function $\zeta(s,\alpha)$, a weighted limit theorem on weakly convergent probability measures in the space of analytic functions will be applied, and such a theorem is the aim of this section. Let $\mathcal{B}(X)$ denote the Borel σ-field of the space X, and let $H(D)$ be the space of analytic functions on the strip D endowed with the topology of uniform convergence on compacta.

Denote by γ the unit circle $\{s \in \mathbb{C} : |s|=1\}$ on the complex plane, and define

$$\Omega = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\gamma_m=\gamma$ for all $m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega,\mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we obtain the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. Denote by $\omega(m)$, $m\in\mathbb{N}_0$, the mth component of the element $\omega\in\Omega$, and, on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$, define the $H(D)$-valued random element $\zeta(s,\alpha,\omega)$ by the formula

$$\zeta(s,\alpha,\omega) = \sum_{m=0}^{\infty} \frac{\omega(m)}{(m+\alpha)^s}.$$

We note that the latter series is uniformly convergent on compact subsets of the strip D for almost all $\omega \in \Omega$ with respect to the measure m_H. Let P_ζ be the distribution of the random element $\zeta(s,\alpha,\omega)$, i.e.,

$$P_\zeta(A) = m_H\{\omega\in\Omega : \zeta(s,\alpha,\omega)\in A\}, \quad A\in\mathcal{B}(H(D)).$$

For $A \in \mathcal{B}(H(D))$, let

$$P_T(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I\left(\{\tau \in [T_0, T] :\ \zeta(s, \alpha) \in A\}\right) \mathrm{d}\tau.$$

THEOREM 3.1. *Suppose that the weight function $w(t)$ and the parameter α are as in Theorem 1.2. Then P_T converges weakly to P_ζ as $T \to \infty$. Moreover, the support of P_ζ is the whole $H(D)$.*

We divide the proof of Theorem 3.1 into lemmas. The firs lemma is a weighted limit theorem on the torus Ω. For $A \in \mathcal{B}(\Omega)$, let

$$Q_T(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] :\ \left((m+\alpha)^{-i\tau} :\ m \in \mathbb{N}_0\right) \in A\right\}\right) \mathrm{d}\tau.$$

LEMMA 3.2. *Suppose that the weight function $w(t)$ and the parameter α are as in Theorem 3.1. Then Q_T converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. We consider the Fourier transform $g_T(\underline{k})$, $\underline{k} = (k_m :\ k_m \in \mathbb{Z},\ m \in \mathbb{N}_0)$. The characters χ of the group Ω have the form

$$\chi(\omega) = \prod_{m \in \mathbb{N}_0} \omega^{k_m}(m),$$

where only a finite number of integers k_m are distinct from zero. Therefore,

$$g_T(\underline{k}) = \int_{\Omega} \prod_{m \in \mathbb{N}_0} \omega^{k_m}(m) \mathrm{d}Q_T.$$

Thus, by the definition of Q_T,

$$g_T(\underline{k}) = \frac{1}{U} \int_{T_0}^{T} w(\tau) \prod_{m \in \mathbb{N}_0} (m+\alpha)^{-ik_m\tau} \mathrm{d}\tau$$

$$= \frac{1}{U} \int_{T_0}^{T} w(\tau) \exp\left\{-i\tau \sum_{m \in \mathbb{N}_0}{}' k_m \log(m+\alpha)\right\} \mathrm{d}\tau, \tag{3.1}$$

where the sum $\sum'$ means that only a finite number of integers k_m are distinct from zero. Obviously,

$$g_T(\underline{0}) = 1. \tag{3.2}$$

Suppose that $\underline{k} \neq \underline{0}$. It is well known that the set $\{\log(m+\alpha) : m \in \mathbb{N}_0\}$ with transcendental α is linearly independent over the field of rational numbers $\mathbb{Q}$. Therefore, in this case,

$$\sum_{m\in\mathbb{N}_0}{}' k_m \log(m+\alpha) \neq 0,$$

and, in view of (3.1),

$$\begin{aligned} g_{T},(\underline{k}) &= -\frac{1}{iU \sum'_{m\in\mathbb{N}_0} k_m \log(m+\alpha)} \int_{T_0}^{T} w(\tau)\mathrm{d}\exp\left\{-i\tau \sum_{m\in\mathbb{N}_0}{}' k_m \log(m+\alpha)\right\} \\ &\ll \left(U\left|\sum_{m\in\mathbb{N}_0}{}' k_m \log(m+\alpha)\right|\right)^{-1} \left(1+\int_{T_0}^{T} |\mathrm{d}w(\tau)|\right) \\ &\ll \left(U\left|\sum_{m\in\mathbb{N}_0}{}' k_m \log(m+\alpha)\right|\right)^{-1} \end{aligned}$$

because the variation of $w(t)$ is bounded. This estimate shows that

$$\lim_{T\to\infty} g_T(\underline{k}) = 0.$$

Thus, taking into account (3.2), we have that

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H, the lemma follows by a continuity theorem for probability measures on compact groups.

The next lemma deals with absolutely convergent Dirichlet series. Let $\theta > \frac{1}{2}$ be a fixed number, and, for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$,

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\theta}\right\}.$$

Define two functions

$$\zeta_n(s,\alpha) = \sum_{m=0}^{\infty} \frac{v_n(m,\alpha)}{(m+\alpha)^s}$$

and

$$\zeta_n(s,\alpha,\omega) = \sum_{m=0}^{\infty} \frac{\omega(m)v_n(m)}{(m+\alpha)^s}.$$

Then it is known [6] that the above series are absolutely convergent in the half-plane $\sigma > \frac{1}{2}$. Consider the function $u_n : \Omega \to H(D)$ given by the formula

$$u_n(\omega) = \zeta_n(s, \alpha, \omega), \quad \omega \in \Omega.$$

The absolute convergence of the series $\zeta_n(s, \alpha, \omega)$ implies the continuity of the function u_n. Thus, the function u_n is $(\mathcal{B}(\Omega), \mathcal{B}(H(D)))$-measurable, and the measure m_H induces the probability measure $\hat{P}_n = m_H u_n^{-1}$ given by

$$\hat{P}_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1}A), \quad A \in \mathcal{B}(H(D)).$$

For $A \in \mathcal{B}(H(D))$, define

$$P_{T,n}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I\left(\{\tau \in [T_0, T] : \ \zeta_n(s + i\tau, \alpha) \in A\}\right) \mathrm{d}\tau.$$

Lemma 3.3. *Suppose that the weight function $w(t)$ and the parameter α are as in Theorem 3.1. Then $P_{T,n}$ converges weakly to the measure $\hat{P}_n$ as $T \to \infty$.*

Proof. By the definition of the function u_n,

$$u_n\left((m + \alpha)^{-i\tau} : \ m \in \mathbb{N}_0\right) = \zeta_n(s + i\tau, \alpha).$$

Hence,

$$\begin{aligned} P_{T,n}(A) &= \frac{1}{U} \int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] : \ \left((m + \alpha)^{-i\tau} : \ m \in \mathbb{N}_0\right) \in u_n^{-1}A\right\}\right) \mathrm{d}\tau \\ &= Q_T(u_n^{-1}A) = Q_T u_n^{-1}(A), \end{aligned}$$

where Q_T is from Lemma 3.2. This equality, the continuity of the function u_n, Lemma 3.2 and Theorem 5.1 of [2] prove the lemma.

The proof of a limit theorem for P_T requires a certain approximation $\zeta(s, \alpha)$ by $\zeta_n(s, \alpha)$. For this, we use the following metric on $H(D)$

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D),$$

where $\{K_l : \ l \in \mathbb{N}\}$ is a sequence of compact subsets of the strip D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if K is a compact set of D, then $K \subset K_l$ for some l. This metric induces the topology of $H(D)$ of uniform convergence on compacta.

LEMMA 3.4. *Suppose that the weight function $w(t)$ is as in Theorem 3.1. Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{U}\int_{T_0}^{T} w(\tau)\rho(\zeta(s+i\tau,\alpha),\zeta_n(s+i\tau,\alpha))\mathrm{d}\tau=0.$$

Proof. Let, as usual, $\Gamma(s)$ denote the Euler gamma-function, and let θ be from the definition of $\zeta_n(s,\alpha)$. Define

$$l_n(s,\alpha)=\frac{s}{\theta}\Gamma\left(\tfrac{s}{\theta}\right)(n+\alpha)^s,\quad n\in\mathbb{N}.$$

Then, for $\zeta_n(s,\alpha)$, the integral representation

$$\zeta_n(s,\alpha)=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z,\alpha)l_n(z,\alpha)\frac{\mathrm{d}z}{z}$$

for $\frac{1}{2}<\sigma<1$ is known [6]. Let $K\subset D$ be a compact set. Using the above representation and a standard contour integration, we find that, as $T\to\infty$,

$$\begin{aligned}&\frac{1}{U}\int_{T_0}^{T} w(\tau)\sup_{s\in K}|\zeta(s+i\tau,\alpha)-\zeta_n(s+i\tau,\alpha)|\mathrm{d}\tau\\&\quad\ll\int_{-\infty}^{\infty}|l_n(\sigma_1+it,\alpha)|\left(\frac{1}{U}\int_{T_0}^{T}w(\tau)\left|\zeta(\sigma+it+i\tau,\alpha)\right|\mathrm{d}\tau\right)\mathrm{d}t+\mathrm{o}(1),\end{aligned}\qquad(3.3)$$

where $\sigma_1<0$, and $\frac{1}{2}<\sigma<1$. Moreover, by Lemma 2.2,

$$\begin{aligned}\int_{T_0}^{T} w(\tau)|\zeta(\sigma+it+i\tau,\alpha)|\mathrm{d}\tau&\ll\left(\int_{T_0}^{T}w(\tau)\mathrm{d}\tau\int_{T_0}^{T}w(\tau)|\zeta(\sigma+it+i\tau,\alpha)|^2\mathrm{d}\tau\right)^{1/2}\\&\ll U\left(1+|t|^2\right)^{1/2}\ll U(1+|t|).\end{aligned}$$

Therefore, in view of (3.3),

$$\frac{1}{U}\int_{T_0}^{T} w(\tau)\sup_{s\in K}|\zeta(s+i\tau,\alpha)-\zeta_n(s+i\tau,\alpha)|\mathrm{d}\tau\ll\int_{-\infty}^{\infty}|l_n(\sigma_1+it,\alpha)|\,(1+|t|)\mathrm{d}t+\mathrm{o}(1)$$

as $T\to\infty$. Since $\sigma_1<0$,

$$\lim_{n\to\infty}l_n(\sigma+it,\alpha)=0,$$

thus,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{U}\int_{T_0}^{T} w(\tau)\sup_{s\in K}|\zeta(s+i\tau,\alpha)-\zeta_n(s+i\tau,\alpha)|\,\mathrm{d}\tau=0.$$

This equality and the definition of the metric ρ prove the lemma.

Now we able to prove Theorem 3.1.

Proof of Theorem 3.1. We will prove that the limit measure $\hat{P}_n$ of Lemma 3.3 converges weakly to a certain probability measure P as $n \to \infty$, and that P_T, as $T \to \infty$, also converges weakly to P.

Let a random variable θ_T is defined on a certain probability space $(\hat{\Omega}, \mathcal{A}, \mathbb{P})$ by

$$\mathbb{P}(\theta_T \in A) = \frac{1}{U} \int_{T_0}^{T} w(t) I_A(t) \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{R}),$$

where I_A is the indicator of the set A. Define the $H(D)$-valued random element

$$X_{T,n} = X_{T,n}(s) = \zeta_n(s + i\theta_T, \alpha).$$

Since $P_{T,n}$, by Lemma 3.3, converges weakly to $\hat{P}_n$ as $T \to \infty$, we have that

$$X_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \hat{X}_n, \tag{3.4}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution, and $\hat{X}_n$ is the $H(D)$-valued random element with the distribution $\hat{P}_n$. Further, we will consider the family of probability measures $\{\hat{P}_n : n \in \mathbb{N}\}$ and will prove that this family is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D)$ such that

$$\hat{P}_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Let the set K_l be from the definition of the metric ρ, and $M_l > 0$. Then we have

$$\begin{aligned} &\mathbb{P}\left(\sup_{s\in K_l} |X_{T,n}(s)| > M_l\right) \\ &= \frac{1}{U}\int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] : \sup_{s\in K_l} |\zeta_n(s+i\tau,\alpha)| > M_l\right\}\right) \mathrm{d}\tau \\ &\ll \frac{1}{M_l U}\int_{T_0}^{T} w(\tau) \sup_{s\in K_l} |\zeta_n(s+i\tau,\alpha)| \mathrm{d}\tau. \end{aligned} \tag{3.5}$$

Since the series for $\zeta_n(s, \alpha)$ is absolutely convergent for $\sigma > \frac{1}{2}$,

$$\lim_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(t) |\zeta_n(s+it,\alpha)|^2 \mathrm{d}t = \sum_{m=0}^{\infty} \frac{v_n^2(m,\alpha)}{(m+\alpha)^{2\sigma}} \leqslant \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}} \leqslant C < \infty.$$

This and the Cauchy integral formula show that

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau) \sup_{s\in K_l} |\zeta_n(s+i\tau,\alpha)| \mathrm{d}\tau \leqslant C_l < \infty. \tag{3.6}$$

We fix $\varepsilon > 0$ and take $M_l = C_l 2^l \varepsilon^{-1}$. Then (3.5) and (3.6) give

$$\limsup_{T\to\infty} \mathbb{P}\left(\sup_{s\in K_l} |X_{T,n}(s)| > M_l\right) \leqslant \frac{\varepsilon}{2^l}$$

for all $l \in \mathbb{N}$ and $n \in \mathbb{N}$. Hence, by (3.4),

$$\mathbb{P}\left(\sup_{s\in K_l} |\hat{X}_n(s)| > M_l\right) \leqslant \frac{\varepsilon}{2^l} \tag{3.7}$$

for all $l \in \mathbb{N}$ and $n \in \mathbb{N}$. Define the set

$$K = K(\varepsilon) = \left\{ g \in H(D) : \sup_{s\in K_l} |g(s)| \leqslant M_l,\ l \in \mathbb{N} \right\}.$$

Then K is uniformly bounded on compact subsets of the strip D, therefore, it is a compact set of the space $H(D)$. Moreover, in view of (3.6),

$$\mathbb{P}\left(\hat{X}_n(s) \in K\right) \geqslant 1 - \varepsilon \sum_{l=1}^{\infty} 2^{-l} = 1 - \varepsilon,$$

or

$$\hat{P}_n(K) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Thus the family $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight.

By the Prokhorov theorem [2, Theorem 6.1], the tightness of the family $\{\hat{P}_n : n \in \mathbb{N}\}$ implies its relative compactness. Therefore, there exists a sequence $\{\hat{P}_{n_k}\} \subset \{\hat{P}_n\}$ weakly convergent to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $k \to \infty$. In other words,

$$\hat{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{3.8}$$

Moreover, an application of Lemma 3.3 shows that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I\left(\{\tau \in [T_0, T] :\ \rho(\zeta(s+i\tau,\alpha), \zeta_n(s+i\tau,\alpha)) \geqslant \varepsilon\}\right) \mathrm{d}\tau \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon U} \int_{T_0}^{T} w(\tau) \rho(\zeta(s+i\tau,\alpha), \zeta_n(s+i\tau,\alpha)) \mathrm{d}\tau = 0.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\left(\rho(X_T, X_{T,n}) \geqslant \varepsilon\right) = 0, \tag{3.9}$$

where $H(D)$-valued random element $X_T = X_T(s)$ is defined by

$$X_T(s) = \zeta(s + i\theta_T, \alpha).$$

Now relations (3.4), (3.8) and (3.9) together with Theorem 4.2 of [2] imply the relation

$$X_T \xrightarrow[T\to\infty]{\mathcal{D}} P, \tag{3.10}$$

and this is equivalent to weak convergence of P_T to P as $T \to \infty$.

For identification of the measure P, we apply the following arguments. Relation (3.10) shows that the measure P is independent of the choice of the subsequence $\hat{P}_{n_k}$. Since the family $\{\hat{P}_n\}$ is relatively compact, from this we find that

$$\hat{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P,$$

or, in other words, $\hat{P}_n$ converges weakly to P. Let

$$R_T(A) = \frac{1}{T}\,\mathrm{meas}\,\{\tau \in [0,T]:\ \zeta(s+i\tau,\alpha) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Then, in [5], it is obtained that if the set $L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$, then R_T, as $T \to \infty$, also converges weakly to the limit measure P of $\hat{P}_n$ as $n \to \infty$, and that $P = P_\zeta$. Thus, since the set $L(\alpha)$, with transcendental α, is linearly independent over $\mathbb{Q}$, we have from above, that P_T also converges weakly to P_ζ as $T \to \infty$. Moreover, the support of P_ζ is the whole $H(D)$. The theorem is proved.

4. Proof of universality

Proof of Theorem 1.2. By the Mergelyan theorem on the approximation of analytic functions by polynomials [7], there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{4.1}$$

Define the set

$$G_\varepsilon = \left\{ g \in H(D) :\ \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Since, by Theorem 3.1, the polynomial $p(s)$ is an element of the support of the measure P_ζ, we have that

$$P_\zeta(G_\varepsilon) > 0. \tag{4.2}$$

Moreover, by Theorem 3.1 again and the equivalent of weak convergence of probability measures in terms of open sets,

$$\liminf_{T\to\infty} P_T(G_\varepsilon) \geqslant P_\zeta(G_\varepsilon).$$

Therefore, by the definition of P_T and (4.2),

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau) I\left(\{\tau\in[T_0,T]:\ \zeta(s+i\tau,\alpha)\in G_\varepsilon\}\right) \mathrm{d}\tau \\
&\quad = \liminf_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau) I\left(\left\{\tau\in[T_0,T]:\ \sup_{s\in K}|\zeta(s+i\tau,\alpha)-p(s)|<\frac{\varepsilon}{2}\right\}\right) \mathrm{d}\tau \\
&\quad > 0.
\end{aligned} \tag{4.3}$$

Inequality (4.1) shows that

$$\begin{aligned}
&\left\{\tau\in[T_0,T]:\ \sup_{s\in K}|\zeta(s+i\tau,\alpha)-p(s)|<\frac{\varepsilon}{2}\right\} \\
&\qquad \subset \left\{\tau\in[T_0,T]:\ \sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&I\left(\left\{\tau\in[T_0,T]:\ \sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}\right) \\
&\qquad \geqslant I\left(\left\{\tau\in[T_0,T]:\ \sup_{s\in K}|\zeta(s+i\tau,\alpha)-p(s)|<\frac{\varepsilon}{2}\right\}\right).
\end{aligned}$$

This and (4.3) prove the theorem.

Proof of Theorem 1.3. Define the set

$$\hat{G}_\varepsilon = \left\{g\in H(D):\ \sup_{s\in K}|g(s)-f(s)|<\varepsilon\right\},$$

and let ∂ be the boundary operator. Then we have that $\partial\hat{G}_{\varepsilon_1}\cap\partial\hat{G}_{\varepsilon_2}=\varnothing$ for $\varepsilon_1\neq\varepsilon_2$. Hence, it follows that the set $\hat{G}_\varepsilon$ is a continuity set of the measure P_ζ, i.e., $P_\zeta(\partial\hat{G}_\varepsilon)=0$, for all but at most countably many $\varepsilon>0$. Therefore, using Theorem 3.1 and the equivalent of weak convergence in terms of continuity sets, we obtain that the limit

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(t) I\left(\left\{\tau\in[T_0,T]:\ \sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}\right) \\
&\quad = P_\zeta(\hat{G}_\varepsilon)
\end{aligned} \tag{4.4}$$

exists for all but at most countably many $\varepsilon>0$. On the other hand, in view of (4.1), we have that $\hat{G}_\varepsilon\supset G_\varepsilon$. Thus, $P_\zeta(\hat{G}_\varepsilon)\geqslant P_\zeta(G_\varepsilon)>0$ by (4.2), and (4.4) proves the theorem.

Remark 4.1. The transcendence of the parameter α in Theorems 1.2 and 1.3 can be replaced by the weaker hypothesis of the linear independence over $\mathbb{Q}$ for the set $L(\alpha)$.

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