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Approximation of Analytic Functions by Shifts of Zeta-Functions of Certain Cusp Forms

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Analizinių funkcijų aproksimavimas parabolinių formų dzeta funkcijos postūmiais

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INTRODUCTION

The approximation of the analytic functions by simpler or more general functions is one of the more challenging tasks for mathematicians. One of the famous approaches proposed by S. Mergelyan states that every function $f(s)$ of a complex variable $s = \sigma + it \in \mathbb{C}$ which is continuous on a compact subset $K \subset \mathbb{C}$ and analytic in the interior of K can be approximated uniformly on K by polynomials in s. This and similar findings enabled the spread of computational methods and also further research in the field of approximation.

The last quarter of the 20th century was marked by the spread of the notion of universality. It was noticed that there exist the so-called universal functions whose shifts can approximate any analytic function in a given area. In 1975, S. M. Voronin proved a wondrous property of universality for the Riemann zeta function $\zeta(s)$, which is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and is analytically continued for the whole complex plane except for the simple pole at $s = 1$. He showed that any analytic function in the complex plane can be approximated with a given accuracy by the shifts $\zeta(s + i\tau), \tau \in \mathbb{R}$. Voronin's discovery inspired further investigations in the field. It turned out that some other zeta and L-functions, as well as certain classes of the Dirichlet series, such as Dirichlet L-functions, Dedekind, Hurwitz, Lerch zeta-functions and others, are universal in the Voronin sense.

During several decades of research, it was shown that there exist different types of universality, such as continuous, discrete, weighted, joint and others, when the composition or different combinations of functions are taken for the approximation of analytic functions. In some cases, shifts can be taken from different types of sets, such as subsets of the real numbers, arithmetic progressions or non-linear sequences. In each case, there still are numerous limitations that prevent the extension of the universality property for more general function classes. For instance, in case of the discrete version of universality, it is still difficult to extend the set of shifts for other than linear sets, while for the continuous cases, the growth rate of the shifts is problematic. In case of joint universality, the dependence of functions becomes a limitation.

In 2001, A. Laurinčikas and K. Matsumoto [28] obtained the universality for zeta-functions $\zeta(s, F)$ attached to certain cusp forms F. Let $F(z)$ be the Hecke-eigen cusp form, i.e., F is a holomorphic cusp form of weight $\kappa \in 2\mathbb{N}$ and it is a simultaneous Hecke-eigen form. Let $c(m), m \in \mathbb{N}$, be the Fourier series coefficients of F. Then $\zeta(s, F)$ is defined by the series

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}, \quad \sigma > (\kappa + 1)/2,$$

and by the analytic continuation elsewhere. The Laurinčikas-Matsumoto theorem states that for $\zeta(s, F)$, the continuous universality property holds.

This thesis is devoted to an extension of the Laurinčikas-Matsumoto theorem and different modifications of the universality theorems for the approximation of analytic functions, taking non-linear shifts $\zeta(s + i\varphi(\tau), F)$, where function φ satisfies some natural growth conditions. In particular, continuous universality, discrete universality, and joint discrete universality cases are considered.

Aims and problems

The aim of the thesis is to provide a generalization of the universality theorem for the zeta function of certain cusp forms $\zeta(s, F)$. This thesis will address the following problems:

1. Define the set of non-linear shifts and identify conditions for which continuous universality holds.
2. Define the set of non-linear shifts and identify conditions for which discrete universality holds.
3. Define the set of non-linear shifts and identify conditions for which joint discrete universality hods.

Actuality

Intensity and diversity in the investigation of the universality property in recent decades prove the importance and actuality of the problem. As the universality property is an interesting investigation topic in itself, and as research in the field opens new methods and techniques in the analytic number theory, it also supports the solution of other problems in the number theory and even mathematical physics. As an example, the universality theorem of the Voronin type can be applied in the value distribution problems for zeta functions. Another issue associated with universality is the hypertranscendence of the zeta function: the universality property could be used to support the statement that there is no non-trivial algebraic differential equations having $\zeta(s)$ as a solution. Also, some strong statements come with the functional independence and universality properties of zeta functions: if $F_0, F_1, ..., F_N$ are continuous functions defined on $\mathbb{C}^{N+1}$ and at least some of them are not identically zero, then

$$\sum_{k=0}^{N} s^k F_k(\zeta(s),\, \zeta'(s), ..., \zeta^{(n)}(s)) \neq 0$$

for some $s \in \mathbb{C}$ (see [53], [51]). The Riemann hypothesis, one of the Millennium Prize Problems, must not be forgotten here, stating that $\zeta(s) \neq 0$ for $\sigma > 1/2$, which was found equivalent to the statement that $\zeta(s)$ can approximate itself inside the strip $1/2 < \sigma < 1$ in the sense of the Voronin theorem.
From the object point of view, the zeta function of the cusp forms was chosen as a recent object with a variety of applications. Interest in the exploration of $\zeta(s, F)$ (in some sources also referred as $L(s, F)$) was inspired by its success in the solution of the last Fermat theorem in 1995 by A. Wiles [55], [52]. Also, it is an important object in the number theory, as E. Hecke has shown a bijection between the modular forms and the Dirichlet series that satisfy a functional equation of the Riemann type [12]. In addition to that, $\zeta(s, F)$ also plays an important role in algebra and group theory.

Finally, the case of non-linear shifts was chosen as problematic and requiring further investments in the investigations. A similar case of non-linear shifts was originally proposed by Ł. Pánkowski in 2016 [47], for the dependent L-functions, and now it is seen as one of the more promising approaches to the generalization of universality sets and the search for new types of universal shift sets.

About the object

The roots of zeta functions were developed in the 19th century with the first results of B. Riemann, inspired by L. Euler's research in the 18th century.

In 1737, L. Euler proved [8] that the set of prime numbers is infinite and dense. In particular, he showed that a series

$$\sum_{p\in\mathbb{P}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + ...,$$

where $\mathbb{P}$ denote the set of prime numbers, diverge. This means that the set of prime numbers is denser than, for instance, the set of squares. Even more, he assumed that prime numbers have the asymptotic density

$$\frac{1}{\log u},$$

where $\log u$ denote the natural logarithm. Although Euler's results were not strict in the sense of formulation or proof, a hundred years later K. Gauss supported this observation. In his letter to a colleague dated around 1849, Gauss stated that the set of prime numbers $\mathbb{P}$ has the asymptotic density equal to $1/\log x$, i.e., if $\pi(x)$ denotes the number of prime numbers no greater than x, then the following asymptotic formula holds

$$\frac{\pi(x)}{x} \sim \frac{1}{\log x}, \quad x \to \infty.$$

According to Gauss, this conjecture was also approved by the observation of the prime numbers compiled until that time by John Lambert [19], [6]. This statement was an inspiration for further research of the relation between $\pi(x)$ and the integral of $1/\log x$ with numerous results by A. M. Legendre, P. Chebyshev and others.

A completely new approach to the solution of this problem was proposed by B. Riemann. In 1859, he proposed the so-called Riemann zeta function $\zeta(s)$ as a function of a complex variable for the analysis of the distribution of the prime numbers. In his article "Über die Anzahl der

Primzahlen unter einer gegebenen Grösse" [50], he expressed $\pi(x)$ as a sum of infinite series with $\int\limits_2^x \frac{du}{\log u}$ as the main element. Moreover, he proposed the function $\zeta(s)$ as an object to analyze the prime numbers.

The Riemann zeta function $\zeta(s)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and by analytic continuation elsewhere, except a simple pole at $s = 1$ with the residue equal to 1. In Riemann's article the functional equation for $\zeta(s)$ was also derived by linking the Riemann zeta function with the Euler gamma function $\Gamma(s)$:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

It was also shown that $\zeta(s)$ has the so-called Euler product over primes, i.e.,

$$\zeta(s) = \prod_{p\in\mathbb{P}}\left(1 - \frac{1}{p^s}\right)^{-1}.$$

These facts about $\zeta(s)$ were later used both for the analysis of the prime numbers and the investigation of other properties of $\zeta(s)$, such as its universality.

The most remarkable fact is that Riemann associated the distribution of zeroes of the function $\zeta(s)$ with the distribution of prime numbers. Together with that, he stated the famous Riemann hypothesis: that the only non trivial zeroes of $\zeta(s)$ lie in the line $\sigma = 1/2$ or

$$\zeta(s) \neq 0, \quad \sigma > 1/2.$$

The Riemann hypothesis is still an open question. However, investigations of $\zeta(s)$ have shown other interesting properties of zeta and L-functions, some of which are discussed in this thesis. An extensive overview of the Riemann zeta function and its relation to the distribution of the prime numbers can be found, for instance, in [19].

Now we will define the zeta functions of cusp forms. Let

$$SL(2,\mathbb{Z}) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}$$

be the full modular group. We say that the function $F(z)$, $z \in \mathbb{C}$, is a holomorphic cusp form of weight $\kappa \in 2\mathbb{N}$ for $SL(2,\mathbb{Z})$ if F is holomorphic (or analytic) for $\mathrm{Im}(z) > 0$, for all $M \in SL(2,\mathbb{Z})$ satisfies the functional equation

$$F(Mz) := F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z), \tag{1}$$

and at infinity has the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}.$$

It is important to mention that the transformation Mz on the left side of the equality (1) allows us to narrow the domain of the function $F(z)$ for the analysis, as it keeps the function argument in the same domain (it translates the elements from the upper halfplane of the complex plane to the same halfplane).

Cusp forms are a subset of modular forms, vanishing in the infinity (or having the Fourier series expansion with coefficient $c(0) = 0$). A classical example of modular forms is the discriminant function

$$\Delta(z) = (2\pi)^{12} e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24},$$

which is a cusp form of weight 12. Indeed, in the definition of the cusp forms, we may take any natural κ, but in cases when κ is odd, modular forms of weight κ are equal to zero. Even more, for cusp forms, it is true that there are no cusp forms with weight less than 12 which are not equal to zero. A good overview of the properties and role of the modular forms can be found in [1] and [9].

The zeta-function $\zeta(s, F)$ associated with the cusp form $F(z)$ of weight

κ is defined, for $\sigma > (\kappa+1)/2$, by absolutely convergent Dirichlet series

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

We assume additionally that $F(z)$ is an eigen form of all Hecke operators

$$T_m F(z) = m^{\kappa-1} \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d^\kappa} \sum_{b \pmod d} F\left(\frac{az+b}{d}\right), \quad m \in \mathbb{N}.$$

We remind that function F is called an eigen form of an operator T_m, if there exists $\lambda(m) \in \mathbb{C}$ such that $T_m F = \lambda(m) F$. If F is an eigen form of all operators $T_m, m \in \mathbb{N}$, then it is called a simultaneous Hecke-eigen form.

With these assumptions, we get that $c(m) \neq 0$, $m \in \mathbb{N}$; therefore, $F(z)$ can be normalized to have the Fourier coefficient $c(1) = 1$.

It is proved [13] that $\zeta(s, F)$ attached to a cusp form is analytically continued to an entire function in the whole complex plane and also has a Riemann-type functional equation

$$(2\pi)^{-s}\Gamma(s)\zeta(s, F) = (-1)^k (2\pi)^{s-2k}\Gamma(2k-s)\zeta(2k-s, F).$$

Moreover, taking the simultaneous Hecke-eigen forms, we get that the Fourier coefficients $c(m)$ are multiplicative; therefore, for $\sigma > (\kappa+1)/2$, the function $\zeta(s, F)$ has the Euler product expansion over primes

$$\zeta(s, F) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers satisfying $\alpha(p) + \beta(p) = c(p)$.

In this thesis, all the results are proved for $\zeta(s, F)$ where F means a normalized simultaneous Hecke-eigen cusp form.

History of the problem

Notion of the universality

Researchers have noticed [25], [46] that the first results in the area of universality already appeared in the beginning of the 20th century. In 1914, M. Fekete demonstrated a real power series

$$\sum_{k=1}^{\infty} a_k x^k$$

such that for every continuous function $f(x)$ in the interval $x \in [-1, 1]$ with $f(0) = 0$, there exists an increasing sequence $\{\lambda_n, n \in \mathbb{N}\} \subset \mathbb{N}$ such that

$$\sup_{x\in[-1,1]} \left| \sum_{k=1}^{\lambda_n} a_k x^k - h(x) \right| \to 0, \quad n \to \infty.$$

In other words, it was demonstrated that a single object can approximate a wide class of different objects.

The first results on universal functions belong to G. D. Birkhoff [5]. In 1929, he proved the existence of an entire function $g(s)$ such that, for every entire function $f(s)$, a compact subset $K \subset \mathbb{C}$, and an arbitrary $\varepsilon > 0$, there exists a number $a \in \mathbb{C}$ such that

$$\sup_{s\in K} |g(s+a) - f(s)| < \varepsilon.$$

This was the first attempt to approximate a class of analytic functions with the shifts of one and the same function g. However, Birkhoff proved only the existence of such a universal function, and neither an example of such a function g nor the evaluation of the constant a was given.

The notion of universality itself is attributed to J. Marcinkiewicz (see [38]). In 1935, he proved that there exists a continuous function $g \in C[a, b]$ which can approximate every finite Lebesgue measurable function f. Let $[a, b] \subset \mathbb{R}$ and let $\{h_n\}$ be a fixed sequence of non-zero real numbers converging to zero. Marcinkiewicz proved that under such conditions there exists a continuous function $g : [a, b] \to \mathbb{R}$ such that for any Lebesgue measurable function $f : [a, b] \to \mathbb{R}$, there is a subsequence $\{h_{n_k}\} \subset \{h_n\}$ for which

$$\lim_{k\to\infty} \frac{g(x + h_{n_k}) - g(x)}{h_{n_k}} = f(x)$$

almost everywhere on $[a, b]$. He stated that such functions g constitute a residual set in $C[a, b]$. Marcinkiewicz named the function g the universal primitive.

In the field of functions of a complex variable, one of the most important results is the famous S. Mergelyan theorem of 1952. It asserts [41] that every function $f(s)$ of a complex variable s, continuous on a compact subset $K \subset \mathbb{C}$ and analytic in the interior of K, can be approximated uniformly on K by a polynomial $p(s)$ such that, for every $\varepsilon > 0$,

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

It was also shown that the hypotheses on $f(s)$ and K cannot be weakened. Therefore, the Mergelyan theorem gives necessary and sufficient conditions for the approximation of analytic functions by polynomials. The Mergelyan theorem will be strictly formulated in Chapter 1 since it plays a significant role in the proof of the defended results.

The first example of a universal function was only found in 1975. S. M. Voronin proved [53] that shifts $\zeta(s + i\tau), \tau \in \mathbb{R}$, of the Riemann zeta function $\zeta(s)$ can uniformly approximate any analytic function on some compact subsets. Later, a similar property was proved for other types of zeta and L-functions. Due to the importance of these results, the next subsection will be dedicated to the findings of Voronin and the evolution of the universality theorem for the zeta functions.

It is worth mentioning that the abstract definition of universality was finally given by K.-G. Grosse-Erdmann in 1999 (see [11]). Let X and Y be two topological spaces and $T_j : X \to Y$, $j \in I$, be continuous mappings. Then an element $x \in X$ is called universal with respect to the family T_j, $j \in I$, if the set $T_j(X) = \{T_j(x) : j \in I\}$, is dense in Y.

By analyzing the Riemann zeta function $\zeta(s)$, Voronin noticed that with certain shifts of one and the same function $\zeta(s)$ a whole class of analytic functions can be approximated. He proved the following statement.

Theorem A (Original Voronin's universality theorem)**.** *Let $0 < r < \frac{1}{4}$. Suppose that $f(s)$ is a continuous non-vanishing function on the disc $|s| \leqslant r$, and analytic for $|s| < r$. Then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s| \leqslant r} \left| \zeta(s + \frac{3}{4} + i\tau) - f(s) \right| < \varepsilon.$$

Roughly speaking, Voronin proved that any non-vanishing holomorphic function can be approximated uniformly by a certain shift of $\zeta(s)$ with a given accuracy.

Voronin's proof non-explicitly included an even stronger fact: that the set of τ for a given function has positive lower density, i.e., that there are infinitely many such τ by which $f(s)$ can be approximated. Further analysis also generalized the disk to a more general set in which the function is approximated. We will formulate the most recent version of Voronin's statement. Let $D(a, b) = \{s \in \mathbb{C} : a < \sigma < b\}$, for $a < b$, $a, b \in \mathbb{R}$, $\mathcal{K}$ be the class of compact subsets in the strip $D(a, b)$ with connected complements, and $H_0(K)$, $K \in \mathcal{K}$, stand for the class of continuous non-vanishing functions on K that are analytic in the interior of K. The Lebesgue measure of a measurable set $A \subset \mathbb{R}$ is denoted by $\operatorname{meas} A$.

Theorem B (Modern Voronin's universality theorem)**.** *Suppose that $\mathcal{K} \subset D(\frac{1}{2}, 1)$, $K \in \mathcal{K}$, $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$, the inequality*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0$$

holds.

Voronin's important discovery gained attention from the number theorists. In over four decades, various generalizations, analogies and refinements of Voronin's theorem were discussed. Already in the first decade of the research (1975-1987), two alternative proofs for the Voronin theorem were constructed. In addition to the original proof proposed by S. M. Voronin, the second was given by A. Good [10] and a third, more probabilistic one, by B. Bagchi [2]. These three proofs are still considered the key approaches to this theorem. At the same time, generalizations to other classes of functions, such as Dirichlet L-functions and Dedekind zeta functions, as well as new types of universality were studied, such as joint universality, strong universality, discrete universality, χ-universality and hybrid universality. It was also proved that some zeta functions without the Euler product are universal. In this case, the target functions $f(s)$ can be taken from a wider class, removing the requirement that $f(s)$ is non-vanishing. We call this type of approximation the strong universality property. Although many results of this initial period remain unpublished, they laid the basis for the development of the universality theory.

Lithuania also has a strong record in the research of universality. The history of the research is primarily associated with Professor A. Laurinčikas and the appearance of his first textbook [20] in 1996, devoted to universality and related topics. This book broke a decade of silence in the field and not only made available the unpublished results of Bagchi, but also paved the way for the new generation of mathematicians who started to work in the area. This is also seen as the beginning of the Lithuanian school.

Recently, the theory of universality for zeta and L-functions has been expansively developed by B. Bagchi, R. Garunkštis, J. Genys, S. M. Gonek, R. Kačinskaitė, J. Kaczorowski, A. Laurinčikas, Y. Lee, R. Macaitienė, K. Matsumoto, H. Mishou, H. Nagoshi, T. Nakamura, Ł. Pánkowski, A. Reich, J. Sander, W. Schwarz, J. Steuding, R. Steuding, D. Šiaučiūnas and others. Current research topics include such issues as mixed universality (simultaneous approximation by zeta functions with and without the Euler product), composite universality (universality given by composition of functions), ergodic universality, weighted universality, and others. In addition, sets of universal shifts and other extensions of the universality inequality are studied. An extensive overview of the results in the analysis of universality of the zeta functions was presented by K. Matsumoto in [39]. Some more in-depth analysis in the topic can also be found in [51] or [25].

We will now present the universality theorems which gave ground for the results of this thesis.

Voronin's theorem provided the basis for research in the continuous case. The discrete version of universality for zeta-functions was proposed by A. Reich. Discrete universality in particular proposes the approximation of the analytic functions with shifts from a discrete set. In [49], Reich obtained a discrete universality theorem for Dedekind zeta-functions. In his theorem, τ takes values from the arithmetic progression $\{kh : k \in \mathbb{N}_0\}$.

Let $\#A$ denote the cardinality of a set A. Let M be the number field, $\zeta_M(s)$ be the associated Dedekind zeta function and $d_M = [M : \mathbb{Q}]$.

Theorem C (Discrete universality)**.** *Suppose K is a compact subset of the region $D(1 - (\max\{2, d_M\})^{-1}, 1)$ with connected complement and $f(s) \in H_0(K)$. Then for any real $h \neq 0$ and any $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta_M(s + ikh) - f(s)| < \varepsilon \right\} > 0.$$

The results of discrete universality are strongly dependent on the arithmetic nature of the parameter h. We give several examples. R. Kačinskaitė proved discrete universality for the Matsumoto zeta functions under the condition that $\exp(2\pi k/h)$ is irrational for any $k \in \mathbb{N}$ [16]. For the Lerch zeta function, the condition that $\exp(2\pi/h)$ is rational was used [14]. The case of the periodic Hurwitz zeta function is often studied under the condition that $\{\log(m + \alpha), m \in \mathbb{N}_0\} \cup \left\{\frac{2\pi}{h}\right\}$ is linearly independent over $\mathbb{Q}$ [43].

There exists a problem to prove analogues of Theorem C for the sets different from the progression $\{kh : k \in \mathbb{N}_0\}$. The first attempt in this direction, in the case of the Riemann zeta-function, was made in [7], where the arithmetical progression was replaced by the set $\{k^\alpha h : k \in \mathbb{N}_0\}$ with a fixed $\alpha, \quad 0 < \alpha < 1$. An analogue of the theorem from [7] for the function $\zeta(s, F)$ was given in [23]. While investigating the joint universality of Dirichlet L-functions, Ł. Pánkowski extended [47] the theorem of [7] for all non-integers $\alpha > 0$ and more general sets of the type $\{hk^\alpha \log^\beta k : k = 2, 3, \dots\}$, where

$$\beta \in \begin{cases} \mathbb{R} \quad \text{if} \quad \alpha \notin \mathbb{Z}, \\ (-\infty, 0] \cup (1, \infty) \quad \text{if} \quad \alpha \in \mathbb{N}. \end{cases}$$

The thesis results are some type of generalization of shifts proposed by Ł. Pánkowski.

Some collections of zeta and L-functions have a joint universality property. In this case, a collection of analytic functions is simultaneously approximated by a collection consisting of shifts of zeta or L-functions. The first joint universality theorem was obtained also by S. M. Voronin. In [54], he proved a joint universality theorem for Dirichlet L-functions $L(s, \chi_1), ..., L(s, \chi_r)$ with pairwise non-equivalent Dirichlet characters (also see [2], [17], [22]). We will formulate a modified version of this theorem.

Theorem D (Joint universality theorem)**.** *Let $K_1, ..., K_r$ be compact subsets of $D(1/2, 1)$, and $f_j \in H_0(K_j)$, $j = 1, ..., r$. Let $\chi_1, ..., \chi_r$ be pairwise non-equivalent Dirichlet characters, and $L(s, \chi_1), ..., L(s, \chi_r)$ the corresponding Dirichlet L-functions. Then, for any $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

A discrete version of the joint universality theorem by using the shifts from the arithmetic progression was proposed by B. Bagchi [2]. Later, many results on the joint universality of zeta and L-functions were obtained, see, for example, [21], [27], [29] and a survey paper [39].

In joint universality theorems, the zeta-functions approximating a collection of analytic functions must be independent in a certain sense. For example, in case of Dirichlet L-functions, this independence is described by the non-equivalence of characters. In case of periodic zeta-functions, some rank conditions are applied. However, if the coefficients of Dirichlet series defining zeta-functions are non-periodic, then the problem of joint universality for those zeta-functions becomes very complicated. This remark also concerns the zeta-functions of cusp forms.

Universality of $\zeta(s, F)$

The very first results on the universality of $\zeta(s, F)$ were discussed by A. Kačėnas and A. Laurinčikas in 1998 [15]. However, since classical approaches required an asymptotic formula for the sum $\sum_{p \leqslant x} |c(p)| p^{-1}$ which is not known, the universality was proved under a very strong assumption.

Therefore, the proof of the universality of $\zeta(s, F)$ is often associated with the work of A. Laurinčikas and K. Matsumoto in 2001 [28]. For the proof of this theorem, Laurinčikas and Matsumoto invented a new method called the positive density method, where the original estimate for the series of primes with the known estimates for the coefficients $\hat{c}(p) = c(p)p^{-(\kappa-1)/2}$ is combined. This method was later applied to prove the universality of other L-functions and certain Dirichlet series with multiplicative coefficients.

A formulation of the Laurinčikas-Matsumoto universality theorem requires several notations. Let $\mathcal{K} = \mathcal{K}_F$ be the class of compact subsets in the strip $D\left(\frac{\kappa}{2}, \frac{\kappa+1}{2}\right)$ with connected complements, and $H_0(K)$, $K \in \mathcal{K}$, stand for the class of continuous non-vanishing functions on K that are analytic in the interior of K.

Theorem E (Laurinčikas-Matsumoto universality theorem)**.** *Suppose that $K \in \mathcal{K}$, $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$, the following inequality*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, F) - f(s)| < \varepsilon \right\} > 0$$

holds.

In this case, shifts τ were uniformly distributed in the interval. However, it was shown that more general shifts can be considered. Generalizations of Theorem E were given in [32] and [26]. In [32], A. Laurinčikas, K. Matsumoto and J. Steuding considered the universality for zeta functions of the so-called new Forms. We will revisit the definition of a new form. Let $SL(2,\mathbb{Z})$ be the full modular group, and, for $q \in \mathbb{Z}$,

$$\Gamma_0(q) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \equiv 0 \pmod q \right\}$$

be the Hecke subgroup. Suppose that $\hat{F}(z)$ is a holomorphic function on the upper half plane $\operatorname{Im} z > 0$, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$ satisfies the functional equation

$$\hat{F}\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} \hat{F}(z), \quad \kappa \in 2\mathbb{N},$$

and is holomorphic and vanishing at cusps. Then $\hat{F}(z)$ is called a cusp form of weight κ and level q, and has the Fourier series expansion at infinity. Denote the space of all cusp forms of weight κ and level q by $S_\kappa(\Gamma_0(q))$. For every $d|q$, the element of the space $S_\kappa(\Gamma_0(d))$ can be also considered as an element of the space $S_\kappa(\Gamma_0(q))$. The form $\hat{F} \in S_\kappa(\Gamma_0(q))$ is called a new form if it is not a cusp form of level less than q, and if it is an eigen-function of all Hecke operators. Zeta (or L-) function of new forms is defined for the new form $\hat{F}$ in the same way as the zeta function for the cusp forms using the Fourier series coefficients. In other words, it is a specific subset of the analyzed zeta functions.

The first discrete universality theorem for $\zeta(s,\hat{F})$ attached to a new form $\hat{F}(z)$ under a certain arithmetical hypothesis for the number h was proven in [33]. In the article, step h was taken to satisfy the condition $\exp(2\pi k/h)$ is irrational for any $k \neq 0$. In [35], this hypothesis was removed, and the following statement was obtained.

Theorem F. *Suppose that $K \in \mathcal{K}$, $f(s) \in H_0(K)$, and $h > 0$ is an arbitrary fixed number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh, \hat{F}) - f(s)| < \varepsilon \right\} > 0.$$

The first result for a pair of zeta-functions of cusp forms belongs to H. Mishou [44]. Let F_1 and F_2 be two different normalized Hecke-eigen cusp forms for the full modular group $SL(2,\mathbb{Z})$, of weights κ_1 and κ_2 and Fourier coefficients $c_1(m)$ and $c_2(m)$, respectively. Define

$$\hat{c}_j(m) = c_j(m) m^{-(\kappa_j - 1)/2}, \quad j = 1, 2,$$

and

$$\hat{\zeta}(s, F_j) = \sum_{m=1}^{\infty} \frac{\hat{c}_j(m)}{m^s}, \quad \sigma > 1, j = 1, 2.$$

Then the Mishou theorem is the following statement.

Theorem G. *For $j = 1, 2$, let K_j be a compact subset of $D\left(\frac{1}{2}, 1\right)$, and $f_j(s)$ be a continuous non-vanishing function on K_j that is analytic in the interior of K_j. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant 2} \sup_{s\in K_j} |\hat{\zeta}(s+i\tau, F_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Theorem G also remains valid for pairs consisting of the Riemann zeta-function, Rankin-Selberg L-functions and symmetric square L-functions [44]. In [37], Y. Lee, T. Nakamura and Ł. Pánkowski proved the joint universality theorem for an arbitrary number of automorphic zeta-functions.

In [24], joint discrete universality theorems for zeta-functions of cusp forms were obtained. Let $F_1, ..., F_r$ be different normalized Hecke-eigen cusp forms of weight $\kappa_1, ..., \kappa_r$, with Fourier coefficients $c_1(m), ..., c_r(m)$, respectively, and let

$$\zeta(s, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m)}{m^s}, \quad \sigma > (\kappa_j + 1)/2, \quad j = 1, ..., r,$$

be the corresponding zeta-functions. For positive numbers $h_j, j = 1, ..., r$, define

$$L(\mathbb{P}; h_1, ..., h_r; \pi) = \{(h_1 \log p : p \in \mathbb{P}), ..., (h_r \log p : p \in \mathbb{P}), 2\pi\}.$$

Let $D_j = D\left(\frac{\kappa_j}{2}, \frac{\kappa_j+1}{2}\right)$, $\mathcal{K}_j$ be the class of compact subset of the strip D_j with connected complements, and let $H_0(K_j), K_j \in \mathcal{K}_j$, denote the class of continuous non-vanishing functions on K_j that are analytic in the interior of K_j, $j = 1, ..., r$. Thus, the following theorem is proven in [24].

Theorem H. *Suppose that the set $L(\mathbb{P}; h_1, ..., h_r; \pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, ..., r$, let $K_j \in \mathcal{K}_j$ and $f_j(s) \in H_0(K_j)$. Then, for any $\varepsilon > 0$, the following inequality holds*

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s + ikh_j, F_j) - f_j(s)| < \varepsilon \right\} > 0.$$

The set $L(\mathbb{P}; h_1, ..., h_r; \pi)$ here is used for the definition of a certain independence of the functions $\zeta(s, F_1), ..., \zeta(s, F_r)$.

In recent years, also other types of universality for the zeta functions of cusp forms were analyzed. S. Račkauskienė in her PhD thesis [48] obtained the mixed joint universality theorem for a zeta-function of normalized Hecke-eigen cusp forms and periodic Hurwitz zeta-functions. A. Laurinčikas, K. Matsumoto and J. Steuding in [34] introduced several wide classes of operators Φ, including the operators from the Lipchitz class, for which composition $\Phi(\zeta(s, F))$ is universal. A. Balčiūnas, V. Franckevič, V. Garbaliauskienė, R. Macaitienė and A. Rimkevičienė in [3] have proven, under a weak form of the Montgomery pair correlation conjecture, that the shifts $\zeta(s + iy_k h, F)$, where $y_1 < y_2 < ...$ is a sequence of imaginary parts of non-trivial zeros of the Riemann zeta function and $h > 0$, can approximate a wide class of analytic functions.

Traditionally, formulations of universality theorems are stated using the lower limit. This statement is strong enough as it already implies that the set of the shifts of the specific zeta or L-function, used for the approximation of the analytic functions, has a positive lower density. The question whether the set of shifts also has a positive density, i.e. can the lower limit be replaced with the limit, remained unsolved for some time. In 2012, S. M. Voronin mentioned that such a replacement is possible for almost all $\varepsilon > 0$ (see [30]). Just after this claim, A. Meška and A. Laurinčikas have demonstrated the relevant results [30]. In particular, they have shown that the following assertion for the Riemann zeta-function is valid.

Theorem I. *Suppose that* $\mathcal{K} \subset D(\frac{1}{2}, 1)$*,* $K \in \mathcal{K}$*,* $f(s) \in H_0(K)$*. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$*.*

In the traditional proof of universality, Meška and Laurinčikas replaced the equivalent of the weak convergence of probability measures in terms of open sets with the equivalent in terms of the continuity sets which gave the desired result. Although there are limitations of the probabilistic nature to prove the universality for all $\varepsilon > 0$, they showed that this approach can be applied for all zeta-functions defined by the Dirichlet series and satisfying some natural growth conditions [31], [42].

As the results demonstrated in the abovementioned papers and the PhD thesis are claimed to be one of the first in this direction, similar results were independently confirmed by the research of J.-L. Mauclaire [40].

In this thesis, the modified versions of the universality theorems similar to Theorem I are proven.

Defended results of the dissertation

This thesis is devoted to the extensions of Theorems E, F and H. In particular, the following results are proven:

1. The continuous universality theorem of the Voronin type and its modification for the approximation of analytic functions by the shifts $\zeta(s+i\varphi(\tau), F)$ of zeta function attached to the normalized simultaneous Hecke-eigen cusp form and certain functions φ are true.

2. The discrete universality theorem of the Voronin type and its modification for the approximation of analytic functions by the shifts $\zeta(s+i\varphi(k), F)$ of zeta function attached to the normalized simultaneous Hecke-eigen cusp form and certain functions φ are true.

3. The joint discrete universality theorem of the Voronin type and its modification for the approximation of analytic functions by the set of shifts $\zeta(s+i\varphi_j(k), F_j)$ of zeta functions attached to the normalized simultaneous Hecke-eigen cusp forms and certain functions φ_j are true.

Methods

The results of the thesis are theoretical. Proof of main results is based on probabilistic methods of the analytic number theory. Similarly to the classical cases, proof contains two main steps: limit theorem for the function $\zeta(s, F)$ and properties of the support of the probability measure. The general approach combines the Euler product for $\zeta(s, F)$, a weak convergence of probability measures, and approximation in the mean and the Mergelyan theorem. In discrete cases, the Weyl criterion and Gallagher's lemma, connecting continuous and discrete mean square results, are applied. For proof of the modified theorems, the method introduced by A. Meška and A. Laurinčikas with the equivalent of weak convergence in terms of continuity sets is applied.

Novelty

All results of the thesis are new. They were primarily presented in publications and conferences related to this research. The results are theoretical and can be applied for further investigation of zeta functions of certain cusp forms as well as for other zeta functions.

Approbation

Results of this thesis were presented in the following seminar and conferences:

1. A. Vaiginytė. About the universality of the zeta functions of certain cusp forms, Seminar of the Department of Probability Theory and Number Theory at Vilnius University, Vilnius, 2018-05-28.
2. A. Vaiginytė. On the extension of universality for zeta-function of certain cusp forms, The 23th International Conference on Mathematical Modelling and Analysis (MMA2018), Sigulda, Latvia, 2018-05-31.
3. A. Vaiginytė. A class of sequences in the theory of the cusp forms, The 59th conference of Lithuanian Mathematical Society, Kaunas, 2018-06-18.
4. A. Vaiginytė. On some properties of zeta-function of certain cusp forms, International conference on Number Theory dedicated to the 70th birthdays of professors Antanas Laurinčikas and Eugenijus Manstavičius, Palanga, 2018-09-13.

5. A. Vaiginytė. A short review on the universality for zeta-function of certain cusp forms, The Sixth International Conference on Analytic Number Theory and Spatial Tessellations, Kiev, Ukraine, 2018-09-25.

6. A. Vaiginytė, A. Laurinčikas, D. Šiaučiūnas. On joint universality of zeta-functions of certain cusp forms, conference "Value distribution of zeta and L-functions and related topics", Tokyo, Japan, 2019-03-21 – 2019-03-27.

Principal publications of the author

1. A. Laurinčikas, D. Šiaučiūnas, A. Vaiginytė, Extension of the discrete universality theorem for zeta-functions of certain cusp forms, *Non-linear analysis: modelling and control*, 23(6), 2018, 961–973.

2. A. Vaiginytė, Extension of the Laurinčikas-Matsumoto theorem, *Chebyshevskii Sbornik*, 20(1), 2019, 82–93.

3. A. Laurinčikas, D. Šiaučiūnas, A. Vaiginytė, On joint approximation of analytic functions by non-linear shifts of zeta-functions of certain cusp forms, *Non-linear analysis: modelling and control*, 25(1), 2020, 108-125.

Outline of the thesis

This thesis consists of an Introduction, three Chapters, Conclusions, Bibliography and Notations. The Introduction contains a short review of the actuality of the research, the aims and problems of the thesis, and a brief overview of the methods used. Also, the history and issues associated with the study of the universality property and the recent results for $\zeta(s, F)$ are presented. The main defended results of the thesis, supported by the novelty aspect and an overview of the introduction of the results to the scientific community, concludes the Introduction. In Chapter 1, theorems about continuous approximation of the analytic functions by nonlinear shifts of $\zeta(s + i\varphi, F)$ are stated and proven. In Chapter 2, the discrete universality cases are presented and proven. Chapter 3 contains the statements and proofs of the simultaneous approximation of functions by a set of functions $\zeta(s, F_j)$. The Conclusions summarize the main results of the thesis and present open research areas for the future.

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1 Continuous Universality Theorems

Research has shown [28] that $\zeta(s+i\tau, F)$ can approximate a given analytic function when shifts τ are taken from a linear set. However, it was shown that more general shifts can be considered. The aim of this Chapter is the generalization of Laurinčikas-Matsumoto theorem (Theorem E) while taking shifts for the universality theorem from a certain class of functions $U(\tau_0)$.

1.1 Statements of the Theorems

We say that function $\varphi(\tau)$ belongs to the class $U(\tau_0)$, $\tau_0 > 0$, if the following conditions are satisfied:

1. $\varphi(\tau)$ is a differentiable real-valued positive increasing function on $[\tau_0, \infty)$;
2. $\varphi'(\tau)$ is monotonic and positive on $[\tau_0, \infty)$, satisfying $\frac{1}{\varphi'(\tau)} = o(\tau)$, $\tau \to \infty$;
3. $\varphi(2\tau) \max\limits_{\tau \leqslant t \leqslant 2\tau} \frac{1}{\varphi'(t)} \ll \tau, \quad \tau \to \infty.$

Let $D = D\left(\frac{\kappa}{2}, \frac{\kappa+1}{2}\right)$, $\mathcal{K} = \mathcal{K}_F$ be the class of compact subsets in the strip D with connected complements, and $H_0(K)$, $K \in \mathcal{K}$, stand for the class of continuous non-vanishing functions on K that are analytic in the interior of K.

Under such conditions, the results of this Chapter are the following statements:

Theorem 1.1.1. *Suppose that $\varphi(\tau) \in U(\tau_0)$, $K \in \mathcal{K}$, $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T-\tau_0} \operatorname{meas}\left\{\tau \in [\tau_0, T] : \sup_{s\in K}|\zeta(s+i\varphi(\tau), F) - f(s)|< \varepsilon\right\} > 0.$$

As we already know from [30], [31] and [42], the lower density in universality theorems can in some cases be replaced by the density. In particular, Theorem 1.1.1 has the following modification, which will be proved in this Chapter.

Theorem 1.1.2. *Suppose that $\varphi(\tau) \in U(\tau_0)$, $K \in \mathcal{K}$, $f(s) \in H_0(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T-\tau_0} \operatorname{meas}\left\{\tau \in [\tau_0, T] : \sup_{s\in K}|\zeta(s+i\varphi(\tau), F) - f(s)|< \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

In the following section, the probabilistic model used for the proof of Theorems 1.1.1 and 1.1.2 will be introduced.

1.2 Probabilistic Model

Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X, and by γ the unit circle on the complex plane. Define

$$\Omega = \prod_{p\in\mathbb{P}}^{\infty} \gamma_p,$$

where $\gamma_p = \gamma$ for all primes $p \in \mathbb{P}$.

By the Tikhonov theorem with product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological

Abelian group. Therefore, the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ can be defined, and so we have a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathbb{P}$, and on probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the $H(D)$-valued random element $\zeta(s, \omega, F)$ by the formula

$$\zeta(s, \omega, F) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}.$$

Denote by $P_{\zeta,F}$ the distribution of $\zeta(s, \omega, F)$, i.e.,

$$P_{\zeta,F}(A) = m_H\{\omega \in \Omega : \zeta(s, \omega, F) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

where $H(D)$ means the space of analytic functions on D endowed with the topology of uniform convergence on compacta.

Proof of the universality theorem is based on the weak convergence for

$$P_{T,F}(A) = \frac{1}{T - \tau_0} \operatorname{meas}\Big\{\tau \in [\tau_0, T] : \zeta(s + i\varphi(\tau), F) \in A\Big\},$$

where $A \in \mathcal{B}(H(D))$.

1.2.1 Auxiliary Lemmas

For convenience, we remind the definition of weak convergence and some related theorems from the measure theory.

Let $P_n, n \in \mathbb{N}$, and P be the probability measures on the probability space $(X, \mathcal{B}(X))$. We say that $P_n, n \to \infty$, converges weakly to P if, for every real continuous bounded function g on X,

$$\lim_{n \to \infty} \int_X g dP_n = \int_X g dP.$$

It is known that several equivalents of weak convergence exist. Here we state part of Theorem 2.1 from [4] that will be used for the proofs.

Lemma 1.2.1 (Equivalents of the weak convergence)**.** *Let $P_n, n \in \mathbb{N}$, and P be the probability measures on $(X, \mathcal{B}(X))$. The following statements are then equivalent:*

1. *P_n converges weakly to P as $n \to \infty$;*
2. *For every open set $G \subset X$,*

$$\liminf_{n \to \infty} P_n(G) \geqslant P(G);$$

3. *For every continuity set A of the measure P (A is a continuity set of P if $P(\partial A) = 0$, where ∂A is the boundary of A),*

$$\lim_{n \to \infty} P_n(A) = P(A).$$

Let S denote a metric space. Subclass $\mathcal{A} \subset S$ is called a separating class if two probability measures that agree on $\mathcal{A}$ necessarily agree also on the whole S. In such case, the values of $P(A)$ for $A \in \mathcal{A}$ are enough to separate measure P from all the other probability measures on S. $\mathcal{A}$ is a separating class if it is a π-system (closed under the formation of finite intersections), generating the σ-field of S.

Lemma 1.2.2 (Theorem 2.3 of [4])**.** *Suppose that $\mathcal{A}_P$ is a π-system and S is a separable space and such that for every $x \in S$ and $\varepsilon > 0$, there is $A \in \mathcal{A}_P$ which is contained in a ball with center in x and radius equal to ε. If $P_n(A) \xrightarrow[n \to \infty]{} P(A)$ for every $A \in \mathcal{A}_P$, then P_n weakly converges to P.*

Now let $\mathcal{P}$ be the family of probability measures in space $(S, \mathcal{B}(S))$; then, the following statement is true.

Lemma 1.2.3 (Prokhorov theorem)**.** *If $\mathcal{P}$ is tight then it is relatively compact (has a weakly convergent subsequence).*

The proof can be found in, for instance, Theorem 5.1 [4].

Let S_0 denote the ball σ-field of S, i.e., the one generated by the open balls. Then the following statement is true.

Lemma 1.2.4 (Theorem 6.1 in [4]). *Suppose P is a probability measure on (S, S_0), $A \in S_0$ is an open set, and $\varepsilon > 0$. Then there exists a closed set $H \in S_0$ and an open set $G \in S_0$ such that $H \subset A \subset G$ and $P(G \setminus H) < \varepsilon$.*

We will also use this property of the continuous functions.

Lemma 1.2.5. *Suppose that function $g : S_1 \to S_2$ is continuous. Then g is also $(\mathcal{B}(S_1), \mathcal{B}(S_2))$-measurable (preimage of any set in $\mathcal{B}(S_2)$ belong to $\mathcal{B}(S_1)$).*

Before the proof of the main theorems, here we remind the famous Mergelyan theorem on the approximation of analytic functions by polynomials.

Lemma 1.2.6 (Mergelyan theorem). *Let K be a compact subset of the complex plane $\mathbb{C}$ with a connected complement, and f is a function continuous on K and holomorphic at the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p = p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

In other words, every f can be approximated uniformly on K by polynomials. Proof of the lemma can be found in [41].

1.2.2 Limit Theorems

Now we state the main theorem that will lead to the proof of Theorem 1.1.1.

Theorem 1.2.7. *Suppose $\varphi(\tau) \in U(\tau_0)$. Then $P_{T,F}$ converges weakly to $P_{\zeta,F}$ as $T \to \infty$. Moreover, the support of $P_{\zeta,F}$ is the set*

$$S_F = \{g \in H(D) : g(s) \neq 0 \quad \textit{or} \quad g(s) \equiv 0\}.$$

We divide the proof of Theorem 1.2.7 to several lemmas. The first of them is a limit theorem on the torus Ω. For the proof of this lemma, properties of the function $\varphi(\tau)$ are needed.

For $A \in \mathcal{B}(\Omega)$, define

$$Q_T(A) = \frac{1}{T-\tau_0} \operatorname{meas} \Big\{ \tau \in [\tau_0, T] : (p^{-i\varphi(\tau)} : p \in \mathbb{P}) \in A \Big\}.$$

Lemma 1.2.8. *Suppose $\varphi(\tau) \in U(\tau_0)$. Then Q_T converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. For the proof, we will apply the Fourier transform method.

It is known that the dual group of Ω is isomorphic to the group

$$\mathcal{D} = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p,$$

where $\mathbb{Z}_p = \mathbb{Z}$ for all $p \in \mathbb{P}$. An element $\underline{k} = \{k_p : k_p \in \mathbb{Z}, p \in \mathbb{P}\}$ of $\mathcal{D}$, where only a finite number of integers k_p are distinct from zero, acts on Ω by

$$\omega \to \omega^{\underline{k}} = {\prod_{p \in \mathbb{P}}}' \omega^{k_p}(p)$$

where the sign " $'$ " means that only a finite number of integers k_p are distinct from zero.

Let $g_T(\underline{k}), \quad \underline{k} = (k_p : k_p \in \mathbb{Z}, \quad p \in \mathbb{P})$, be the Fourier transform of Q_T, i.e.,

$$g_T(\underline{k}) = \int_{\Omega} \left({\prod_{p \in \mathbb{P}}}' \omega^{k_p}(p) \right) dQ_T.$$

Thus, from the definition of Q_T, we have

$$
\begin{aligned}
g_T(\underline{k}) &= \frac{1}{T-\tau_0}\int\limits_{\tau_0}^{T}\left(\prod_{p\in\mathbb{P}}{}' p^{-ik_p\varphi(\tau)}\right)d\tau \\
&= \frac{1}{T-\tau_0}\int\limits_{\tau_0}^{T}\exp\left\{-i\varphi(\tau)\sum_{p\in\mathbb{P}}{}' k_p\log p\right\}d\tau.
\end{aligned} \tag{1.1}
$$

Obviously,

$$
g_T(\underline{0}) = 1. \tag{1.2}
$$

Since the elements of the set $\{\log p : p \in \mathbb{P}\}$ are linearly independent over the field of rational numbers $\mathbb{Q}$, we have that

$$
a := \sum_{p\in\mathbb{P}}{}' k_p \log p \neq 0
$$

for all $\underline{k} \neq \underline{0}$.

Suppose that $\varphi'(\tau)$ is decreasing. Then, $\frac{1}{\varphi'(\tau)}$ is increasing, and, therefore,

$$
\int\limits_{\tau_0}^{T}\exp\{-ia\varphi(\tau)\}\,d\tau = \int\limits_{\tau_0}^{T}\cos(a\varphi(\tau))d\tau - i\int\limits_{\tau_0}^{T}\sin(a\varphi(\tau))d\tau. \tag{1.3}
$$

Thus, by the mean value theorem,

$$
\int\limits_{\tau_0}^{T}\cos(a\varphi(\tau))d\tau = \frac{1}{a}\int\limits_{\tau_0}^{T}\frac{a\,\varphi'(\tau)\cos(a\varphi(\tau))}{\varphi'(\tau)}d\tau
$$

$$
= \frac{1}{a\,\varphi'(T)}\int\limits_{\xi}^{T} a\,\varphi'(\tau)\cos(a\varphi(\tau))d\tau = \frac{1}{a\,\varphi'(T)}\int\limits_{\xi}^{T} d\sin(a\varphi(\tau)) = o(T),
$$

as $T \to \infty$ and $\tau_0 \leqslant \xi \leqslant T$, is true for the second integral in (1.3).

Therefore, by (1.3),

$$\int_{\tau_0}^{T} \exp\{-ia\varphi(\tau)\}\, d\tau = o(T), \qquad \text{as} \quad T \to \infty. \tag{1.4}$$

Similarly, if $\varphi'(\tau)$ is increasing, then

$$\int_{\tau_0}^{T} \exp\{-ia\varphi(\tau)\}\, d\tau \ll \frac{1}{|a|\varphi(\tau_0)}. \tag{1.5}$$

From (1.4) and (1.5) together with (1.1), we get that

$$\lim_{T\to\infty} g_T(\underline{k}) = 0,$$

whenever $\underline{k} \neq \underline{0}$. Therefore, with respect to (1.2),

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases} \tag{1.6}$$

On the right-hand side of the equality we see the Fourier transformation of the Haar measure m_H. Therefore, the lemma is proved using the continuity theorem for probability measures on compact groups. □

Lemma 1.2.8 implies a limit theorem for probability measures on the space of analytic functions defined by means of absolutely convergent Dirichlet series. This theorem is quite standard but plays an important role in further proof.

Now, some absolutely convergent Dirichlet series will be analysed. Let $\theta > \frac{1}{2}$ be a fixed number, and $m, n \in \mathbb{N}$. We define series

$$\zeta_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^s}$$

and

$$\zeta_n(s, \omega, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega(m)v_n(m)}{m^s},$$

where

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\theta}\right\} \quad \text{and} \quad \omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

The latter series are absolutely convergent for $\sigma > \kappa/2$ [28]. Define the function $u_{n,F} : \Omega \to H(D)$ by the formula $u_{n,F}(\omega) = \zeta_n(s, \omega, F)$. Due to the absolute convergence of $\zeta_n(s, \omega, F)$, we have that the function $u_{n,F}(\omega)$ is continuous and $(\mathcal{B}(\Omega), \mathcal{B}(H(D)))$-measurable. Therefore, the Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ induces the unique probability measure $\hat{P}_{n,F}$ on $(H(D), \mathcal{B}(H(D)))$ defined by

$$\hat{P}_{n,F}(A) = m_H u_{n,F}^{-1}(A) = m_H(u_{n,F}^{-1} A), \quad A \in \mathcal{B}(H(D)).$$

Lemma 1.2.9. *Suppose $\varphi(\tau) \in U(\tau_0)$. Then*

$$P_{T,n,F} := \frac{1}{T - \tau_0} \operatorname{meas}\left\{\tau \in [\tau_0, T] : \zeta_n(s + i\varphi(\tau), F) \in A\right\}, A \in \mathcal{B}(H(D))),$$

converges weakly to $\hat{P}_{n,F}$ as $T \to \infty$.

Proof of Lemma 1.2.9 comes with standard arguments from Lemma 1.2.8 and the continuity of the function $u_{n,F}$.

1.2.3 Approximation in the Mean

Our aim is to prove that $P_{T,F}$ converges weakly to the limit measure P_F of the measure $\hat{P}_{n,F}$, as $n \to \infty$. For the proof of Theorem 1.2.7, approximation in the mean of $\zeta(s, F)$ by $\zeta_n(s, F)$ is used.

Lemma 1.2.10. *Suppose that $\varphi(\tau) \in U(\tau_0)$, and σ, $\kappa/2 < \sigma < (\kappa + 1)/2$, is fixed. Then for all $t \in \mathbb{R}$*

$$\int_{\tau_0}^{T} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\tau \ll T(1 + |t|).$$

Proof. It is known that, for fixed σ, $\kappa/2 < \sigma < (\kappa+1)/2$, we have

$$\int_{\tau_0}^{T} |\zeta(\sigma+it,F)|^2 dt \ll T. \tag{1.7}$$

For $X > \tau_0$, we get

$$\begin{aligned}
\int_X^{2X} |\zeta(\sigma+it+i\varphi(\tau),F)|^2 d\tau &= \int_X^{2X} \frac{1}{\varphi'(\tau)} |\zeta(\sigma+it+i\varphi(\tau),F)|^2 d\varphi(\tau) \\
&\ll \max_{X\leqslant\tau\leqslant 2X} \frac{1}{\varphi'(\tau)} \int_X^{2X} d\left(\int_0^{|t|+\varphi(\tau)} |\zeta(\sigma+iu,F)|^2 du\right) \\
&= \max_{X\leqslant\tau\leqslant 2X} \frac{1}{\varphi'(\tau)} \left(\int_0^{|t|+\varphi(\tau)} |\zeta(\sigma+iu,F)|^2 du\right)\Bigg|_X^{2X}.
\end{aligned}$$

Consequently,

$$\int_0^{|t|+\varphi(\tau)} |\zeta(\sigma+iu,F)|^2 du\Big|_X^{2X} \ll_\sigma |t|+\varphi(2X),$$

and thus,

$$\begin{aligned}
\int_X^{2X} |\zeta(\sigma+it+i\varphi(\tau),F)|^2 d\tau &\ll (|t|+\varphi(2X)) \max_{X\leqslant\tau\leqslant 2X} \frac{1}{\varphi'(\tau)} \\
&\ll X + |t| \max_{X\leqslant\tau\leqslant 2X} \frac{1}{\varphi'(\tau)} \ll X(1+|t|).
\end{aligned}$$

Taking $X = 2^{-k-1}T$ and summing over $k = 0, 1, ..$ prove the lemma. □

Now we can approximate $\zeta(s,F)$ by $\zeta_n(s,F)$ in the mean. For $g_1, g_2 \in H(D)$, take

$$\rho(g_1,g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}, \tag{1.8}$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then ρ is the metric in $H(D)$, inducing its topology of uniform convergence on compacta.

Lemma 1.2.11. *Suppose that $\varphi(\tau) \in U(\tau_0)$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-\tau_0} \int_{\tau_0}^{T} \rho\Big(\zeta(s+i\varphi(\tau), F), \zeta_n(s+i\varphi(\tau), F)\Big) d\tau = 0.$$

Proof. Let θ be as in the definition of $v_n(m)$, and

$$l_n(s) = \frac{s}{\theta}\Gamma\Big(\frac{s}{\theta}\Big)n^s, \quad n \in \mathbb{N},$$

where $\Gamma(s)$ denotes the Euler gamma-function. Then, as we know from [28], the function $\zeta_n(s, F)$ has the representation

$$\zeta_n(s, F) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s+z, F) l_n(z) \frac{dz}{z}, \quad \sigma > \kappa/2. \tag{1.9}$$

Let K be an arbitrary compact subset of D. Then, from the residue theorem and the above equality, we get

$$\frac{1}{T-\tau_0} \int_{\tau_0}^{T} \sup_{s\in K} \Big(\zeta(s+i\varphi(\tau), F), \zeta_n(s+i\varphi(\tau), F)\Big) d\tau$$
$$\ll \int_{-\infty}^{\infty} |l_n(\hat{\sigma}+iu)| \left(\frac{1}{T-\tau_0} \int_{\tau_0}^{T} |\zeta(s+it+iu+i\varphi(\tau), F)| d\tau \right) du,$$

as $T \to \infty$, where $\hat{\sigma} < 0$, $\frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}$, and t is bounded by a constant depending on K. Lemma 1.2.10 implies that, with $t \in \mathbb{R}$, for

$\frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}$,

$$\int_{\tau_0}^{T} |\zeta(s+it+i\varphi(\tau), F)| d\tau \ll \Big(T \int_{\tau_0}^{T} |\zeta(s+i\varphi(\tau), F)| d\tau\Big)^{1/2} \ll_\sigma T(1+|t|).$$

Therefore,

$$\frac{1}{T-\tau_0} \int_{\tau_0}^{T} \sup_{s\,\in K} \Big(\zeta(s+i\varphi(\tau), F), \zeta_n(s+i\varphi(\tau), F)\Big) d\tau$$
$$\ll_{\sigma,K} \int_{-\infty}^{\infty} |l_n(\hat{\sigma}+iu)|(1+|t|),$$

as $T \to \infty$. Hence,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-\tau_0} \int_{\tau_0}^{T} \sup_{s\in K} \Big(\zeta(s+i\varphi(\tau), F), \zeta_n(s+i\varphi(\tau), F)\Big) d\tau = 0.$$

So, the lemma follows from the definition of the metric ρ. □

1.2.4 Proof of the Weak Convergence Theorem

Proof of Theorem 1.2.7. Let θ be a random variable uniformly distributed on $[0,1]$ and defined on a certain probability space with measure μ. Define the $H(D)$-valued random element $X_{T,n,F}$ by the formula

$$X_{T,n,F} = X_{T,n,F}(s) = \zeta_n(s+i\varphi(\theta T), F).$$

Then the assertion of Lemma 1.2.9 can be written as

$$X_{T,n,F} \xrightarrow[T\to\infty]{\mathcal{D}} \hat{X}_{n,F}, \tag{1.10}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution, and $\hat{X}_{n,F}$ is the $H(D)$-valued random element with the distribution $\hat{P}_{n,F}$. Here $\hat{P}_{n,F}$ is the same limit probability measure as in Lemma 1.2.9.

Now we will prove that the family $\{\hat{P}_{n,F} : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D)$ such that $\hat{X}_{n,F}(K) > 1 - \varepsilon$ for all $n \in \mathbb{N}$. We will apply the method used in [20].

Let $K \subset D$ be a compact set. Then, by the integral Cauchy formula,

$$\sup_{s\in K}|\zeta(s+i\varphi(\tau),F)| \ll \frac{1}{\delta_K}\int_{L_K}|\zeta(z+i\varphi(\tau),F)||dz|,$$

where L_K is a simple closed contour lying in D and enclosing the set K, and δ_K is the distance of L_K from the set K. Hence,

$$\int_{\tau_0}^{T}\sup_{s\in K}|\zeta(s+i\varphi(\tau),F)|d\tau$$
$$\ll \frac{1}{\delta_K}\int_{L_K}|dz|\int_{\tau_0}^{T}|\zeta(\mathrm{Re}(z)+\mathrm{Im}(z)+i\varphi(\tau),F)|d\tau \ll_K T.$$

This with Lemma 1.2.11 show that

$$\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T-\tau_0}\int_{\tau_0}^{T}\sup_{s\in K_l}|\zeta_n(s+i\varphi(\tau),F)|d\tau \leqslant C_l < \infty, \qquad (1.11)$$

where $\{K_l : l \in \mathbb{N}\}$ is the sequence of compact subsets of D from the definition of metric ρ.

Now let ε be an arbitrary positive number, and $M_l = M_l(\varepsilon) = C_l 2^l \varepsilon^{-1}$. Then, from (1.11), we have

$$\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\mu\{\sup_{s\in K_l}|X_{T,n,F}(s)|> \varepsilon\}$$
$$\leqslant \sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T-\tau_0}\int_{\tau_0}^{T}\sup_{s\in K_l}|\zeta_n(s+i\varphi(\tau),F)|d\tau \leqslant \frac{\varepsilon}{2},$$

and, by (1.10),

$$\mu\{\sup_{s\in K_l}|X_{n,F}(s)|> \varepsilon\} \leqslant \frac{\varepsilon}{2^l} \qquad (1.12)$$

for all $n \in \mathbb{N}$. Define the set

$$K = K(\varepsilon) = \{g \in H(D) : \quad \sup_{s \in K_l} |g(s)| \leqslant M_l, \quad l \in \mathbb{N}\}.$$

Then K is a compact set in $H(D)$, and, by (1.12),

$$\mu\{X_{n,F} \in K\} \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$, or, by definition of $X_{n,F}$,

$$\hat{P}_{n,F}(K) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$; thus, the family $\{\hat{P}_{n,F} : n \in \mathbb{N}\}$ is tight. Therefore, by the Prokhorov theorem (see Lemma 1.2.3), it is relatively compact, i.e., every sequence of $\{\hat{P}_{n,F}\}$ contains a weakly convergent subsequence. Thus, there exists $\{\hat{P}_{n_r,F}\} \subset \{\hat{P}_{n,F}\}$ such that $\{\hat{P}_{n_r,F}\}$ converges weakly to a certain probability measure P_F on $(H(D), \mathcal{B}(H(D)))$, as $r \to \infty$. In terms of convergence in distribution, we say

$$\hat{X}_{n_r,F} \xrightarrow[r \to \infty]{\mathcal{D}} P_F. \tag{1.13}$$

Define one more $H(D)$-valued random element

$$X_{T,F} = X_{T,F}(s) = \zeta(s + \varphi(\theta T), F).$$

Then, in view of Lemma 1.2.11, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n \to \infty} \limsup_{T \to \infty} \mu\{\rho(X_{T,F}, X_{T,n,F}| \geqslant \varepsilon\} \\
&= \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T - \tau_0} \operatorname{meas} \Big\{\tau \in [\tau_0, T] : \rho(\zeta(s + i\varphi(\tau), F), \\
&\zeta_n(s + i\varphi(\tau), F)) \geqslant \varepsilon\Big\} \\
&\leqslant \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{(T - \tau_0)\varepsilon} \int_{\tau_0}^{T} \rho(\zeta(s + i\varphi(\tau), F), \zeta_n(s + i\varphi(\tau), F)) d\tau = 0.
\end{aligned}$$

This, together with (1.10) and (1.13), show that all hypotheses of Theorem 4.2 of [4] are fulfilled, and therefore,

$$X_{T,F} \xrightarrow[T\to\infty]{\mathcal{D}} P_F,$$

or $P_{T,F}$ converges weakly to the limit measure P_F of $\hat{P}_{n,F}$, as $T \to \infty$.

The final step is to identify the measure P_F. For this, we will use simple observation. It is known [7], [20] that

$$\frac{1}{T}\operatorname{meas}\Big\{\tau \in [0,T] : \zeta(s+i\tau, F) \in A\Big\}, \qquad A \in \mathcal{B}(H(D))),$$

as $T \to \infty$, converges weakly to the limit measure P_F of $\hat{P}_{n,F}$, and that $P_F = P_{\zeta,F}$. Moreover, the support of $P_{\zeta,F}$ is the set S_F. Therefore, $P_{T,F}$ also converges weakly to $P_{\zeta,F}$ as $T \to \infty$. □

1.3 Proof of the Universality Theorems

Proof of Theorem 1.1.1. Define the set

$$G_\varepsilon = \left\{ g \in H(D) : \quad \sup_{s\in K} |g(s) - e^{p(s)}| < \frac{\varepsilon}{2} \right\},$$

where $p(s)$ is a polynomial satisfying

$$\sup_{s\in K} |f(s) - e^{p(s)}| < \frac{\varepsilon}{2}. \tag{1.14}$$

The existence of $p(s)$ follows from the Mergelyan theorem (Lemma 1.2.6).

By the second part of Theorem 1.2.7, the function $e^{p(s)}$ belongs to the support of the measure $P_{\zeta,F}$. Therefore,

$$P_{\zeta,F}(G_\varepsilon) > 0. \tag{1.15}$$

Since G_ε is an open set, by the first part of Theorem 1.2.7 and the equivalent of weak convergence of probability measures in terms of open sets (Lemma 1.2.1), we have that

$$\liminf_{T\to\infty} P_{T,F}(G_\varepsilon) \geqslant P_{\zeta,F}(G_\varepsilon).$$

This, the definition of $P_{T,F}$ and inequality (1.15) give

$$\liminf_{T\to\infty} \frac{1}{T-\tau_0} \operatorname{meas}\left\{\tau \in [\tau_0, T] : \sup_{s\in K} |\zeta(s+i\varphi(\tau), F) - e^{p(s)}| < \frac{\varepsilon}{2}\right\} > 0.$$

This together with (1.14) proves the theorem. □

Proof of Theorem 1.1.2. Define the set

$$\hat{G}_\varepsilon = \left\{ g \in H(D) : \quad \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then the boundary $\partial\hat{G}_\varepsilon$ of $\hat{G}_\varepsilon$ lies in the set

$$\left\{ g \in H(D) : \quad \sup_{s\in K} |g(s) - f(s)| = \varepsilon \right\}.$$

Therefore, $\partial\hat{G}_{\varepsilon_1} \cap \partial\hat{G}_{\varepsilon_2} = \emptyset$ for $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_1, \varepsilon_2 > 0$. Hence, for at most countably many $\varepsilon > 0$, the sets $\partial\hat{G}_\varepsilon$ have a positive $P_{\zeta,F}$ measure. Using Theorem 1.2.7 and the equivalent of the weak convergence of probability measures in terms of continuity sets (Lemma 1.2.1), we obtain that

$$\lim_{T\to\infty} P_{T,F}(\hat{G}_\varepsilon) = P_{\zeta,F}(\hat{G}_\varepsilon) \tag{1.16}$$

for all but at most countably many $\varepsilon > 0$. Let G_ε be from the proof of Theorem 1.1.1. Then, in view of (1.14), we obtain that $G_\varepsilon \subset \hat{G}_\varepsilon$, and thus, by (1.15), $P_{\zeta,F}(\hat{G}_\varepsilon) > 0$. This, the definition of $P_{T,F}$, and (1.16) prove the theorem. □

2 Discrete Universality Theorems

From Theorem F we have seen that discrete shifts of zeta functions can also be used for the approximation of analytic functions. The aim of this Chapter is to prove a discrete universality theorem for the function $\zeta(s, F)$ when τ in $\zeta(s + i\tau, F)$ runs over some general discrete sequence of real numbers. The statement of this Chapter contain the corrected results presented in [36].

2.1 Statements of the Theorems

For the definition of a class of sequences for τ, we will use similar conditions as in Chapter 1 and the notion of uniform distribution modulo 1. Let $\{u\}$ denote the fractional part of $u \in \mathbb{R}$, and let χ_I be the indicator function of the set I. We remind that the sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for every interval $I = [a, b) \subset [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I(\{x_k\}) = b - a.$$

Let $k_0 \in \mathbb{N}$. We say that a function $\varphi \in U(k_0)$, $k_0 > 0$, if the following conditions are satisfied:

1. $\varphi(\tau)$ is a differentiable real-valued positive increasing function on $[k_0 - \frac{1}{2}, \infty)$;

2. $\varphi'(\tau)$ satisfy the estimate

$$\varphi(2\tau)\left(\max_{\tau\leqslant t\leqslant 2\tau}\frac{1}{\varphi'(t)}+\max_{\tau\leqslant t\leqslant 2\tau}\varphi'(t)\right)\ll\tau,\quad \tau\to\infty;$$

3. The sequence $\{a\varphi(k):k\geqslant k_0\}\subset\mathbb{R}$ with every $a\in\mathbb{R}\setminus\{0\}$ is uniformly distributed modulo 1.

Let $D=D_F=D\left(\frac{\kappa}{2},\frac{\kappa+1}{2}\right)$, $\mathcal{K}=\mathcal{K}_F$ be the class of compact subsets in the strip D with connected complements, and $H_0(K)$, $K\in\mathcal{K}$, stand for the class of continuous non-vanishing functions on K that are analytic in the interior of K.

Under these conditions, the main results of this Chapter are the following two theorems.

Theorem 2.1.1. *Suppose that $\varphi\in U(k_0)$, $K\in\mathcal{K}_F$, $\in H_0(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N-k_0+1}\#\left\{k_0\leqslant k\leqslant N:\sup_{s\in K}|\zeta(s+i\varphi(k),F)-f(s)|<\varepsilon\right\}>0.$$

Similarly to the continuous case, Theorem 2.1.1 has the following modification which will be proved in this Chapter.

Theorem 2.1.2. *Suppose that $\varphi\in U(k_0)$, $K\in\mathcal{K}_F$, $\in H_0(K)$. Then the limit*

$$\lim_{N\to\infty}\frac{1}{N-k_0+1}\#\left\{k_0\leqslant k\leqslant N:\sup_{s\in K}|\zeta(s+i\varphi(k),F)-f(s)|<\varepsilon\right\}>0$$

exists for all but at most countably many $\varepsilon>0$.

2.2 Probabilistic Model

We will follow a similar approach as in the continuous case. As in Chapter 1,

$$\Omega = \prod_{p\in\mathbb{P}}^{\infty} \gamma_p$$

stands for a torus defined by the product of complex unit circles γ_p, $\mathcal{B}(\Omega)$ is the Borel σ-field of the space Ω, and $(\Omega, \mathcal{B}(\Omega), m_H)$ stands for a probability space with the Haar measure m_H.

Again, denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathbb{P}$, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the $H(D)$-valued random element $\zeta(s, \omega, F)$ by the formula

$$\zeta(s, \omega, F) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}.$$

Denote by $P_{\zeta,F}$, the distribution of $\zeta(s, \omega, F)$, i.e.,

$$P_{\zeta,F}(A) = m_H\{\omega \in \Omega : \zeta(s, \omega, F) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

where $H(D)$ means the space of analytic functions on D endowed with the topology of uniform convergence on compacta. As we know from Theorem 1.2.7, the support of $P_{\zeta,F}$ is the set

$$S_F = \{g \in H(D) : g(s) \neq 0 \quad \text{or} \quad g(s) \equiv 0\}.$$

The main result of this section is the weak convergence for the following measure, as $N \to \infty$:

$$P_{N,F}(A) = \frac{1}{N - k_0 + 1} \#\Big\{k_0 \leqslant k \leqslant N : \zeta(s + i\varphi(k), F) \in A\Big\},$$

where $A \in \mathcal{B}(H(D))$.

Theorem 2.2.1. *Suppose that $\varphi(k) \in U(k_0)$. Then $P_{N,F}$ converges weakly to $P_{\zeta,F}$, as $N \to \infty$.*

We separate the proof of Theorem 2.2.1 into several lemmas. We start with the Weyl criterion on distribution modulo 1 that will noticeably facilitate our proof.

2.2.1 Limit Theorems

Lemma 2.2.2 (Weyl criterion)**.** *A sequence* $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ *is uniformly distributed modulo* 1 *if, and only if, for all* $m \in \mathbb{Z} \setminus \{0\}$ *,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i m x_k} = 0.$$

Proof of the lemma can be found, for example, in [18].

For $A \in \mathcal{B}(\Omega)$, define

$$Q_N(A) = \frac{1}{N - k_0 + 1} \#\Big\{k_0 \leqslant k \leqslant N : (p^{-i\varphi(k)} : p \in \mathbb{P}) \in A\Big\}.$$

Lemma 2.2.3. *Suppose* $\varphi \in U(k_0)$*. Then* Q_N *converges weakly to the Haar measure* m_H *as* $N \to \infty$*.*

Proof. For the proof, we will apply the Fourier transform method. The Fourier transform $g_N(\underline{k})$ of Q_N with $\underline{k} = \{k_p : k_p \in \mathbb{Z}, p \in \mathbb{P}\}$ is given by the formula

$$g_N(\underline{k}) = \int\limits_{\Omega} \left({\prod_{p\in\mathbb{P}}}' \omega^{k_p}(p) \right) dQ_N,$$

where the sign " $'$ " means that only a finite number of k_p are distinct from zero. Thus, from the definition of Q_N, we have

$$\begin{aligned} g_N(\underline{k}) &= \frac{1}{N - k_0 + 1} \sum_{k=k_0}^{N} {\prod_{p\in\mathbb{P}}}' p^{-ik_p\varphi(k)} \\ &= \frac{1}{N - k_0 + 1} \sum_{k=k_0}^{N} \exp\left\{ -i\varphi(k) {\sum_{p\in\mathbb{P}}}' k_p \log p \right\}. \end{aligned} \tag{2.1}$$

Obviously,

$$g_N(\underline{0}) = 1. \tag{2.2}$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$, we have that $\sum'_{p\in\mathbb{P}} k_p \log p \neq 0$ for all $\underline{k} \neq \underline{0}$. Therefore, since $\varphi \in U(k_0)$, in the case $\underline{k} \neq \underline{0}$, the sequence

$$\left\{ \frac{\varphi(k)}{2\pi} \sum_{p\in\mathbb{P}}{}' k_p \log p : k \geqslant k_0 \right\}$$

is uniformly distributed modulo 1. Thus, by the Weyl criterion (Lemma 2.2.2), with $m = -1$ and (2.1), we find that, for $\underline{k} \neq \underline{0}$,

$$\lim_{N\to\infty} g_N(\underline{k}) = 0.$$

This and (2.2) show that $g_N(\underline{k})$, as $N \to \infty$, converges to the Fourier transform of the Haar measure m_H (see equality (1.6) in Chapter 1), and the lemma is the consequence of a continuity theorem for probability measures on compact groups. $\square$

Lemma 2.2.3 implies a limit theorem in the space of analytic functions for a certain absolutely convergent Dirichlet series. This theorem is very important to proving Theorem 2.2.1.

Similarly to the continuous case, we extend the functions $\omega(p), p \in \mathbb{P}$, to the set $\mathbb{N}$ by

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Let $\theta > \frac{1}{2}$ be a fixed number, and $m, n \in \mathbb{N}$. We define the series

$$\zeta_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^s} \quad \text{and} \quad \zeta_n(s, \omega, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega(m)v_n(m)}{m^s},$$

where

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\theta} \right\}.$$

The latter series are absolutely convergent for $\sigma > \kappa/2$. Define the function $u_{n,F} : \Omega \to H(D)$ by the formula $u_{n,F}(\omega) = \zeta_n(s, \omega, F)$. Due to the absolute convergence of $\zeta_n(s, \omega, F)$, we have that the function $u_{n,F}(\omega)$ is continuous and $(\mathcal{B}(\Omega), \mathcal{B}(H(D)))$-measurable (see Lemma 1.2.5 in Chapter 1). Therefore, the Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ induces the unique probability measure $\hat{P}_{n,F}$ on $(H(D), \mathcal{B}(H(D)))$ defined by

$$\hat{P}_{n,F}(A) = m_H u_{n,F}^{-1}(A) = m_H(u_{n,F}^{-1}A), \qquad A \in \mathcal{B}(H(D)).$$

For $A \in \mathcal{B}(H(D))$, define

$$P_{N,n,F}(A) = \frac{1}{N - k_0 + 1} \# \Big\{ k_0 \leqslant k \leqslant N : \zeta_n(s + i\varphi(k), F)) \in A \Big\}.$$

The Prokhorov theorem (see Lemma 1.2.3), Lemma 2.2.3 and the above remarks lead to the following.

Lemma 2.2.4. *Suppose $\varphi \in U(k_0)$. Then $P_{N,n,F}$ converges weakly to $\hat{P}_{n,F}$ as $N \to \infty$.*

2.2.2 Discrete Mean Square Estimates

Our next aim is to prove that $P_{N,F}$ converges weakly to the limit measure P_F of the measure $\hat{P}_{n,F}$, as $n \to \infty$. For this, approximation in the mean of $\zeta(s, F)$ by $\zeta_n(s, F)$ is used. Thus, the following estimate of the mean square is needed.

Lemma 2.2.5. *Suppose $\varphi \in U(k_0)$, and σ, $\kappa/2 < \sigma < (\kappa + 1)/2$, is fixed. Then, for all $t \in \mathbb{R}$,*

$$\int_{k_0 - 1/2}^{T} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\tau \ll T(1 + |t|).$$

Proof. It is known (see (1.7)) that, for fixed σ, $\kappa/2 < \sigma < (\kappa + 1)/2$, we have

$$\int_{\tau_0}^{T} |\zeta(\sigma + it, F)|^2 dt \ll T.$$

Let $X > 1$. Since the function φ is increasing and continuously differentiable, we have that

$$\int_X^{2X} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\tau = \int_X^{2X} \frac{1}{\varphi'(\tau)} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\varphi(\tau)$$

$$\ll \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\varphi'(\tau)} \int_X^{2X} d \left(\int_0^{|t|+\varphi(\tau)} |\zeta(\sigma + iu, F)|^2 du \right)$$

$$= \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\varphi'(\tau)} \left(\int_0^{|t|+\varphi(\tau)} |\zeta(\sigma + iu, F)|^2 du \right) \Bigg|_X^{2X}.$$

By estimate (1.7), we have

$$\int_0^{|t|+\varphi(\tau)} |\zeta(\sigma + iu, F)|^2 du \Big|_X^{2X} \ll_\sigma |t| + \varphi(2X),$$

and thus,

$$\int_X^{2X} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\tau \ll (|t| + \varphi(2X)) \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\varphi'(\tau)}$$

$$\ll X + |t| \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\varphi'(\tau)} \ll X(1 + |t|).$$

Taking $X = 2^{-k-1}T$ and summing over $k = 0, 1, ..$ proves the lemma. □

For the estimate of the discrete mean square

$$I_N(\sigma, t, F) = \sum_{k=k_0}^{N} |\zeta(\sigma + it + i\varphi(k), F)|^2,$$

we need to apply both Lemma 2.2.5 and Gallagher's lemma, which connects the continuous and discrete mean squares of some functions.

For convenience, we state the Gallagher's lemma (see Lemma 1.4 in

[45]).

Lemma 2.2.6 (Gallagher's lemma)**.** *Suppose that $T_0, T \geqslant \delta > 0$ are real numbers and $\mathcal{T} \neq \emptyset$ is a finite set in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1.$$

Let $S(x)$ be a complex-valued continuous function on $[T_0, T_0 + T]$ with a continuous derivative on $(T_0, T_0 + T)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leqslant \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(x)|^2 dx + \left(\int_{T_0}^{T_0+T} |S(x)|^2 dx \int_{T_0}^{T_0+T} |S'(x)|^2 dx \right)^{1/2}.$$

Lemma 2.2.7. *Suppose $\varphi \in U(k_0)$, and σ, $\kappa/2 < \sigma < (\kappa+1)/2$, is fixed. Then for all $t \in \mathbb{R}$*

$$I_N(\sigma, t, F) \ll N(1 + |t|).$$

Proof. For proof of the lemma, we will apply Lemma 2.2.5 together with the Cauchy integral formula for the derivative of $\zeta(s, F)$.

We take a circle $L \subset D$ with a center σ. In view of the Cauchy integral formula,

$$\zeta'(\sigma + it + i\varphi(\tau), F) = \frac{1}{2\pi i} \int_L \frac{\zeta(z + it + i\varphi(\tau), F)}{(z-\sigma)^2} dz.$$

Accordingly,

$$|\zeta'(\sigma + it + i\varphi(\tau), F)|^2 = \frac{1}{4\pi^2} \left| \int_L \frac{\zeta(z + it + i\varphi(\tau), F)}{(z-\sigma)^2} dz \right|^2$$

$$\ll \int_L \frac{|dz|}{|z-\sigma|^4} \int_L |\zeta(z + it + i\varphi(\tau), F)|^2 |dz| \ll \int_L |\zeta(z + it + i\varphi(\tau), F)|^2 |dz|.$$

With respect to the Lemma 2.2.5, we have

$$\int\limits_{k_0-1/2}^{N+1/2} |\,\zeta'(\sigma+it+i\varphi(\tau),F)|^2 d\tau$$

$$\ll \int\limits_{L} |dz| \int\limits_{k_0-1/2}^{N+1/2} \zeta(\operatorname{Re} z + i\operatorname{Im} z + it + i\varphi(\tau),F)|^2 d\tau \ll N(1+|t|). \tag{2.3}$$

Similarly, we obtain

$$\begin{aligned}
&\int\limits_{X}^{2X} (\,\varphi'(\tau))^2 |\,\zeta'(\sigma+it+i\varphi(\tau),F)|^2 d\tau \\
&= \int\limits_{X}^{2X} \varphi'(\tau)|\,\zeta'(\sigma+it+i\varphi(\tau),F)|^2 d\varphi(\tau) \\
&\ll \max_{X\leqslant\tau\leqslant 2X} \varphi'(\tau) \int\limits_{X}^{2X} d\left(\int\limits_{0}^{|t|+\varphi(\tau)} |\zeta(\sigma+iu,F)|^2 du \right) \\
&\ll (|t|+\varphi(2X)) \max_{X\leqslant\tau\leqslant 2X} \varphi'(\tau) \ll X(1+|t|).
\end{aligned}$$

Therefore, for $\kappa/2 < \sigma < (\kappa+1)/2$, and $t \in \mathbb{R}$, we have

$$\int\limits_{k_0-1/2}^{T} (\,\varphi'(\tau))^2 |\zeta(\sigma+it+i\varphi(\tau),F)|^2 d\tau \ll T(1+|t|) \tag{2.4}$$

Further, we apply Gallagher's lemma with $\mathcal{T} = \{k : k \in \mathbb{N}, k_0 \leqslant k \leqslant N\}$, $\quad T_0 = k_0 - 1/2$, $\quad T = N - k_0 + 1$, and $\delta = 1$. Then we get $N_\delta(x) = 1$.

Taking $S(\tau) = \zeta(\sigma + it + i\varphi(\tau), F)$, we have

$$
\begin{aligned}
I_N(\sigma, t, F) \ll & \int\limits_{k_0-1/2}^{N+1/2} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\tau \\
& + \left(\int\limits_{k_0-1/2}^{N+1/2} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\tau \right. \\
& \left. \times \int\limits_{k_0-1/2}^{N+1/2} (\varphi'(\tau))^2 |\zeta'(\sigma + it + i\varphi(\tau), F)|^2 d\tau \right)^{1/2}.
\end{aligned}
$$

This, Lemma 2.2.5, and estimates (2.3), (2.4) prove the lemma. □

2.2.3 Approximation in the mean

Now we will approximate $\zeta(s, F)$ by $\zeta_n(s, F)$ in the mean. Let metric ρ be defined and satisfy the same conditions as in (1.8) in Chapter 1.

Lemma 2.2.8. *Suppose $\varphi \in U(k_0)$. Then*

$$
\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N - k_0 + 1} \sum_{k=k_0}^{N} \rho\Big(\zeta(s + i\varphi(k), F), \zeta_n(s + i\varphi(k), F)\Big) = 0.
$$

Proof. Let $\theta > 1/2$ as in the definition of $v_n(m)$, and

$$
l_n(s) = \frac{s}{\theta}\Gamma\Big(\frac{s}{\theta}\Big)n^s, \quad n \in \mathbb{N}.
$$

Then, the function $\zeta_n(s, F)$ has the representation (1.9).

Let K be an arbitrary compact subset of D. Then, from the residue theorem and the above equality, we get

$$\frac{1}{N-k_0+1}\sum_{k\,=k_0}^{N}\sup_{s\,\in K}\Big(\zeta(s+i\varphi(k),F),\zeta_n(s+i\varphi(k),F)\Big)$$
$$\ll \int_{-\infty}^{\infty}|l_n(\hat{\sigma}+iu)|\left(\frac{1}{N-k_0+1}\sum_{k=k_0}^{N}|\zeta(s+it+iu+i\varphi(k),F)|\right)du, \tag{2.5}$$

where $\hat{\sigma}<0,\quad \kappa/2<\sigma<(\kappa+1)/2$, and t is bounded by a constant depending on K. Lemma 2.2.7 and (2.5) implies that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N-k_0+1}\sum_{k=k_0}^{N}\sup_{s\in K}\Big(\zeta(s+i\varphi(k),F),\zeta_n(s+i\varphi(k),F)\Big)=0.$$

This and the definition of the metric ρ prove the lemma. □

2.2.4 Proof of the Weak Convergence Theorem

Proof of Theorem 2.2.1. Let θ_N be a random variable defined on a certain probability space with measure μ and distribution

$$\mu\{\theta_N=\varphi(k)\}=\frac{1}{N-k_0+1},\quad k=k_0,...,N.$$

Consider the $H(D)$-valued random element $X_{N,n,F}$ defined by the formula

$$X_{N,n,F}=X_{N,n,F}(s)=\zeta_n(s+i\theta_N),F).$$

Then, the assertion of Lemma 2.2.4 can be written as

$$X_{N,n,F}\xrightarrow[N\to\infty]{\mathcal{D}}\hat{X}_{n,F}, \tag{2.6}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution, and $\hat{X}_{n,F}$ is the $H(D)$-valued random element with the distribution $\hat{P}_{n,F}$. Here $\hat{P}_{n,F}$ is the same limit probability measure as in Lemma 2.2.4.

As we have shown in Chapter 1 subsection 1.2.4, the family $\{\hat{P}_{n,F} : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D)$ such that $\hat{X}_{n,F}(K) > 1 - \varepsilon$ for all $n \in \mathbb{N}$.

Therefore, by the Prokhorov theorem (Lemma 1.2.3 in Chapter 1), it is relatively compact, i.e., every sequence of $\{\hat{P}_{n,F}\}$ contains a weakly convergent subsequence. Thus, there exists $\{\hat{P}_{n_r,F}\} \subset \{\hat{P}_{n,F}\}$ such that $\{\hat{P}_{n_r,F}\}$ converges weakly to a certain probability measure P_F on $(H(D), \mathcal{B}(H(D)))$, as $r \to \infty$. Thus,

$$\hat{X}_{n_r,F} \xrightarrow[r\to\infty]{\mathcal{D}} P_F. \tag{2.7}$$

On the probability space of the random variable θ_N, define one more $H(D)$-valued random element

$$X_{N,F} = X_{N,F}(s) = \zeta(s + i\theta_N, F).$$

Then, in view of Lemma 2.2.8, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mu\{\rho(X_{N,F}, X_{N,n,F}) \geqslant \varepsilon\} \\
&= \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N - k_0 + 1} \#\Big\{k_0 \leqslant k \leqslant N : \\
&\quad \rho(\zeta(s + i\varphi(k), F), \zeta_n(s + i\varphi(k), F)) \geqslant \varepsilon\Big\} \\
&\leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N - k_0 + 1)\varepsilon} \sum_{k=k_0}^{N} \rho(\zeta(s + i\varphi(k), F), \zeta_n(s + i\varphi(k), F)) \\
&= 0.
\end{aligned}$$

This together with (2.6) and (2.7) show that all hypotheses of Theorem 4.2 of [4] are fulfilled. Therefore,

$$X_{N,F} \xrightarrow[N\to\infty]{\mathcal{D}} P_F, \tag{2.8}$$

or $P_{N,F}$ converges weakly to the limit measure P_F of $\hat{P}_{n,F}$, as $N \to \infty$. On the other hand, (2.8) shows that the measure P_F is independent of the sequence $\{\hat{P}_{n_r,F}\}$. Since the family $\{\hat{P}_{n,F}\}$ is relatively compact, hence we have, by Theorem 2.3 of [4] (see Lemma 1.2.2 in Chapter 1), that

$$\hat{X}_{n,F} \xrightarrow[n \to \infty]{\mathcal{D}} P_F,$$

or equivalently, $\{\hat{P}_{n,F}\}$ converges weakly to P_F as $n \to \infty$.

Finally, we identify the measure P_F. For this, we will use some elements of the ergodic theory. We remind (see Chapter 1) that

$$\frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \zeta(s + i\tau, F) \in A\Big\}, \qquad A \in \mathcal{B}(H(D))),$$

as $T \to \infty$, converges weakly to the limit measure P_F of $\hat{P}_{n,F}$, and that $P_F = P_{\zeta,F}$. Moreover, the support of $P_{\zeta,F}$ is the set S_F. Therefore, $P_{N,F}$ also converges weakly to $P_{\zeta,F}$ as $N \to \infty$. □

2.3 Proof of the Universality Theorems

Proof of Theorem 2.1.1. Define the set

$$G_\varepsilon = \left\{ g \in H(D) : \quad \sup_{s \in K} |g(s) - e^{p(s)}| < \frac{\varepsilon}{2} \right\},$$

where $p(s)$ is a polynomial. By Theorem 2.2.1, the function $e^{p(s)}$ is an element of the support of the measure $P_{\zeta,F}$. Therefore,

$$P_{\zeta,F}(G_\varepsilon) > 0. \tag{2.9}$$

Based on Theorem 2.2.1 and the equivalent of the weak convergence of probability measures in terms of open sets (see Lemma 1.2.1 or Theorem 2.1 in [4]),

$$\liminf_{N \to \infty} P_{N,F}(G_\varepsilon) \geqslant P_{\zeta,F}(G_\varepsilon).$$

This, the definitions of $P_{N,F}$ and G_ε, and inequality (2.9) give

$$\liminf_{N \to\infty} \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N \ : \sup_{s\in K} |\zeta(s+i\varphi(k), F) \quad (2.10)$$
$$-e^{p(s)}| \ < \frac{\varepsilon}{2}\Big\} > 0.$$

Based on the Mergelyan theorem (Lemma 1.2.6), we can choose the polynomial $p(s)$ to satisfy the inequality

$$\sup_{s\in K}|f(s)-e^{p(s)}|< \frac{\varepsilon}{2}. \tag{2.11}$$

This, together with (2.10), prove the theorem. □

Proof of Theorem 2.1.2. Define the set

$$\hat{G}_\varepsilon = \left\{ g \in H(D): \quad \sup_{s\in K}|g(s)-f(s)|<\varepsilon \right\}.$$

Then the boundary $\partial\hat{G}_\varepsilon$ of $\hat{G}_\varepsilon$ lies in the set

$$\left\{ g \in H(D): \quad \sup_{s\in K}|g(s)-f(s)|=\varepsilon \right\}.$$

Therefore, $\partial\hat{G}_{\varepsilon_1} \cap \partial\hat{G}_{\varepsilon_2} = \emptyset$ for $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_1, \varepsilon_2 > 0$. Hence, for at most countably many $\varepsilon > 0$, the sets $\partial\hat{G}_\varepsilon$ have a positive $P_{\zeta,F}$ measure. Using Theorem 2.2.1 and the equivalent of weak convergence of probability measures in terms of continuity sets (Lemma 1.2.1), we obtain that

$$\lim_{N\to\infty} P_{N,F}(\hat{G}_\varepsilon) = P_{\zeta,F}(\hat{G}_\varepsilon) \tag{2.12}$$

or, in other words, $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,F}$ for all but at most countably many $\varepsilon > 0$. Let G_ε be from the proof of Theorem 2.1.1. Then, in view of (2.11), we obtain that $G_\varepsilon \subset \hat{G}_\varepsilon$, and thus by (2.9), $P_{\zeta,F}(\hat{G}_\varepsilon) > 0$. This, the definition of $P_{N,F}$, and (2.12) prove the theorem. □

3 Joint Universality Theorems

In Theorem H, the shits $\zeta(s+ikh_j, F_j)$ for the joint approximation of analytic functions were taken from the linear sets $\{kh_j\}$, $j = 1, \dots, r$. The aim of this Chapter is to obtain a version of Theorem H by using more complicated nonlinear sets in place of $\{kh_j\}$. Theorems of this Chapter can also be seen as a generalization of the theorems from Chapter 2.

3.1 Statements of the Theorems

Let $k_0 \in \mathbb{N}$. We say that functions $\varphi_1(\tau), \dots, \varphi_r(\tau)$ belong to class $U_r(k_0)$ if the following conditions are satisfied:

1. $(\varphi_1, \dots, \varphi_r)$ are real-valued positive increasing continuously differentiable functions on $[k_0 - \frac{1}{2}, \infty)$;
2. Derivatives $\varphi'_1(\tau), \dots, \varphi'_r(\tau)$, on $[k_0 - \frac{1}{2}, \infty)$, satisfy the estimate

$$\varphi_j(2\tau)\left(\max_{\tau\leqslant t\leqslant 2\tau}\frac{1}{\varphi'_j(t)} + \max_{\tau\leqslant t\leqslant 2\tau}\varphi'_j(t)\right) \ll \tau, \quad \tau\to\infty, j = 1, \dots, r; \tag{3.1}$$

3. The sequence $\{a_1\varphi_1(k) + \dots + a_r\varphi_r(k) : k \geqslant k_0\} \subset \mathbb{R}$ is uniformly distributed modulo 1 with every $a_1, \dots, a_r \in \mathbb{R}$, where a_j, $j = 1, \dots, r$, are not all zeroes.

We remind that the definition of uniform distribution modulo 1 can be found in Chapter 2.

Let $D_j = D(\kappa_j/2, (\kappa_j+1)/2)$, $\mathcal{K}_j$ be the class of compact subset of the strip D_j with connected complements, and let $H_0(K_j), K_j \in \mathcal{K}_j$ denote the class of continuous non-vanishing functions on K_j that are analytic in the interior of K_j, $j = 1, \dots, r$.

Under such conditions, the results of this Chapter are the following two theorems.

Theorem 3.1.1. *Suppose that $(\varphi_1, \dots, \varphi_r) \in U_r(k_0)$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}_j$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$, the following inequality is true*

$$\liminf_{N \to \infty} \frac{1}{N-k_0+1} \#\Big\{ k_0 \leqslant k \leqslant N \ : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s+i\varphi_j(k), F_j) - f_j(s)| < \varepsilon \Big\} > 0.$$

An important note is that the cusp forms $F_1, \dots, F_r$ used for simultaneous approximation are not necessarily different.

Theorem 3.1.1 has the following modification, which will also be proved in this Chapter.

Theorem 3.1.2. *Suppose that $(\varphi_1, \dots, \varphi_r) \in U_r(k_0)$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}_j$ and $f_j(s) \in H_0(K_j)$. Then the limit*

$$\lim_{N \to \infty} \frac{1}{N-k_0+1} \#\Big\{ k_0 \leqslant k \leqslant N \ : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s+i\varphi_j(k), F_j) - f_j(s)| < \varepsilon \Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Theorems 3.1.1 and 3.1.2 in some sense are joint generalizations of the corresponding one-dimensional theorems from Chapter 2.

3.2 Probabilistic Model

We will start the proof with the definition of the probability space. Let again γ be the unit circle on the complex plane, and

$$\Omega = \prod_{p \in \mathbb{P}}^{\infty} \gamma_p,$$

where $\gamma_p = \gamma$ for all primes $p \in \mathbb{P}$. The torus Ω, with the product topology and pointwise multiplication, is a compact topological Abelian group. Define

$$\underline{\Omega} = \Omega_1 \times \ldots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \ldots, r$. Then again, $\underline{\Omega}$ is a compact topological Abelian group. Therefore, on space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$ where $\mathcal{B}(\underline{\Omega})$ is the Borel σ-field of the space $\underline{\Omega}$, the probability Haar measure m_H exists; thus, we obtain the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$. Denote the elements of $\underline{\Omega}$ by $\underline{\omega} = (\omega_1, \ldots, \omega_r)$, where $\omega_j \in \Omega_j, j = 1, \ldots, r$. We start with a limit theorem for probability measures on$(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$. The main result of this section is the weak convergence theorem (Theorem 3.2.4). However, we will first prove some auxiliary lemmas.

3.2.1 Limit Theorems

For $A \in \mathcal{B}(\underline{\Omega})$, define

$$\begin{aligned} Q_N(A) \; &= \frac{1}{N - k_0 + 1} \# \Big\{ k_0 \leqslant k \leqslant N : \\ &((p^{-i\varphi_1(k)} : p \in \mathbb{P}), \ldots, (p^{-i\varphi_r(k)} : p \in \mathbb{P})) \; \in A \Big\}. \end{aligned}$$

Lemma 3.2.1. *Suppose that the sequence* $\{a_1\varphi_1(k) + \ldots + a_r\varphi_r(k) : k \geqslant k_0\}$ *is uniformly distributed modulo* 1 *with* $a_1, \ldots, a_r \in \mathbb{R}$ *where* $a_j, j = 1, \ldots, r,$ *are not all zeroes. Then,* Q_N *converges weakly to the Haar measure* m_H *as* $N \to \infty$.

Proof. We apply the uniform distribution modulo 1 for the investigation of the Fourier transformation $g_N(\underline{k}_1, \ldots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \ldots, r$, of Q_N. We have that the dual group of $\underline{\Omega}$ is isomorphic to the group

$$\bigoplus_{j=1}^{r} \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_{jp},$$

where $\mathbb{Z}_{jp} = \mathbb{Z}$ for all $j = 1, \ldots, r$, $p \in \mathbb{P}$. Therefore,

$$g_N(\underline{k}_1, \ldots, \underline{k}_r) = \int_{\underline{\Omega}} \left(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}' \omega^{k_{jp}}(p) \right) dQ_N,$$

where the sign " ′ " means that only a finite number of integers k_{jp} are distinct from zero. Thus, from the definition of Q_N, we have

$$\begin{aligned} g_N(\underline{k}_1, \ldots, \underline{k}_r) &= \frac{1}{N - k_0 + 1} \sum_{k=k_0}^{N} \prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}' p^{-ik_{jp}\varphi_j(k)} \\ &= \frac{1}{N - k_0 + 1} \sum_{k=k_0}^{N} \exp\left\{ -i \sum_{j=1}^{r} \varphi_j(k) \sum_{p \in \mathbb{P}}{}' k_{jp} \log p \right\}. \end{aligned} \tag{3.2}$$

If $(\underline{k}_1, \ldots, \underline{k}_r) = (\underline{0}, \ldots, \underline{0})$, then, clearly,

$$g_N(\underline{k}_1, \ldots, \underline{k}_r) = 1. \tag{3.3}$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$, we have that

$$\sum_{p \in \mathbb{P}}{}' k_{jp} \log p \neq 0 \quad \text{for all} \quad \underline{k}_j \neq \underline{0}, \quad j = 1, \ldots, r.$$

Therefore, by hypothesis of the lemma on the uniform distribution, the Weyl criterion (Lemma 2.2.2 in Chapter 2), and (3.2), we find that for $(\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0})$,

$$\lim_{N \to \infty} g_N(\underline{k}_1, \ldots, \underline{k}_r) = 0.$$

This and (3.3) give

$$\lim_{N\to\infty} g_N(\underline{k}_1,\dots,\underline{k}_r) = \begin{cases} 1 & \text{if} \quad (\underline{k}_1,\dots,\underline{k}_r) = (\underline{0},\dots,\underline{0}), \\ 0 & \text{if} \quad (\underline{k}_1,\dots,\underline{k}_r) \neq (\underline{0},\dots,\underline{0}). \end{cases}$$

This, together with the continuity theorem for the probability measures on compact groups, shows that $g_N(\underline{k})$, as $N \to \infty$, converges to the Fourier transformation of the Haar measure m_H (see equality (1.6) in Chapter 1). □

Denote by $H(D_j)$ the space of analytic functions on D_j endowed with the topology of uniform convergence on compacta, $j = 1,\dots,r$, and let $H(D_1,\dots,D_r) = H(D_1) \times \dots \times H(D_r)$. Let $\theta > \frac{1}{2}$ be a fixed number, and $m, n \in \mathbb{N}$. We define, for $j = 1,\dots,r$, the series

$$\zeta_n(s, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m) v_n(m)}{m^s}$$

and

$$\zeta_n(s, \omega_j, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m)\omega_j(m) v_n(m)}{m^s},$$

where

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\theta} \right\},$$

and

$$\omega_j(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad m \in \mathbb{N}.$$

The latter series are absolutely convergent for $\sigma > \kappa_j/2$ (see [28]). For brevity, we will denote $\underline{s} = (s_1,\dots,s_r), \underline{F} = (F_1,\dots,F_r), \underline{\varphi}(k) = (\varphi_1(k),\dots,\varphi_r(k))$, and

$$\underline{\zeta}_n(\underline{s} + i\underline{\varphi}(k), \underline{F}) = (\zeta_n(s_1 + i\varphi_1(k), F_1),\dots,\zeta_n(s_r + i\varphi_r(k), F_r))$$

and

$$\underline{\zeta}_n(\underline{s}, \underline{\omega}, \underline{F}) = (\zeta_n(s_1, \omega_1, F_1),\dots,\zeta_n(s_r, \omega_r F_r)).$$

For $A \in \mathcal{B}(H(D_1, \ldots, D_r))$, define

$$P_{N,n}(A) = \frac{1}{N - k_0 + 1} \# \Big\{ k_0 \leqslant k \leqslant N : \underline{\zeta}_n(\underline{s} + i\underline{\varphi}(k), \underline{F}) \in A \Big\}.$$

Lemma 3.2.2. *Suppose that the sequence* $\{a_1\varphi_1(k) + \ldots + a_r\varphi_r(k) : k \geqslant k_0\}$ *is uniformly distributed modulo* 1 *with* $a_1, \ldots, a_r \in \mathbb{R}$ *where* $a_j, j = 1, \ldots, r$, *are not all zeroes. Then, on* $(H(D_1, \ldots, D_r), \mathcal{B}(H(D_1, \ldots, D_r)))$, *there exists a probability measure* $\hat{P}_n$ *such that* $P_{N,n}$ *converges weakly to* $\hat{P}_n$ *as* $N \to \infty$.

Proof. Define the function $u_n : \underline{\Omega} \to H(D_1, \ldots, D_r)$ by the formula

$$u_n(\omega) = \underline{\zeta}_n(\underline{s}, \underline{\omega}, \underline{F}).$$

Due to absolute convergence of $\zeta_n(s_j, \omega_j, F_j)$ for $\sigma_j > \kappa_j/2, j = 1, \ldots, r$, we have that the function u_n is continuous.

Moreover, for $A \in \mathcal{B}(H(D_1, \ldots, D_r))$,

$$P_{N,n}(A) = \frac{1}{N - k_0 + 1} \# \Big\{ k_0 \leqslant k \leqslant N : ((p^{-i\varphi_1(k)} : p \in \mathbb{P}), \ldots, (p^{-i\varphi_r(k)} : p \in \mathbb{P})) \in u_n^{-1}A \Big\} = Q_N(u_n^{-1}A)$$

because

$$u_n((p^{-i\varphi_1(k)} : p \in \mathbb{P}), \ldots, (p^{-i\varphi_r(k)} : p \in \mathbb{P})) = \underline{\zeta}_n(\underline{s} + i\underline{\varphi}(k), \underline{\omega}, \underline{F}).$$

Therefore, we have $P_{N,n} = Q_N u_n^{-1}$, where

$$Q_N u_n^{-1}(A) = Q_N(u_n^{-1}A), \quad A \in \mathcal{B}(H(D_1, \ldots, D_r)).$$

The above equality, Lemma 3.2.1, the continuity of u_n, and Lemma 1.2.3 imply the weak convergence of $P_{N,n}$ to $\hat{P}_n = m_H u_n^{-1}$ as $N \to \infty$. □

Now we define the metric in $H(D_1, \ldots, D_r)$. As in previous cases, for $j = 1, \ldots, r$ and $g_1, g_2 \in H(D_j)$, let

$$\rho_j(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_{jl}} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_{jl}} |g_1(s) - g_2(s)|},$$

where $\{K_{jl} : l \in \mathbb{N}\} \subset D_j$ is a sequence of compact subsets such that

$$D_j = \bigcup_{l=1}^{\infty} K_{jl},$$

$K_{jl} \subset K_{j(l+1)}$ for all $l \in \mathbb{N}$, and if $K \subset D_j$ is a compact subset, then $K \subset K_{jl}$ for some $l \in \mathbb{N}$. Then ρ_j is the metric in $H(D_j)$, inducing its topology of uniform convergence on compacta. Taking $\underline{g}_1 = (g_{11}, \ldots, g_{1r}), \underline{g}_2 = (g_{21}, \ldots, g_{2r}) \in H(D_1, \ldots, D_r)$ and

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1\leqslant j\leqslant r} \rho_j(g_{1j}, g_{2j}),$$

we get the metric in the space $H(D_1, \ldots, D_r)$ that induces the product topology.

Now, we are able to approximate the collection

$$\underline{\zeta}(\underline{s} + i\underline{\varphi}(k), \underline{F}) = (\zeta(s_1 + i\varphi_1(k), F_1), \ldots, \zeta(s_r + i\varphi_r(k), F_r))$$

by $\underline{\zeta}_n(\underline{s} + i\underline{\varphi}(k), \underline{F})$.

Lemma 3.2.3. *The following equality is true*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N - k_0 + 1} \sum_{k=k_0}^{N} \underline{\rho}\Big(\underline{\zeta}(\underline{s} + i\underline{\varphi}(k), \underline{F}), \underline{\zeta}_n(\underline{s} + i\underline{\varphi}(k), \underline{F})\Big) = 0.$$

Proof. From the definition of the metrics ρ_j and $\underline{\rho}$ it follows that it suffices to prove that, for every compact sets $K_j \subset D_j$,

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \sup_{s_j \in K_j} |\zeta(s_j + i\varphi_j(k), F_j) - \zeta_n(s_j + i\varphi_j(k), F_j)| = 0,$$

$j = 1, \dots, r$. Thus, let F be a normalized Hecke-eigen cusp form of weight κ, $\zeta(s, F)$ be the corresponding zeta-function, and let φ have the properties of the class $U_r(k_0)$.

From Lemma 2.2.7 in Chapter 2 we have

$$\begin{aligned} \sum_{k=k_0}^{N} |\zeta(\sigma + it + i\varphi(k), F)|^2 &\ll \int_{k_0-1/2}^{N+1/2} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\tau \\ &+ \left(\int_{k_0-1/2}^{N+1/2} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\tau \right. \\ &\times \left. \int_{k_0-1/2}^{N+1/2} (\varphi'(\tau))^2 |\zeta'(\sigma + it + i\varphi(\tau), F)|^2 d\tau \right)^{1/2} \ll N(1+|t|), \end{aligned} \tag{3.4}$$

for a fixed σ, $\kappa/2 < \sigma < (\kappa+1)/2$, and $t \in \mathbb{R}$.

Let $\theta > 1/2$ be as in the definition of $v_n(m)$. Then, as we know from (1.9), the function $\zeta_n(s, F)$ for $\sigma > \kappa/2$, has the representation

$$\zeta_n(s, F) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s+z, F) l_n(z) \frac{dz}{z},$$

where

$$l_n(s) = \frac{s}{\theta} \Gamma\Big(\frac{s}{\theta}\Big) n^s, \quad n \in \mathbb{N}.$$

Let K be an arbitrary compact subset of the strip $\{\kappa/2 < \sigma < (\kappa+1)/2\}$. We take $\varepsilon > 0$ such that

$$\frac{\kappa}{2} + 2\varepsilon \leqslant \sigma \leqslant \frac{\kappa+1}{2} - \varepsilon$$

for point $s \in K$. In (1.9), replace θ by $-\hat{\theta}$, where $\hat{\theta} > 0$. This gives

$$\zeta_n(s,F) - \zeta(s,F) = \frac{1}{2\pi i} \int\limits_{-\hat{\theta}-i\infty}^{-\hat{\theta}+i\infty} \zeta(s+z,F) l_n(z) \frac{dz}{z}. \tag{3.5}$$

Denote points of the set K by $s = \sigma + iv$ and take

$$\hat{\theta} = \sigma - \varepsilon - \frac{\kappa}{2}, \quad \theta = \frac{1}{2} + \varepsilon.$$

In view of (3.5), we get

$$\begin{aligned}&|\zeta_n(s+i\varphi(k),F) - \zeta(s+i\varphi(k),F)| \\ &\leqslant \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \zeta(s+i\varphi(k) - \hat{\theta} + it), F) \frac{|l_n(-\hat{\theta}+it)|}{|-\hat{\theta}+it|} dt\end{aligned}$$

Now, taking the shift $t + v \mapsto t$, we get

$$\begin{aligned}&|\zeta_n(s+i\varphi(k),F) - \zeta(s+i\varphi(k),F)| \\ &\leqslant \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \zeta(\frac{\kappa}{2} + \varepsilon + i(t+\varphi(k)), F) \frac{|l_n(\frac{\kappa}{2}+\varepsilon - s + it)|}{|\frac{\kappa}{2}+\varepsilon - s + it|} dt.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \sup_{s \in K} |\zeta(s+i\varphi(k),F) - \zeta_n(s+i\varphi(k),F)| \\ &\leqslant \frac{1}{2\pi(N-k_0+1)} \int\limits_{-\infty}^{\infty} \left(\sum_{k=k_0}^{N} \left| \zeta(\frac{\kappa}{2} + \varepsilon + i(t+\varphi(k)), F) \right| \right) \\ &\quad \times \sup_{s \in K} \frac{|l_n(\frac{\kappa}{2}+\varepsilon - s + it)|}{|\frac{\kappa}{2}+\varepsilon - s + it|} dt =: J.\end{aligned} \tag{3.6}$$

It is known that

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0,$$

uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. Thus, from definition of $l_n(s)$,

$$\frac{|l_n(\frac{\kappa}{2}+\varepsilon-s+it)|}{|\frac{\kappa}{2}+\varepsilon-s+it|} \leqslant \frac{n^{\kappa/2+\varepsilon-\sigma}}{\theta} \exp\left\{-\frac{c}{\theta}|t-v|\right\} \leqslant_K n^{-\varepsilon} \exp\{-c|t|\}.$$

With respect to (3.4),

$$J \ll_K n^{-\varepsilon} \int\limits_{-\infty}^{\infty} (1+|t|)^{1/2} \exp\{-c|t|\} dt \ll_K n^{-\varepsilon}.$$

This and (3.6) show that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \sup_{s\in K} \Big(\zeta(s+i\varphi(k), F), \zeta_n(s+i\varphi(k), F)\Big) = 0,$$

and the lemma is thus proven. □

3.2.3 Proof of the Weak Convergence Theorem

Now we will prove the discrete limit theorem for the collection $\underline{\zeta}(\underline{s} + i\underline{\varphi}(k), \underline{F})$. For $A \in \mathcal{B}(H(D_1, \ldots, D_r))$, define

$$P_N(A) = \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N : \underline{\zeta}(\underline{s} + i\underline{\varphi}(k), \underline{F}) \in A\Big\}.$$

On the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$ define the $H(D_1, \ldots, D_r)$-valued random element $\zeta(\underline{s}, \underline{\omega}, \underline{F})$ by the formula

$$\zeta(\underline{s}, \underline{\omega}, \underline{F}) = (\zeta(s_1, \omega_1, F_1), \ldots, \zeta(s_r, \omega_r, F_r)),$$

where

$$\zeta(s_j, \omega_j, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m)\omega_j(m)}{m_j^s}, \quad j = 1, \ldots, r.$$

Denote by $P_{\underline{\zeta}}$ the distribution of $\zeta(\underline{s}, \underline{\omega}, \underline{F})$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H\{\underline{\omega} \in \underline{\Omega} : \zeta(\underline{s}, \underline{\omega}, \underline{F}) \in A\}, \quad A \in \mathcal{B}(H(D_1, \ldots, D_r)).$$

Under such conditions, the following statement is true.

Theorem 3.2.4. *Suppose that* $(\varphi_1, \ldots, \varphi_r) \in U_r(k_0)$*. Then* P_N *converges weakly to* $P_{\underline{\zeta}}$*, as* $N \to \infty$*. Moreover, the support of the measure* $P_{\underline{\zeta}}$ *is the set* $S = S_1 \times \ldots \times S_r$*, where*

$$S_j = \{g \in H(D_j) : g(s) \neq 0 \quad or \quad g(s) \equiv 0\}, \quad j = 1, \ldots, r.$$

Proof. Let $\hat{P}_n$ be the same limit probability measure as in Lemma 3.2.2. We may show that the family $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D_1, \ldots, D_r)$ such that

$$\hat{P}_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Indeed, let $\hat{P}_{nj}, \quad j = 1, \ldots, r$, be the marginal measures of $\hat{P}_n$. Then it is known that the sequences $\{\hat{P}_{nj} : n \in \mathbb{N}\}$ are tight, $j = 1, \ldots, r$ (see [51], [28] or [35] for reference). Therefore, for every $\varepsilon > 0$, there exists a compact set $K_j \subset H(D_j)$ such that

$$\hat{P}_{nj}(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \ldots, r, \tag{3.7}$$

for all $n \in \mathbb{N}$. The set $K = K_1 \times \ldots \times K_r$ is compact in the space $H(D_1, \ldots, D_r)$ and, by (3.7),

$$\hat{P}_n(H(D_1, \ldots, D_r) \setminus K) \leqslant \sum_{j=1}^{r} \hat{P}_{nj}(H(D_j) \setminus K_j) < \varepsilon$$

for all $n \in \mathbb{N}$. Therefore, $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight.

By the Prokhorov theorem (Lemma 1.2.3 in Chapter 1), $\{\hat{P}_n\}$ is relatively compact, i.e., every sequence of $\{\hat{P}_n\}$ contains a weakly convergent subsequence $\{\hat{P}_{n_l}\} \subset \{\hat{P}_n\}$ such that $\{\hat{P}_{n_l}\}$ converges weakly to a certain probability measure P on $(H(D_1, \ldots, D_r), \mathcal{B}(H(D_1, \ldots, D_r)))$, as $l \to \infty$.

Let θ_N be a discrete random variable defined on a certain probability space with measure μ and distribution

$$\mu\{\theta_N = \underline{\varphi}(k)\} = \frac{1}{N - k_0 + 1}, \quad k = k_0, ..., N.$$

Consider the $H(D_1, \ldots, D_r)$-valued random element $\underline{X}_{N,n}$ defined by the formula

$$\underline{X}_{N,n} = \underline{X}_{N,n}(s) = \underline{\zeta}_n(\underline{s} + i\theta_N), \underline{F}).$$

Then, the assertion of Lemma 3.2.2 can be written as

$$\underline{X}_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} \underline{\hat{X}}_n, \tag{3.8}$$

where $\underline{\hat{X}}_n = \underline{\hat{X}}_n(\underline{s})$ is the $H(D_1, \ldots, D_r)$-valued random element with the distribution $\hat{P}_n$. Respectively,

$$\hat{X}_{n_l} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{3.9}$$

On the probability space of the random variable θ_N, define one more $H(D_1, \ldots, D_r)$-valued random element

$$\underline{X}_N = \underline{X}_N(s) = \underline{\zeta}(\underline{s} + i\theta_N, \underline{F}).$$

Then, with respect to Lemma 3.2.3, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n \to\infty} \limsup_{N \to\infty} \mu\{\underline{\rho}(\underline{X}_N, \underline{X}_{N,n}) \geqslant \varepsilon\} \\
&\leqslant \lim_{n \to\infty} \limsup_{N \to\infty} \frac{1}{(N - k_0 + 1)\varepsilon} \sum_{k = k_0}^{N} \underline{\rho}(\underline{\zeta}(\underline{s} + i\underline{\varphi}(k), \underline{F}), \underline{\zeta}_n(\underline{s} \\
&\qquad + i\underline{\varphi}(k), \underline{F})) = 0.
\end{aligned}$$

This, together with (3.8) and (3.9), show that all hypotheses of Theorem 4.2 of [4] are fulfilled, and therefore,

$$X_N \xrightarrow[N \to\infty]{\mathcal{D}} P, \tag{3.10}$$

or P_N converges weakly to the limit measure P of $\hat{P}_n$, as $N \to \infty$. On the other hand, (3.10) shows that the measure P is independent of the sequence $\{\hat{P}_{n_l}\}$. Since the family $\{\hat{P}_n\}$ is relatively compact, we have, by Theorem 2.3 of [4] (see Lemma 1.2.2 in Chapter 1), that

$$\hat{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P,$$

or equivalently, $\{\hat{P}_n\}$ converges weakly to P as $n \to \infty$.

Finally, we identify the measure P. In [24] it was obtained that the measure P coincides with $P_{\underline{\zeta}}$. Moreover, the support of $P_{\underline{\zeta}}$ is the set S. In [24], the observation that $\mathcal{B}(H(D_1, \ldots, D_r)) = \mathcal{B}(H(D_1) \times \ldots \times \mathcal{B}(H(D_r))$ is used. In such case, the Haar measure m_H is the product of the Haar measures on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \ldots, r$. $\square$

3.3 Proofs of the Universality Theorems

With all the auxiliary results and the Mergelyan theorem, we can prove the main statements of the Chapter.

Proof of Theorem 3.1.1. Define the set

$$\underline{G}_\varepsilon = \left\{(g_1, \ldots, g_r) \in H(D_1, \ldots, D_r) : \quad \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2}\right\},$$

where $p_j(s)$ is a polynomial, $j = 1, \ldots, r$. By Theorem 3.2.4, the collection $(e^{p_1(s)}, \ldots, e^{p_r(s)})$ is an element of the support of the measure $P_{\underline{\zeta}}$. Therefore, the set $\underline{G}_\varepsilon$ is an open neighbourhood of an element of the support of $P_{\underline{\zeta}}$ and

$$P_{\underline{\zeta}}(\underline{G}_\varepsilon) > 0. \tag{3.11}$$

By Theorem 3.2.4 and the equivalent of the weak convergence of probability measures in terms of open sets (see Lemma 1.2.1 in Chapter 1),

$$\liminf_{N\to\infty} P_N(\underline{G}_\varepsilon) \geqslant P_{\underline{\zeta}}(\underline{G}_\varepsilon) > 0.$$

This, the definitions of P_N and $\underline{G}_\varepsilon$, and inequality (3.11) give

$$\liminf_{N\to\infty} \frac{1}{N-k_0+1}\#\Big\{k_0 \leqslant k \leqslant N \ : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s_j+i\varphi_j(k),F_j)-e^{p_j(s)}| < \frac{\varepsilon}{2}\Big\} > 0. \tag{3.12}$$

Based on the Mergelyan theorem (Lemma 1.2.6 in Chapter 1), we can choose the polynomials $p_1(s),\dots,p_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s)-e^{p_j(s)}|< \frac{\varepsilon}{2}. \tag{3.13}$$

This, together with (3.12), proves the theorem. □

Proof of Theorem 3.1.2. Define the set

$$\underline{\hat{G}}_\varepsilon = \left\{(g_1,\dots,g_r)\in H(D_1,\dots,D_r): \quad \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|g_j(s)-f_j(s)|<\varepsilon\right\}.$$

Then the boundary $\partial\underline{\hat{G}}_\varepsilon$ of $\underline{\hat{G}}_\varepsilon$ lies in the set

$$\left\{(g_1,\dots,g_r)\in H(D_1,\dots,D_r): \quad \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|g_j(s)-f_j(s)|=\varepsilon\right\}.$$

Therefore, $\partial\underline{\hat{G}}_{\varepsilon_1}\cap\partial\underline{\hat{G}}_{\varepsilon_2}=\emptyset$ for $\varepsilon_1\neq\varepsilon_2$, $\varepsilon_1,\varepsilon_2>0$. Hence, for at most countably many $\varepsilon>0$, the sets $\partial\underline{\hat{G}}_\varepsilon$ have a positive $P_{\underline{\zeta}}$ measure. Therefore, by Theorem 3.2.4 and the equivalent of the weak convergence of probability measures in terms of continuity sets (Lemma 1.2.1), we obtain that

$$\lim_{N\to\infty} P_N(\underline{\hat{G}}_\varepsilon)=P_{\underline{\zeta}}(\underline{\hat{G}}_\varepsilon) \tag{3.14}$$

or, in other words, $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,F}$ for all but at most countably many $\varepsilon>0$. On the other hand, the definitions of $\underline{G}_\varepsilon$ and $\underline{\hat{G}}_\varepsilon$, together with (3.13), imply that $\underline{G}_\varepsilon\subset\underline{\hat{G}}_\varepsilon$. Thus, by (3.11), $P_{\underline{\zeta}}(\underline{\hat{G}}_\varepsilon)>0$. This, the definition of P_N, and (3.14) prove the theorem.

□

CONCLUSIONS

In this thesis, three different universality theorems for zeta-function $\zeta(s, F)$ associated with the normalized simultaneous Hecke-eigen cusp form $F(z)$ of weight κ and their modifications were proven. Let $\zeta(s, F)$ be defined for $\sigma > (\kappa + 1)/2$, by absolutely convergent Dirichlet series

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

and analytically continued to the whole complex plain. Then the following statements are true:

1. If $\varphi \in U(\tau_0)$ is real-valued function with certain growth conditions, then holomorphic non-vanishing functions $f(s)$ can be uniformly approximated with given accuracy by continuous shifts $\zeta(s + i\varphi(\tau), F)$, and the lower density of such shifts is positive, i.e.,

   $$\liminf_{T \to \infty} \frac{1}{T - \tau_0} \operatorname{meas} \Big\{ \tau \in [\tau_0, T] : \sup_{s \in K} |\zeta(s + i\varphi(\tau), F) - f(s)| < \varepsilon \Big\} > 0.$$

   The density of such shifts is positive for all but at most countably many $\varepsilon > 0$.

2. If $\varphi \in U(k_0)$ is a real-valued function with certain growth conditions, which is a uniformly distributed modulo 1, then holomorphic non-vanishing functions $f(s)$ can be uniformly approximated with given accuracy by discrete shifts $\zeta(s+i\varphi(k), F)$, while

the lower density of such shifts is positive, i.e.,

$$\liminf_{N\to\infty} \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N \ : \\ \sup_{s\in K} |\zeta(s+i\varphi(k),F) - f(s)| < \varepsilon\Big\} > 0.$$

The density of such shifts is positive for all but at most countably many $\varepsilon > 0$.

3. If $(\varphi_1, \ldots, \varphi_r) \in U_r(k_0)$ are real-valued functions with certain growth conditions, which are uniformly distributed modulo 1, then a set of holomorphic non-vanishing functions $f_1(s), \ldots, f_r(s)$ can be simultaneously uniformly approximated with given accuracy by discrete shifts $\zeta(s + i\varphi_j(k), F_j)$, where the cusp forms $F_1, \ldots, F_r$ are not necessarily different. The lower density of such shifts is positive, i.e.,

$$\liminf_{N\to\infty} \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N \ : \\ \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\varphi_j(k),F_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

The density of such shifts is positive for all but at most countably many $\varepsilon > 0$.

As universality of $\zeta(s, F)$ is already analyzed quite well, there are some certain open areas that could be addressed in future research. Firstly, the extension of the class of functions φ to more general or more complicated cases would be valuable. Secondly, an exploration of other subgroups of the full modular group for the definition of $\zeta(s, F)$ could be conducted. Thirdly, a solution for the effectivization problem of the given universality theorems, i.e., since the universality theorems in the thesis are non-effective in the sense that we cannot indicate any specific shifts for the approximation of a given function, research enabling the choice of such shifts is needed.

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Notation

$s = \sigma + it \in \mathbb{C}$	the complex plane
$\mathbb{N}, \mathbb{N}_0$	set of natural numbers, $\mathbb{N}_0 \bigcup 0$
$\mathbb{Z}$	set of integers
$\mathbb{R}$	set of real numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{P}$	set of prime numbers
k, m, n	natural numbers
$meas.$	the Lebesgue measure
$\#\{.\}$	cardinality of a set
$\mathcal{B}(.)$	Borel σ-algebra
m_H	Haar measure
γ	complex unit circle
Ω	space defined by product of γ by primes
ω, ω_j	elements of Ω
ε	small positive number
$U(\tau_0), U(k_0), U_r(k_0)$	class of functions with specified conditions
$SL(2, \mathbb{Z})$	full modular group
F	cusp form
κ, κ_j	weight of the cusp form
$\zeta(s, F)$	zeta function attached to a certain cusp form
$P, P_n, ...$	probability measures
$\underline{x}$	vectors
$\rho(g_1, g_2)$	metric in a given space
$\Gamma(s)$	Euler gamma-function
$\bigoplus_m Z_m$	direct sum of sets Z_m

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