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CONSTRUCTION OF SOLITARY SOLUTIONS TO DIFFERENTIAL EQUATIONS VIA OPERATOR TECHNIQUES

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1. INTRODUCTION

1.1. Included papers and co-authors' contribution to papers

Copies of five papers listed below that are the basis of this thesis. Papers X1, X2, X3, X5 have been published in Elsevier Publishing journals, thus the author retains the right to reprint them in a thesis. Paper X4 has been published Open Access, thus the author needs no permission from the publisher to reprint.

Paper X1. Navickas, Zenonas; Ragulskis, Minvydas Kazys; **Telksnys, Tadas**. Existence of solitary solutions in a class of nonlinear differential equations with polynomial nonlinearity // Applied mathematics and computation. New York, NY: Elsevier. ISSN 0096-3003. eISSN 1873-5649. 2016, vol. 283, p. 333-338. DOI: 10.1016/j.amc.2016.02.049.

Paper X2. **Telksnys, T**.; Navickas, Z.; Marcinkevicius, R.; Ragulskis, M. Existence of solitary solutions in systems of PDEs with multiplicative polynomial coupling // Applied mathematics and computation. New York: Elsevier. ISSN 0096-3003. eISSN 1873-5649. 2018, Vol. 320, p. 380-388. DOI: 10.1016/j.amc.2017.09.051.

Paper X3. Navickas, Z.; Ragulskis, M.; Marcinkevicius, R.; **Telksnys, T**. Kink solitary solutions to generalized Riccati equations with polynomial coefficients // Journal of mathematical analysis and applications. Atlanta, GA: Elsevier. ISSN 0022-247X. 2017, vol. 448, iss. 1, p. 156-170. DOI: 10.1016/j.jmaa.2016.11.011.

Paper X4. **Telksnys, Tadas**; Navickas, Zenonas; Marcinkevicius, Romas; Cao, Maosen; Ragulskis, Minvydas. Homoclinic and heteroclinic solutions to a hepatitis C evolution model // Open mathematics. Warsaw: De Gruyter. eISSN 2391-5455. 2017, vol. 16, iss. 1, p. 1537-1555. DOI: 10.1515/math-2018-0130.

Paper X5. Navickas, Zenonas; **Telksnys, Tadas**; Marcinkevičius, Romas; Ragulskis, Minvydas Kazys. Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations // Chaos, solitons and fractals. Oxford: Pergamon-Elsevier Science. ISSN 0960-0779. eISSN 1873-2887. 2017, vol. 104, p. 625-634. DOI: 10.1016/j.chaos.2017.09.026.

Listed below if there is any contribution by co-author(s) to the papers pertaining to the thesis:

1) Paper X1: Z. Navickas, M. Ragulskis, T. Telksnys. Existence of solitary solutions in a class of differential equations with polynomial nonlinearity.
    - Z. Navickas conceived the idea of the inverse balancing method and drew up the initial drafts of the required proofs;

- M. Ragulskis, as the leader of the research group, oversaw the writing of the manuscript and organized group seminars;
- T. Telksnys expanded upon the inverse balancing concept conceived by Z. Navickas, performed all the necessary analytical derivations and numerical computations needed for the paper and wrote the manuscript text. He also corresponded with the editors of the journal.

2) Paper X2: T. Telksnys, Z. Navickas, R. Marcinkevičius, M. Ragulskis. Existence of solitary solutions in systems of PDEs with multiplicative polynomial coupling.
   - T. Telksnys conceived the idea for the paper as an extension of the previous publication. He also performed the necessary analytical derivations and numerical computations, co-wrote the text of the paper and corresponded with the editorial office;
   - Z. Navickas advised on the topic of the extension of the inverse balancing to a higher number of equations. He also offered some refinements to the derivations done by T. Telksnys;
   - R. Marcinkevičius advised on the presentation of examples included in the manuscript text and assisted T. Telksnys with computer algebra computations;
   - M. Ragulskis, as the leader of the research group, oversaw the writing of the manuscript, organized seminars and offered advice on the text of the manuscript.
3) Paper X3: Z. Navickas, M. Ragulskis, R. Marcinkevičius, T. Telksnys. Kink solitary solutions to generalized Riccati equations with polynomial coefficients.
   - Z. Navickas presented the idea for the direct balancing technique included in the paper and did the initial draft of mathematical proofs that were required;
   - M. Ragulskis, as the leader of the research group, suggested this topic of research, organized seminars and oversaw the writing of the paper;
   - R. Marcinkevičius performed the initial computations needed to showcase the paper’s idea and advised on the application of computer algebra;
   - T. Telksnys performed all of the numerical and computer algebra computations presented in the text. He also redid some of the proofs and statements presented by Z. Navickas to make them better suited

for presentation. He also wrote the entire manuscript text and corresponded with the editorial office.

4) Paper X4: T. Telksnys, Z. Navickas, R. Marcinkevičius, M. Cao, M. Ragulskis. Homoclinic and heteroclinic solutions to a hepatitis C evolution model.
    - T. Telksnys performed most of the numerical and all of the required computer algebra computations for this study. He was also responsible for the mathematical proofs related to the existence of solitary solutions for biologically viable parameter values. T. Telksnys wrote the entire text of the manuscript and corresponded with the editorial office;
    - Z. Navickas performed the required mathematical proofs in relation to the generalized differential operator method and presented the initial draft of the derivation of necessary and sufficient existence conditions for solitary solutions in the studied system;
    - R. Marcinkevičius advised on the application of computer algebra for the realization of the generalized differential operator method;
    - M. Cao, together with M. Ragulskis, suggested the topic of the research and performed a part of the literature review;
    - M. Ragulskis, as the leader of the group, coordinated the writing of the manuscript and organized seminars.
5) Paper X5: Z. Navickas, T. Telksnys, R. Marcinkevičius, M. Ragulskis. Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations.
    - Z. Navickas was responsible for most of the mathematical derivations required to adapt operator methods for fractional order differential equations. He also presented the initial draft of the properties of the new operators for further refinement;
    - T. Telksnys, together with Z. Navickas, worked on presenting the properties of fractional differential operators and was responsible for most of the proofs pertaining to such properties. He also performed computer algebra and numerical computations present in the paper, wrote the entire manuscript text and corresponded with the editorial office;
    - R. Marcinkevičius advised on the adaptation of the already existing computer algebra algorithms for fractional differential equations and performed some initial computations on the Riccati equation;
    - M. Ragulskis, as the leader of the research group, organized seminars and oversaw the writing of the paper. He also came up with the idea to adapt existing algorithms for the research of

fractional differential equations and suggested the Riccati equation as a fitting illustration for the refined operator scheme.

1.2. Main objective and scientific novelty of the thesis

The main objective of this thesis is to present a novel mathematical and computational framework for the construction of solitary solutions to various types of nonlinear differential equations, including ordinary differential equations (ODE), partial differential equations (PDE), and fractional differential equations (FDE).

The implementation of this objective was accomplished by completing several tasks:

1) **Investigation of the existence of solitary solutions in various classes of partial differential equations and their systems** (papers X1 and X2).
This step is necessary to determine the types of differential equations that may admit solitary solutions. Furthermore, the techniques implemented for the completion of this task were later applied to determine necessary existence conditions of solitary solutions and, in a few select cases, to directly construct these solutions.
2) **Development of direct techniques for the construction of solitary solutions** (paper X3).
Direct techniques allow to construct the solution by considering the solitary solution as an ansatz and by using computer algebra systems to determine the solution parameters. The drawbacks of such techniques and the limits of their application(s) are discussed.
3) **Development of algebraic operator techniques for the construction of solitary solutions to systems of ordinary and partial differential equations** (paper X4).
Building on the results of the first two tasks, more advanced techniques for the construction of solitary solutions can be created. These techniques are based on algebraic operator methods, most prominently, the generalized differential operator method. In conjunction with computer algebra, it allows the construction of solitary solutions to a plethora of different models.
4) **Extension of algebraic operator techniques for the construction of solitary solutions to fractional differential equations** (paper X5).
Fractional differential equations possess fundamental differences compared to their non-fractional counterparts and thus require extensive adaptation of techniques developed while completing the previous task to construct solitary solutions.

The scientific novelty of the research presented in this thesis is as follows. In papers X1 and X2, novel techniques were created and used to identify and formulate previously unknown necessary existence conditions for solitary solutions in systems of partial differential equations. In paper X3, a new technique is used to construct previously unknown analytical solutions to a class of Riccati equations with variable coefficients. In paper X4, a hepatitis C evolution model is considered. Novel solitary solutions to the model were constructed and shown to correspond to a set of biomedically relevant system parameters. In paper X5, solitary solutions to nonlinear fractional differential equations are considered. To adapt the algebraic operator-based approach to this class of problems, a number of classical definitions are restated by using operator calculus. Furthermore, it is shown that a class of Riccati fractional differential equations can be transformed into ordinary differential equations under special conditions that are derived by using the described novel technique.

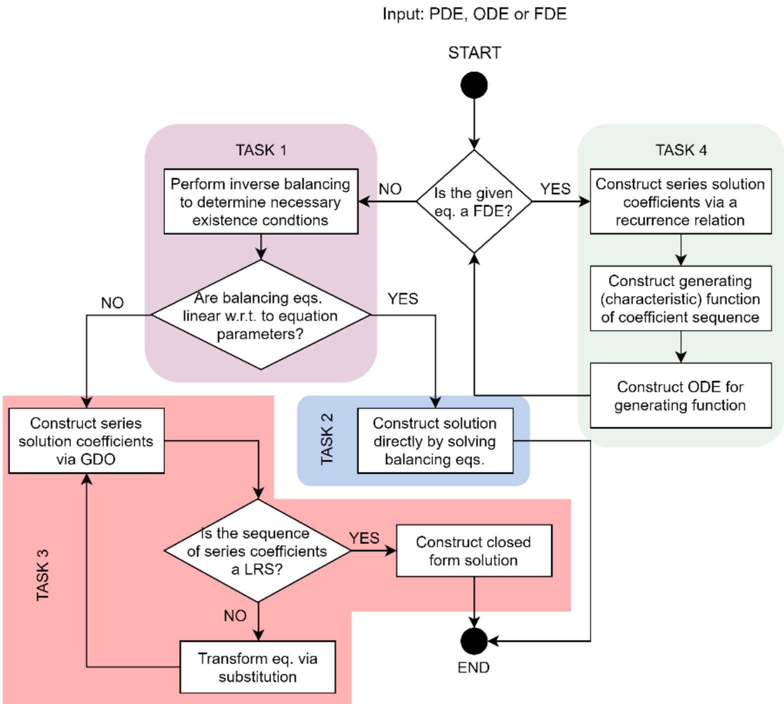


Fig. 1.2.1. Schematic diagram depicting the relations between the four tasks posed in this thesis. Differently colored backgrounds correspond to different tasks.

A schematic diagram is depicted in Fig. 1.2.1 that shows how the four tasks are related and showcases the entirety of the proposed computational scheme.

To summarize, this thesis contains novel techniques that use operator calculus in conjunction with computer algebra for the construction of analytical solutions to various types of differential equations. This computational framework was applied to various models, including real-world systems, to construct novel solitary solutions which provide more insight into the characteristics of the considered system.

1.3. Literature review

Solitary solutions and their applications. With recent advances in computer algebra software power and availability over recent years, interest in analytical solitary solution construction to differential equations has greatly increased. While observations of the phenomena of solitary solutions (also called solitons) date back to John Scott Russel's paper published in the nineteenth century (Russell, 1844) as well as Korteweg and de Vries' early mathematical description (Korteweg & De Vries, 1895), the importance of such phenomena was underlined only in the middle of the twentieth century in seminal publications by Zabusky and Kruskal (Zabusky & Kruskal, 1965) and Gardner (Gardner, Greene, Kruskal, & Miura, 1974).

A solitary solution is a nonlinear wave that satisfies the two following properties: 1) It maintains its shape as it moves at a constant velocity; 2) After colliding with another solitary solution, it emerges from the collision with no change, except for a phase shift (Scott, 2006). These properties make solitary solution phenomena important to identify in a number of different fields of research. A review of recent works in this field by foreign scientists is given below. To the best of the author's knowledge, there is no research being performed in Lithuania on the construction of analytical, non-numerical solitary solutions.

In this thesis, the solitary solution is considered an analytical solution to a nonlinear differential equation of any type, including ordinary, partial or fractional order equations which has the following form:

$$w(\xi) = \sigma \frac{\prod_{j=1}^{l}\left(\exp\left(\eta(\xi - c)\right) - y_j\right)}{\prod_{j=1}^{l}\left(\exp\left(\eta(\xi - c)\right) - x_j\right)}, \tag{1.3.1}$$

where η, c are constants, and parameters y_j, x_j depend on the initial conditions formulated for the considered differential equation. Variable ξ can be an independent variable if the considered differential equation is ordinary or single-variable fractional order equation. If the considered model is a partial differential equation, ξ is a linear combination of all independent variables and is called the wave variable.

Even though the effect of solitary solutions was first observed as a water wave, one of most vibrant fields of applications of such solutions is nonlinear optics. Stable localized pulses propagating in optical fibers are described by solitary solutions in (Kruglov & Harvey, 2018). Solitary solutions to a nonlinear Schrodinger equation that describes attosecond pulses in optical fibers are obtained in (Yang, Gao, Su, Zuo, & Feng, 2017). Solitary solutions to the Sasa-Satsuma equation which model the behavior of ultrashort pulses in optical fibers under Raman scattering effects are discussed in (L. Liu, Tian, Chai, & Yuan, 2017). Dark soliton solutions of the coupled higher-order Schrodinger equations that describe ultrashort pulse propagation in the birefringent or two-mode fiber are investigated in (Sun, Tian, Wang, & Zhen, 2015) and (Yuan, Tian, Liu, Sun, & Du, 2018). Solitary solutions to the Ginzburg-Landau equation which model light propagation in the presence of a detuning factor are studied for several cases of nonlinear optical fibers in (Inc, Aliyu, Yusuf, & Baleanu, 2018). Experimental observation of the real-time behavior of a soliton particle in ultrafast fiber laser is described in (M. Liu *et al.*, 2018).

There has been significant research on solitary solutions in a plethora of other fields, for example, biomedicine. It is shown that populations of bacteria are able to break through a circle-shaped corral by launching solitary waves (Morris *et al.*, 2017). The spatio-temporal dynamics of systems of differential equations describing interacting populations is studied by constructing solitary solutions in (Vitanov, Jordanov, & Dimitrova, 2009). The transmissions of signals between neurons is modeled via soliton-like impulses in (Poznanski *et al.*, 2017). A new model for the propagation of nerve impulses based on solitary solutions is proposed in (Engelbrecht, Peets, Tamm, Laasmaa, & Vendelin, 2018).

Beginning with the Gross-Pitaevskii model (Gross, 1961), solitary solutions have been one of the main tools in the analysis of Bose-Einstein condensates. The scattering of a two-soliton particle by Gaussian potential is studied in (Umarov, Aklan, Baizakov, & Abdullaev, 2016). Dark and gray solitary-like states have been shown to emerge in Bose gas placed in a toroidal trap (Shamailov & Brand, 2019). The stability of non-autonomous bright solitons in Rabi coupled Bose-Einstein condensates is considered in (Kanna, Mareeswaran, & Mertens, 2017).

Methods for the construction of solitary solutions. The first mathematical model that possesses solitary solutions was studied by (Korteweg & De Vries, 1895) who suggested the following partial differential equation:

$$\frac{\partial w}{\partial t}+\frac{\partial^3 w}{\partial x^3}-6w\frac{\partial w}{\partial x}=0. \tag{1.3.2}$$

Korteweg and de Vries constructed the following solution to (1.3.2):

$$w(t,x)=-\frac{1}{2}c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct-a)\right), \tag{1.3.3}$$

where c, a are constants. We note that the solution can be also be written by denoting $\xi = x - ct$ as the wave variable:

$$w(\xi)=-\frac{1}{2}c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(\xi-a)\right). \tag{1.3.4}$$

In this thesis, we consider a generalization of (1.3.3), which is one the best-known analytical forms of solitary solutions (Scott, 2006):

$$w(\xi)=\sigma\frac{\prod_{j=1}^{l}\left(\exp\left(\eta(\xi-c)\right)-y_j\right)}{\prod_{j=1}^{l}\left(\exp\left(\eta(\xi-c)\right)-x_j\right)}, \tag{1.3.5}$$

where η, c are constants, and parameters y_j, x_j depend on the initial conditions.

A plethora of methods for the construction of (1.3.5) exist. They include classical methods, such as the inverse scattering transformation (Biondini & Kovačič, 2014), the Darboux transformation (Shi & Zhao, 2018), and the Backlund transformation (Carillo, 2019). These methods arise mainly from physical considerations and require comparatively little computation to be used. However, they are not as powerful as the new generation of techniques which are based mainly on the application of computer algebra. Such techniques include the Exp-function method (Ayub, Khan, & Mahmood-Ul-Hassan, 2017), the tanh method (Hu, Li, & Zhu, 2018), the $\frac{G'}{G}$ expansion method (Foroutan, Manafian, & Ranjbaran, 2018) and the simplest equation method (Vitanov, Dimitrova, & Vitanov, 2015).

These approaches are direct methods as they are ansatz-based, i.e., the form of the solution is taken to be (1.3.5), then the solution is inserted into the differential equation, and various algorithms are used in conjunction with computer algebra to determine the solution parameters. While it is possible to use this framework to construct solitary solutions, such methods have received a large amount of criticism for not considering conditions of the existence of a solution (1.3.5) in either the space of the initial conditions or in terms of the equation parameters (Popovych & Vaneeva, 2010), (Kudryashov, 2009), (Navickas & Ragulskis, 2009).

In this thesis, the techniques that utilize computer algebra but are not ansatz-based are constructed – the analytical form of the solitary solution is not guessed but is instead derived via operator methods during the construction of the solution.

2. REVIEW OF PAPERS

2.1. Review of "Existence of solitary solutions in a class of differential equations with polynomial nonlinearity"

Input from T. Telksnys: T. Telksnys performed the symbolic and numerical computations, as well as most of the mathematical derivations given in the paper. He also prepared the text together with the co-authors and corresponded with the editor of the journal.

Summary of the paper: The paper is concerned with determining the necessary existence conditions of solitary solutions to a family of partial differential equations with polynomial nonlinearity:

$$\frac{\partial^m u}{\partial t^m} + \sum_{s=1}^{m-1} \sum_{\substack{j+k=s \\ j,k\geq 0}} A_{j,k} \frac{\partial^s u}{\partial t^j \partial z^k} = a_n u^n + \cdots + a_0; \tag{2.1.1}$$

where $u = u(t, z)$ is the unknown function; $m, n \in \mathbb{N}$ and $A_{j,k}; a_l$ are any real coefficients.

It is convenient to transform (2.1.1) via the independent variable substitution $x = \kappa t + \omega z$, where κ, ω are non-zero real numbers. This procedure yields:

$$y_x^{(m)} + b_{m-1} y_x^{(m-1)} + \cdots + b_1 y_x' = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_0, \qquad (2.1.2)$$

where $y = y(\kappa t + \omega z) = u(t, z)$ is the transformed function and coefficients $b_l; l = 1, \dots, m-1$ are derived from the parameters of (2.1.1). Let us consider the following solitary solutions to (2.1.2):

$$y_0(x) = \sigma \frac{\prod_{j=1}^{l} \left(\exp\left(\eta(x - c)\right) - y_j\right)}{\prod_{j=1}^{l} \left(\exp\left(\eta(x - c)\right) - x_j\right)}, \qquad (2.1.3)$$

where $l \in \mathbb{N}; \sigma, \eta, c \in \mathbb{R}; \sigma, \eta \neq 0; y_j, x_j \in \mathbb{C}, j = 1, \dots l$. Parameter l denotes the order of the solitary solution; furthermore it is assumed that $x_j \neq y_k$ for any j, k.

The main objective of this paper is to determine the necessary existence conditions for (2.1.3) in (2.1.1) in terms of the solitary solution parameters. This goal is achieved by utilizing the inverse balancing technique which is first presented in this paper. The main idea of this technique is to invert the standard ansatz methods: solution expression (2.1.3) is inserted into equation (2.1.2), which yields a polynomial equation in $\hat{x} = \exp\left(\eta(x - c)\right)$ (the expressions for its coefficients α_j, β_j are given in paper X1):

$$\alpha_{(m+1)l} \hat{x}^{(m+1)l} + \cdots + \alpha_1 \hat{x} + \alpha_0 = \beta_{nl} \hat{x}^{nl} + \cdots + \beta_0. \qquad (2.1.4)$$

This polynomial equation can only hold true if the degree of polynomials on the left- and right-hand sides are balanced, thus yielding the first existence condition:

$$m + 1 = n \qquad (2.1.5)$$

Assuming (2.1.5) holds true, the coefficients of (2.1.4) yield a system of equations that depend linearly on the parameters of equation (2.1.2) $a_0, \dots, a_n; b_1, \dots, b_{n-1}$ and nonlinearly on solution parameters y_j, x_j. Thus it is not feasible to obtain a general solution for the solution parameters in terms of the equation parameters. However, the inverse is not true: equation parameters $a_0, \dots, a_n; b_1, \dots, b_{n-1}$ can be determined in terms of y_j, x_j by solving a system of linear equations. If this system is not consistent, it means that (2.1.1) cannot admit solitary solutions (2.1.3). It is derived in X1 that the necessary condition for the consistency of the aforementioned system of linear equations reads:

$$(m + 1)l \leq 2(l + m + 1). \qquad (2.1.6)$$

Thus the necessary existence conditions for solitary solutions to (2.1.1) are (2.1.5)–(2.1.6). It is notable that the left-hand side of (2.1.6) depends nonlinearly on l and m, while the opposite is true for the right-hand side, which leads to the conclusion that higher-order equations do not necessarily admit higher-order solutions. An outlook of possible solution and equation order pairs is given in Table 2.1.

Table 2.1. Table of necessary existence conditions of solitary solutions (2.1.3) in (2.1.1). Symbol $\exists$ denotes existence with any parameter values; $\exists^*$denotes existence with some parameters subject to constraints, and $\nexists$ denotes nonexistence.

l \ (n,m)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)	(7,6)	(8,7)
1	$\exists$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$

2	$\nexists$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$
3	$\nexists$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\nexists$	$\nexists$
4	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
5	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$

As seen in Table 2.1, some equations admit solutions only when the parameters of either the differential equation or solitary solution satisfy some additional constraints. It is not possible to provide a general expression of these constraints, but their derivation is equivalent to achieving a non-singular matrix for the determination of equation parameters when using the inverse balancing technique.

Conclusions: A computational framework for the determination of necessary existence conditions of solitary solutions to a class of partial differential equations with polynomial nonlinearity is presented in the paper. By applying the inverse balancing technique, it is proven that as the order of the equations increases, higher-order solitary solutions do not appear, but the opposite is true – the considered class of differential equations can only admit solitary solutions of order $l = 1{,}2{,}3$.

It is clear that the techniques presented in this paper provide a solid foundation for both direct and indirect solitary solution construction methods as the developed concept provides a simple way of verifying the necessary existence conditions for such solutions. Furthermore, the presented computations also demonstrate the derivation of additional constraints on the parameters of the equation that can be used to formulate not only necessary, but also sufficient solitary solution existence conditions. This idea is expanded upon in the other articles presented below.

2.2. Review of “Existence of solitary solutions in systems of PDEs with multiplicative polynomial coupling”

Input from T. Telksnys: T. Telksnys conceived the idea for the paper, performed the symbolic and numerical computations, as well as most of the mathematical derivations given in the paper. He also prepared the text together with co-authors and corresponded with the editor of the journal.

Summary of the paper: Results reviewed in Section 2.1. are extended to systems of partial differential equations in this paper. The following class of systems of partial differential equations is considered:

$$\sum_{j=1}^{n}\sum_{s=0}^{j} a_{s,j-s}\frac{\partial^j U}{\partial x^s \partial t^{j-s}} = \sum_{j=0}^{k}\sum_{s=0}^{j} b_{j-s,s} U^{j-s} V^s ;$$
$$\sum_{j=1}^{m}\sum_{s=0}^{j} c_{s,j-s}\frac{\partial^j V}{\partial x^s \partial t^{j-s}} = \sum_{j=0}^{l}\sum_{s=0}^{j} d_{j-s,s} U^{j-s} V^s , \qquad (2.2.1)$$

where $U = U(t,x); V = V(t,x)$ are unknown functions, and $a_{s,j-s}, b_{j-s,s}, c_{s,j-s}, d_{j-s,s}$ are arbitrary real coefficients considered to be non-zero at the highest derivative order. We note that system (2.2.1) is an extension to (2.1.1) presented in the previous section: the left-hand sides remain unchanged, while the right-hand sides exhibit multiplicative coupling. As noted earlier, this type of coupling has seen extensive use in modeling biological real-world systems (Kraenkel, Manikandan, & Senthilvelan, 2013).

The wave variable substitution $z = \alpha t + \beta x$ is used to transform the system of PDEs (2.2.1) into the following ODE system:

$$u_z^{(n)} + a_{n-1} u_z^{(n-1)} + \cdots + a_1 u_z' = \sum_{j=0}^{k} \sum_{s=0}^{j} b_{j-s,s} u^{j-s} v^s ;$$
$$v_z^{(m)} + c_{m-1} v_z^{(m-1)} + \cdots + c_1 v_z' = \sum_{j=0}^{k} \sum_{s=0}^{j} d_{j-s,s} u^{j-s} v^s , \qquad (2.2.2)$$

where $u = u(z) = U(t,x); v = v(z) = V(t,x)$, and coefficients a_j, c_j are linear combinations of $a_{j-s,s}$ and $c_{j-s,s}$, respectively. As in the previous paper, solitary solutions of the following form are sought:

$$u_0(z) = \sigma \frac{\prod_{j=1}^{N} \left(\exp\left(\eta(z - z_0)\right) - u_j\right)}{\prod_{j=1}^{N} \left(\exp\left(\eta(z - z_0)\right) - z_j\right)};$$
$$v_0(z) = \gamma \frac{\prod_{j=1}^{N} \left(\exp\left(\eta(z - z_0)\right) - v_j\right)}{\prod_{j=1}^{N} \left(\exp\left(\eta(z - z_0)\right) - z_j\right)}. \qquad (2.2.3)$$

Here, σ, γ, η are real constants, and parameters u_j, v_j, z_j may depend on the initial conditions imposed on (2.2.2). We note that since this study is concerned primarily with the existence of solutions, we do not consider the set viable initial conditions – this will be discussed in further sections. The main objective of this paper is to determine the necessary existence conditions of solitary solutions (2.2.3) in (2.2.1).

By using a version of the inverse balancing technique described in Section 2.1 that is modified for systems of differential equations, the following balancing conditions between the multiplicative nonlinearity order and the highest derivative order are computed:

$$k = n + 1; \quad l = m + 1 \qquad (2.2.4)$$

We note that these conditions are analogous to (2.1.5), which means that the relation between derivative and highest order nonlinear terms remains unchanged when comparing systems of PDEs with a single PDE. However, computer algebra computations using the inverse balancing technique provide the following necessary existence conditions for (2.2.3) in (2.2.1):

$$2(n + m - 1)N - 2 \leq (n + 2)^2 + (m + 2)^2 + 3(n + m). \qquad (2.2.5)$$

It is clear that the nature of condition (2.2.5) is fundamentally different from that of (2.1.6) as it allows for the growth of solitary solution order because the order of the PDE system increases, as illustrated in Fig. 2.2.1. Furthermore, if (2.2.5) holds together with the inequality:

$$2(n+m+2)N+4 \leq (n+2)^2+(m+2)^2+3(n+m), \tag{2.2.6}$$

then, solitary solutions exist without the need to impose additional constraints on the parameters of the solution (as discussed in Section 2.1). We note that since the most common case of system (2.2.1) occurs when the differential equations are of the same order, that is, $m=n$, conditions (2.2.5) and (2.2.6) can be simplified to:

$$(2n-1)N-1 \leq (n+2)^2+3n;$$
$$(2n+2)N+2 \leq (n+2)^2+3n \tag{2.2.7}$$

respectively. A graphical representation of these conditions is given in Fig. 2.2.2.

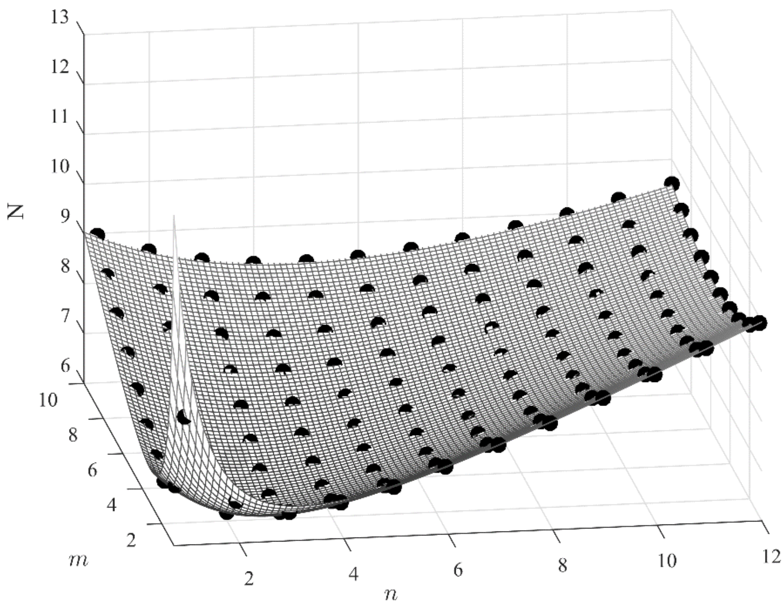


Fig. 2.2.1. Plot illustrating condition (2.2.5). For given orders n, m of the considered differential equations, the necessary existence conditions of solitary solutions can only be satisfied if the solution order N lies at or below the depicted surface. Black circles denote the maximum value of N for integer values of n, m.

Conclusions. The necessary conditions of existence for solitary solutions in systems of partial differential equations coupled with multiplicative terms have been derived by using the inverse balancing technique.

While uncoupled PDEs with equivalent nonlinear terms can only admit solitary solutions of the order no greater than $N=3$, this result does not hold true for systems of coupled PDEs: as the system order increases, the order of solitary solutions that satisfy the necessary existence conditions also grows, thus revealing more complex phenomena in the considered equations. For example, first order coupled systems may admit solitary solutions of up to order $N = 13$, whereas second order coupled systems may admit solitary solutions of up to order $N = 7$.

Some higher order solutions may require additional constraints to be imposed on the parameters of the solutions in order to satisfy the necessary existence conditions. A general form for these conditions does not exist, however, the inverse balancing technique can be used to derive them case-by-case, as shown in paper X2.

Thus it is clear that the presented framework is a solid foundation that can be employed in direct solution construction methods as *a priori* consideration of the necessary existence conditions can greatly reduce the complexity of the required computations.

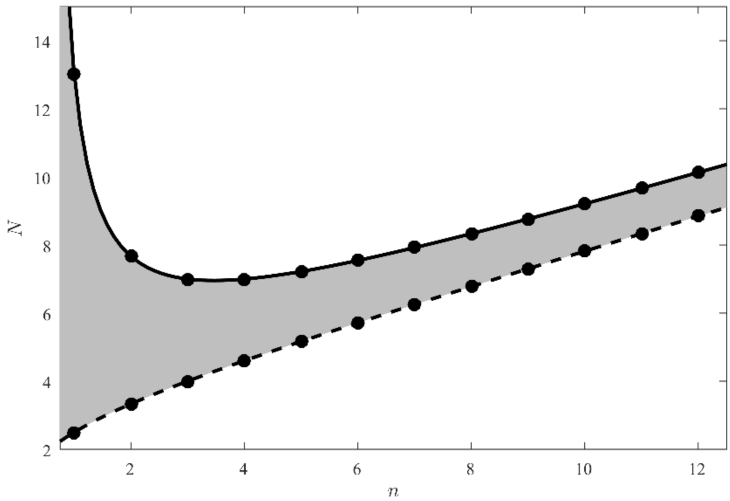


Fig. 2.2.2. Plot illustrating conditions (2.2.7). The necessary existence condition of solitary solutions is only satisfied if N does not lie above the solid black curve. Furthermore, if N lies below the dashed line, no additional constraints on the parameters of the solitary solutions need to be imposed. Solitary solutions can only exist with additional constraints on their parameters for values of N lying inside the shaded region. Black circles indicate integer values of equation order n.

2.3. Review of "Kink solitary solutions to generalized Riccati equations with polynomial coefficients"

Input from T. Telksnys: T. Telksnys performed all of the numerical and the computer algebra computations required to perform this study. He was also responsible for a significant part of mathematical proofs and derivation presented in the paper. He co-wrote the paper text and corresponded with the editors of the journal.

Summary of the paper: The previous two sections discuss articles where the main focus is to determine the necessary existence conditions of solitary solutions to differential equations. The next question that can be posed is: what techniques could be used to construct such solutions? As mentioned in the introduction, several direct solution construction methods have been described in literature, but they have been heavily criticized for not considering the existence conditions of the constructed solution.

In this paper, in addition to the inverse balancing technique presented in the previous sections, a direct balancing approach is used to construct kink solitary solutions to generalized Riccati equations with polynomial coefficients. As well as constructing the solutions, this paper also showcases the limitations of direct solution construction methods as the equations used to determine the parameters of solitary solutions must remain linear in order to ensure that all the possible solutions have been discussed.

The following class n-th order Riccati-like ordinary differential equations is considered:

$$\left(\sum_{k=0}^{r} b_k x^k\right) y_x^{(m)} = \sum_{l=0}^{n} a_l y^l\,;\ \ r, m \in \mathbb{N} \tag{2.3.1}$$

where a_l, b_k are real coefficients, and $y = y(x)$ is the unknown function. Kink solitary solutions have the following form:

$$y_0(x) = \sigma \frac{\exp\big(\eta(x - x_0)\big) - y_1}{\exp\big(\eta(x - x_0)\big) - x_1}, \tag{2.3.2}$$

where η, σ are constants, and y_1, x_1 may depend on the initial conditions. We note that we do not seek solutions of the form (2.3.2) to (2.3.1). Equations of the form (2.3.1) arise after applying substitution $\hat{x} = \exp\,(\eta x)$. Thus it is possible to consider the transformed form of (2.3.2) which is a rational function:

$$y_0(x) = \frac{\alpha_1 x - \alpha_0}{\beta_1 x - \beta_0}\,;\ \alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}. \tag{2.3.3}$$

While solutions (2.3.3) seem simple, the physical and mathematical properties of both (2.3.3) and its untransformed form (2.3.2) have received attention in recent publications (Yamaleev, 2014).

As the inverse balancing technique has already been discussed in Sections 2.1 and 2.2, presently, we only summarize that inserting solution (2.3.3) into (2.3.1) and solving for coefficients a_l, b_k yields the following necessary conditions for the existence of (2.3.3) in (2.3.1):

$$n = m + 1; \quad\ 0 \le r \le m + 1. \tag{2.3.4}$$

The direct balancing approach for equation (2.3.1) is similar to the inverse balancing technique; however, in this case, we seek to determine the parameters of solution (2.3.3). We note that (2.3.1) can be rearranged as:

$$b_r \prod_{k=1}^{r} (x - x_k)\, y_x^{(m)} = \prod_{l=1}^{m+1} (y - y_l)\,, \tag{2.3.5}$$

where x_k, y_l denote the roots of the right and left side polynomials in (2.3.1). Inserting solution (2.3.3) into (2.3.5) yields equations that are linear in both the equation coefficients and the solution parameters. This is very important to note – as otherwise direct balancing would be impossible. Explicit expression of the obtained equations reads:

$$\boldsymbol{A}(\tau)\,\boldsymbol{p}(\tau) = \boldsymbol{0}, \tag{2.3.6}$$

where

$$\boldsymbol{A}(\tau) = \begin{bmatrix} x_1 & -1 & -x_1 y_{\tau(1)} & y_{\tau(1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_r & -1 & -x_r y_{\tau(r)} & y_{\tau(r)} \\ 1 & 0 & -y_{\tau(r+1)} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & -y_{\tau(m+1)} & 0 \end{bmatrix};\ \boldsymbol{p}(\tau) = \begin{bmatrix} \alpha_1(\tau) \\ \alpha_0(\tau) \\ \beta_1(\tau) \\ \beta_0(\tau) \end{bmatrix}, \tag{2.3.7}$$

where τ is an arbitrary permutation of order $m+1$. We note that the number of linear systems is equal to $(m+1)!$, which coincides with the maximum number of solitary solutions (2.3.3) to (2.3.1). An additional constraint that must be satisfied by the scaling coefficient b_r is also obtained by the direct balancing technique:

$$b_r = \frac{(-1)^{1-r}}{m!\,(\alpha_1\beta_0 - \alpha_0\beta_1)\beta_1^{m-1}} \left(\prod_{l=1}^{r} \left(\alpha_1 - y_{\tau(l)}\beta_1\right)\right)\left(\prod_{j=r+1}^{m+1} \left(\alpha_0 - y_{\tau(j)}\beta_0\right)\right). \tag{2.3.8}$$

Thus the generalized Riccati equation with polynomial coefficients (2.3.1) admits kink solitary solutions (2.3.3) if and only if conditions (2.3.4), (2.3.6) and (2.3.8) hold for some permutation τ.

Conclusions: It is shown that the generalized Riccati equation admits kink solitary solutions under the derived conditions. The maximum number of possible kink solutions is $(m+1)!$ and depends on the order of the generalized Riccati equation.

The results have been obtained by using inverse and direct balancing techniques. It was possible to apply direct balancing because the balancing equations were linear with respect to both equation coefficients and solution parameters. Attempting to construct higher order solitary solutions would result in high-dimensional systems of nonlinear algebraic equation thus making the determination of the solution parameters problematic. The solution to this problem is given in the next section by considering operator methods to obtain more complex solitary solutions not only to single differential equations but also to systems of differential equations.

2.4. Review of "Homoclinic and heteroclinic solutions to a hepatitis C evolution model"

Input from T. Telksnys: T. Telksnys performed most of the numerical and all of computer algebra computations required to conduct this study. He was also responsible for a significant part of mathematical proofs and derivation present in the paper. He co-wrote the paper text and corresponded with the editors of the journal.

Summary of the paper: In this paper, a real-world problem is investigated: given a clinically verified model of hepatitis C virus (HCV) evolution (Reluga, Dahari, & Perelson, 2009), we construct higher order than kink (as discussed in the previous section) solitary solutions, which would provide insight into the evolution of healthy and infected cell population during hepatitis C virus treatment.

A note on terminology: in this paper, solitary solutions are referred to as homoclinic (a solution that corresponds to a trajectory which joins a saddle point in the phase space to itself) and heteroclinic (a solution that corresponds to a trajectory which joins any two equilibrium points in the phase space) solutions. This was done (at a reviewer's request) to further emphasize the focus that was put into the analysis of the phase plane of the considered system. However, the solutions constructed in the paper are solitary solutions, and, to avoid confusion, we shall refer to them as such in this brief review.

The following non-dimensionalized system for hepatitis C virus evolution is considered:

$$\begin{aligned} x'_\tau &= a_0 + a_1 x + a_2 x^2 + a_3 xy + a_4 y; \qquad x(c) = u; \\ y'_\tau &= b_0 + b_1 y + b_2 y^2 + b_3 xy + b_4 x; \qquad y(c) = v, \end{aligned} \tag{2.4.1}$$

where $a_0, \ldots, a_4, b_0, \ldots, b_4$ are arbitrary real coefficients; functions $x(\tau), y(\tau)$ represent dimensionless state variables for uninfected hepatocites and infected cells, respectively; c, u, v denote the initial conditions. We note that system (2.4.1) is more general than the hepatitis C virus evolution model presented in (Reluga *et al.*, 2009) and thus can be applied to other fields where systems of form (2.4.1) are studied.

The following second order (also called bright/dark (Scott, 2006)) solitary solutions to (2.4.1) are considered:

$$\begin{aligned} x(\tau; c, u, v) &= \sigma \frac{\left(\exp\left(\eta(\tau-c)\right) - x_1\right)\left(\exp\left(\eta(\tau-c)\right) - x_2\right)}{\left(\exp\left(\eta(\tau-c)\right) - \tau_1^{(x)}\right)\left(\exp\left(\eta(\tau-c)\right) - \tau_2^{(x)}\right)}; \\ y(\tau; c, u, v) &= \gamma \frac{\left(\exp\left(\eta(\tau-c)\right) - y_1\right)\left(\exp\left(\eta(\tau-c)\right) - y_2\right)}{\left(\exp\left(\eta(\tau-c)\right) - \tau_1^{(y)}\right)\left(\exp\left(\eta(\tau-c)\right) - \tau_2^{(y)}\right)}, \end{aligned} \tag{2.4.2}$$

where σ, γ, η are constants, and parameters $x_1, x_2, y_1, y_2, \tau_1^{(x)}, \tau_2^{(x)}, \tau_1^{(y)}, \tau_2^{(y)}$ can depend only on initial conditions u, v (but not on c as (2.4.1) is autonomous).

As in the previous publications, the inverse balancing technique is used to determine the necessary existence conditions for (2.4.2) in (2.4.1). It is obtained that the denominators of x and y must be equal in order for the solution to exist:

$$\tau_1^{(x)} = \tau_1^{(y)}; \quad \tau_2^{(x)} = \tau_2^{(y)}. \tag{2.4.3}$$

Furthermore, the coefficients of system (2.4.1) must also obey the following relations:

$$a_3 = b_2; \quad a_2 = b_3. \tag{2.4.4}$$

Next, generalized differential operator techniques are used to determine the sufficient existence conditions of solitary solutions (2.4.2) to any system of ordinary differential equations. We shall omit the statement of these conditions for the sake of brevity (a detailed description can be found in paper X4) and instead summarize the necessary and sufficient existence conditions in the following theorem:

Theorem 2.4.1. Hepatitis C virus evolution model (2.4.1) admits second order solitary solutions (2.4.2) if and only if (2.4.3), (2.4.4) hold true and:

$$\begin{aligned} &9a_0a_1a_2 + 9b_0b_1b_2 - 18a_0a_2b_1 - 18b_0b_2a_1 + 3a_1b_1^2 \\ &+3b_1a_1^2 - 2a_1^3 - 2b_1^3 - 9a_1a_4b_4 - 9b_1b_4a_4 + 27a_0b_2b_4 \\ &+27b_0a_2a_4 = 0. \end{aligned} \quad (2.4.5)$$

It must be noted that condition (2.4.5) requires a significant effort of symbolic (non-numerical) computation to derive. The computer algebra system *Maple* was used to perform these computations.

If the conditions of this theorem hold true, all solutions with any initial conditions c, u, v to (2.4.1) are second order solitary solutions. It is shown that their phase trajectories take the form of conic sections:

$$Ax^2 + Bxy + Cy^2 + Ex + Fy + G = 0, \quad (2.4.6)$$

where coefficients A, B, C, E, F, G depend on the initial conditions.

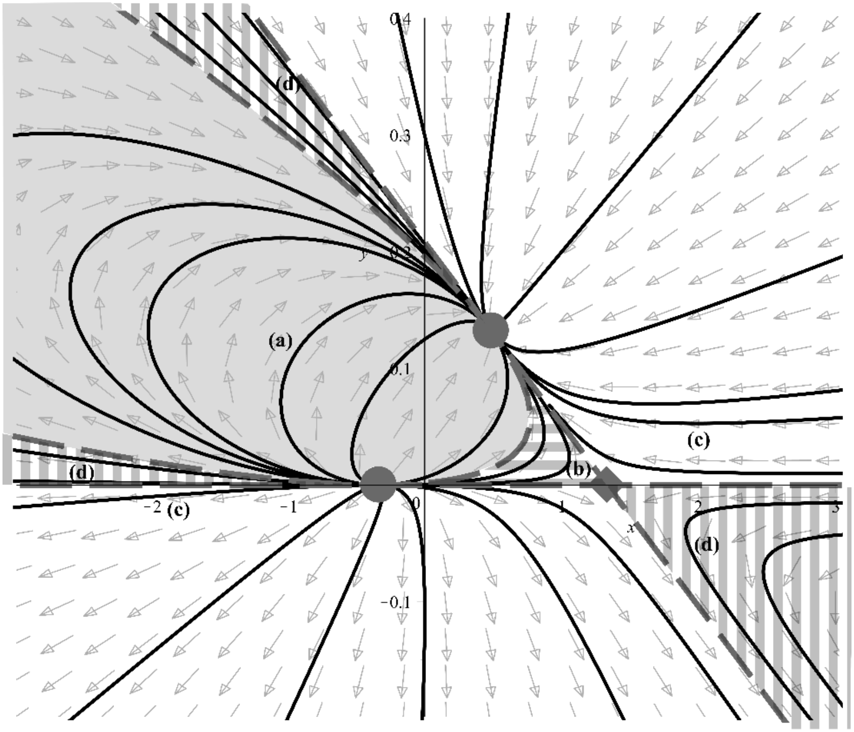


Fig. 2.4.1. Phase portrait of system (2.4.1) for the case of biologically relevant parameter values. Grey circles denote stable and unstable nodes. The diamond denotes the saddle point. Solid black lines denote solution trajectories which are conic sections and correspond to a pair of second order solitary solutions. Labels (a), (b), (c), (d) correspond to respective solutions in Fig. 2.4.2. The solutions lying in the grey-shaded region have elliptic trajectories and contain no singularities. Solutions in the horizontally striped region have hyperbolic trajectories and

also contain no singular points. Solutions in the unshaded region have a single singularity point. Solutions in the vertically striped regions have two singular points.

By using the results of Theorem 2.4.1 and the parameter values that correspond to biologically meaningful cases of model (2.4.1) (provided in *(Reluga et al*., 2009)), the phase portrait of system (2.4.1) given in Fig. 2.4.1 is obtained. The evolution of solutions in time is depicted in Fig. 2.4.2. There exist three distinct types of solitary solutions – solutions that have no singularities (denoted as (a) in Figs. 2.4.1 and 2.4.2); solutions with one singular point (denoted as (b)); and solutions with two singularities (denoted as (c)). There are three equilibria in the considered system – an unstable node, a stable node, and a saddle point. We note that the stable and unstable manifolds of the saddle point (straight lines), as well as the separatrix between the singular and non-singular solutions (a rotated parabola) are computed exactly from the equation parameters by using computer algebra.

Conclusions: Second order solitary solutions to the model hepatitis C virus evolution have been constructed. The necessary and sufficient second order solitary solution existence conditions with respect to the model and solution parameters have been derived by using the inverse balancing and the generalized differential operator techniques. It has been shown that the obtained solutions exist for all parameter values and that their trajectories correspond to conic sections. Exact equations for the stable and unstable manifolds and the separatrix between singular and non-singular solutions are also derived by using the presented techniques.

The obtained solutions have some important properties that may help to better understand the phenomena of hepatitis C virus evolution. Even though the initial treatment reduces the number of infected cells (which can be seen by comparing the beginning and end of the treatment in Fig. 2.4.2), in some cases (see Fig. 2.4.2 (b)), the number of the infected cells initially increases during the treatment, reaches a maximum point, and then declines to its final state. Analogous observations may be made for the population of healthy cells – their number may initially decrease and reach a minima before increasing and reaching their stable state. Such observations provide a deeper insight into the processes governed by model (2.4.1) and may help in planning patient treatment.

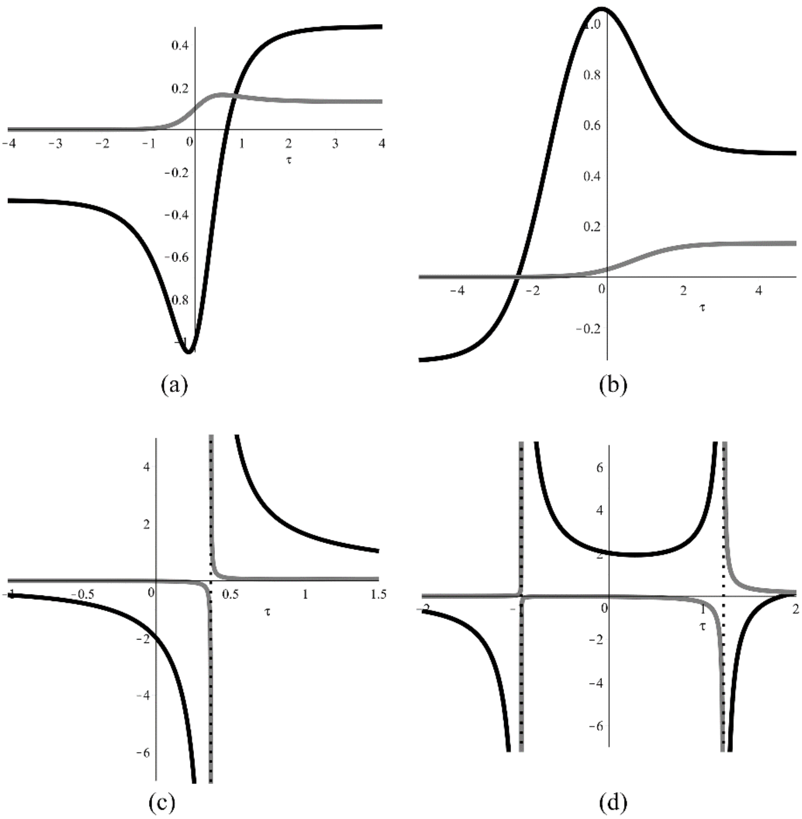


Fig. 2.4.2. Second order solitary solutions corresponding to the labeled solutions in phase portrait Fig. 2.4.1. The black line corresponds to $x(\tau)$, the grey line corresponds to $y(\tau)$. Thin dotted lines denote the singularity points of solutions (c), (d).

2.5. Review of "Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations"

Input from T. Telksnys: T. Telksnys performed all the computer algebra and numerical computations presented in the paper. He also authored some of the proofs that are given. On top of that, he wrote the paper text and corresponded with the editors of the journal.

Summary of the paper: In this paper, the concept of soliton solutions is extended beyond the standard notions of ordinary and partial differential equations. An approach for constructing solitary solutions to fractional order differential equations is discussed. We note that as the paper is the first in a series concerning fractional differential equations, the order of differentiation is fixed at $\alpha = \frac{1}{2}$. However, the presented definitions and proofs are easily generalized to general α, which was actually done in subsequent papers (Navickas, Telksnys, Timofejeva, Marcinkevičius, & Ragulskis, 2018).

One of the main concepts in use is the fractional power series that reads:

$$f = \sum_{j=1}^{+\infty} c_j z_j ; \quad z_j = \frac{x^{\frac{j-1}{2}}}{\Gamma\left(\frac{j+1}{2}\right)}. \tag{2.5.1}$$

Functions z_j are called the basis functions, and the set of all f described by (2.5.1) is called the set of Caputo fractional power series and denoted ${}^C\mathbb{F}$. The conventional definition of the Caputo fractional derivative reads:

$$ {}^c\mathbf{D}^{\left(\frac{1}{2}\right)} f(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x \frac{f'(t)}{(x-t)^{\frac{1}{2}}} \mathrm{d}x. \tag{2.5.2}$$

We propose an alternative but equivalent definition based on (2.5.1):

$$ {}^c\mathbf{D}^{\left(\frac{1}{2}\right)} f = \sum_{j=1}^{+\infty} c_{j+1} z_j, \tag{2.5.3}$$

where ${}^c\mathbf{D}^{\left(\frac{1}{2}\right)} z_j = z_{j-1}$ for $j = 2,3, \ldots$ and ${}^c\mathbf{D}^{\left(\frac{1}{2}\right)} z_1 = 0$. Properties of operator ${}^c\mathbf{D}^{\left(\frac{1}{2}\right)}$ are described in detail in paper X5.

The main idea in identifying solitary solutions to nonlinear fractional differential equations is to consider the equivalent of well-known non-fractional differential equations that admit solitary solutions. The simplest model that possesses this property is the Riccati differential equation. A fractional form of this equation is considered:

$$ z_{k+1} \left({}^c\mathbf{D}^{\left(\frac{1}{2}\right)} \right)^k y = A_0 + A_1 y + A_2 y^2, \tag{2.5.4}$$

where $A_0, A_1, A_2 \in \mathbb{C}$ are, in general, complex coefficients; the solution is $y \in {}^C\mathbb{F}$, and $k \in \mathbb{N}$.

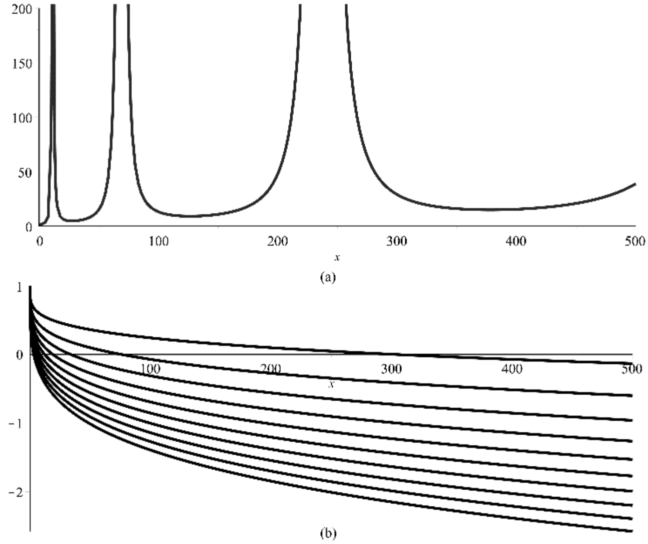


Fig. 2.5.1. Solution (2.5.8) with $\alpha = 1$ (part (a)) and $\alpha = -10, -9, \ldots, -1$ (part (b)). We note that the solution is non-singular only for the negative values of the initial condition alpha.

Initial conditions on (2.5.4) read:

$$ {}^{c}\mathbf{D}^{\left(\frac{1}{2}\right)} y|_{x=0} = \alpha \in \mathbb{R}. \tag{2.5.5} $$

By using computer algebra in conjunction with operator techniques based on definition (2.5.3), three solutions for different values of k are obtained. In the case $k = 1$, the Riccati-like fractional equation (2.5.4) admits solution:

$$ y = \frac{1 - A_1}{2A_2} + \frac{2}{A_2 \pi} \sum_{j=1}^{+\infty} j \nu_j^{(1)} \left(\frac{2A_2\alpha}{\sqrt{\pi}} \right)^j x^{\frac{j}{2}}, \tag{2.5.6} $$

where coefficients $\nu_j^{(1)}$ are solutions to the recurrence relation:

$$ \nu_j^{(1)} = \frac{1}{\binom{j/2}{1/2} - 1} \sum_{k=1}^{m} \nu_k^{(1)} \nu_{j-k}^{(1)}; \quad \nu_1^{(1)} = 1. \tag{2.5.7} $$

We note that the closed form of solution (2.5.6) could not be obtained. However, the fractional power series (2.5.6) gives the exact analytical solution. In the case $k = 2$, equation (2.5.4) does not take the form of an ordinary differential equation, as, for $y \in {}^{C}\mathbb{F}$, operator $\left({}^{c}\mathbf{D}^{\left(\frac{1}{2}\right)}\right)^2$ is not equivalent to $\frac{\mathrm{d}}{\mathrm{d}x}$ (equivalence would only hold if every coefficient corresponding to a fractional power of x in power series (2.5.1) would be zero). Inserting the series solution $y = \sum_{j=1}^{+\infty} c_j z_j$ into (2.5.4) with $k = 2$ and rearranging both sides of the obtained equation yields coefficients for the series solution which can then be written as:

$$y = -\frac{A_1}{2A_2} + \frac{1}{A_2\pi}\sum_{j=1}^{+\infty} j\nu_j^{(2)}\left(\frac{4A_2\alpha}{\sqrt{\pi}}\right)^j x^{\frac{j}{2}}, \tag{2.5.8}$$

where $\nu_j^{(2)}$ satisfies the following recurrence relation:

$$\nu_1^{(2)} = 1; \quad \nu_j^{(2)} = \frac{1}{j}\sum_{k=1}^{j-1} \nu_k^{(2)}\nu_{j-k}^{(2)}. \tag{2.5.9}$$

Let us consider the generating function of coefficients $\nu_j^{(2)}$:

$$w = w(t) = \sum_{j=0}^{+\infty} \nu_j^{(2)} t^j; \quad \nu_0^{(2)} = 0. \tag{2.5.10}$$

Differentiating (2.5.10) results in:

$$t\frac{\mathrm{d}w}{\mathrm{d}t} = w^2 + t, \tag{2.5.11}$$

which has the general solution:

$$w(t) = \frac{\sqrt{t}\left(CY_1(2\sqrt{t}) + J_1(2\sqrt{t})\right)}{CY_0(2\sqrt{t}) + J_0(2\sqrt{t})}, \tag{2.5.12}$$

where $J_\beta(t)$ are Bessel functions of the first kind:

$$J_\beta(t) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\;\Gamma(m+\beta+1)}\left(\frac{t}{2}\right)^{2m+\beta}; \tag{2.5.13}$$

and $Y_\beta(t)$ are Bessel functions of the second kind:

$$Y_\beta(t) = \frac{J_\beta(t)\cos(\beta\pi) - J_{-\beta}(t)}{\sin(\beta\pi)}. \tag{2.5.14}$$

Considering the initial condition $\left.\frac{w^2}{t}\right|_{t=0} = 0$ (which is obtained from noting that $\nu_1^{(2)} = 1, \nu_0^{(2)} = 0$) yields $C = 0$ in (2.5.12). We note that taking $t = \frac{4A_2\alpha}{\sqrt{\pi}}\sqrt{x}, a_0 = -\frac{A_1}{2A_2}, a_1 = \frac{1}{A_2\pi}$ yields $y = y(x) = a_0 + a_1 t\frac{\mathrm{d}w}{\mathrm{d}t}$.

Finally, the closed-form solution can be written in terms of Bessel functions:

$$y = \tilde{y}\left(\frac{4A_2\alpha}{\sqrt{\pi}}\sqrt{x}\right); \quad \tilde{y}(t) = -\frac{A_1}{2A_2} + \frac{1}{A_2\pi}\frac{t\left(J_0^2(2\sqrt{t}) + J_1^2(2\sqrt{t})\right)}{J_0^2(2\sqrt{t})}, \tag{2.5.15}$$

A plot of solution (2.5.15) for various values of α is given in Fig. 2.5.1. We note that the solution is non-singular only for $\alpha < 0$.

In the cases $k = 3,4,\ldots$, equation (2.5.4) admits only trivial solutions of the form:

$$y = -\frac{A_1}{2A_2\sqrt{\pi x}}. \tag{2.5.16}$$

Conclusions: An operator-based approach for the construction of closed form solutions to fractional order nonlinear differential equations is presented in this paper. The presented derivations are founded in fractional power series (2.5.1) and the definition of Caputo fractional differentiation operator (2.5.3). It has been shown that, by using these concepts, it is possible to construct both fractional power series and closed-form solutions to nonlinear fractional order differential equations.

As solution (2.5.15) is obtained from a Riccati-like fractional differential equation and as the non-fractional equation of the same form is known to admit solitary solutions, it is natural to assume that soliton-like effects are exhibited by solution (2.5.15). Since the research into this topic is still in its beginning stages, the identification of soliton-like effects in fractional differential equations is an important step towards understanding the processes described by such models.

3. CONCLUSIONS

The following conclusions corresponding to the objectives that have been raised in Section 1.2 are made:

1) By using the inverse balancing technique, it is proven that uncoupled partial differential equations with polynomial nonlinearity can only admit solitary solutions of an order no greater than 3, while the maximum possible order of solitary solutions to coupled partial differential equations with polynomial nonlinearity grows as the order of the system is increased.
2) In the case that the inverse balancing equations are linear in both the solution and equation parameters, the solutions can be constructed directly. However, the class of equations that satisfies this condition is very limited.
3) The system of hepatitis C virus evolution admits bright/dark solitary solutions when its parameters describe a biologically meaningful regime of virus evolution. The solutions and conditions for their existence in the space of system parameters are constructed by using the generalized differential operator and inverse balancing techniques.
4) The operator techniques used to construct solitary solutions to ordinary and partial differential equations can be extended and used for fractional order differential equations. The obtained solutions are expressible not by exponential, but by Bessel functions.

4. SANTRAUKA

4.1. Nagrinėjama mokslo problema

Šiame darbe nagrinėjami įvairių diferencialinių lygčių, įskaitant paprastąsias, dalinių išvestinių bei trupmenines, solitoniniai sprendiniai. Šie sprendiniai – dažnas tyrimų objektas įvairiose mokslo srityse, kadangi jie pasižymi specialiomis fizikinėmis savybėmis – solitoninis sprendinys (arba solitonas) yra netiesinė banga, kuri tenkina dvi prielaidas (Scott, 2006):

1) Bangos forma nekinta, kai ji juda pastoviu greičiu (taigi neprarandama energija);
2) Po susidūrimo su kita to paties tipo banga solitoninis sprendinys nepakinta, neskaitant fazės postūmio.

Nors pirmąkart tokio tipo sprendiniai pastebėti dar XIX a. kaip vandens bangos (Russell, 1844), didesnį susidomėjimą šiuo reiškiniu paskatino XX a. viduryje atlikti tyrimai, kurie parodė, kad solitoniniai sprendiniai egzistuoja ir aprašo svarbius procesus ir kitose mokslo srityse (Gardner et al., 1974; Zabusky & Kruskal, 1965), kurių trumpa apžvalga pristatoma žemiau.

Pati aktyviausia solitoninių reiškinių tyrimo sritis – netiesinė optika, kadangi čia solitoniniai impulsai gali būti pritaikomi informacijai perduoti. Stabilūs lokalizuoti impulsai, sklindantys optinio pluošto kabeliais, gali būti aprašomi solitoninių sprendinių pavidalu (Kruglov & Harvey, 2018). Publikacijoje (Inc et al., 2018) pademonstruojama, kad Ginzburg-Landau lygtis, kuria modeliuojamas šviesos sklidimas veikiant trikdžiams, turi solitoninius sprendinius kelių trukdžių atveju. Eksperimentiniai dalelės su solitoninio sprendinio savybėmis tyrimai ultra-greitame lazeryje aprašomi (M. Liu et al., 2018).

Kadangi solitonai naudojami aprašant informacijos perdavimą optiniais kabeliais, nenuostabu, kad sukurtas ne vienas solitoniniais sprendiniais paremtas modelis, leidžiantis modeliuoti neuronų veiklą. Solitoniniai impulsai naudojami neuronų komunikacijai aprašyti (Poznanski et al., 2017) bei (Engelbrecht et al., 2018). Šie sprendiniai taip pat sutinkami kitose biomedicinos srityse, pavyzdžiui, populiacijų dinamikoje. (Vitanov et al., 2009) parodoma, kad tarpusavyje konkuruojančių populiacijų modelis gali būti tiriamas taikant solitoninius sprendinius. Bakterijų populiacijos sugeba išsilaisvinti iš apskritimo formos užtvaro, sukurdamos solitonines bangas (Morris et al., 2017).

Dar viena aktyvi solitoninių sprendinių taikymų sritis – Boze-Einšteino kondensatų studijos. Pradedant nuo Gross-Pitajevskio modelio (Gross, 1961), įvykdyta aibė tyrimų, parodančių solitoninių sprendinių aktualumą šioje srityje. Solitonų savybes turinčių dalelių išsisklaidymas veikiant Gausinio tipo potencialui ištirtas (Umarov et al., 2016). Neautonomiai šviesaus tipo solitonai pastebėti Boze-Einšteino kondensatuose su Rabi surišimu (Kanna et al., 2017).

Solitoninių sprendinių paplitimas įvairiose mokslinėse srityse paaiškina tyrimų aktualumą – šių sprendinių sukonstravimas nėra trivialus uždavinys ir yra svarbus tiek teorine, tiek praktine ir taikomąja prasme.

4.2. Darbo tikslai ir uždaviniai

Pagrindinis šio darbo tikslas – sukurti ir pristatyti naują matematinį ir algoritminį įrankį, leidžiantį sukonstruoti analizinius solitoninius sprendinius įvairaus tipo netiesinėms diferencialinėms lygtims, įskaitant paprastąsias, dalinių išvestinių bei trupmeninės eilės diferencialines lygtis.

Šiam tikslui įgyvendinti iškelti keturi uždaviniai:

1) **Solitoninių sprendinių egzistavimo įvairiose netiesinių dalinių išvestinių diferencialinių lygčių klasėse tyrimas**. Šis žingsnis reikalingas, kad būtų galima nustatyti, ar skirtingi diferencialinių lygčių tipai gali turėti solitoninius sprendinius. Be to, šio uždavinio sprendimo metu sukurtos technikos vėliau pritaikomos būtinoms solitoninių sprendinių egzistavimo sąlygoms nustatyti ir keliais atskirais atvejais –tokius sprendinius sukonstruoti tiesiogiai.
2) **Tiesioginių solitoninių sprendinių konstravimo technikų sukūrimas**. Tiesioginės technikos leidžia sukonstruoti solitoninį sprendinį, spėjant, kad sprendinio forma yra solitonas, ir tuomet, įstačius sprendinį į modelį, nustatant sprendinio parametrus. Šio uždavinio sprendimo metu aptariami tokių metodikų pranašumai ir minusai bei apibrėžiamos lygčių klasės, kurioms galima taikyti šiuos metodus.
3) **Algebrinių operatorinių technikų solitoninių sprendinių konstravimui paprastosioms bei dalinių išvestinių diferencialinėms lygtims sukūrimas.** Pirmojo bei antrojo uždavinio rezultatus imant kaip pagrindą, gali būti konstruojamos sudėtingesnės solitoninių sprendinių radimo metodikos. Šios technikos paremtos algebrinių operatorių sąvokomis, tarp kurių svarbiausia – apibendrintojo diferencialinio operatoriaus sąvoka. Derinant operatorines technikas su kompiuterinės algebros priemonėmis, gaunama metodika, leidžianti konstruoti solitoninius sprendinius plačiai diferencialinių lygčių klasei.
4) **Algebrinių metodų išplėtimas, leidžiantis konstruoti solitoninius sprendinius trupmeninės eilės diferencialinėms lygtims.** Trupmeninės eilės diferencialinės lygtys turi esminių skirtumų, palyginti su paprastosiomis ar dalinių išvestinių diferencialinėmis lygtimis, ir todėl reikalauja naujų sprendimų, norint pritaikyti jau sukonstruotas technikas trupmeninės eilės uždaviniams. Šis pritaikymas ir yra ketvirtojo uždavinio tikslas.

4.3. Mokslinis naujumas ir metodika

Darbo mokslinis naujumas glaudžiai susijęs su metodika, kadangi pagrindinis darbo tikslas – sukurti naują solitoninių sprendinių konstravimo metodiką. Straipsniuose X1 ir X2 pristatoma nauja technika, kuri pavadinta atvirkštiniu balansavimu, ji leidžia nustatyti, ar nagrinėjama netiesinė dalinių išvestinių diferencialinė lygtis gali turėti solitoninių sprendinių, ir, jeigu taip, kokias būtinas egzistavimo sąlygas turi tenkinti lygties bei sprendinio parametrai. Pagrindinė šios technikos idėja primena *ansatz* metodus, kai spėjama, kad sprendinio analizinė forma atitinka solitoninį sprendinį, ir ši funkcija įstatoma į diferencialinę lygtį. Skirtingai nei tiesioginiuose sprendinio konstravimo metoduose (tiesioginis balansavimas), šiuo atveju ieškome ne sprendinio, o lygties parametrų – tokiu būdu gautos balansavimo lygtys yra tiesinės, o jas atitinkančios matricos naudojamos būtinoms solitoninių sprendinių egzistavimo sąlygoms išvesti.

Balansavimo metodika toliau plėtojama straipsnyje X3, kuriame naujos tiesioginės balansavimo technikos naudojamos solitoniniams sprendiniams sukonstruoti Rikati tipo diferencialinių lygčių klasei. Pagrindinis šio tyrimo tikslas – ištirti atvejus, kai tiesioginės balansavimo technikos gali būti taikomos solitoniniams sprendiniams konstruoti ir kai jos nėra pakankamos šiam uždaviniui išspręsti.

Straipsnyje X3 ištyrus tiesioginių metodų trūkumus ir privalumus, tolesniuose darbuose kuriama nauja algebriniais operatoriais paremta solitoninių sprendinių konstravimo netiesinėms paprastosioms bei dalinių išvestinių diferencialinėms lygtims metodika. Šią metodiką sudaro du žingsniai. Pirmajame žingsnyje, naudojant apibendrintąjį diferencialinį operatorių, sukonstruojamas eilutės formos sprendinys. Kiekvienas sprendinio eilutės koeficientas gaunamas pakėlus apibendrintąjį diferencialinį operatorių tam tikru laipsniu, taigi yra žinoma šių koeficientų analizinė išraiška. Antrajame žingsnyje, naudojantis jau žinomais algoritmais, siekiama pademonstruoti, kad nagrinėjamos eilutės koeficientų seka (arba šios sekos transformacija) yra tiesiškai rekurentinė. Įrodžius, kad seka – tiesiškai rekurentinė, eilutės formos sprendinys perrašomas į uždarą formą, išreikštą elementariosiomis funkcijos. Gautas rezultatas atitinka solitoninį sprendinį. Straipsnyje X4 šios technikos taikymas iliustruojamas pasinaudojant hepatito C viruso evoliucijos modeliu – surandami nauji solitoniniai sprendiniai, tenkinantys šią sistemą.

Vienas iš darbo uždavinių – sukurti solitoninių sprendinių konstravimo metodiką ir trupmeninės eilės diferencialinėms lygtims. Šios diferencialinės lygtys turi fundamentalių skirtumų, palyginti su sveikų išvestinių diferencialinėmis lygtimis, taigi darbe sukurti metodai turi būti adaptuojami dirbti su jomis. Straipsnyje X5 remiantis literatūra apibrėžiama algebrinė trupmeninės eilės diferencijavimo operatoriaus samprata, reikalinga sukurtiems metodams adaptuoti. Pagrindinis darbo naujumas gaunamas naudojantis Rikati tipo trupmeninės eilės diferencialine lygtimi, kai parodoma, kad, esant tam tikroms sąlygoms, kurios išvedamos naudojantis balansavimo metodika, trupmeninės eilės diferencialinę lygtį galima atvaizduoti į ekvivalenčią, tačiau sudėtingesnę paprastąją diferencialinę lygtį. Šios lygties sprendinys po tam tikrų transformacijų sutampa su trupmeninės eilės diferencialinės lygties sprendiniu.

4.4. Svarbiausi darbo rezultatai

Kadangi disertacija ginama straipsnių rinkinio pagrindu, toliau pristatomi svarbiausi darbo rezultatai poskyriuose, atitinkančiuose disertaciją sudarančiose publikacijose.

4.4.1. Straipsnio „Existence of solitary solutions in a class of differential equations with polynomial nonlinearity“ rezultatų santrauka

Šiame straipsnyje nagrinėjama ši netiesinių dalinių išvestinių diferencialinių lygčių klasė:

$$\frac{\partial^m u}{\partial t^m} + \sum_{s=1}^{m-1} \sum_{\substack{j+k=s \\ j,k\geq 0}} A_{j,k} \frac{\partial^s u}{\partial t^j \partial z^k} = a_n u^n + \cdots + a_0, \tag{4.4.1}$$

čia: $u(t, z)$ – nežinoma funkcija; $m, n \in \mathbb{N}$, o $A_{j,k}, a_j$ – realūs koeficientai. Straipsnio tikslas – surasti parametrus, kuriems esant lygtis (4.4.1) turi solitoninius sprendinius, išreiškiamus tokiu būdu:

$$y_0(x) = \sigma \frac{\prod_{j=1}^{l}\left(\exp\left(\eta(x-c)\right) - y_j\right)}{\prod_{j=1}^{l}\left(\exp\left(\eta(x-c)\right) - x_j\right)}, \tag{4.4.2}$$

čia: $\sigma, \eta \in \mathbb{R}; l \in \mathbb{N}$; $\sigma, \eta \neq 0; y_j, x_j \in \mathbb{C}, j = 1, \ldots, l$ – sprendinio parametrai, o $x = \kappa t + \omega z$ – banginis nepriklausomas kintamasis. Parametras l vadinamas solitoninio sprendinio eile.

Būtinos sprendinio (4.4.2) egzistavimo lygtyje (4.4.1) sąlygos gaunamos straipsnyje pristatytu atvirkštinio balansavimo metodu. Šio metodo esmė – transformuoti dalinių išvestinių lygtį (4.4.1) į paprastųjų išvestinių lygtį minėtu banginiu keitiniu $x = \kappa t + \omega z$, o tuomet įstatyti solitoninį sprendinį (4.4.2) į gautą diferencialinę lygtį. Kaip šio žingsnio rezultatas gaunamas kintamojo $\hat{x} = \exp\left(\eta(x-c)\right)$ daugianaris, kuris privalo būti tapatingai lygus nuliui, kad (4.4.2) tenkintų (4.4.1).

Verta pastebėti, kad sprendinio parametrų iš minėto daugianario koeficientų apskaičiavimas – sudėtingas, kadangi reikia spręsti aukštos dimensijos netiesinių algebrinių lygčių sistemą. Tačiau lygties parametrus $A_{j,k}, a_j$ galima apskaičiuoti išsprendus tiesinių lygčių sistemą. Natūralu, kad jeigu lygtis turi solitoninius sprendinius, šios sistemos matrica yra neišsigimusi, priešingu atveju reikia įvesti apribojimus sprendinio (4.4.2) parametrams $y_j, x_j, j = 1, \ldots, l$. Šiuos rezultatus naudojantis kompiuterine algebra galima apibendrinti išvedant būtinas sprendinio (4.4.2) egzistavimo sąlygas lygtyje (4.4.1):

$$m + 1 = n; \tag{4.4.3}$$

$$(m + 1)l \leq 2(l + m + 1). \tag{4.4.4}$$

Iš sąlygos (4.4.4) matome, kad solitoninio sprendinio eilei l didėjant, net ir didinant lygties eilę m, naujų sprendinių neatsiranda, kaip iliustruota lentelėje:

4.1 lentelė. Būtinų solitoninio sprendinio (4.4.2) egzistavimo lygtyje (4.4.1) sąlygų lentelė. Simbolis $\exists$ žymi egzistavimą prie bet kokių parametrų reikšmių; $\exists^*$žymi egzistavimą prie tam tikrų papildomų sąlygų, kuriomis suvaržomi lygties arba sprendinio parametrai; $\nexists$ žymi sprendinio neegzistavimą

l \ (n,m)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)	(7,6)	(8,7)
1	$\exists$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$
2	$\nexists$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$
3	$\nexists$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\nexists$	$\nexists$
4	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
5	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$

Straipsnyje pateikta atvirkštinio balansavimo technika leidžia nustatyti, ar duota lygtis gali turėti solitoninį sprendinį, ir leidžia surasti būtinas tokio sprendinio egzistavimo sąlygas. Be to, lygčių klasėje (4.4.1), didėjant lygties eilei m, naujų aukštos eilės solitoninių sprendinių neatsiranda – aukščiausia sprendinio eilė yra $l = 3$.

Straipsnyje pristatyta metodika (kurią nesunkiai galima taikyti ir platesnei lygčių klasei nei (4.4.1)) sukuria tvirtą pagrindą tiesioginio bei netiesioginio tipo solitoninių sprendinių konstravimo algoritmams plėtoti, kadangi leidžia paprastu būdu patikrinti, ar nagrinėjama lygtis gali turėti solitoninį sprendinį, ir nustatyti jo egzistavimo būtinąsias sąlygas.

4.4.2. Straipsnio „Existence of solitary solutions in systems of PDEs with multiplicative polynomial coupling“ rezultatų santrauka

Šioje publikacijoje nagrinėjama netiesinių dalinių išvestinių diferencialinių lygčių sistema, kurioje lygtys surištos multiplikatyviuoju ryšiu:

$$\sum_{j=1}^{n}\sum_{s=0}^{j} a_{s,j-s}\frac{\partial^j U}{\partial x^s \partial t^{j-s}} = \sum_{j=0}^{k}\sum_{s=0}^{j} b_{j-s,s}U^{j-s}V^s\,;$$

$$\sum_{j=1}^{m}\sum_{s=0}^{j} c_{s,j-s}\frac{\partial^j V}{\partial x^s \partial t^{j-s}} = \sum_{j=0}^{l}\sum_{s=0}^{j} d_{j-s,s}U^{j-s}V^s\,, \tag{4.4.5}$$

čia: $a_{s,j-s}, c_{s,j-s}, b_{j-s,s}, d_{j-s,s}$ – realieji koeficientai, nelygūs nuliui esant didžiausioms indekso reikšmėms; $U(t,x)$ ir $V(t,x)$ – nežinomos funkcijos. Galima pastebėti, kad (4.4.5) yra jau aptartos lygties (4.4.1) bendresnis atvejis, išplečiant lygtį į lygčių sistemą.

Tyrimo tikslas – surasti būtinąsias solitoninių sprendinių

$$u_0(z) = \sigma \frac{\prod_{j=1}^{N}\left(\exp\left(\eta(z-z_0)\right)-u_j\right)}{\prod_{j=1}^{N}\left(\exp\left(\eta(z-z_0)\right)-z_j\right)};$$

$$v_0(z) = \gamma \frac{\prod_{j=1}^{N}\left(\exp\left(\eta(z-z_0)\right)-v_j\right)}{\prod_{j=1}^{N}\left(\exp\left(\eta(z-z_0)\right)-z_j\right)}; \qquad (4.4.6)$$

egzistavimo sąlygas. Sprendinio parametrai, kaip minėta skyriuje 4.4.1, įgyja šias reikšmes: σ, γ, η – realiosios konstantos, u_j, v_j, z_j gali priklausyti nuo suformuluotų pradinių sąlygų.

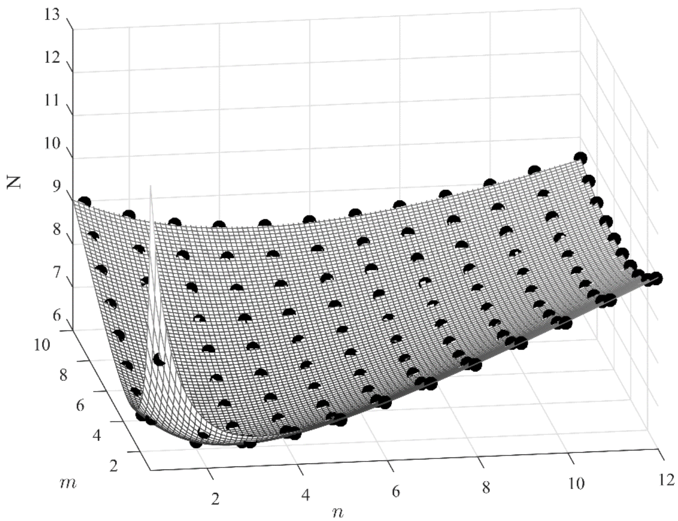
Kaip minėta aukščiau, šiam uždaviniui spręsti gali būti naudojama atvirkštinio balansavimo technika. Jos taikymas sistemai (4.4.5) – kiek sudėtingesnis nei lygčiai (4.4.1), tačiau principas išlieka toks pats – ieškoma lygčių sistemos parametrų $a_{s,j-s}, c_{s,j-s}, b_{j-s,s}, d_{j-s,s}$ išraiškų per solitoninių sprendinių (4.4.6) parametrus.

Gaunamos tokios būtinos egzistavimo sąlygos:

$$k = n+1; \;\; l = m+1; \qquad (4.4.7)$$

$$2(n+m-1)N - 2 \leq (n+2)^2 + (m+2)^2 + 3(n+m). \qquad (4.4.8)$$

Kaip parodyta 4.1 pav., solitoninių sprendinių elgesys sistemoje (4.4.5) fundamentaliai skiriasi nuo atvejo, kai nagrinėjama viena lygtis (4.4.1), kadangi, augant lygties eilei, būtinąsias egzistavimo sąlygas tenkina vis aukštesnės eilės solitoniniai sprendiniai. Pavyzdžiui, pirmos eilės sistema (4.4.5) gali turėti iki $N = 13$-tosios eilės solitoninius sprendinius, antros eilės sistemos sprendiniai gali siekti $N = 7$ eilę, o lygtyje be multiplikatyviųjų ryšių sprendinių eilė neviršija $N = 3$.



4.1 pav. Būtinosios solitonų egzistavimo sąlygos (4.4.8) iliustracija. Duotoms lygčių eilėms n, m būtinosios solitoninio sprendinio egzistavimo sąlygos tenkinamos tik tuomet, jeigu sprendinio eilė N yra žemiau pavaizduoto paviršiaus. Juodi taškai žymi maksimalias N reikšmes esant natūraliosioms n, m reikšmėms

Galima padaryti išvadą, kad multiplikatyvusis ryšys smarkiai padidina solitoninių procesų kompleksiškumą tiriamuose modeliuose. Atvirkštinio balansavimo technika išlieka patikimas būdas aukščiausiai galimai solitoninių sprendinių eilei tokio pobūdžio sistemose nustatyti.

4.4.3. Straipsnio „Kink solitary solutions to generalized Riccati equations with polynomial coefficients“ rezultatų santrauka

Pirmosios dvi publikacijos, pristatytos skyriuose 4.4.1 ir 4.4.2, sufokusuotos į būtinųjų solitoninių sprendinių egzistavimo sąlygų išvedimą. Išsprendus šį uždavinį, natūralu kelti klausimą apie technikas, kurias galima panaudoti konstruojant šiuos sprendinius.

Tiesioginiai sprendinių konstravimo metodai, kai sprendinio išraiška įstatoma į lygtį ir ieškoma sprendinio parametrų sprendžiant netiesines algebrines lygtis, įgijo nemažą populiarumą augant kompiuterinės algebros efektyvumui, tačiau buvo gausiai kritikuoti dėl sprendinio egzistavimo sąlygų nepaisymo (Kudryashov, 2009; Navickas & Ragulskis, 2009; Popovych & Vaneeva, 2010).

Šiame straipsnyje iškeltas uždavinys panaudoti tiesiogines solitoninio sprendinio konstravimo technikas kartu su jau pristatyta atvirkštinio balansavimo technika ne tik norint sukonstruoti solitoninius sprendinius, bet ir siekiant pademonstruoti tokių metodų limitus.

Tiriama n-tos eilės Rikati su kintamais koeficientais diferencialinė lygtis:

$$\left(\sum_{k=0}^{r} b_k x^k\right) y_x^{(m)} = \sum_{l=0}^{n} a_l y^l\,;\ r, m \in \mathbb{N}, \tag{4.4.8}$$

čia: b_k, a_l – realieji koeficientai.

Verta pastebėti, kad lygtys, kurios kintamieji koeficientai – daugianariai, dažnai gaunamos pritaikius keitinį $\hat{x} = \exp(\eta x)$. Analogiškas keitinys taikomas norint suvesti pirmos eilės solitoninio tipo sprendinį, vadinamą kinku, į paprastesnę formą:

$$y_0(x) = \frac{\alpha_1 x - \alpha_0}{\beta_1 x - \beta_0}\,;\ \alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}. \tag{4.4.9}$$

Nors sprendinys (4.4.9) atrodo labai paprastas, tiek transformuotas sprendinys (4.4.9), tiek jo solitoninis atitikmuo sulaukė dėmesio neseniai paskelbtuose tyrimuose (Yamaleev, 2014).

Pritaikius jau aptartą atvirkštinio balansavimo techniką, gaunamos būtinosios (4.4.9) egzistavimo lygtyje (4.4.8) sąlygos:

$$n = m + 1;\ \ 0 \le r \le m + 1. \tag{4.4.10}$$

Tiesioginis balansavimas vykdomas panašiu būdu, kaip jau aprašytas atvirkštinis balansavimas, tik šiuo atveju, prilyginus daugianario, gauto įstačius (4.4.9) į (4.4.8), koeficientus nuliui, ieškoma sprendinio parametrų $\alpha_0, \alpha_1, \beta_0, \beta_1$. Tai įmanoma todėl, kad šių parametrų atžvilgiu balansavimo lygtys yra tiesinės. Gaunamos šios parametrų reikšmės:

$$\boldsymbol{A}(\tau)\,\boldsymbol{p}(\tau) = \boldsymbol{0}, \tag{4.4.11}$$

kur

$$\boldsymbol{A}(\tau) = \begin{bmatrix} x_1 & -1 & -x_1 y_{\tau(1)} & y_{\tau(1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_r & -1 & -x_r y_{\tau(r)} & y_{\tau(r)} \\ 1 & 0 & -y_{\tau(r+1)} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & -y_{\tau(m+1)} & 0 \end{bmatrix}; \ \boldsymbol{p}(\tau) = \begin{bmatrix} \alpha_1(\tau) \\ \alpha_0(\tau) \\ \beta_1(\tau) \\ \beta_0(\tau) \end{bmatrix}, \tag{4.4.12}$$

čia: τ yra bet koks $m+1$ eilės kėlinys. Išraiška rodo, kad maksimalus sprendinių, tenkinančių (4.4.8), yra $(m+1)!$. Balansavimo technika taip pat gaunama pakankama sprendinio egzistavimo sąlyga:

$$b_r = \frac{(-1)^{1-r}}{m!\,(\alpha_1\beta_0 - \alpha_0\beta_1)\beta_1^{m-1}} \left(\prod_{l=1}^{r} \left(\alpha_1 - y_{\tau(l)}\beta_1\right) \right) \left(\prod_{j=r+1}^{m+1} \left(\alpha_0 - y_{\tau(j)}\beta_0\right) \right). \tag{4.4.13}$$

Taigi sprendinys (4.4.9) tenkina lygtį (4.4.8) tada ir tik tada, kai galioja sąlygos (4.4.10), (4.4.11) ir (4.4.13).

Pristatyti rezultatai, gauti atvirkštinio bei tiesioginio balansavimo technikomis, taip pat atskleidžia tiesioginio balansavimo apribojimus – jeigu algebrinė lygčių sistema, naudojama sprendinio parametrams surasti, yra tiesinė, kaip buvo šiuo atveju, sprendinio konstravimo uždaviniui pakanka tiesioginio balansavimo metodo. Priešingu atveju, naudojantis kompiuterine algebra sprendžiant netiesinių algebrinių lygčių sistemą, ypač jei ši – aukštos dimensijos, sprendinius konstruoti sudėtinga, ir uždavinio sprendimas reikalauja netiesioginių metodų, tokių kaip apibendrintojo diferencialinio operatoriaus metodas, aptariamas tolesniame skyriuje.

4.4.4. Straipsnio „Homoclinic and heteroclinic solutions to a hepatitis C evolution model“ rezultatų santrauka

Šioje publikacijoje sprendžiamas realaus pasaulio uždavinys: duotai hepatito C viruso (HCV) evoliuciją aprašančiai diferencialinių lygčių sistemai konstruojami bent antros eilės (sudėtingesni nei jau minėti kinkai) solitoniniai sprendiniai. Tiriamas kliniškai validuotas HCV evoliucijos modelis (Reluga et al., 2009):

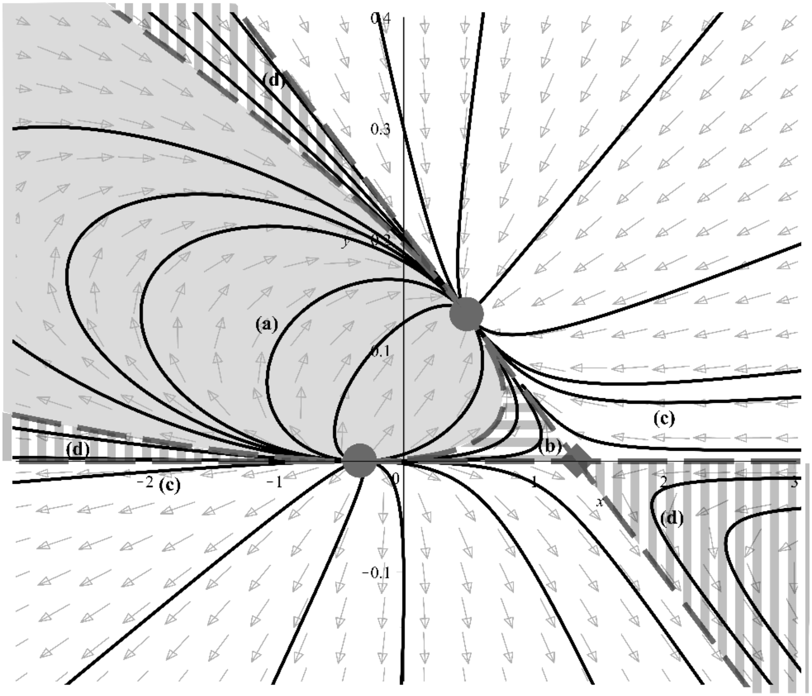
$$\begin{aligned} x'_\tau &= a_0 + a_1 x + a_2 x^2 + a_3 xy + a_4 y; \quad x(c) = u; \\ y'_\tau &= b_0 + b_1 y + b_2 y^2 + b_3 xy + b_4 x; \quad y(c) = v, \end{aligned} \tag{4.4.14}$$

čia: $x(\tau), y(\tau)$ aprašo bedimensinius kintamuosius, atitinkančius atitinkamai sveikų ir infekuotų hepatocitų populiacijų dydį; $a_0, \dots, a_4; b_0, \dots, b_4$ – realieji koeficientai, o c, u, v žymi pradines sąlygas.

Siekiama sukonstruoti aukštesnės eilės nei kinkus solitoninius sprendinius, kadangi šie sugeba geriau aprašyti procesų evoliuciją, įskaitant laikiną populiacijos sumažėjimą ar išaugimą (sprendinys gali turėti minimumą arba maksimumą, o tai nėra įmanoma žemesnės eilės solitoninio sprendinio atveju). Sprendinių analizinės išraiškos yra šios:

$$x(\tau; c, u, v) = \sigma \frac{\left(\exp(\eta(\tau - c)) - x_1\right)\left(\exp(\eta(\tau - c)) - x_2\right)}{\left(\exp(\eta(\tau - c)) - \tau_1^{(x)}\right)\left(\exp(\eta(\tau - c)) - \tau_2^{(x)}\right)};$$
$$y(\tau; c, u, v) = \gamma \frac{\left(\exp(\eta(\tau - c)) - y_1\right)\left(\exp(\eta(\tau - c)) - y_2\right)}{\left(\exp(\eta(\tau - c)) - \tau_1^{(y)}\right)\left(\exp(\eta(\tau - c)) - \tau_2^{(y)}\right)}. \qquad (4.4.15)$$

Sprendinių parametrų priklausomybės analogiškos aptartoms ankstesniuose skyriuose. Pasinaudojant aprašyta atvirkštinio balansavimo technika bei apibendrintuoju diferencialiniu operatoriumi, išvedamos bendros būtinos bei pakankamos sąlygos, kai antros eilės diferencialinių lygčių sistema turi sprendinį (4.4.15). Šių sąlygų išvedimą praleidžiame vardan glaustumo, jį galima rasti straipsnyje X4.



4.2 pav. Sistemos (4.4.14) fazinis portretas, esant biologiškai interpretuojamoms parametrų reikšmėms. Pilki skrituliai atitinka stabilų ir nestabilų mazgus. Pilkas rombas atitinka balno tašką. Juodos linijos žymi solitoninių sprendinių trajektorijas. Juodos punktyrinės linijos žymi separatrisę tarp trūkio bei netrūkių sprendinių ir balno taško daugdaras. Žymės (a), (b), (c) ir (d) atitinka sprendinių poras, pavaizduotas 4.3 pav. Sprendiniai, esantys pilkai nuspalvintoje srityje, neturi trūkio taškų ir turi elipsines fazines trajektorijas. Sprendiniai, esantys horizontaliai dryžuotoje srityje, neturi trūkio taškų ir turi hiperbolines fazines trajektorijas; sprendiniai, esantys nenuspalvintoje srityje, turi vieną trūkio tašką ir turi hiperbolines fazines trajektorijas; sprendiniai, esantys vertikaliai dryžuotoje srityje, turi du trūkio taškus bei hiperbolines fazines trajektorijas

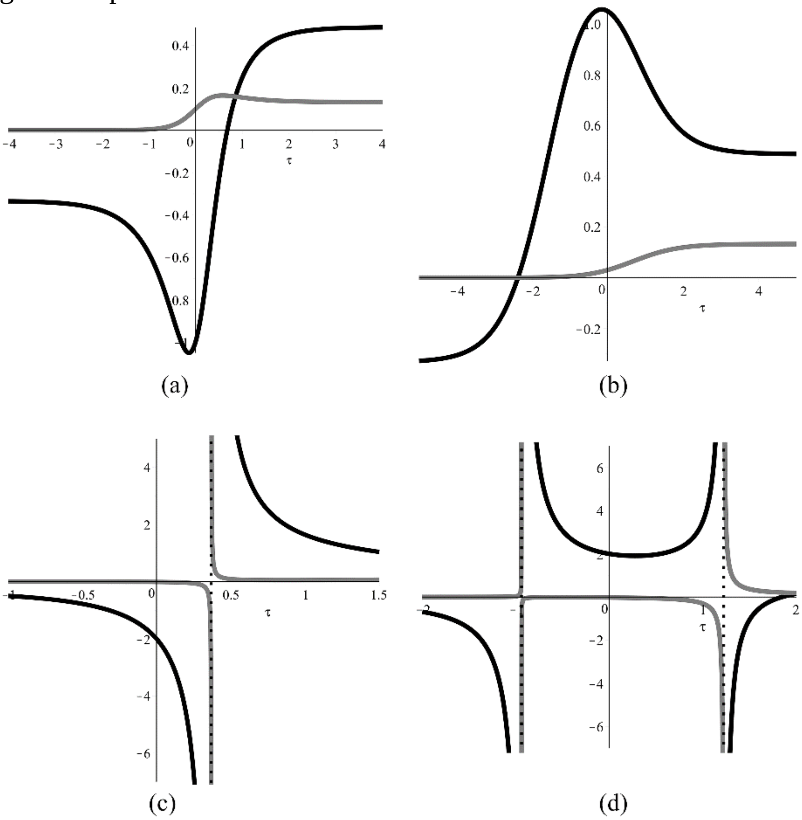
HCV evoliuciją aprašančioje sistemoje solitoniniai sprendiniai (4.4.15) egzistuoja tada ir tik tada, kai tenkinamos trys sąlygos:

$$\tau_1^{(x)} = \tau_1^{(y)}; \quad \tau_2^{(x)} = \tau_2^{(y)}. \tag{4.4.16}$$

$$a_3 = b_2; \quad a_2 = b_3. \tag{4.4.17}$$

$$\begin{gathered}9a_0a_1a_2 + 9b_0b_1b_2 - 18a_0a_2b_1 - 18b_0b_2a_1 + 3a_1b_1^2 \\ +3b_1a_1^2 - 2a_1^3 - 2b_1^3 - 9a_1a_4b_4 - 9b_1b_4a_4 + 27a_0b_2b_4 \\ +27b_0a_2a_4 = 0.\end{gathered} \tag{4.4.18}$$

Svarbu pažymėti, kad aukščiau pateiktoms sąlygoms išvesti reikalingas didelis kiekis simbolinių (ne skaitinių) skaičiavimų. Šiam darbui atlikti buvo panaudotas Maple programinis paketas.



4.3 pav. Antros eilės solitoninių sprendinių evoliucija laike. Juoda ir pilka linijos žymi atitinkamai $x(\tau), y(\tau)$. Žymės (a), (b), (c) ir (d) atitinka fazines trajektorijas, pavaizduotas 4.2 pav.

Toliau pademonstruojama, kad sąlygos (4.4.16)–(4.4.18) tenkinamos biologiškai reikšmingos sistemos (4.4.14) parametrų reikšmėms, pateiktoms (Reluga et al., 2009). Tuomet galima sukonstruoti analizinę bendrojo sistemos sprendinio išraišką (4.4.15) ir pateikti fazinį portretą, leidžiantį analizuoti HCV evoliuciją.

Kaip matome, yra trijų tipų solitoniniai sprendiniai – neturintys trūkio taškų (pažymėti (a) 4.2, 4.3 pav.); su vienu trūkio tašku (pažymėti (b) 4.2, 4.3 pav.) bei su dviem trūkio taškais (pažymėti (c) 4.2, 4.3 pav.). Sistema (4.4.14) turi tris pusiausvyros taškus – nestabilų bei stabilų mazgus, žymimus skrituliais 4.2 pav., bei balno tašką, pažymėtą rombu 4.3 pav. Panaudojant analizinės bendrojo sprendinio išraišką, galima ne tik identifikuoti tikslias balno taško daugdarų išraiškas, bet ir separatrisę – kreivę, skiriančią solitoninius sprendinius be singuliarumo taškų nuo juos turinčių sprendinių.

Gauti sprendiniai leidžia giliau pažvelgti į tam tikrus HCV evoliucijos aspektus, pavyzdžiui, galime pastebėti efektą, kai gydymas galų gale sumažina infekuotų ląstelių skaičių, tačiau gydymo pradžioje jų skaičius išauga, pasiekia maksimumą ir tik tuomet pradeda mažėti iki galutinės vertės. Šie pastebėjimai leidžia geriau suprasti ne tik HCV evoliuciją, bet ir kitus procesus, kurie gali būti aprašomi naudojantis nagrinėjama sistema.

4.4.5. Straipsnio „Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations“ rezultatų santrauka

Pagrindinis šios publikacijos tikslas – pristatyti galimą solitoninių sprendinių koncepciją trupmeninės eilės diferencialinėms lygtims, panaudojant pristatytų algebrinių operatorių modifikacijas. Tyrimo esmė – pristatyti ir iliustruoti naujas sąvokas, taigi užfiksuota trupmeninio diferencijavimo eilė $\alpha = \frac{1}{2}$. Vėlesnėse publikacijose, kurios į šį darbų rinkinį neįtrauktos, tyrimai apibendrinti ir bet kokiai racionaliai išvestinės eilei (Navickas et al., 2018).

Viena iš svarbiausių koncepcijų, reikalingų operatorinėms technikoms trupmeninės eilės diferencialinėms lygtims pritaikyti, – trupmeninių laipsnių eilutės f:

$$f = \sum_{j=1}^{+\infty} c_j z_j ; \quad z_j = \frac{x^{\frac{j-1}{2}}}{\Gamma\left(\frac{j+1}{2}\right)}; \quad c_j \in \mathbb{R}, \tag{4.4.19}$$

čia: z_j – bazinės funkcijos, o f – trupmeninė laipsninė eilutė Kaputo prasme. Visų tokių laipsninių eilučių aibė žymima ${}^{C}\mathbb{F}$. Naudojant taip apibrėžtas eilutes ir įprastą Kaputo trupmeninio diferencijavimo operatorių, galima apibrėžti trupmeninį diferencijavimą šių eilučių aibėje:

$$ {}^{c}\mathbf{D}^{\left(\frac{1}{2}\right)} f = \sum_{j=1}^{+\infty} c_{j+1} z_j, \tag{4.4.20} $$

kur ${}^{c}\mathbf{D}^{\left(\frac{1}{2}\right)} z_j = z_{j-1}$ kai $j = 2,3, \dots$ ir ${}^{c}\mathbf{D}^{\left(\frac{1}{2}\right)} z_1 = 0$. Operatoriaus ${}^{c}\mathbf{D}^{\left(\frac{1}{2}\right)}$ savybės detaliai išdėstytos straipsnyje X5.

Norint pasiekti pagrindinį darbo tikslą – identifikuoti solitoninius sprendinius trupmeninės eilės diferencialinėse lygtyse, analizuojamas paprasčiausio modelio, turinčio šiuos sprendinius, – Rikati lygties – trupmeninės eilės ekvivalentas:

$$ z_{k+1} \left({}^{c}\mathbf{D}^{\left(\frac{1}{2}\right)} \right)^{k} y = A_0 + A_1 y + A_2 y^2, \tag{4.4.21} $$

čia: $A_0, A_1, A_2 \in \mathbb{C}$, $k \in \mathbb{N}$, o sprendinys $y \in {}^{C}\mathbb{F}$. Lygčiai suformuluotos tokios pradinės sąlygos:

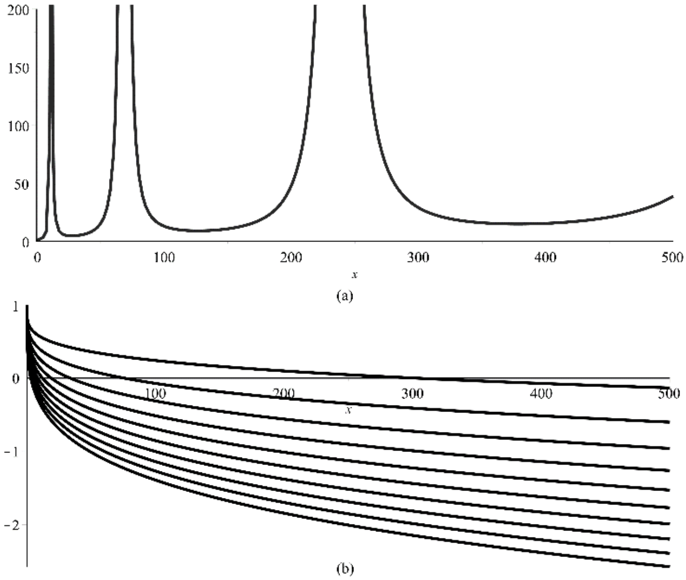
$$ {}^{c}\mathbf{D}^{\left(\frac{1}{2}\right)} y|_{x=0} = \alpha \in \mathbb{R}. \tag{4.4.22} $$

Ištyrus lygtį (4.4.21) naudojant apibrėžtus operatorius bei kompiuterinę algebrą, uždaros formos sprendinys gaunamas vieninteliu atveju, kai $k = 2$. Svarbu pažymėti, kad operatorius $\left({}^{c}\mathbf{D}^{\left(\frac{1}{2}\right)} \right)^{2}$ nėra ekvivalentus netrupmeninės eilės diferencijavimui $\frac{d}{dx}$, ir šis teiginys būtų teisingas tik tuomet, jeigu visi sprendinio koeficientai esant trupmeninės eilės laipsniams lygūs nuliui. Gaunamas lygties sprendinys išreiškiamas Beselio funkcijomis:

$$ y = \tilde{y}\left(\frac{4A_2\alpha}{\sqrt{\pi}} \sqrt{x} \right); \;\; \tilde{y}(t) = -\frac{A_1}{2A_2} + \frac{1}{A_2\pi} \frac{t\left(J_0^2\left(2\sqrt{t}\right) + J_1^2\left(2\sqrt{t}\right) \right)}{J_0^2\left(2\sqrt{t}\right)}, \tag{4.4.23} $$

čia: $J_\beta(t)$ žymi pirmojo tipo Beselio funkciją:

$$ J_\beta(t) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\ \Gamma(m+\beta+1)} \left(\frac{t}{2} \right)^{2m+\beta}. \tag{4.4.24} $$



4.4 pav. Sprendinys (4.4.23) su $\alpha = 1$ ((a) dalis) ir $\alpha = -10, -9, \dots, -1$ ((b) dalis). Verta pastebėti, kad sprendinys nėra trūkus tik neigiamoms pradinės sąlygos reikšmėms

Sprendinio grafikas esant skirtingoms parametrų reikšmėms pavaizduotas 4.4 pav. Kadangi gautasis sprendinys tenkina Rikati tipo trupmeninės eilės lygtį, kurios netrupmeninis ekvivalentas turi solitoninius sprendinius, natūralu samprotauti, kad trupmeninės eilės lygtis taip pat pasižymi solitoniniais procesais. Kadangi tyrimai šia tema dar tik pradedami, šių procesų identifikavimas trupmeninės eilės diferencialinėse lygtyse yra svarbus žingsnis, norint geriau suprasti reiškinius, aprašomus trupmeninės eilės modeliais.

4.5. Išvados

Remiantis darbo rezultatais, padarytos keturios iškeltus uždavinius atitinkančios išvados:

1) Taikant atvirkštinio balansavimo techniką įrodyta, kad nesurištos dalinių išvestinių diferencialinės lygtys su daugianario tipo netiesiškumu gali turėti solitoninius sprendinius, kurių eilė neviršija 3, o maksimali solitoninių sprendinių eilė šių lygčių sistemose, kai lygtys surištos multiplikatyviuoju ryšiu, auga didinant lygties eilę.
2) Kai atvirkštinio balansavimo metodu gautos lygtys yra tiesinės tiek sprendinio, tiek lygties parametrų atžvilgiu, solitoninius sprendinius galima

sukonstruoti tiesiogiai. Pažymėtina, kad klasė uždavinių, kuriems šis teiginys galioja, yra nedidelė.

3) Hepatito C viruso evoliuciją aprašanti diferencialinių lygčių sistema turi antros eilės solitoninius sprendinius, kai jos parametrai apibrėžia biologiškai prasmingą evoliucijos režimą. Solitoniniai sprendiniai, jų egzistavimo sąlygos sistemos parametrų aibėje konstruojami pritaikius apibendrintojo diferencialinio operatoriaus ir atvirkštinio balansavimo technikas.
4) Operatorinės technikos, naudojamos solitoniniams sprendiniams paprastosioms bei dalinių išvestinių diferencialinėms lygtims konstruoti, gali būti išplėstos ir taikomos trupmeninės eilės diferencialinėms lygtims. Gauti sprendiniai išreiškiami nebe eksponentinėmis, o Beselio funkcijomis.

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6. AUTHOR'S CURRICULUM VITAE

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7. AUTHOR'S LIST OF PUBLICATIONS DURING DOCTORAL STUDIES

Indexed in the Web of Science with Impact Factor International Publishers

1. **Telksnys, Tadas**; Navickas, Zenonas; Timofejeva, Inga; Marcinkevicius, Romas; Ragulskis, Minvydas. Symmetry breaking in solitary solutions to the Hodgkin-Huxley model // Nonlinear Dynamics. Dordrecht: Springer Nature. ISSN 0924-090X. eISSN 1573-269X. 2019, vol. 97, iss. 1, p. 571-582. DOI: 10.1007/s11071-019-04998-4. [Science Citation Index Expanded (Web of Science); Scopus] [IF: 4.604; AIF: 2.847; IF/AIF: 1.617; Q1 (2018, InCites JCR SCIE)] [FOR: N001] [Input: 0.200]
2. Navickas, Zenonas; **Telksnys, Tadas**; Timofejeva, Inga; Ragulskis, Minvydas; Marcinkevičius, Romas. An analytical scheme for the analysis of multi-hump solitons // Advances in complex systems. Singapore : World Scientific. ISSN 0219-5259. eISSN 1793-6802. 2019, vol. 22, iss. 1, art. no. 1850027, p. 1-3. DOI: 10.1142/S0219525918500273. [Science Citation Index Expanded (Web of Science); Scopus] [IF: 0.867; AIF: 3.654; IF/AIF: 0.237; Q3 (2018, InCites JCR SCIE)] [FOR: N001] [Input: 0.200]
3. **(Paper X4) Telksnys, Tadas**; Navickas, Zenonas; Marcinkevicius, Romas; Cao, Maosen; Ragulskis, Minvydas. Homoclinic and heteroclinic solutions to a hepatitis C evolution model // Open mathematics. Warsaw: De Gruyter. eISSN 2391-5455. 2017, vol. 16, iss. 1, p. 1537-1555. DOI: 10.1515/math-2018-0130. [Science Citation Index Expanded (Web of Science); Scopus; DOAJ] [IF: 0.726; AIF: 0.918; IF/AIF: 0.790; Q2 (2017, InCites JCR SCIE)] [FOR: N001] [Input: 0.200]
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5. **(Paper X5)** Navickas, Zenonas; **Telksnys, Tadas**; Marcinkevičius, Romas; Ragulskis, Minvydas Kazys. Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations // Chaos, solitons and fractals. Oxford : Pergamon-Elsevier Science. ISSN 0960-0779. eISSN 1873-2887. 2017, vol. 104, p. 625-634. DOI: 10.1016/j.chaos.2017.09.026. [Science Citation Index Expanded (Web of Science); Scopus; Academic Search

Complete] [IF: 2.213; AIF: 2.230; IF/AIF: 0.992; Q1 (2017, InCites JCR SCIE)] [FOR: N009] [Input: 0.250]

6. Navickas, Zenonas; Ragulskis, Minvydas, Kazys; Listopadskis, Narimantas; **Telksnys, Tadas**. Comments on “Soliton solutions to fractional-order nonlinear differential equations based on the exp-function method" // Optik. Munich: Elsevier. ISSN 0030-4026. 2017, vol. 132, p. 223-231. DOI: 10.1016/j.ijleo.2016.12.030. [Science Citation Index Expanded (Web of Science); Scopus] [IF: 1.191; AIF: 2.581; IF/AIF: 0.461; Q3 (2017, InCites JCR SCIE)] [FOR: N009] [Input: 0.250]

7. **(Paper X3)** Navickas, Z.; Ragulskis, M.; Marcinkevicius, R.; **Telksnys, T**. Kink solitary solutions to generalized Riccati equations with polynomial coefficients // Journal of mathematical analysis and applications. Atlanta, GA: Elsevier. ISSN 0022-247X. 2017, vol. 448, iss. 1, p. 156-170. DOI: 10.1016/j.jmaa.2016.11.011. [Science Citation Index Expanded (Web of Science); Scopus] [IF: 1.138; AIF: 1.077; IF/AIF: 1.056; Q1 (2017, InCites JCR SCIE)] [FOR: N001, N009] [Input: 0.250]

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9. Navickas, Z.; Marcinkevicius, R.; **Telksnys, T.**; Ragulskis, M. Existence of second order solitary solutions to Riccati differential equations coupled with a multiplicative term // IMA journal of applied mathematics. Oxford: Oxford University Press. ISSN 0272-4960. eISSN 1464-3634. 2016, vol. 81, iss. 6, p. 1163-1190. DOI: 10.1093/imamat/hxw050. [Science Citation Index Expanded (Web of Science); Scopus] [IF: 0.945; AIF: 1.191; IF/AIF: 0.793; Q2 (2016, InCites JCR SCIE)] [FOR: N009] [Input: 0.250]

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2. Navickas, Zenonas; Ragulskis, Minvydas; Marcinkevicius, Romas; **Telksnys, Tadas.** Generalized solitary waves in nonintegrable KdV equations // Journal of vibroengineering. Kaunas: JVE Journals. ISSN 1392-8716. 2016, vol. 18, iss. 2, p. 1270-1279. DOI: 10.21595/jve.2016.16941. [Science Citation Index Expanded (Web of Science); Academic Search Complete; Computers & Applied Sciences Complete] [IF: 0.398; AIF: 2.527; IF/AIF: 0.157; Q4 (2016, InCites JCR SCIE)] [FOR: N009] [Input: 0.250]

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