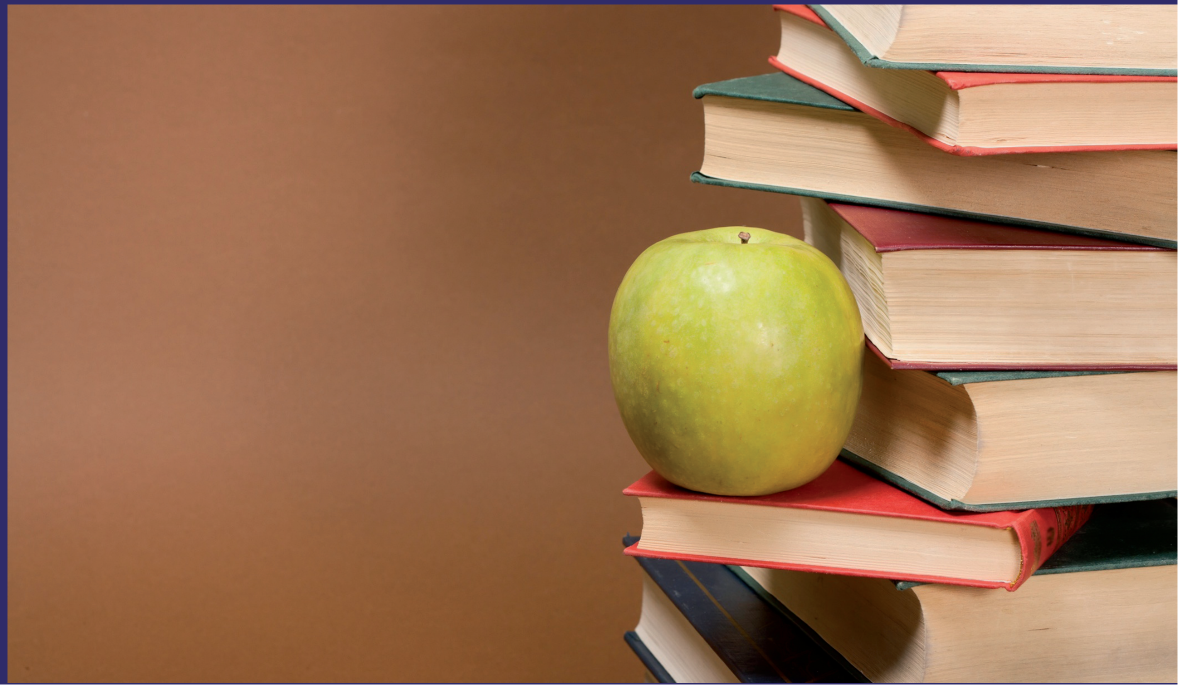
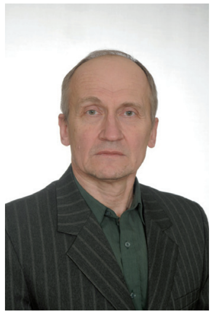
Characterization theorems in probability theory and mathematical statistics are such theorems that establish a connection between the type of the distribution of random variables or random vectors and certain general properties of functions in them. For example, the assumption that two linear (or non-linear) statistics are identically distributed (or independent, or have a constancy regression and so on) can be used to characterize various populations. Verification of conditions of this or that characterization theorem in practice is possible only with some error, i.e., only to a certain degree of accuracy. Such a situation is observed, for instance, in the cases where a sample of finite size is considered. That is why there arises the following natural question. Suppose that the conditions of the characterization theorem are fulfilled not exactly but only approximately. May we assert that the conclusion of the theorem is also fulfilled approximately? Questions of this kind give rise to a following problem: determine the degree of realizability of the conclusions of mathematical statements in the case of approximate validity of conditions.





Romanas Yanushkevichius

R. Yanushkevichius (by passport - R. Januškevičius) is professor of Lithuanian University of Educational Sciences (LEU), head of Department of Mathematics from 2001 till now. From 1993 - Doctor Habilitus, from 2002 - Professor. The main field of scientific interest - stability problems of characterization models.

978-3-659-25389-8

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Impressum / Imprint
Bibliografische Information der Deutschen Nationalbibliothek: Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über http://dnb.d-nb.de abrufbar.

Bibliographic information published by the Deutsche Nationalbibliothek: The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.d-nb.de.

Coverbild / Cover image: www.ingimage.com

Verlag / Publisher:
LAP LAMBERT Academic Publishing
ist ein Imprint der / is a trademark of
OmniScriptum GmbH & Co. KG
Heinrich-Böcking-Str. 6-8, 66121 Saarbrücken, Deutschland / Germany
Email: info@lap-publishing.com

Herstellung: siehe letzte Seite /
Printed at: see last page
ISBN: 978-3-659-25389-8

Stability characterizations of some probability distributions

R. Yanushkevichius

LAP LAMBERT Academic Publishing

AMS Subject Classification: primary 62E10, secondary 60E07

Key words and phrases: *Characterization theorems, stability of characterization, stability theorems, weak metric, stability problems, convolution equations, lack of memory property (memoryless property), uniform (Kolmogorov) metric, Lévy metric,* λ_0*-metric,* λ*-metric, Polya's characterization, identically distributed statistics, Weibull distribution, Stable distribution, Normal distribution*.

ISBN 978-3-659-25389-8

Contents

Introduction

Characterization theorems in probability theory and mathematical statistics are such theorems that establish a connection between the type of the distribution of random variables or random vectors and certain general properties of functions in them. For example, according to G. Polya's [13] characterization theorem, if X_1 and X_2 are independent identically distributed random variables with finite variance, then statistics $S_1 = X_1$ and $S_2 = (X_1 + X_2)/\sqrt{2}$ are identically distributed if and only if X_1 and X_2 have the normal distribution with zero mean. The assumption that two linear (or non-linear) statistics are identically distributed (or independent, or have a constancy regression and so on) can be used to characterize various populations.

Verification of conditions of this or that characterization theorem in practice is possible only with some error ε, i.e., only to a certain degree of accuracy. Such a situation is observed, for instance, in the cases where a sample of finite size is considered. That is why there arises the following natural question. Suppose that the conditions of the characterization theorem are fulfilled not exactly but only approximately. May we assert that the conclusion of the theorem is also fulfilled approximately?

Questions of this kind give rise to a following problem: determine the degree of realizability of the conclusions of mathematical statements in the case of approximate validity of conditions. The theorems in which the problems of this kind are considered are called *stability theorems.*

The first monograph on stability characterization of distributions was the book of R. Yanushkevichius [20]. Now in this scientific monograph "Stability Characterizations of Some Probability Distributions" we continue the investigation of stability estimations of characterization theorems. This investigation is based on the works [19] – [29].

Chapter 1

Stability characterization of the Weibull distribution

1.1 Characterization of the Weibull distribution

Let X be a Weibull random variable (r.v.) with the distribution

$$\mathbf{P}\left(X < x\right) = 1 - \exp\left(-\lambda x^{\alpha}\right), \quad \alpha > 0,\ \lambda > 0,\ x \geq 0;$$
$$\mathbf{P}\left(X < 0\right) = 0.$$

An interesting and useful characterization of X is the lack of memory property (of order α). It can be stated as

$$\mathbf{P}\left(X \geq \sqrt[\alpha]{x^{\alpha} + y^{\alpha}} \mid X \geq y\right) = \mathbf{P}\left(X \geq x\right) \quad \text{for all } x, y \geq 0 \tag{1.1}$$

and was studied by Y. H. Wang (in his paper [17] instead of the inequality $\geq$ in (1.1) the inequality $>$ is used).

Theorem 1.1 (Y. H. Wang [17]). *Let $\alpha > 0$ and X be non-degenerate r.v. with $\mathbf{P}(X \geq 0) = 1$. Then X is a Weibull r.v. if and only if X satisfies* (1.1).

To our mind, three comments are necessary here.

Comment 1. One should probably comprehend the conditional probability in relation (1) as follows: it has a sense for all $y \geq 0$, therefore $\mathbf{P}(X \geq y) > 0$ for all $y \geq 0$. This suggest that, in this context, condition (1.1) can be weakened in the following way:

$$\mathbf{P}\left(X \geq \sqrt[\alpha]{x^\alpha + y^\alpha} \,|\, X \geq y\right) = \mathbf{P}(X \geq x) \quad \text{for all } x \geq 0 \text{ and } y \in \mathcal{U}_+, \tag{1.2}$$

where $\mathcal{U}_+ = \{y \geq 0 \,|\, \mathbf{P}(X \geq y) > 0\}$.

To found this proposition it suffices to prove that

$$\mathcal{U}_+ = [0, \infty). \tag{1.3}$$

Indeed, the assumptions of the theorem require that X be a non-degenerate r.v., therefore $\mathbf{P}(X = 0) \neq 1$. But then there exists a number $y_0 > 0$ such that $\mathbf{P}(X \geq y_0) > 0$, i.e. $y_0 \in \mathcal{U}_+$. We get from (1.2) that

$$\mathbf{P}\left(X \geq \sqrt[\alpha]{2} y_0\right) = \mathbf{P}^2(X \geq y_0) > 0$$

and, analogously, $\mathbf{P}(X \geq \sqrt[\alpha]{n} y_0) > 0$ for any natural n. But then

$$\mathbf{P}(X \geq y) > 0 \text{ for any real } y \geq 0,$$

i.e. formula (1.3) holds.

Comment 2. One can relax the condition for the r.v. X to be nonnegative in Theorem 1.1, i.e. the requirement that $\mathbf{P}(X \geq 0) = 1$.

Really, since (1.1) is valid for all $x \geq 0$, $y \geq 0$, it is also valid at the point $x = y = 0$. Since a conditional probability has to exist at this point, we have **P**$(X \geq 0) > 0$ and we get from (1.1) that at the same time

$$\mathbf{P}\left(X \geq 0\right) = \mathbf{P}\left(X \geq 0\right)\mathbf{P}\left(X \geq 0\right) = \mathbf{P}^2\left(X \geq 0\right).$$

But this is possible only when **P**$(X \geq 0) = 1,$ because **P**$(X \geq 0) \neq 0.$

Comment 3. Theorem 1.1 holds even in the case where it is required to fulfill relation (1) not on the entire semi-axis $\{y\,|\,y \geq 0\}$, but only at two incommensurable points y_1 and y_2. The points y_1 and y_2 are called incommensurable if their ratio y_1/y_2 is irrational.

In fact this has been already observed by M. Eaton [2], Y. H. Wang [17], G. Marsaglia and A. Tubilla [10]. The proof of the latter was simplified by O. Yanushkevichiene [18]. In our case this simplification consists in what follows.

Assume that (1.1) is satisfied only at two incommensurable positive points y_1 and y_2, i.e.

$$\mathbf{P}\left(X \geq \sqrt[\alpha]{x^\alpha + y_i^\alpha}\right) = \mathbf{P}\left(X \geq x\right)\mathbf{P}\left(X \geq y_i\right) \quad \text{for all } x \geq 0,\ i = 1, 2. \tag{1.4}$$

Denote

$$\mathbf{P}\left(X \geq \sqrt[\alpha]{x}\right) = \exp\left(-\lambda_i x\right)\varphi_i\left(x\right),\ i = 1, 2, \tag{1.5}$$

where $\lambda_i = -\ln$**P**$(X \geq y_i)\,/y_i^\alpha$. Then it follows from (1.4) that

$$\varphi_i\left(x^\alpha + y_i^\alpha\right) = \varphi_i\left(x^\alpha\right)\varphi_i\left(y_i^\alpha\right) \quad \text{for all } \ x \geq 0,\ i = 1, 2. \tag{1.6}$$

Making use of the definitions of λ_i, we note that

$$\begin{aligned} \varphi_i\left(y_i^\alpha\right) &= \exp\left(\lambda_i y_i^\alpha\right)\mathbf{P}\left(X \geq y_i\right) \\ &= \exp\left\{-\ln\mathbf{P}\left(X \geq y_i\right)\right\}\mathbf{P}\left(X \geq y_i\right) = 1, \quad i = 1, 2, \end{aligned}$$

therefore it follows from (1.6) that

$$\varphi_i\left(x^\alpha + y_i^\alpha\right) = \varphi_i\left(x^\alpha\right)\ \forall x \geq 0,\ i = 1, 2.$$

Consequently, $\varphi_1(x)$ and $\varphi_2(x)$ are periodic functions with the periods y_1^α and y_2^α respectively.

Let us prove now that $\lambda_1 = \lambda_2$. Indeed, let, for example, $\lambda_1 > \lambda_2$. By virtue of (1.5) we have

$$\varphi_1(x) = \exp((\lambda_1 - \lambda_2)x)\,\varphi_2(x);$$

however, due to the periodicity of φ_1 it is impossible.

Thus, $\varphi(x) = \varphi_1(x) = \varphi_2(x)$ and the function $\varphi(x)$ has two incommensurable periods. That is possible only in the case, when $\varphi(x)$ is a constant. According to Comment 2, **P**$(X \geq 0) = 1$, therefore from (1.5) we obtain that this constant is equal to 1 and thus for all $x \geq 0$

$$\mathbf{P}(X \geq x) = \exp(-\lambda x^\alpha),$$

where $\lambda = -\ln\mathbf{P}(X \geq y_1)/y_1^\alpha = -\ln\mathbf{P}(X \geq y_2)/y_2^\alpha$.

Taking into consideration all the three comments, we can reformulate Theorem 1.1 as follows:

Theorem 1.2 (R. Yanuskevichius, O. Yanushkevichiene [27]). *Let* $\alpha > 0$. *The random variable has the Weibull distribution*

$$F(x) = 1 - \exp(-\lambda x^\alpha)\ \ for\, all\, x \geq 0 \tag{1.7}$$

if and only if X *satisfies*

$$\mathbf{P}\left(X \geq \sqrt[\alpha]{x^\alpha + y_i^\alpha}\,|\,X \geq y_i\right) = \mathbf{P}(X \geq x)\ \ for\, all\, x \geq 0 \tag{1.8}$$

at least at two incommensurable points y_1 *and* y_2.

Besides, if the r.v. X satisfies relation (1.8), then the parameter λ in formula (1.7) is defined as follows:

$$\lambda = -\ln\mathbf{P}(X \geq y_1)/y_1^\alpha = -\ln\mathbf{P}(X \geq y_2)/y_2^\alpha. \tag{1.9}$$

The exponential distribution is associated with a very substantial characterizations literature. It is known that if a property characterizes $X \sim Exp(\lambda)$ and $Y = h(X)$, then a characterization of Y is available via $h(\cdot)$ under some additional conditions on $h(\cdot)$ (see, for example, [19] and [6]). In the Weibull case $h(\cdot)$ is comparatively simple, $h(x) = x^{1/\alpha}$, therefore the method of convolution proposed below is, naturally, not a single possible way of proof.

1.2 Stability problems

It was noted in the Introduction that verification of conditions of this or that characterization theorem in practice is possible only with some error ε, i.e., only to a certain degree of accuracy. But if the assumptions of the characterization theorem are fulfilled not exactly but only approximately, then may we state that the conclusion of this characterization is also fulfilled approximately? Theorems, in which this kind of problems are considered, are called the stability theorems.

It ought to be noted that solution of a characterization problem is frequently reduced to the solution of functional equations of a particular kind. The majority of these equations and their perturbed analogues turned out to be transformable into characteristic equations of the convolution type with the kernels $\{k_{j1}\}$, $\{k_{j2}\}$:

$$\sum_{j=0}^{n}\left\{ \gamma_j f^{(j)}(t) + \int_0^{\infty} f^{(j)}(t-s)\, dk_{j1}(s) + \int_{-\infty}^{0} f^{(j)}(t-s)\, dk_{j2}(s) \right\} = r(t)\,. \tag{1.10}$$

For example, it is well known that if preliminary use of a device in no way influences the remaining time of its operation, then this device has an exponential distribution of the time of first failure.

More precisely, the non-negative r.v. X has the lack of memory property if for all $x \geq 0$ and for $y \geq 0$ such that $\mathbf{P}(X \geq y) > 0$,

$$\mathbf{P}(X \geq x + y | X \geq y) = \mathbf{P}(X \geq x). \tag{1.11}$$

The lack of memory property (1.11) characterizes the exponential distribution.

It is not difficult prove that (1.11) can be rewritten for **all** $y \geq 0$ as follows:

$$\mathbf{P}(X \geq x + y) = \mathbf{P}(X \geq x)\,\mathbf{P}(X \geq y), \ \forall x \geq 0, \ \forall y \geq 0.$$

By integrating with respect to y we see that

$$\bar{F}(x) = \lambda \int_0^\infty \overline{F}(x + y)\, dy, \tag{1.12}$$

where

$$\overline{F}(x) = \mathbf{P}(X \geq x), \ \lambda^{-1} = \int_0^\infty \overline{F}(x)\, dx, \tag{1.13}$$

i.e. we obtain the convolution equation (1.10).

Equation (1.12) and many other equations, as shown by R.Yanushkevichius in [20] and [19], represent the Wiener-Hopf equation with the kernel from $\mathbf{L}_1(-\infty, \infty)$. As it is known, such are equations

$$\chi_r(t) - \int_0^\infty \chi_r(s)\, q(t - s)\, ds = r(t), \quad \forall t \geq 0, \tag{1.14}$$

$$\chi(t) - \int_0^\infty \chi(s)\, q(t - s)\, ds = 0, \quad \forall t \geq 0. \tag{1.15}$$

Finally, we need the following theorem which shows that if we know that equation (1.15) is fulfilled only with some error $r\,(t)$, $|r\,(t)\,| \leq \varepsilon$, i.e. equation (1.14) holds, then there exists a solution χ_0 of a homogeneous equation (1.15) such that approximates the solution χ_r of non-homogeneous equation (1.14) quite well.

Theorem 1.3 (R. Yanuskevichius [19]]). *Assume that χ_r is the solution of equation (1.14) in the space $\mathbf{L}_\infty(0,\infty)$ and $q \in \mathbf{L}_1(-\infty,\infty)$, $|r\,(t)\,| \leq \varepsilon$ for $t \geq 0$, $Q(y) = \int_{-\infty}^{\infty} \exp\,(ity)\,q\,(t)\,dt \neq 1$, $\forall y \in (-\infty,\infty)$. Let equation (1.15) have at least one non-trivial solution. Then in the space $\mathbf{L}_\infty\,(0,\infty)$ there exists a solution χ_0 of equation (1.15) such that*

$$\sup_{t\geq 0} |\chi_r\,(t) - \chi_0\,(t)\,| \leq C\varepsilon, \tag{1.16}$$

where C may depend only on $q\,(\cdot)$.

1.3 Stability estimation of the Wang characterization

However, the proximity of $\chi_r\,(t)$ to $\chi_0\,(t)$ in the sense of (1.16) does not mean at all that $\chi_r\,(t)$ possesses properties that are close in some sense to the properties of the solution $\chi_0\,(t)$ of the homogeneous equation (1.15).

For example, if $\chi_0\,(t)$ is an exponential distribution that has moments of all orders, this does not imply anyway that $\chi_r\,(t)$ satisfying relation (1.16) bears such a property.

However, as we shall see below, by exactly using the information, contained in the perturbed characterization problem itself, one can discover additional useful information on the properties of the solution $\chi_r\,(t)$ of non-homogeneous equation (1.14).

In the case under consideration, a very favorable circumstance is that, under the conditions of Theorem 1.4, there exist moments of all orders of the r.v. X.

The most important result of this theorem, however, is that assuming the Weibull distribution characterization conditions (i.e. conditions of Theorem 1.1) to be fulfilled only with a certain error ε, we obtain that the conclusions of this characterization are also valid with the same error ε (accurate to the constant).

Theorem 1.4 (R. Yanuskevichius, O. Yanushkevichiene [27]). *Let X be a non-negative r.v. If*

$$\mathbf{P}\left(X \geq \sqrt[\alpha]{x^\alpha + y^\alpha} | X \geq y\right) = \mathbf{P}\left(X \geq x\right) + r\left(x, y\right), \quad \left|r\left(x, y\right)\right| \leq \varepsilon, \tag{1.17}$$

$\forall x \geq 0, \forall y \in \mathcal{U}_+ = \left\{y \geq 0 | \mathbf{P}\left(X \geq y\right) > 0\right\}$, *then the r.v. X has the moments of all orders, $\mathcal{U}_+ = [0, \infty)$ and there exists $\lambda > 0$ such that for all $\varepsilon \geq 0$*

$$\left|\mathbf{P}\left(X \geq x\right) - \exp\left(-\lambda x^\alpha\right)\right| \leq 2\varepsilon, \ \forall x \geq 0. \tag{1.18}$$

Proof. Since $\forall x \geq 0$

$$\left|\mathbf{P}\left(X \geq x\right) - \exp\left(-\lambda x^\alpha\right)\right| \leq \max\left\{\mathbf{P}\left(X \geq x\right), \exp\left(-\lambda x^\alpha\right)\right\} \leq 1, \tag{1.19}$$

then inequality (1.17) is obvious for $\varepsilon \geq 1/2$, therefore it suffices to consider the case $\varepsilon < 1/2$.

We shall prove that, either there exists $y_0 = y_0\left(\varepsilon\right) > 0$ such that $\mathbf{P}\left(X \geq y_0\right) > \varepsilon$, or the r.v. X is close (in the sense of (1.17)) to a r.v. degenerate at a point 0.

Indeed, if $\forall y_0 > 0\, \mathbf{P}\left(X \geq y_0\right) \leq \varepsilon$, then

$$\sup_{x>0} \left|\mathbf{P}\left(X \geq x\right) - E_0\left(x\right)\right| \leq \varepsilon. \tag{1.20}$$

Here $1 - E_0\left(x\right)$ is a distribution of degenerate at the point 0 r.v.,

$$E_0\left(x\right) = \begin{cases} 0 & \text{for } x > 0, \\ 1 & \text{for } x \leq 0. \end{cases}$$

But (1.20) implies that formula (1.18) holds for $\lambda = +\infty$.

Thus, let now exist $y_0 > 0$ such that the relation

$$\mathbf{P}\left(X \geq y_0\right) > \varepsilon \tag{1.21}$$

be valid.

On the basis of relation (1.17) we see now that not only for the values of the set $\mathcal{U}_+$, but also for all non-negative y

$$\begin{aligned} \mathbf{P}\left(X \geq \sqrt[\alpha]{x^\alpha + y^\alpha}\right) &= \mathbf{P}\left(X \geq x\right)\mathbf{P}\left(X \geq y\right) \\ &\quad + R\left(x, y\right), \quad \forall x \geq 0,\ \forall y \geq 0, \end{aligned} \tag{1.22}$$

where $R\left(x, y\right) = r\left(x, y\right)\mathbf{P}\left(X \geq y\right)$.

For this let us take $x = y = y_0$ in formula (1.17). Since according to (1.21) $\mathbf{P}\left(X \geq y_0\right) > 0$, hence we derive that

$$\mathbf{P}\left(\sqrt[\alpha]{2}y_0\right) = \mathbf{P}^2\left(X \geq y_0\right) + \mathbf{P}\left(X \geq y_0\right) r\left(y_0, y_0\right).$$

Making use of (1.21) once more, we obtain:

$$\mathbf{P}\left(X \geq \sqrt[\alpha]{2}y_0\right) \geq \mathbf{P}\left(X \geq y_0\right)\left(\mathbf{P}\left(X \geq y_0\right) - \varepsilon\right) > 0. \tag{1.23}$$

Analogously, putting $x = y_0$, $y = \sqrt[\alpha]{2}\, y_0$ in formula (1.17), we get first from (1.23) and afterwards from (1.21) and again from (1.23) that

$$\begin{aligned}\mathbf{P}\left(X \geq \sqrt[\alpha]{3} y_0\right) &\geq \mathbf{P}(X \geq y_0)\,\mathbf{P}\left(X \geq \sqrt[\alpha]{2} y_0\right) \\ &\quad + r\left(y_0,\ \sqrt[\alpha]{2} y_0\right)\mathbf{P}\left(X \geq \sqrt[\alpha]{2} y_0\right) \\ &\geq \mathbf{P}\left(X \geq \sqrt[\alpha]{2} y_0\right)\left(\mathbf{P}(X \geq y_0) - \varepsilon\right) > 0.\end{aligned}$$

Applying the mathematical induction method, we can easily obtain that for any natural n

$$\mathbf{P}\left(X \geq \sqrt[\alpha]{n} y_0\right) > 0. \tag{1.24}$$

This means that for any real $y \geq 0$

$$\mathbf{P}(X \geq y) > 0. \tag{1.25}$$

We conclude from (1.25) that $\mathcal{U}_+ = [0, \infty)$, and formulas (1.17) and (1.22) are equivalent.

We verify that the r.v. X has the moments of all orders. Using the notation $\overline{F}(x) = \mathbf{P}(X \geq x)$ again and following T.Azlarov, N.Volodin [1], we choose a point $x_0 > 0$ so that the condition

$$\overline{F}(x_0) \leq 1/3 \tag{1.26}$$

be satisfied. From (1.22), analogously to the formula (1.24) proved above, we have that

$$\begin{aligned}\overline{F}\left(\sqrt[\alpha]{n}\, x_0\right) &\leq \overline{F}\left(\sqrt[\alpha]{n-1}\, x_0\right)\left(\overline{F}(x_0) + \varepsilon\right) \leq \ldots \\ &\leq \overline{F}(x_0)\left(\overline{F}(x_0) + \varepsilon\right)^{n-1}.\end{aligned}$$

Since $\overline{F}(x) = 1 - F(x)$ is a non-increasing function, then

$$\overline{F}(x) \leq \overline{F}\left(\sqrt[\alpha]{n}\, x_0\right) \leq \overline{F}(x_0)\left(\overline{F}(x_0) + \varepsilon\right)^{n-1}$$

for all $x \in \left[\sqrt[\alpha]{n}x_0, \sqrt[\alpha]{n+1}x_0\right)$. Noting that $\varepsilon < 1/2$, hence and from (1.26) we obtain that

$$\overline{F}(x) < \frac{1}{3}\left(\frac{1}{3}+\frac{1}{2}\right)^{n-1} < \left(\frac{5}{6}\right)^{n}, \quad x \in \left[\sqrt[\alpha]{n}x_0, \sqrt[\alpha]{n+1}x_0\right).$$

Since $n \geq (x/x_0)^{\alpha} - 1$, it follows

$$\overline{F}(x) < \left(\frac{6}{5}\right)\left(\frac{5}{6}\right)^{n} = \frac{6}{5}h^{x^{\alpha}},$$

where $h = (5/6)^{1/x_0} < 1$.

Consequently, in our conditions the r.v. X has got finite moments of all orders. In particular,

$$EX^{\alpha} = -\int_0^{\infty} xd\mathbf{P}(X^{\alpha} \geq x) = \int_0^{\infty}\mathbf{P}(X^{\alpha} \geq x)\,dx < \infty. \tag{1.27}$$

Since X is a non-negative r.v. and $\alpha > 0$, we can rewrite (1.22) as follows:

$$\begin{aligned} \mathbf{P}(X^{\alpha} \geq x^{\alpha}+y^{\alpha}) &= \mathbf{P}(X^{\alpha} \geq x^{\alpha})\,\mathbf{P}(X^{\alpha} \geq y^{\alpha}) \\ &\quad + R(x,y), \forall x \geq 0, \forall y \geq 0, \end{aligned} \tag{1.28}$$

Denoting

$$G(x) = \mathbf{P}(X^{\alpha} \geq x),\, u = x^{\alpha},\, v = y^{\alpha},$$

we can easily get from (1.28) that

$$G(u+v) = G(u)\,G(v) + R_1(u,v), \; \forall u \geq 0, \; \forall v \geq 0, \tag{1.29}$$

where

$$\begin{aligned} R_1(u,v) &= r\left(u^{1/\alpha}, v^{1/\alpha}\right)G(v), \\ |R_1(u,v)| &\leq \varepsilon G(v), \quad \forall u \geq 0, \; \forall v \geq 0. \end{aligned}$$

Let us denote now

$$H(\delta) = 1/\int_0^\infty G(x)\exp(-\delta x)\,dx.$$

It follows from (1.27) that $H(0) > 0$. Let us define $E(x) = x$. Since $E(0) = 0$ and the functions H and E are continuous, there exists $\delta_0 > 0$ such that $E(\delta_0) < H(\delta_0)$, i.e.

$$\delta_0 < H(\delta_0). \tag{1.30}$$

Multiplying (1.29) by $\exp(-\delta_0 v)$ and afterwards integrating with respect to v we obtain that $\forall u \geq 0$

$$\begin{aligned}\int_0^\infty G(u+v)\,e^{-\delta_0 v}dv &= G(u)\int_0^\infty G(v)\,e^{-\delta_0 v}dv \\ &\quad + \int_0^\infty R_1(u,v)\,e^{-\delta_0 v}dv. \end{aligned}\tag{1.31}$$

Denote

$$\delta_1 = 1/\int_0^\infty G(v)\,e^{-\delta_0 v}dv = H(\delta_0),$$

$$q(u) = \begin{cases} 0 & \text{for } u > 0, \\ \delta_1 \exp(\delta_0 u) & \text{for } u \leq 0. \end{cases}$$

Then we can rewrite (1.31) for all $u \geq 0$ as follows:

$$\int_0^\infty G(v)\,q(u-v)\,dv = G(u) + \delta_1 \int_0^\infty R_1(u,v)\,e^{-\delta_0 v}dv, \tag{1.32}$$

besides $q \in \mathbf{L}_1(-\infty,\infty)$.

In order that we could apply the theorem on convolution (i.e. Theorem 1.3), we need to get convinced that the Fourier transform $Q(t)$ of the kernel $q(x)$ of convolution equation (1.32) does not acquire any value equal to 1 on the entire real axis. For this note that

$$Q(t) = \int_{-\infty}^\infty e^{itx} q(x)\,dx = \delta_1 \int_0^\infty e^{-itx} e^{-\delta_0 x} dx = \frac{\delta_1}{\delta_0 + it}.$$

Since $\delta_1 = H(\delta_0)$, we get from (1.30) that $\delta_0 < \delta_1$ and therefore

$$Q(t) \neq 1 \quad \forall t \in (-\infty, \infty).$$

Recall that if the function $M(\lambda)$ is the boundary value of an analytical function, excluding only a finite number poles in the upper (lower) half-plane, then according to the argument principle the equality

$$ind\, M(\lambda) = \pm (N - P) \tag{1.33}$$

holds, where N is the number of zeros and P is the number of poles in the respective half-plane (multiple zeros or poles are counted according to their multiplicity).

We have from (1.33) that the index ν of an equation (1.32) is equal 1, because

$$\nu = -ind\,(1 - Q(t)) = -(0 - 1) = 1.$$

It means that the basis of the set of solutions of a homogeneous equation corresponding to (1.32) consists of a single function. It is easy to see directly that this function is exponential. We see from (1.29) (assuming $R_1(x, y) \equiv 0$) that this exponential function has no discontinuity at the initial point. Making use of Theorem 1.3 we obtain from (1.32) that there exists $\lambda > 0$ such that

$$\sup_{x \geq 0} |G(x) - \exp(-\lambda x)| \leq C\varepsilon, \tag{1.34}$$

where C can depend only on the kernel $q(\cdot)$.

Since $G(x) = \overline{F}(\sqrt[\alpha]{x})$, we get from (1.34) that

$$\left|\overline{F}(x) - \exp(-\lambda x^{\alpha})\right| \leq C\varepsilon, \quad \forall x \geq 0. \tag{1.35}$$

According to [1], the constant C in relation (1.35) admits the estimate $C \leq 2$. Hence follows (1.18).

The Theorem is proved.

Chapter 2

Stability characterization of the Stable distribution

2.1 Characterization of the symmetric Stable distribution

The assumption that two linear statistics are identically distributed can be used to characterize various populations.

The first work on the investigation of the stability of characterizations by the identically distributed linear forms is that of L. D. Meshalkin [12], devoted to the estimation of stability in the historically first characterization problem – G. Polya's theorem [13].

As it was mentioned in Introduction, according to G. Polya's [13] characterization theorem if X_1 and X_2 are independent identically distributed random variables with finite variance, and statistics $S_1 = X_1$ and $S_2 = (X_1 + X_2)/\sqrt{2}$ are identically distributed, then the investigated population has normal distribution function with zero mean, and vice versa.

Throughout this book, we shall write $\mathcal{L}(W) = \mathcal{L}(Z)$ to meaning that the random variables W and Z have the same distribution. So, $\mathcal{L}(X_1) = \mathcal{L}((X_1 + X_2)/\sqrt{2})$, if and only if X_i has the normal distribution function with zero mean.

According to Eaton's characterization theorem, if under the additional conditions the two linear statistics $S_1 = (X_1 + ... + X_{k_1})/k_1^{1/\alpha}$ and $S_2 = (X_1 + ... + X_{k_2})/k_2^{1/\alpha}$ have the same distribution as the monomial X_1, then this monomial has a symmetric stable distribution of order α.

Theorem 2.1 (M.L. Eaton [2]). *Let $X, X_1, \ldots, X_{k_1}, \ldots, X_{k_2}$ be independent identically distributed (i.i.d.) symmetric random variables. If $0 < \alpha \leq 2$, and k_1 and k_2 are integers such that $\theta = \log k_1 / \log k_2 \quad (2 \leq k_1 < k_2)$ is irrational, and*

$$\mathcal{L}(X) = \mathcal{L}(k_1^{-1/\alpha} \sum_{i=1}^{k_1} X_i) = \mathcal{L}(k_2^{-1/\alpha} \sum_{i=1}^{k_2} X_i), \tag{2.1}$$

then X has a symmetric stable distribution of order α .

If in (2.1) $\alpha = 1$, we have such a characterization of Cauchy law without the symmetry condition:

Theorem 2.2 (B. Ramachandran, C.R. Rao [14]). *Let $X, X_1, X_2, \ldots, X_n$ be i.i.d. random variables. If X and sample mean $\overline{X}_{(n)} = \frac{1}{n}(X_1 + \ldots + X_n)$ have the same distribution for two values k_1 and k_2 of n such that $\theta = \log k_1 / \log k_2 \quad (2 \leq k_1 < k_2 \leq n)$ is irrational, then X has a Cauchy distribution.*

However, it is important to emphasize that authors of this theorem successfully avoided the condition of symmetry in it only by using the condition $\alpha = 1$ essentially.

But in the general case the way to avoid the condition of symmetry in Eaton's theorem is still not found. As one can see from [5], for $0 < \alpha < 1$ or $1 < \alpha < 2$ the condition of symmetry can be avoided only under the additional condition on the existence of negative numbers among the coefficients of linear statistics L_1 and L_2 (see Theorem 13.7.2 in [5]).

Therefore Eaton's theorem 1 has preserved its actuality and is of special interest at present.

Verification of this or that characterization theorem in practice is possible only with some error ε, i.e., only to a certain degree of accuracy. Such a situation is observed, for instance, in the cases where a sample of finite size is considered. That is why there arises a following natural question. Suppose that the conditions of the theorem are fulfilled not exactly but only approximately. May we assert, that the conclusion of the theorem is also fulfilled approximately?

We discuss the conditions in which sense the assumptions of the characterization theorem is fulfilled not exactly but only approximately in the next section.

2.2 Stability problems

Let now the conditions (2.1) of Eaton's theorem be fulfilled only approximately, with some error ε. In the main theorem – Theorem 2.3 – ε is any positive number. The parameter ε express the proximity of the considered in formula (2.1) statistics in the λ_0-metric defined below.

Our aim is to get convinced that in a certain sense the characteristic function $f(t)$ of the random variable X is close to the characteristic function of a symmetric stable law.

For 'measurements' of the error of fulfillment of conditions (2.1) we choose a metric λ_0 that is defined in the class of characteristic functions by analogy with a uniform (Kolmogorov) metric ρ defined in the class of distributions:

$$\lambda_0(X,Y) = \lambda_0(f_X, f_Y) = \sup_t |f_X(t) - f_Y(t)|, \tag{2.2}$$

where $f_X(t) = E\exp(itX)$, $f_Y(t) = E\exp(itY)$. Analogously as the uniform metric ρ is invariant with respect to the multiplier, i.e. $\rho(cX, cY) = \rho(X,Y)$ for any real constant $c \neq 0$, the metric λ_0 defined by formula (2.2) is also invariant with respect to the multiplier:

$$\lambda_0(cX, cY) = \lambda_0(X,Y).$$

The λ_0-metric is convenient to express the essence of stability problems of characterization theorems, since the latter are frequently proved by analyzing the equations considered in the space of characteristic functions. Some aspects of this problem are analyzed by R. Yanushkevichius [19].

Now we are ready for the statement of our main theorem.

Theorem 2.3 (R. Yanuskevichius [21]). *Let $X, X_1, \ldots, X_{k_1}, \ldots, X_{k_2}$ be k_2 symmetric i.i.d. random variables, where k_1 and k_2 are integers such that $\theta = \log k_1 / \log k_2$ $(2 \leq k_1 < k_2)$ is irrational. If there exists $\alpha \in (0,2]$ such that for $j = 1, 2$ the relations*

$$\lambda_0\left(X, k_j^{-1/\alpha} \sum_{i=1}^{k_j} X_i\right) \leq \varepsilon \tag{2.3}$$

are fulfilled, then there exist a random variable Y with the symmetric stable distribution of order α and constants C_1, δ depending only on α and k_1, k_2

such that

$$\lambda_0 (X, Y) \leq C_1 \varepsilon^{\delta}. \tag{2.4}$$

The next lemma, proved together with O.Yanushkevichiene, probably of independent interest, is generalization of Lemma 3 in [25] and is very useful in the following:

Lemma 2.1 *Let* k_1 *and* k_2 $(2 \leq k_1 < k_2)$ *be integers such that the ratio of their logarithms* $\log k_1 / \log k_2$ *is irrational, and for some* $\alpha \in (0, 2]$ *there exists a characteristic function* $f(t)$ *such that for* $j = 1, 2$

$$\left| f(t) - f^{k_j}\left(t/k_j^{1/\alpha}\right)\right| \leq \varepsilon, \quad \forall t \in [-1, 1]. \tag{2.5}$$

In addition, if

$$|f(t)| \geq 1/2 \quad for \quad \forall t \in [-1, 1], \tag{2.6}$$

then there exist constants C_2 *and* Δ *depending only on* α *and* k_1, k_2 *such that for* $|t| \leq 1$

$$|f(t) - \exp\{-|D| \exp(iQ \operatorname{sign} t) |t|^{\alpha}\}| \leq C_2 \varepsilon^{\Delta}, \tag{2.7}$$

where $D = k_1 \log f\left(k_1^{-1/\alpha}\right)$ *and* $Q = \arctan(\operatorname{Im} D / \operatorname{Re} D)$.

2.3 Proof of the main Theorem

Note that the proof is non-trivial only when ε is a small positive number. It can be appreciated as follows: there exists a small positive number ε_0, depending only on α, k_1 and k_2, such that (2.4) is valid for all $\varepsilon \in (0, \varepsilon_0]$. In all the other cases Theorem 2.3 is trivial.

Indeed, according to the definition of λ_0-metric, for any X and Y

$$0 \leq \lambda_0(X, Y) \leq 2. \tag{2.8}$$

Hence and from condition (2.3) we obtain that $\varepsilon \geq 0$. In case $\varepsilon = 0$, we get Theorem 2.3. Finally, if $\varepsilon > \varepsilon_0$, then C_1 in formula (2.4) is chosen in such a way: $C_1 = 2\varepsilon_0^{-\delta}$. In this case

$$\lambda_0(X,Y) \leq 2 = C_1\varepsilon_0^{\delta} < C_1\varepsilon^{\delta} \qquad \text{if} \quad \varepsilon > \varepsilon_0, \tag{2.9}$$

i.e. (2.4) is trivial for $\varepsilon > \varepsilon_0$.

From condition (2.3) we have that

$$|f(t) - f^{k_j}(t/k_j^{1/\alpha})| \leq \varepsilon \qquad \text{for} \quad |t| \leq \infty, \quad j = 1, 2. \tag{2.10}$$

Since $f(t)$ is a continuous function such that $f(0) = 1$, and, besides, $f(t)$ is real as a characteristic function of a symmetric random variable, there exists p_0 such that $\inf\{|t|:\ f(t) = 1/2\} = p_0(f) = p_0 > 0$. Let $p = \min(p_0, 1)$.

Instead of the characteristic function $f(t)$, we introduce the characteristic function $f_p(t) = f(pt)$, for which

$$\begin{aligned} p_0(f_p) &= \inf\left\{|t|:\ f_p(t) = 1/2\right\} = \inf\left\{|t|:\ f(pt) = 1/2\right\} \\ &= \frac{1}{p}\inf\left\{|u|:\ f(u) = \frac{1}{2}\right\} = \frac{1}{p}\cdot p_0 \geq 1. \end{aligned}$$

If $f(t)$ satisfies (2.10) for $|t| \leq \infty$, then $f_p(t)$ satisfies (2.10) for $|t| \leq \infty$ also.

So, for the characteristic function $f_p(t)$ we have that

$$|f_p(t) - f_p^{k_j}(t/k_j^{1/\alpha})| \leq \varepsilon \qquad \forall t \in [-1, 1], \tag{2.11}$$

$$\inf\{|t|:\ f_p(t) = 1/2\} \geq 1, \quad \text{i.e.} \quad |f_p(t)| \geq 1/2 \quad \forall t \in [1, 1]. \tag{2.12}$$

Applying Lemma 4 and having in mind that $f_p(t)$ is real, we get that

$$\max_{|t|\leq 1} |f_p(t) - \exp\{-|A_p||t|^{\alpha}\}| \leq C_2\varepsilon^{\triangle}, \tag{2.13}$$

where $A_p = k_1 \log f_p(k_1^{-1/\alpha})$.

Thus, it remains to consider the domain $1 < |t| \leq \infty$.

Note that the method for extending the estimate of type (2.13) from the interval $\mid t \mid \leq 1$ to a considerably wider interval was first applied by author in [25] for a particular case $k_1 = 2$ and $k_2 = 3$.

We denote

$$r_j(t) = f_p(t) - f_p^{k_j}(t/k_j^{1/\alpha}), \tag{2.14}$$

$$h(t) = f_p(t) - \exp\{-|A_p||t|^{1/\alpha}\}. \tag{2.15}$$

According to (2.10), $|r_j(t)| \leq \varepsilon$ for $|t| \leq \infty$. And according to (2.14), (2.15) we have that, for $|t| \leq \infty$,

$$\begin{aligned} &h(t) + \exp\{-|A_p||t|^{\alpha}\} \\ &= \left(h(t/k_j^{1/\alpha}) + \exp\Big\{-|A_p|\frac{|t|^{\alpha}}{k_j}\Big\}\right)^{k_j} + r_j(t) \\ &= h^{k_j}\big(t/k_j^{1/\alpha}\big) + \sum_{i=1}^{k_j-1} \mathbf{C}_{k_j}^i \exp\Big\{-\frac{i|A_p|}{k_j}|t|^{\alpha}\Big\} h^{k_j-i}(t/k_j^{1/\alpha}) \\ &\quad + \exp\{-|A_p||t|^{\alpha}\} + r_j(t), \end{aligned}$$

where $\mathbf{C}_k^i$ is a binomial coefficient. So,

$$h(t) = h^{k_j}\big(t/k_j^{1/\alpha}\big) + \sum_{i=1}^{k_j-1} \mathbf{C}_{k_j}^i \exp\Big\{-\frac{i|A_p|}{k_j}|t|^{\alpha}\Big\} h^{k_j-i}\big(t/k_j^{1/\alpha}\big) + r_j(t) \tag{2.16}$$

for $|t| \leq \infty, \quad j = 1, 2$.

Having assumed that for some $t_0 \in [1, \infty]$

$$\sup_{|t|<t_0} |h(t)|^{k_j} \leq \varepsilon^{\triangle}, \tag{2.17}$$

we prove that the estimate of the same type is also true in the interval $k_j^{1/\alpha}$ times wider, i.e. we prove that

$$\sup_{|t|<k_j^{1/\alpha}t_0} |h(t)|^{k_j} \leq \varepsilon^{\triangle}. \tag{2.18}$$

We note at first that $k_j \geq 2$. If $C_2\varepsilon^{\triangle} \leq 1$, then we have from (2.13) that

$$\sup_{|t|<1} |h(t)|^{k_j} \leq (\sup_{|t|<1} |h(t)|)^{k_j} \leq (C_2\varepsilon^{\triangle})^{k_j} \leq (C_2\varepsilon^{\triangle})^2$$

$$\leq (C_2^2\varepsilon^{\triangle})\varepsilon^{\triangle}. \tag{2.19}$$

Since, as mentioned at the beginning of this paper, we are interested only in small enough $\varepsilon > 0$, let us consider only those ε for which $C_2^2\varepsilon^{\triangle} \leq 1$, i.e. $\varepsilon \leq C_2^{-2/\triangle}$.

Consequently, we obtain from (2.19) that $\sup_{|t|<1} |h(t)|^{k_j} \leq \varepsilon^{\triangle}$, i.e. in the interval $[1, \infty]$ there exists t_0 such that relation (2.17) holds.

Thus, if we have assumption (2.17), we shall prove (2.18).

Since according to the assumption (2.17), $|h(t)|^{k_j} \leq \varepsilon^{\triangle}$ in the interval $|t| < t_0$, for any natural m and $n, \quad n \geq 2$,

$$|h(d_j^m t)|^n \leq \varepsilon^{\triangle} \tag{2.20}$$

for $|t| < t_0/d_j$, where $d_j = 1/k_j^{1/\alpha}$. Using (2.16), (2.17) and (2.20), in the interval $|t| < t_0/d_j$ we obtain:

$$|h(d_j^{m-1}t)| \leq \sum_{i=1}^{k_j-1} \mathbf{C}_{k_j}^i \exp\{-i|A_p||t|^\alpha/k_j^m\}|h(d_j^m t)|^{k_j-i} + \varepsilon^{\triangle} + 2\varepsilon.$$

Since $\varepsilon^{\triangle} < 1$, it is obvious that among all the members of the type $\exp\{-i|A_p||t|^\alpha/k_j^m\}|h(d_j^m t)|^{k_j-i}$ under summation sign, the first member is the largest one, therefore

$$\begin{aligned}|h(d_j^{m-1}t)| &\le \exp\{-|A_p||t|^\alpha/k_j^m\}|h(d_j^m t)|^{k_j-1}\sum_{i=1}^{k_j-1}\mathbf{C}_{k_j}^i+\varepsilon^\triangle+2\varepsilon\\ &= (2^{k_j}-2)\exp\{-|A_p||t|^\alpha/k_j^m\}|h(d_j^m t)|^{k_j-1}+\varepsilon^\triangle+2\varepsilon\\ &\le (2^{k_j}-2)|h(d_j^m t)|\exp\{-|A_p||t|^\alpha/k_j^m\}+\varepsilon^\triangle+2\varepsilon\,.\end{aligned}$$

So, if we make $s+1$ steps we shall get the next result in the interval $|t|<t_0/d_j$:

$$\begin{aligned}|h(t)| &\le (2^{k_j}-2)|h(d_jt)|\exp\{-|A_p|k_j^{-1}|t|^\alpha\}+\varepsilon^\triangle+2\varepsilon\\ &\le (2^{k_j}-2)^2|h(d_j^2t)|\exp\{-|A_p|(k_j^{-1}+k_j^{-2})|t|^\alpha)\}\\ &\quad +(2^{k_j}-2)(\varepsilon^\triangle+2\varepsilon)\exp\{-|A_p|k_j^{-1}|t|^\alpha\}+\varepsilon^\triangle+2\varepsilon\le\dots\\ &\le (2^{k_j}-2)^{s+1}|h(d_j^{s+1}t)|\exp\{-|A_p|k_j^{-1}|t|^\alpha\}\\ &+ (2^{k_j}-2)^{s+1}(\varepsilon^\triangle+2\varepsilon)\exp\{-|A_p|k_j^{-1}|t|^\alpha)\}+\varepsilon^\triangle+2\varepsilon.\end{aligned} \tag{2.21}$$

Let us define s as follows: $s=1+[\alpha\log_{k_j}t_0]$. Note that for $|t|<t_0/d_j$,

$$d_j^{s+1}|t|<1. \tag{2.22}$$

Having denoted

$$F(t)=(2^{k_j}-2)^2|t|^{\alpha/\log_{2^{k_j}-2}k_j}\exp\{-|A_p|k_j^{-1}|t|^\alpha\}$$

we see that, for $|t|\ge t_0$,

$$(2^{k_j}-2)^{s+1}\exp\{-|A_p|k_j^{-1}|t|^\alpha\}\le F(t_0). \tag{2.23}$$

Since $F(t)$ is even, it is easy to verify that the maximum $F(t)$ is attained at the points t^* and $-t^*$, where $t^*=k_j/(|A_p|\log_{2^{k_j}-2}k_j)^{1/\alpha}$. Hence and from relations (2.21), (2.23) we obtain for $t_0\le|t|<t_0/d_j$

$$|h(t)|\le F(t^*)(|h(d_j^{s+1}t)|+\varepsilon^\triangle+2\varepsilon)+\varepsilon^\triangle+2\varepsilon.$$

This implies that there exists a constant $C_3 = C_3(k_j)$ such that for $t_0 \leq |t| < t_0 k_j^{1/\alpha}$

$$|h(t)| \leq C_3 |A_p|^{-1/C_4}(|h(d_j^{s+1}t)| + \varepsilon^{\triangle} + 2\varepsilon) + \varepsilon^{\triangle} + 2\varepsilon, \qquad (2.24)$$

where $C_4 = \log_{2^{k_j}-2} k_j$.

If $p = p_0$ (the case $p = 1$ is trivial) then according to (2.12) $f_p(1) = 1/2$ and from (2.11) we have that $f_p^{k_j}(d_j) \leq 1/2 + \varepsilon$. Recalling the definition of A_p, we obtain for $\varepsilon \leq 1/4$, that

$$|A_p|^{-1} \leq |\log(1/2 + \varepsilon)|^{-1} \leq |\log(3/4)|^{-1} \leq 3.5. \qquad (2.25)$$

From (2.24) and (2.25) we conclude that for $t_0 \leq |t| < t_0/d_j$

$$|h(t)| \leq C_5 |h(d_j^{s+1}t)| + (C_5 + 1)(\varepsilon^{\triangle} + 2\varepsilon), \qquad (2.26)$$

where $C_5 = C_5(k_1, k_2)$ is a constant. By virtue of (2.22) and (2.13), from (2.26) we derive that for $t_0 \leq |t| < t_0/d_j$

$$|h(t)| \leq C_2 C_5 \varepsilon^{\triangle} + (C_5 + 1)(\varepsilon^{\triangle} + 2\varepsilon) \leq C_6 \varepsilon^{\triangle}, \qquad (2.27)$$

where $C_6 = C_2 C_5 + 3(C_5 + 1)$.

Consequently, if

$$\varepsilon^{1-\triangle/k_j} \leq C_6^{-1},$$

we obtain that $|h(t)| \leq \varepsilon^{\triangle/k_j}$ for $|t| < t_0/d_j$, i.e., assuming that (2.17) is true in the interval $|t| < t_0$, we have proved that (2.17) is true in the wider interval $|t| < t_0/d_j$, and simultaneously, as mentioned above, in the whole interval $|t| \leq \infty$, if $\varepsilon \leq \varepsilon_0$, where $\varepsilon_0 = C_6^{-1/(1-\triangle/k_j)}$. Thus, by virtue of (2.26), (2.22) and (2.13), we get (2.27) in the whole t axis:

$$|f_p(t) - \exp\{-|A_p||t|^{\alpha}\} \leq C_6 \varepsilon^{\triangle},$$

i.e. if $\delta = \triangle$ we have that

$$|f(u) - \exp\{-|A_p|p^{-\alpha}|u|^{\alpha}\} \leq C_6\varepsilon^{\delta}$$

for any real u.

According to the definition of the λ_0-metric, it means that relation (2.4) is proved.

Chapter 3

Stability characterization of the Normal distribution

3.1 Characterization of the Normal distribution

We consider a sample $X_1, X_2, ..., X_n$ of n independent observations drawn from a population. Let $2 \leq k_1 < k_2 \leq n$ and $\log k_1 / \log k_2$ be irrational. According to Eaton's theorem (see Chapter 2), if, under additional conditions, the two linear statistics $S_1 = (X_1 + X_2 + ... + X_{k_1})/k_1^{1/\alpha}$ and $S_2 = (X_1 + X_2 + ... + X_{k_2})/k_2^{1/\alpha}$ have the same distribution as the monomial X_1, then this monomial has a symmetric stable distribution of order α.

As it was mentioned in Chapter 2, we shall write $\mathcal{L}(W_1) = \mathcal{L}(W_2)$ to mean that the random variables W_1 and W_2 have the same distribution. So, Eaton's theorem describes conditions under which the relations

$\mathcal{L}(S_1) = \mathcal{L}(X_1)$ and $\mathcal{L}(S_2) = \mathcal{L}(X_1)$ characterize the class of symmetrical stable distributions (of order α).

The case $\alpha = 2$ is particular because, instead of fulfilling the two relations $\mathcal{L}(S_1) = \mathcal{L}(X_1)$ and $\mathcal{L}(S_2) = \mathcal{L}(X_1)$, it is suffices to require only one: $\mathcal{L}(S_1) = \mathcal{L}(X_1)$. One can read more about this, for example, in [28] and [24].

Linnik [8] (see [5] also) has obtained a necessary and sufficient condition for the characterization of the normal distribution by the property of identically distributed linear statistics

$$L_1 = \sum_{i=1}^{k_1} a_i X_i, \quad L_2 = \sum_{i=1}^{k_2} b_i X_i, \tag{3.1}$$

where $X_1, X_2, ...$ are non-degenerate independent identically distributed (i.i.d.) random variables and, for real coefficients $a_1, a_2, ...$ and $b_1, b_2, ...$, the relation

$$\sum_{i=1}^{k_1} a_i^2 = \sum_{i=1}^{k_2} b_i^2 \tag{3.2}$$

is satisfied.

If in (3.1) $k_2 = 1$ and $\boldsymbol{E}X_i = 0$, $\boldsymbol{E}X_i^2 = 1$, and in (3.2) $a_i \neq 0$ for $i = 1, 2, ..., k_1$, then, according to Polya's theorem [13], $X_1, X_2, ...$ are normal random variables.

3.2 Stability problems

Let now the assumptions of Polya's characterization be fulfilled not exactly, but only approximately, with some error ε, where ε is any positive number. For measurements of the error, we choose the metric λ_0 defined in the class of characteristic functions by analogy to a uniform (Kolmogorov) metric ρ defined in the class of distributions:

$$\lambda_0 \left(f_X, f_Y \right) = \lambda_0 \left(X, Y \right) = \sup_t \left| f_X \left(t \right) - f_Y \left(t \right) \right|, \tag{3.3}$$

where $f_X \left(t \right) = \boldsymbol{E} \exp \left(itX \right), \ f_Y \left(t \right) = \boldsymbol{E} \exp \left(itY \right)$. Analogously as the uniform metric ρ is invariant with respect to the multiplier, i.e. $\rho \left(cX, cY \right) = \rho \left(X, Y \right)$ for any real constant $c \neq 0$, the metric λ_0, defined by formula (3.3), is also invariant with respect to the multiplier:

$$\lambda_0 \left(cX, cY \right) = \lambda_0 \left(X, Y \right).$$

It is convenient to express the essence of stability problems of characterization theorems by the λ_0-metric, since the latter are frequently proved by analyzing the equations, considered in the space of characteristic functions. Some aspects of this problem are analyzed by R. Yanushkevichius [21].

Let $a_1, a_2, ...a_k$ be real coefficients such that

$$\sum_{i=1}^{k} a_i^2 = 1, \quad a = \max \left\{ |a_i| : i = 1, 2, ..., k \right\} < 1. \tag{3.4}$$

Now we are ready for the statement of our main theorem. Note, that a distinctive feature of this theorem is the fact that we don't request any conditions of symmetry in comparison with [28] and [21] and, in addition, we weaken the moment conditions in comparison with [24] and [13].

Theorem 3.1 (R. Yanuskevichius [22]). *Let* $X, X_1, X_2, ..., X_k$ *be independent identically distributed random variables and the linear statistic* $L = \sum_{i=1}^{k} a_i X_i$ *and monomial* X *is almost identically distributed in such a sense*

$$\lambda_0\left(X,L\right)=\lambda_0\left(X,\sum_{i=1}^{k}a_iX_i\right)\leq\varepsilon \tag{3.5}$$

and additionally

$$\int_{-\infty}^{\infty}xd\mathtt{S}(x)=\int_{-\infty}^{\infty}x^2d\mathtt{S}(x)=0, \tag{3.6}$$

$$\int_{-\infty}^{\infty}\left|x\right|^{2+\delta}d\left|\mathtt{S}\right|(x)\leq M<\infty, \tag{3.7}$$

where δ *is a constant from the interval* $(0,1]$, $\mathtt{S}(x)=\boldsymbol{P}(L<x)-\boldsymbol{P}(X<x)$ *and* $\left|\mathtt{S}\right|(x)$ *be the total variation of the function* $\mathtt{S}(x)$*. Then there exist a normal random variable* Z *and a constant* $C=C(M,k,a)$ *such that*

$$\lambda_0\left(X,Z\right)\leq C\varepsilon^{1/\beta}. \tag{3.8}$$

Here $\beta=1-\log k/\left(\log\Sigma_{i=1}^{k}\left|a_i\right|^{2+\delta}\right)>0.$

Let us compare this theorem with known Meshalkin's theorem [12]. In this theorem in (3.4) there are $k=2,\ a_1=a_2=1/\sqrt{2}$, instead of (3.6) and (3.7) there are more restrictive conditions $\boldsymbol{E}X=0$, $\boldsymbol{E}X^2=1$ and $\boldsymbol{E}\left|X\right|^3\leq M$, correspondingly. So, Meshalkin's theorem may be reformulated in the metric λ_0 in the following manner. If X_1,X_2 are i.i.d. random variables, $\boldsymbol{E}X_i=0$, $\boldsymbol{E}X_i^2=1$ and $\boldsymbol{E}\left|X_i\right|^3\leq M$ and $\lambda_0\left(X_1,\left(X_1+X_2\right)/\sqrt{2}\right)$
$\leq\varepsilon$, then

$$\lambda_0\left(X_i,Z\right)\leq C\varepsilon^{1/3}\text{ for }i=1,2, \tag{3.9}$$

where Z is a standard normal random variable. Let us compare estimations (3.8) and (3.9). To this end it suffices to find the value of parameter β :

$$\beta = 1 - \log 2 / \left(\log \left(\left(\frac{1}{\sqrt{2}} \right)^3 + \left(\frac{1}{\sqrt{2}} \right)^3 \right) \right) = 3,$$

i.e., the order of stability in formulas (3.8) and (3.9) is the same. Taking into account that the conditions in our main Theorem 3.1 are essentially weaker, we may confirm that our result generalizes Meshalkin's theorem.

3.3 Proof of the main Theorem

Note that the proof of this theorem is nontrivial only when ε is a small positive number. It can be appreciated as follows: there exists a small positive number ε_0, depending only on M, k and a, such that inequality (3.8) is valid for all $\varepsilon \in (0, \varepsilon_0]$. In all the other cases Theorem 3.1 is trivial.

Indeed, by definition of λ_0 metric, for any random variables X and Y , $0 \leq \lambda_0(X, Y) \leq 2$. Hence and from condition (3.5) we obtain that $\varepsilon \geq 0$. In case $\varepsilon = 0$, we get the characterization theorem. Finally, if $\varepsilon > \varepsilon_0$, then C in formula (3.8) is chosen in such a way: $C = 2\varepsilon_0^{-1/\beta}$. In this case

$$\lambda_0(X, Z) \leq 2 = C\varepsilon_0^{1/\beta} < C\varepsilon^{1/\beta} \text{ if } \varepsilon > \varepsilon_0,$$

i.e., (3.8) is trivial for $\varepsilon > \varepsilon_0$.

Let $\varphi(t)$ be the characteristic function of random variable X and let $\Psi(t)$ be the Fourier-Stieltjes transform of $\mathtt{S}(x)$. Since $\mathtt{S}(x) = \boldsymbol{P}(L < x) - \boldsymbol{P}(X < x)$, we have

$$\varphi(t) = \prod_{i=1}^{k} \varphi(a_i t) + \Psi(t), \quad t \in (-\infty, \infty). \tag{3.10}$$

Assumptions (3.6) and (3.7) imply

$$\Psi(t) = M\omega \left|t\right|^{2+\delta}, \tag{3.11}$$

where the symbol ω denotes a quantity bounded by 1.

From (3.10) and (3.11) and relations (7)–(15) in the paper of R. Shimizu [16], we conclude that there exist the first two moments of X, i.e.

$$\boldsymbol{E}X = 0, \quad \boldsymbol{E}X^2 = \sigma^2 < \infty \tag{3.12}$$

and, in addition, for $|t| < 1$

$$\left|\varphi(t) - \exp\left(-\frac{\sigma^2 t^2}{2}\right)\right| \le \frac{M\,|t|^{2+\delta}}{1-a}, \tag{3.13}$$

where a is defined in (3.4).

If X and Y are i.i.d. random variables and $F(x)$ is the distribution function of $X - Y$, then by (3.12)

$$1 - |\varphi(t)|^2 \le \frac{\sigma^2 t^2}{2}. \tag{3.14}$$

Since the metric λ_0 is invariant with respect to the multiplier, i.e.,

$$\lambda_0\left(cX, cY\right) = \lambda_0\left(X, Y\right) \tag{3.15}$$

for any real constant $c \neq 0$, without loss of generality we can assert that $\sigma = 1$ in (3.12). Consequently, from (3.10) and (3.14) we get for $|t| < 1$ that

$$\log\varphi(t) = \sum_{i=1}^{k} \log\varphi(a_i t) + \Psi_1(t), \tag{3.16}$$

where

$$\Psi_1(t) = \log\left(1 + \Psi(t) / \prod_{i=1}^{k} \varphi(a_i t)\right).$$

From (3.5) we derive that $|\Psi(t)| \leq \varepsilon$ for all real t. Let us denote

$$\eta(t) = \frac{2}{t^2} \log \varphi(t), \quad \Psi_2(t) = 2\Psi_1(t)/t^2.$$

Then, from (3.16) we get

$$\eta(t) = \sum_{i=1}^{k} a_i^2 \eta(a_i t) + \Psi_2(t), \quad |t| < 1. \tag{3.17}$$

Consequently,

$$\eta(a_i t) = \sum_{j=1}^{k} a_j^2 \eta(a_j a_i t) + \Psi_2(a_i t), \quad |t| < 1$$

and by (3.17) we have:

$$\begin{aligned} \eta(t) &= \sum_{i=1}^{k} a_i^2 \left(\sum_{j=1}^{k} a_j^2 \eta(a_j a_i t) + \Psi_2(a_i t) \right) + \Psi_2(t) = \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} a_i^2 a_j^2 \eta(a_j a_i t) + \sum_{i=1}^{k} a_i^2 \Psi_2(a_i t) + \Psi_2(t), \quad |t| < 1. \end{aligned}$$

By proceeding this procedure, we get convinced that, for $|t| < 1$,

$$\eta(t) = \sum_{i_1=1}^{k} \sum_{i_2=1}^{k} \dots \sum_{i_s=1}^{k} a_{i_1}^2 a_{i_2}^2 \dots a_{i_s}^2 \eta(a_{i_1} a_{i_2} \dots a_{i_s} t) + \tag{3.18}$$

$$+ \sum_{l=2}^{s} \left(\sum_{i_1=1}^{k} \dots \sum_{i_{l-1}=1}^{k} a_{i_1}^2 \dots a_{i_{l-1}}^2 \Psi_2(a_{i_1} a_{i_2} \dots a_{i_{l-1}} t) \right) + \Psi_2(t).$$

Since according to (3.4)

$$\sum_{i_1=1}^{k} \sum_{i_2=1}^{k} \dots \sum_{i_s=1}^{k} a_{i_1}^2 a_{i_2}^2 \dots a_{i_s}^2 = \left(a_1^2 + a_2^2 + \dots + a_k^2\right)^s = 1,$$

we can insert $\eta + 1$ in (3.18) instead of η :

$$\eta(t)+1=\sum_{i_1=1}^{k}\sum_{i_2=1}^{k}...\sum_{i_s=1}^{k}a_{i_1}^2a_{i_2}^2...a_{i_s}^2\left(\eta(a_{i_1}a_{i_2}...a_{i_s}t)+1\right)+ \tag{3.19}$$

$$+\sum_{l=2}^{s}\left(\sum_{i_1=1}^{k}...\sum_{i_{l-1}=1}^{k}a_{i_1}^2...a_{i_{l-1}}^2\Psi_2(a_{i_1}a_{i_2}...a_{i_{l-1}}t)\right)+\Psi_2(t),\quad |t|<1.$$

Let us estimate the first summand on the right-hand side of (3.19). To this end denote $n_s = n\left(i_1,...,i_s\right) = a_{i_1}^{-2}a_{i_2}^{-2}...a_{i_s}^{-2}$. Then, for $|t| < 1$, it follows from (3.13) that

$$\left|\eta(a_{i_1}a_{i_2}...a_{i_s}t)+1\right|=\left|\frac{2}{a_{i_1}^2a_{i_2}^2...a_{i_s}^2t^2}\log\varphi(a_{i_1}a_{i_2}...a_{i_s}t)+1\right|=$$

$$=2\left|t\right|^{-2}\left|n_s\log\varphi(\frac{t}{\sqrt{n_s}})+\frac{t^2}{2}\right|\le C\left|t\right|^{\delta}n_s^{-\delta/2}. \tag{3.20}$$

Since $n_s^{-\delta/2}=\left|a_{i_1}\right|^{\delta}\left|a_{i_2}\right|^{\delta}...\left|a_{i_s}\right|^{\delta}$, we obtain from (3.20) that for $|t|<1$

$$\left|\sum_{i_1=1}^{k}\sum_{i_2=1}^{k}...\sum_{i_s=1}^{k}a_{i_1}^2a_{i_2}^2...a_{i_s}^2\left(\eta(a_{i_1}a_{i_2}...a_{i_s}t)+1\right)\right|\le$$

$$\le C\left|t\right|^{\delta}\sum_{i_1=1}^{k}\sum_{i_2=1}^{k}...\sum_{i_s=1}^{k}a_{i_1}^2a_{i_2}^2...a_{i_s}^2n_s^{-\delta/2}=C\left|t\right|^{\delta}\left(\sum_{i=1}^{k}\left|a_i\right|^{2+\delta}\right)^{s}. \tag{3.21}$$

Now let us proceed with the estimation of the second and third summands on the right-hand side of (3.19). By (3.14), $\left|\varphi(t)\right|^2 \ge 1/2$ for $|t| < 1$. Therefore

$$\left|\Psi_1(t)\right|=\left|\log(1+\frac{\Psi(t)}{\prod_{i=1}^{k}\varphi(a_it)})\right|\le\frac{2\left|\Psi(t)\right|}{\left|\prod_{i=1}^{k}\varphi(a_it)\right|}\le 2^{k/2+1}\varepsilon \text{ for } |t|<1.$$

From this estimation and definition of $\Psi_2(t)$ it follows that, for $|t| < 1,$

$$\left|\sum_{l=2}^{s}\left(\sum_{i_1=1}^{k} ... \sum_{i_{l-1}=1}^{k} a_{i_1}^2 ... a_{i_{l-1}}^2 \Psi_2(a_{i_1}a_{i_2}...a_{i_{l-1}}t)\right)\right| + |\Psi_2(t)| \leq$$

$$\leq \frac{2}{t^2}\sum_{l=2}^{s}\left(\sum_{i_1=1}^{k} ... \sum_{i_{l-1}=1}^{k}\left|\Psi_1(a_{i_1}a_{i_2}...a_{i_{l-1}}t)\right|\right) + \frac{2}{t^2}|\Psi_1(t)| \leq$$

$$\leq \varepsilon t^{-2} 2^{k/2+2}\sum_{l=1}^{s} k^{l-1} \leq Ck^s \varepsilon t^{-2}. \tag{3.22}$$

Consequently, from (3.19) - (3.22) we obtain that, for $|t| < 1,$

$$|\eta(t)+1| \leq C\,|t|^{\delta}\left(\sum_{i=1}^{k}|a_i|^{2+\delta}\right)^{s} + Ck^s\varepsilon t^{-2}. \tag{3.23}$$

Let us denote $\Xi(h) = \sum_{i=1}^{k}|a_i|^h$. Since by (3.4) $\Xi(2) = 1,$ note that $|\Xi(h)| < 1$ for $h > 2.$ So, choose the parameter s in (3.23) in the following manner: $s = [\varkappa \log 1/\varepsilon],$ where $\varkappa$ is defined below. Then, for $|t| < 1,$

$$|\eta(t)+1| \leq C\,|t|^{\delta}\,(\Xi(2+\delta))^{-1}\,\varepsilon^{\varkappa\log(1/\Xi(2+\delta))} + C\varepsilon^{1-\varkappa\log k}t^{-2}.$$

Now, let us define $\varkappa = 1/\log(k/\Xi(2+\delta)).$ Then, for $|t| < 1,$

$$|\eta(t)+1| \leq \left(C\,|t|^{\delta}\,(\Xi(2+\delta))^{-1} + Ct^{-2}\right)\varepsilon^{\gamma}, \quad \gamma = \frac{\log \Xi(2+\delta)}{\log \Xi(2+\delta)k^{-1}}. \tag{3.24}$$

Since

$$\left|\varphi(t) - \exp\left(-t^2/2\right)\right| \leq t^2\,|\eta(t)+1|\,, \tag{3.25}$$

we derive from (3.24) that for $|t| < 1$

$$\left|\varphi(t)-\exp\left(-t^2/2\right)\right| \le \left(C\left|t\right|^{2+\delta}\left(\Xi(2+\delta)\right)^{-1}+C\right)\varepsilon^{\gamma}. \qquad (3.26)$$

By Lemma 3 in [24] and (3.26) we obtain (3.8). The essence of this lemma can be expressed in the following words. If in the study of stability problem there appears equation (3.10), in which the coefficients $a_1, a_2, ...a_k$ satisfy the condition (3.4), then deterioration of the order of stability can be only in the interval $(-1, 1)$. Note that Lemma 3 in [24] was generalized in [21].

Chapter 4

Stability estimations of a characterization of the Stable distribution in weak metric

4.1 Introduction and statement of the problem

We consider a population with the distribution function $F(x)$ and a sample $X_1, X_2, ..., X_n$ of n independent observations drawn from this population.

According to Feller [3], the distribution function of X_1 is called strictly stable if it is not concentrated at zero and there exist $\alpha \in (0, 2]$ such that for each n $\mathcal{L}(X_1) = \mathcal{L}((X_1 + ... + X_n)/n^{1/\alpha})$, i.e.,

$$f(t) = f^n(t/n^{1/\alpha}), \quad \alpha \in (0, 2], \quad n = 2, 3, 4, ... \tag{4.1}$$

By choosing $\alpha = 2$ and comparing (1.1) and (4.1), we see that as $\alpha = 2$ for a strict stability it suffices to fulfill (4.1) only for $n = 2$, i.e., the requirement to fulfill (4.1) for $n = 3, 4, ...$ is unnecessary. Maybe, an analogous conclusion is also true for $\alpha \in (0, 2)$?

Unfortunately, it is not. According to P. Lévy's example (Feller [3], Chap. 17), the realization of (4.1) only for $n = 2$ is not yet sufficient for the characterization of the class of strictly stable distributions, because the characteristic function

$$f(t) = \exp\left\{2 \sum_{k=-\infty}^{\infty} (\cos 2^k t - 1)\right\}$$

satisfies (4.1) for $n = 2, \alpha = 1$, i.e. $f(t) = f^2(t/2)$, but $f(t)$ is not strictly stable.

On the other hand, P. Lévy has proved that $f(t)$ is strictly stable, if (4.1) is realized at $n = 2$ and $n = 3$. The stability of this P. Lévy's characterization theorem was investigated by R. Yanushkevichius and O. Yanushkevichiene [25, 26].

The characterization of symmetric stable laws of order α with the characteristic function $f(t) = \exp\{-\lambda |t|^\alpha\}$, i.e. a subclass of strictly stable laws has been considered by Lukacs [9] assuming the identical distribution of the monomial S_1 and the linear form S_2. In that paper the following result is proved:

Theorem 4.1 (E.Lukacs [9]). *Let $X, X_1, X_2, \ldots, X_n, \ldots$ be independent identically distributed (i.i.d.) random variables. For every choice of n and every choice of $(a_1, ..., a_n)$ from $\mathbb{C}_\alpha^n = \{(a_1, \ldots, a_n) : \ a_i \in \mathbb{R}^1, \quad i = 1, 2, \ldots, n, \quad \Sigma_{i=1}^n |a_i|^\alpha = 1\}$*

$$\mathcal{L}(X) = \mathcal{L}(\sum_{i=1}^n a_i X_i) \tag{4.2}$$

if and only if the distribution of X is a symmetric stable law of order $\alpha \in (0, 2]$.

Many authors (see A. Kagan, Yu. Linnik, C.R. Rao monograph [5], R. Yanushkevichius monograph [20] and U. Rösler paper [15]) analyzed relations (4.2) under different conditions for the coefficients and random variables considered. U. Rösler [15] considered (4.2) generalizations as $n = \infty$ and $a_1, a_2, \ldots$ are random variables. In [15], the relation

$$\mathcal{L}(X) = \mathcal{L}\left(\sum_{i=1}^{n} T_i X_i + T_0\right)$$

is called distributional fixed point equation. Here the joint distribution of T_0 and $(T_1, T_2, \ldots)$ is supposed to be known.

Since the point $(n^{-1/\alpha}, ..., n^{-1/\alpha}) \in \mathbb{C}_\alpha^n$, from (4.2) we see that equations (4.1) are satisfied.

The condition "for every choice of n" was omitted in [9], therefore Y.H. Wang [17] wrote about this result: "The assumption – every choice of n – is indispensable in the proof of Lukacs because he used the result due to Lévy ([7], p.95) showing that if ψ is the logarithm of a characteristic function satisfying $n\psi(t) = \psi(a_n t)$ for all n, where $\{a_n\}$ is a sequence of real numbers, then

$$\psi(t) = \left(-c_0 + ic_1 \frac{t}{|t|}\right) |t|^\alpha \tag{4.3}$$

with $c_0 > 0, \alpha > 0$".

Eaton [2] succeeded in avoiding the condition "for every choice of n". For fixed n, $n \geq 2$ he has proven that if $X, X_1, X_2, \ldots, X_n$ are real valued i.i.d. random variables and condition (4.2), where a_i are nonzero and $\sum_{i=1}^{n} a_i^2 \geq 1$ is satisfied, then the distribution of X is infinitely divisible.

It should be stressed that not only n, but also the set of coefficients $a_1, \ldots, a_n$ are fixed here. Condition (4.2), where both – n and the set $a_1, \ldots, a_n$ – are fixed, has been considered in detail in ([5], chapter 13).

Y.H. Wang [17] also considers condition (4.2) for the fixed sample size n, but the coefficients $a_1, \dots, a_n$, however, are not fixed here.

Theorem 4.2 (Y.H. Wang [17]). *Let $X, X_1, X_2, \dots, X_n$ be non-degenerate i.i.d. random variables. If* $0 < \alpha \le 2$, *then X is symmetric stable if and only if condition* (4.2) *is satisfied for some fixed* n, $n \ge 2$ *and all* $(a_1, \dots, a_n) \in \mathbb{C}_\alpha^n$.

In [17] it has been shown that if $g(t)$ is the characteristic function of random variables $X, X_1, X_2, \dots, X_n$, then in the case $n = 2$

$$g\left(\sqrt[\alpha]{|a_1 t|^\alpha + |a_2 t|^\alpha}\right) = g(|a_1 t|)\, g(|a_2 t|) \quad \text{for all} \quad t \neq 0, \tag{4.4}$$

$$|a_1|^\alpha + |a_2|^\alpha = 1 \tag{4.5}$$

(formula (4.7) in [17]). Denoting $x = |a_1 t|$, $y = |a_2 t|$, it is easy to see that (4.4) is the Cauchy functional equation:

$$g(\sqrt[\alpha]{x^\alpha + y^\alpha}) = g(x)g(y) \quad \text{for all} \quad x, y \ge 0. \tag{4.6}$$

In Y.H.Wang's paper [17] the analysis of equation (4.6) is made by applying pacing $y = (m-1)^{1/\alpha} x$, where $m = 2, 3, \dots$ But in such a case, from (4.5) it follows that

$$y = |a_2 t| = (m-1)^{1/\alpha} x = (m-1)^{1/\alpha} |a_1 t|,$$

$$|a_2|^\alpha = (m-1)\,|a_1|^\alpha, \qquad 1 - |a_1|^\alpha = (m-1)\,|a_1|^\alpha.$$

Consequently, for $m = 2, 3, ...$

$$|a_1|^\alpha = 1/m, \qquad |a_2|^\alpha = 1 - 1/m.$$

It means that in Wang's paper [17] variation of the coefficients a_1, a_2 is exploited substantially.

One could avoid this owing to Eaton's work [2]. True, in this case, it would be necessary to refuse the symmetry condition present in the Eaton's paper [2], however, it makes no difficulty if the conditions of Theorem 4.2 are fulfilled.

Theorem 4.3 (M.L. Eaton [2]). *Let* $X, X_1, \dots, X_{k_1}, \dots, X_{k_2}$ *be* k_2 *symmetric i.i.d. random variables. If* $0 < \alpha \le 2$, *and* k_1 *and* k_2 *are integers such that* $\theta = \log k_1 / \log k_2 \quad (2 \le k_1 < k_2)$ *is irrational, and*

$$\mathcal{L}(X) = \mathcal{L}(k_1^{-1/\alpha} \sum_{i=1}^{k_1} X_i) = \mathcal{L}(k_2^{-1/\alpha} \sum_{i=1}^{k_2} X_i), \tag{4.7}$$

then X has a symmetric stable distribution of order α *.*

It is easy to see, that condition (4.7) is equivalent to condition (4.1) if the latter is satisfied not for all natural n, but only for two n values: $n = k_1$ and $n = k_2$.

Note that the points y_1 and y_2 are called *incommensurable* if their ratio y_1/y_2 is irrational.

Condition (4.1) for $n = k_1$ and $n = k_2$ consists of two equations in the space of characteristic functions. Similar, in some sense, two equations for incommensurable points in the space of distribution functions and the related stability problems are analyzed in [27].

If in (4.7) $\alpha = 1$, we have such a characterization of Cauchy law without the symmetry condition:

Theorem 4.4 (B. Ramachandran, C.R. Rao [14]). *Let* $X, X_1, X_2, \dots, X_n$ *be i.i.d. random variables. If* X *and sample mean* $\overline{X}_{(n)} = \frac{1}{n}(X_1 + \ldots + X_n)$ *have the same distribution for two values* k_1 *and* k_2 *of* n *such that*

$\theta = \log k_1 / \log k_2 \quad (2 \le k_1 < k_2 \le n)$ *is irrational, then* X *has a Cauchy distribution.*

However, it is important to emphasize that authors of this theorem successfully avoided the condition of symmetry in it only by using the condition $\alpha = 1$ essentially. Indeed, from relation (4.7) in the case $\alpha = 1$ it is easy to get that the Lévy representation for log f is of the form

$$\log f(t) = i\mu t - c|t| \left\{ 1 + \frac{2}{\pi} ib \frac{t}{|t|} \log |t| \right\}$$

for $t \neq 0$ and for some $c > 0$ and real b $(|b| \le 1)$. By substituting it into (4.1) as $n = k_1$ and $n = k_2$ we find that $b = 0$, i.e., the stable law differs from the symmetric one only by a shift. Therefore $f(t)$ is the Cauchy characteristic function.

However in the general case the way to avoid the condition of symmetry in Eaton's theorem is still not found. As one can see from [5], for $0 < \alpha < 1$ or $1 < \alpha < 2$ the condition of symmetry can be avoided only under the additional condition on the existence of negative numbers among the coefficients $a_1, ..., a_n$ in condition (4.2) (see Theorem 13.7.2 in [5]).

Therefore Eaton's theorem 4.3 has preserved its actuality and is of special interest at present.

Verification of this or that characterization theorem in practice is possible only with some error ε, i.e., only to a certain degree of accuracy. That is why there arises a following natural question. Suppose that the conditions of the theorem are fulfilled not exactly but only approximately. May we assert, that the conclusion of the theorem is also fulfilled approximately?

We discuss the conditions in which sense the assumptions of the characterization theorem is fulfilled not exactly but only approximately in the next section.

4.2 The main Theorem and two auxiliary Lemmas

Let now the conditions (4.7) of Eaton's theorem be fulfilled only approximately, with some error ε. In the main theorem - Theorem 4.5 - ε is any positive number. The parameter ε express the proximity of the considered in formula (4.7) statistics in the λ-metric defined below. However, it should be noted that only the case where ε is small positive member is of mathematical interest. Why? We shall discuss that immediately after formulating Theorem 4.5.

Our aim is to get convinced that in a certain sense the characteristic function $f(t)$ of the random variable X is close to the characteristic function of a symmetric stable law.

For 'measurements' of the error of fulfillment of conditions (4.7) we chose a weak metric λ, i.e. we investigate the stability of Eaton's theorem 4.3 in the metric λ. We remind the definition of this metric.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. We consider a set $\mathcal{X} = \{X : \Omega \longrightarrow \mathbb{R}\}$ of real $\mathcal{F}$-measurable functions. In the probability theory the functions $X(\omega)$ are interpreted as random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We define a λ-metric between arbitrary random variables X and Y from $\mathcal{X}$ as follows:

$$\lambda(X,Y) = \min\left\{ \max\left\{ \frac{1}{2}\max(|f_X(t) - \right.\right.$$
$$\left.\left. - f_Y(t)| :\ |t| \le T), \frac{1}{T} \right\} :\ T > 0 \right\},$$

where $f_X,\ f_Y$ denote characteristic functions of the random variables X and Y, respectively.

The λ-metric is equivalent to the Lévy metric L in the sense that L-convergence of the sequence $\{X_n\}$ from $\mathcal{X}$ implies the convergence of this sequence in the λ-metric, and vice versa. Two-sided estimations of this metric are studied by V. Zolotarev and V. Senatov [32].

The λ-metric is convenient to express the essence of stability problems of characterization theorems, since the latter are frequently proved by analyzing the equations considered in the space of characteristic functions. Some aspects of this problem are analyzed by R. Yanushkevichius [19].

Some additional results will be necessary from the Diophantine approximation theory.

It is known that there exist constants $b = b(k_1, k_2)$ and $b' = b'(k_1, k_2)$ such that for any natural r and k and any integers k_1 and k_2 with irrational $\log k_1 / \log k_2$ the inequality

$$|r \log k_1 - k \log k_2| > b' r^{-b} \tag{4.8}$$

holds.

Let us take some comments on the constants b and b'.

From E.M. Matveev [11] it follows a lower estimation of a linear form $r \log k_1 - k \log k_2$:

$$|r \log k_1 - k \log k_2| > (3r)^{\left(-2^{30} \log k_1 \log k_2\right)}.$$

This estimation may be improved by using Corollary 2.3 from N.Gouillon [4]:

$$|r \log k_1 - k \log k_2| > (23r)^{-36821 \log k_1 \log k_2}.$$

It means that in formula (4.8) constants b and b' can be selected as follows:

$$b' := 23^{-36821 \log k_1 \log k_2}, \quad b := 36821 \log k_1 \log k_2. \tag{4.9}$$

We fix these values of b and b' throughout the paper.

Now we are ready for the statement of the main Theorem.

Theorem 4.5 (R. Yanushkevichius, O. Yanushkevichiene [28]). *Let* $X, X_1,$ $X_2, \ldots, X_{k_1}, \ldots, X_{k_2}$ *be* k_2 *symmetric i.i.d. random variables, where* k_1 *and* k_2 *are integers such that* $\theta = \log k_1 / \log k_2$ $(2 \le k_1 < k_2)$ *is irrational. If there exists* $\alpha \in (0, 2]$ *such that for* $j = 1, 2$ *the relations*

$$\lambda \left(X, k_j^{-1/\alpha} \sum_{i=1}^{k_j} X_i \right) \le \varepsilon \tag{4.10}$$

are fulfilled, then there exists a random variable Y *with the symmetric stable distribution of order* α *such that*

$$\lambda\left(X, Y\right) \le C_1 \varepsilon^{\Delta}, \tag{4.11}$$

where $\Delta = 1/(b + \max(1, \alpha))$, C_1 *is a constant, depending only on* $\alpha, k_1, k_2,$ *and* b *is a constant, depending only on* k_1 *and* k_2.

By taking an equality sign instead of the inequality one in formula (4.10), we get a simple illustration of the parameter ε, i.e., the distance between monomial X and the linear statistic $k_j^{-1/\alpha} \sum_{i=1}^{k_j} X_i$ in the λ metric.

Note that the proof of this theorem (as well as of Lemma 3.1) is nontrivial only when ε is a small positive number. It can be appreciated as follows: there exists a small positive number ε_0, depending only on α, k_1 and k_2, such that (4.11) is valid for all $\varepsilon \in (0, \varepsilon_0]$. In all the other cases Theorem 4.5 is trivial.

Indeed, according to the definition of λ metric, for any X and Y $0 \leq \lambda(X,Y) \leq 1$. Hence and from condition (4.10) we obtain that $\varepsilon \geq 0$. In case $\varepsilon = 0$, we get Theorem 4.3. Finally, if $\varepsilon > \varepsilon_0$, then C_1 in formula (4.11) is chosen in such a way: $C_1 = \varepsilon_0^{-\Delta}$. In this case $\lambda(X,Y) \leq 1 = C_1\varepsilon_0^{\Delta} < C_1\varepsilon^{\Delta}$ if $\varepsilon > \varepsilon_0$, i.e., (4.11) is trivial for $\varepsilon > \varepsilon_0$.

Everything is ready now for the formulation of the first auxiliary lemma.

Lemma 4.1 (O. Yanushkevichiene [18]). *Let m be an arbitrary integer and κ be an arbitrary positive constant. Then there exist an integer m' and an integer n' corresponding to m' such that for an arbitrary small positive number $\varepsilon > 0$*

$$|m'\alpha_2 - n'\alpha_1| < \varepsilon^{\kappa} \tag{4.12}$$

and

$$0 \leq m - m' < M\varepsilon^{\kappa b}, \tag{4.13}$$

where $\alpha_1 = -\alpha^{-1}\log k_1$, $\alpha_2 = -\alpha^{-1}\log k_2$, $M = 2(3^b/((2^{1/b}-1)b'))$.

The next lemma, proved together with L. Klebanov, probably of independent interest, is useful in the following:

Lemma 4.2 (R. Yanushkevichius, O. Yanushkevichiene [28]). *Let* $L(t)$ *be a complex-valued function and let there exist positive numbers* α, ε_* *and natural ones* $k \geq 2$ *such that*

$$|L(t)| \geq k\left|L\left(t/k^{1/\alpha}\right)\right| - \varepsilon,\ \forall\varepsilon \in (0,\varepsilon_*],\ \forall t \in [-1,1].$$

Then for any δ *in the interval* $(0,1]$ *and for any* ε *in the interval* $(0,\varepsilon_*]$

$$\sup_{|t|\leq\delta}|L(t)| \leq k\delta^\alpha \sup_{|t|\leq 1}|L(t)| + \varepsilon.$$

Proof of Lemma 4.2. If $S(\delta) = \sup_{|t|\leq\delta}|L(t)|$, then according to the condition of Lemma 2.3, for any ε in the interval $(0,\varepsilon_*]$,

$$S(\delta) \geq kS\left(\delta/k^{1/\alpha}\right) - \varepsilon.$$

Consequently,

$$S\left(\delta/k^{1/\alpha}\right) \leq S(\delta)/k + \varepsilon/k, \qquad S\left(k^{-1/\alpha}\right) \leq S(1)/k + \varepsilon/k.$$

Proceeding in a similar way we get that

$$S(k^{-2/\alpha}) \leq S(k^{-1/\alpha})/k + \varepsilon/k \leq S(1)/k^2 + \varepsilon/k^2 + \varepsilon/k,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S(k^{-j/\alpha}) \leq S(1)/k^j + \varepsilon/k^j + \ldots + \varepsilon/k^2 + \varepsilon/k. \tag{4.14}$$

It is easy to see that for any $\delta \in (0,1)$ there exists a nonnegative integer $K = K(\delta,\alpha,k)$ such that $k^{-K/\alpha} \geq \delta \geq k^{-(K+1)/\alpha}$. From the monotonicity of the function $S(\delta)$ and (4.14) we have that

$$S(\delta) \leq S(k^{-K/\alpha}) \leq S(1)/k^K + \varepsilon \leq kS(1)\delta^\alpha + \varepsilon,$$

which was to be proved.

4.3 The main Lemma

Lemma 4.3 (R. Yanushkevichius, O. Yanushkevichiene [28]). *Let k_1 and k_2 $(2 \le k_1 < k_2)$ be integers such that the ratio of their logarithms $\log k_1/\log k_2$ is irrational, and for some $\alpha \in (0,2]$ there exists a characteristic function $f(t)$ such that*

$$\left|f(t) - f^{k_i}\left(t/k_i^{1/\alpha}\right)\right| \le \varepsilon, \quad \forall t \in [-1,1]. \tag{4.15}$$

In addition, if

$$|f(t)| \ge 1/2 \quad for \quad \forall t \in [-1,1], \tag{4.16}$$

then there exists a constant C_2 depending only on α and k_1, k_2 such that for $|t| \le 1$

$$|f(t) - \exp\{-|D|\exp(iQ\operatorname{sign} t)|t|^\alpha\}| \le C_2\varepsilon^\Delta, \tag{4.17}$$

where $D = k_1 \log f\left(k_1^{-1/\alpha}\right),\ \Delta = 1/(b + \max(1,\alpha)),$

$$Q = \arctan(\operatorname{Im} D/\operatorname{Re} D)$$

and b is a constant, defined by formula (4.9) *.*

Proof. First we reduce condition (4.15) to the additive form. For that let us denote

$$u = \log t, \quad H(\log t) = \log f(t).$$

From (4.15) we obtain for $0 < t \le 1$ and $i = 1,2$

$$H(u) = k_i H(u + \alpha_i) + R_i(e^u), \tag{4.18}$$

where $R_i(t) = \log(1 + r(t)f^{-k_i}(t/k_i^{1/\alpha}))$, $r(t) = f(t) - f^{k_i}\left(t/k_i^{1/\alpha}\right)$, $\alpha_i = -\frac{1}{\alpha}\log k_i$.

In further considerations it is useful to avoid the multiplier k_i in formula (4.18). The following notation is helpful for this purpose:

$$\psi(u) = H(u)\exp(-\alpha u). \tag{4.19}$$

Using (4.18) from (4.19) for $u \in \mathbb{R}_- = (-\infty, 0]$ we obtain:

$$\psi(u) = \psi(u+\alpha_i) + R_i(e^u)\, e^{-\alpha u}, \quad i = 1, 2. \tag{4.20}$$

The next notation is necessary for the following proof. Let $\mathbb{Z}$ be set of all integers, $\mathbb{Z}_+$ be a set of all nonnegative integers and let us consider the following sets: $\mathcal{M} = \{u\colon\ u = n\alpha_1 + m\alpha_2 \le 0;\quad n \in \mathbb{Z}, m \in \mathbb{Z}\}$, $\mathcal{N} = \{u\colon\ u = n\alpha_1 + m\alpha_2;\ n \in \mathbb{Z}_+, m \in \mathbb{Z}_+\}$, $\mathcal{P} = \{u\colon\ \alpha_1 < u = n\alpha_1 + m\alpha_2 \le 0;\ n \in \mathbb{Z}, m \in \mathbb{Z}\}$.

Note that we consider the main equation - equation (4.20) - in the interval $\mathbb{R}_-$. The set $\mathcal{M}$ is everywhere dense in $\mathbb{R}_-$. On the other hand, it is not difficult (see the authors' paper [25]) to present any element of $\mathcal{M}$ as a sum of an element from the infinite lattice $\mathcal{N}$ in the interval $\mathbb{R}_-$ and an element from a 'small' set $\mathcal{P}$, everywhere dense in the interval $(\alpha_1, 0]$: for all $u \in \mathcal{M}$

$$u = u_1 + u_2, \qquad u_1 \in \mathcal{N}, \quad u_2 \in \mathcal{P}. \tag{4.21}$$

Next we prove the following statement, which is very useful in the sequel. If the function ψ satisfies relation (4.20), then, in a certain sense, it is close enough to the 'initial point of pacing' $\psi(\alpha_1)$.

To be more precise, both for small and large enough (by absolute value) u in the pacing lattice $\mathcal{N}$ the estimate

$$|\psi(u) - \psi(\alpha_1)| \le C_3 \varepsilon \exp(-\alpha u), \quad u \in \mathcal{N} \tag{4.22}$$

is valid, where $C_3 = C_3(k_1, k_2)$ is a constant.

It is quite the other way if we consider the limited set $\mathcal{P}$, everywhere dense in the interval $(\alpha_1, 0]$. We shall prove that

$$|\psi(u) - \psi(\alpha_1)| \le C\varepsilon^{\eta} \qquad \text{for} \quad u \in \mathcal{P}, \tag{4.23}$$

where C is a constant defined below, and

$$\eta = \begin{cases} 1/(b+1) & \text{for } \ 0 < \alpha \le 1, \\ 1/(b+\alpha) & \text{for } \ 1 \le \alpha \le 2. \end{cases}$$

Here constant b, as usual, is defined by (4.9).

In view of representation (4.21), according to which it is possible to present an arbitrary $u \in \mathcal{M}$ as a sum $u_1 + u_2$, where $u_1 \in \mathcal{N}$ and $u_2 \in \mathcal{P}$, on the basis of estimates (4.22) and (4.23) we shall prove finally that

$$|\psi(u) - \psi(\alpha_1)| \le C_3\varepsilon \exp(-\alpha u) + C\varepsilon^{\eta} \qquad \text{for} \quad u \in \mathcal{M}. \tag{4.24}$$

Thus, we go over to the proof of relation (4.22). From (4.20) we obtain that for $u = n\alpha_1 + m\alpha_2 \in \mathcal{N}$

$$\begin{aligned} &\psi(u) = \psi(n\alpha_1 + m\alpha_2) = \psi((n-1)\alpha_1 + m\alpha_2) \\ &- R_1(\exp\{(n-1)\alpha_1 + m\alpha_2\}) \exp\{-\alpha((n-1)\alpha_1 + m\alpha_2)\} = \dots \\ &= \psi(\alpha_1 + m\alpha_2) - \\ &\qquad\qquad - \sum_{j=1}^{n-1} R_1(\exp(j\alpha_1 + m\alpha_2)) \exp(-\alpha(j\alpha_1 + m\alpha_2)) = \dots \\ &= \psi(\alpha_1) - \sum_{j=0}^{m-1} R_2(\exp(\alpha_1 + j\alpha_2)) \exp(-\alpha(\alpha_1 + j\alpha_2)) \\ &- \sum_{j=1}^{n-1} R_1(\exp(j\alpha_1 + m\alpha_2)) \exp(-\alpha(j\alpha_1 + m\alpha_2)). \end{aligned} \tag{4.25}$$

Note that by virtue of condition (4.16) for $\varepsilon \le \min(2^{-(k_1+1)}, 2^{-(k_2+1)}) = 2^{-(k_2+1)}$

$$|R_i(t)| \le 2|r(t)|\,|f^{-k_i}\left(t/k_i^{1/\alpha}\right)| \le 2^{k_i+1}\varepsilon. \tag{4.26}$$

We note that it suffices to prove Lemma 3.1 only for $\varepsilon \le \varepsilon_0 := 2^{-(k_2+1)}$. Indeed, for $\varepsilon > \varepsilon_0 = 2^{-(k_2+1)}$ the statement of Lemma 3.1 is trivial if we will choose $C_2 = C_2(\alpha, k_1, k_2)$ such that $C_2\varepsilon_0^{\Delta} = 2$, i.e. $C_2 = 2\varepsilon_0^{-\Delta} = 2^{\Delta(k_2+1)+1}$.

So, by virtue of (4.26) for $\varepsilon \le 2^{-(k_2+1)}$

$$\begin{aligned}
&\left|\sum_{j=1}^{n-1} R_1(\exp(j\alpha_1 + m\alpha_2))\exp(-\alpha(j\alpha_1 + m\alpha_2))\right| \\
&\quad \le 2^{k_1+1}\varepsilon(\exp(-\alpha(n\alpha_1 + m\alpha_2)) - \exp(-\alpha(\alpha_1 + m\alpha_2))) \\
&\quad \le 2^{k_1+1}\varepsilon\exp(-\alpha u).
\end{aligned} \tag{4.27}$$

If $n \ge 1$, then

$$\begin{aligned}
&\left|\sum_{j=0}^{m-1} R_2(\exp(\alpha_1 + j\alpha_2))\exp(-\alpha(\alpha_1 + j\alpha_2))\right| \\
&\quad \le 2^{k_2+1}\varepsilon\exp(-\alpha(\alpha_1 + m\alpha_2)) \le 2^{k_2+1}\varepsilon\exp(-\alpha(n\alpha_1 + m\alpha_2)) \\
&\quad = 2^{k_2+1}\varepsilon\exp(-\alpha u).
\end{aligned} \tag{4.28}$$

By (4.25), (4.27) and (4.28) we conclude that for $\varepsilon \le 2^{-(k_2+1)}$ relation (4.22) holds, where $C_3 = 2^{k_1+1} + 2^{k_2+1}$.

Let us proceed to the proof of relation (4.23). Let now $u \in \mathcal{P}$. Then $u \in (\alpha_1, 0]$ and $u = n\alpha_1 + m\alpha_2$, where n, m are integers. Both numbers n, m cannot be negative at the same time because $\alpha_1 < 0$, $\alpha_2 < 0$ and in that case (i.e., in case of $n < 0$ and $m < 0$) $\quad n\alpha_1 + m\alpha_2 > 0$, which contradicts the condition $u \in (\alpha_1, 0]$.

Both numbers n, m cannot be positive at the same time as well, because in that case $n\alpha_1 + m\alpha_2 < \alpha_1$.

Consequently, either $n \geq 0$ and $m \leq 0$, or $m \geq 0$ and $n \leq 0$, i.e., either $u = n\alpha_1 - m\alpha_2$ or $u = m\alpha_2 - n\alpha_1$, where n, m are nonnegative integers. Both cases are considered in the same manner, therefore assumption

$$u \in \mathcal{P}, \quad u = n\alpha_1 - m\alpha_2, \quad n \geq 0, \quad m \geq 0 \tag{4.29}$$

does not diminish the generality of reasoning. Define

$$m_i = m_{i-1} - 1 = m_{i-2} - 2 = ... = m - i; \quad n_0 = n, \tag{4.30}$$

$$n_i = \min\{n_* : \quad n_*\alpha_1 - m_i\alpha_2 \leq 0, \quad n_* - \text{natural}\}. \tag{4.31}$$

It is not difficult to prove that n_i can be changed within the following bounds:

$$[m_i\alpha_2/\alpha_1] \leq n_i \leq [m_i\alpha_2/\alpha_1] + 1, \tag{4.32}$$

where the integer part of A is denoted by $[A]$.

Basing on (4.20) and in view of (4.29), (4.30), after the first step with respect to the coefficient of α_2 we take next steps (their number is equal to $n_0 - n_1)$ with respect to the coefficient of α_1, and so on:

$$\begin{aligned}
&\psi(u) = \psi(n\alpha_1 - m\alpha_2) = \psi(n\alpha_1 - (m-1)\alpha_2) \\
&+ R_2(\exp(n\alpha_1 - m\alpha_2))\exp(-\alpha(n\alpha_1 - m\alpha_2)) \\
&= \psi(n_1\alpha_1 - m_1\alpha_2) + \\
&+ R_2(\exp(n\alpha_1 - m\alpha_2))\exp(-\alpha(n\alpha_1 - m\alpha_2)) \\
&- \sum_{j=n_1}^{n_0-1} R_1(\exp(j\alpha_1 - m_1\alpha_2))\exp(-\alpha(j\alpha_1 - m_1\alpha_2)) = \dots \\
&= \psi(n_i\alpha_1 - m_i\alpha_2) \\
&+ \sum_{h=0}^{i-1} R_2(\exp(n_h\alpha_1 - m_h\alpha_2))\exp(-\alpha(n_h\alpha_1 - m_h\alpha_2)) - \\
&- \sum_{h=1}^{i} \sum_{j=n_h}^{n_{h-1}-1} R_1(\exp(j\alpha_1 - m_h\alpha_2))\exp(-\alpha(j\alpha_1 - m_h\alpha_2). \qquad (4.33)
\end{aligned}$$

On the basis of (4.12) and (4.13) in Lemma 4.1 we note that there exists i such that

$$i < M\varepsilon^{-\kappa b}, \qquad (4.34)$$

and, in addition,

$$|m_i\alpha_2 - n_i\alpha_1 - \alpha_1| \le \varepsilon^{\kappa}. \qquad (4.35)$$

By (4.32), in the last expression of the equalities in (4.33) the number of summands other than $\psi(n_i\alpha_1 - m_i\alpha_2)$ does not exceed the number $(3 + [\alpha_2/\alpha_1])i$.

Recalling (4.32), we find that the exponential multipliers in (4.33) are bounded: for $j = n_h, \dots, n_{h-1} - 1$

$$\begin{aligned}
\exp(-\alpha(j\alpha_1 - m_h\alpha_2)) &\le \exp(-\alpha(n_{h-1} - 1)\alpha_1 - m_h\alpha_2) \\
&\le \exp(-\alpha(\alpha_1 + \alpha_2)) = k_1k_2. \qquad (4.36)
\end{aligned}$$

It follows from (4.33) and (4.26), (4.34), (4.36) that for $u = n\alpha_1 - m\alpha_2 \in \mathcal{P}$

$$\begin{aligned} |\psi(u) - \psi(n_i\alpha_1 - m_i\alpha_2)| &= |\psi(n\alpha_1 - m\alpha_2) - \psi(n_i\alpha_1 - m_i\alpha_2)| \\ &\leq (3 + [\alpha_2/\alpha_1]k_1k_2i2\varepsilon(2^{k_1} + 2^{k_2}) \\ &< C_4\varepsilon^{1-\kappa b}, \end{aligned} \tag{4.37}$$

where $C_4 = 2(3 + [\log k_2/\log k_1])k_1k_2(2^{k_1} + 2^{k_2})M.$

Now, making use of (4.35), we estimate $|\psi(n_i\alpha_1 - m_i\alpha_2) - \psi(\alpha_1)|$, and simultaneously $|\psi(u) - \psi(\alpha_1)|$ for $u \in \mathcal{P}$.

Owing to this note that for $0 < t_j \leq 1$ and $u_j = \log t_j, \quad j = 1, 2,$

$$\begin{aligned} |\psi(u_1) - \psi(u_2)| &= |H(u_1)\exp(-\alpha u_1) - H(u_2)\exp(-\alpha u_2)| \\ &= |t_1^{-\alpha}H(\log t_1) - t_2^{-\alpha}H(\log t_2)| \\ &= |t_1^{-\alpha}\log f(t_1) - t_1^{-\alpha}\log f(t_2) + \\ &+ t_1^{-\alpha}\log f(t_2) - t_2^{-\alpha}\log f(t_2)| \\ &\leq t_1^{-\alpha}|\log f(t_1) - \log f(t_2)| + \\ &+ |\log f(t_2)||t_1^{-\alpha} - t_2^{-\alpha}|. \end{aligned} \tag{4.38}$$

Now we need Lemma 4.2. (4.15) implies that $\forall t \in [-1, 1]$ the relation

$$f(t) = f^{k_i}(t/k_i^{1/\alpha}) + r(t), \qquad |r(t)| \leq \varepsilon, \quad i = 1, 2$$

holds. Hence it follows that

$$\log f(t) = k_i \log f(t/k_i^{1/\alpha}) + r_{*i}(t), \tag{4.39}$$

where $r_{*i}(t) = \log(1 + r(t)/f^{k_i}(t/k_i^{1/\alpha}))$.

From this and (4.16) we notice that $\forall t \in [-1, 1]$

$$|r_{*i}(t)| \leq 2^{k_i+1}\varepsilon. \tag{4.40}$$

In formula (4.39), by taking first $t = t_1$, afterwards $t = t_2$, and subtracting from the first relation obtained the second one, we obtain

$$\log f(t_2) - \log f(t_1) = k_i(\log f(t_2/k_i^{1/\alpha}) - \log f(t_1/k_i^{1/\alpha})) + r_{*i}(t_2) - r_{*i}(t_1).$$

Denote $L(t_2 - t_1) = \log f(t_2) - \log f(t_1)$. Then

$$L(t_2 - t_1) = k_i L\left(\frac{t_2 - t_1}{k_i^{1/\alpha}}\right) + r_{*i}(t_2) - r_{*i}(t_1). \qquad (4.41)$$

Next, let $t = t_2 - t_1$, $S(\delta) = \sup_{|t|\le\delta} |L(t)|$. From (4.41) we see that the function $L(t)$ satisfies the conditions of Lemma 4.2, therefore $S(\delta) \le k_i\delta^\alpha \sup_{|t|\le 1} |L(t)| + \varepsilon_i$, where $\varepsilon_i = 2\sup_{|t|\le 1} |r_{*i}(t)|$.

(4.40) yields

$$\varepsilon_i = 2\sup_{|t|\le 1} |\ln(1 + r(t)/f^{k_i}(t/k_i^{1/\alpha}))| \le 2^{k_i+2}\varepsilon.$$

Since $|f(t)| \ge 1/2$ for $|t| \le 1$, we have

$$\begin{aligned} \sup_{|t|\le 1} |L(t)| &\le \sup_{|t_1|\le 1, |t_2|\le 1|} (|\log f(t_2)| + |\log f(t_1)|) \\ &\le 2\sup_{|t|\le 1} |\log f(t)| \le 4\sup_{|t|\le 1} |1 - f(t)| \le 8. \end{aligned}$$

Therefore for any δ from the interval $(0, 1]$

$$S(\delta) \le 8k_i\delta^\alpha + 2^{k_i+2}\varepsilon. \qquad (4.42)$$

Choosing $t_1 = \exp\alpha_1 = k_1^{-1/\alpha} \in (0, 1]$ and a point $t_2 = \exp(n_i\alpha_1 - m_i\alpha_2)$, close to it (in the sense of relation (4.35)), we obtain

$$|\ln f(t_1) - \ln f(t_2)| \le C_5|t_1 - t_2|^\alpha + 2^{k_1+2}\varepsilon, \qquad (4.43)$$

where $C_5 = 8k_1$. It follows from (4.38) and (4.43) that

$$|\psi(u_1) - \psi(u_2)| \le k_1(C_5|t_1 - t_2|^\alpha + 2^{k_1+2}\varepsilon) + 4k_1|e^{\alpha(u_2-u_1)} - 1|. \qquad (4.44)$$

Since $|t_1 - t_2| = t_2|e^{u_1-u_2} - 1|$, from (4.35) and (4.44) we derive that

$$|\psi(u_1) - \psi(u_2)| \le C_5 k_1 t_2^{\alpha} |e^{u_1-u_2} - 1|^{\alpha} + 4k_1|e^{\alpha(u_1-u_2)} - 1| + \\ +2^{k_1+2}k_1\varepsilon \le 2^{\alpha} C_5 k_1 |u_1 - u_2|^{\alpha} + 2^{k_1+2}k_1\varepsilon + 8k_1\alpha|u_1 - u_2| \\ \le 2^{\alpha} C_5 k_1 \varepsilon^{\kappa\alpha} + 9k_1\alpha\varepsilon^{\kappa}, \tag{4.45}$$

if, in addition, $\varepsilon^{\kappa} \max(1, \alpha) \le 1/2.$

Since $\psi(u_1) = \psi(\alpha_1)$, for $u \in \mathcal{P}$ (4.37) and (4.45) yield

$$|\psi(u) - \psi(\alpha_1)| \le C_4 \varepsilon^{1-\kappa b} + 2^{\alpha} C_5 k_1 \varepsilon^{\kappa\alpha} + 9k_1\alpha\varepsilon^{\kappa}. \tag{4.46}$$

We select now the constant κ in Lemma 4.1 so that the summands in (4.46) have approximately 'the same weight' with respect to ε, i.e., on the one hand, that $1 - \kappa b \sim \kappa\alpha$, and on the other hand, that $1 - \kappa b \sim \kappa$. Hence and from (4.46) we conclude that

$$|\psi(u) - \psi(\alpha_1)| \le C\varepsilon^{\eta},$$

where $C = C_4 + 2^{\alpha} C_5 k_1 + 9k_1\alpha$, and $\eta = 1/(b+1)$ for $0 < \alpha \le 1$ and $\eta = 1/(b+\alpha)$ for $1 \le \alpha \le 2$. Thus, relation (4.23) is completely proved.

Since $\triangle$ in (4.11) is defined by formula $\triangle = 1/(b + \max(1, \alpha))$, it is obvious that $\eta = \triangle$.

Finally, let $u \in \mathcal{M}$. Let us go over to the proof of relation (4.24). Expressing u in the form (4.21), let us repeat (4.25):

$$\begin{aligned} \psi(u) &= \psi(u_1 + u_2) = \psi(n\alpha_1 + m\alpha_2 + u_2) \\ &= \psi(u_2) - \sum_{j=0}^{m-1} R_2(\exp(\alpha_1 + j\alpha_2 + u_2)) \exp(-\alpha(\alpha_1 + j\alpha_2 + u_2)) \\ &\quad - \sum_{j=0}^{n-1} R_1(\exp(j\alpha_1 + m\alpha_2 + u_2)) \exp(-\alpha(j\alpha_1 + m\alpha_2 + u_2)). \end{aligned}$$

Consequently, as in (4.22), $|\psi(u)-\psi(u_2)| \le C_3\varepsilon\exp(-\alpha u)$. Hence, as in (4.23), we conclude

$$|\psi(u)-\psi(\alpha_1)| \le C_3\varepsilon\exp(-\alpha u)+C\varepsilon^\eta, \quad u\in\mathcal{M}. \tag{4.47}$$

At the end of the proof in the case $0\le t\le 1$ it remains to note that for any $u\in(-\infty,0]$

$$|\psi(u)-\psi(\alpha_1)| = \lim_{j\uparrow\infty}|\psi(u_j)-\psi(\alpha_1)|,$$

because relation $u=\lim_{j\uparrow\infty}u_j, \quad u_j\in\mathcal{M}$ is valid for any $u\in(-\infty,0]$ and $\psi(u)$ is continuous on $(-\infty,\infty)$, i.e., (4.47) is valid for an arbitrary $u\in(-\infty,0]$. Since $\psi(u)=H(u)\exp(-\alpha u), \quad H(\log t)=\log f(t)$, (4.47) means that for $u\in(-\infty,0]$

$$\begin{aligned} |H(u)\exp(-\alpha u)-H(\alpha_1)\exp(-\alpha\alpha_1)| &\le C_3\varepsilon\exp(-\alpha u)+C\varepsilon^\eta, \\ |\log f(t)-k_1H(\alpha_1)t^\alpha| &\le C_3\varepsilon+C\varepsilon^\eta t^\alpha, \\ |\log f(t)-k_1\log f(k_1^{-1/\alpha})t^\alpha| &\le C_3\varepsilon+C\varepsilon^\eta t^\alpha \end{aligned} \tag{4.48}$$

for $0\le t\le 1$.

Now let $-1\le t<0$. Letting $t=-u$, $0<u\le 1$, we rewrite (4.48) in the following form: $\log f(u)=Au^\alpha+R_3(u), \qquad 0<u\le 1$, where

$$|R_3(u)|\le C_3\varepsilon+C\varepsilon^\eta u^\alpha, \qquad A=k_1\ln f(k_1^{-1/\alpha}). \tag{4.49}$$

Note that

$$f(t)=f(-u)=\overline{f(u)}=\exp(\overline{A}u^\alpha)\exp\overline{R_3(u)}. \tag{4.50}$$

If $\tan Q=\operatorname{Im}D/\operatorname{Re}D$, then, as it is known,

$$D=-|D|\exp(iQ). \tag{4.51}$$

Since obviously Im$\overline{D}$=-iImD and ReD=Re $\overline{D}$, from (4.48)–(4.51) for $|t| \leq 1$ we obtain the following relation:

$$f(t) = \exp\{-|D| \exp(iQ \operatorname{sign} t)|t|^{\alpha}\} \exp R_4(t),$$

where for $|R_4(t)|$ estimation (4.49) holds as it does for $|R_3(t)|$. This means that for $|t| \leq 1$

$$|f(t) - \exp\{-|D| \exp(iQ \operatorname{sign} t)|t|^{\alpha}\}| \leq C_3\varepsilon + C\varepsilon^{\eta}|t|^{\alpha}. \tag{4.52}$$

Lemma 3.1 is proved.

4.4 Proof of the main Theorem

Let us consider two functions of the positive variable T:

$$M_1(T) = \frac{1}{2} \max_{|t| \leq T} |f_X(t) - f_Y(t)|, \qquad M_2(T) = 1/T,$$

where $f_X(t)$ and $f_Y(t)$ are characteristic functions of the random variables X and Y, respectively. Since $M_1(T)$ is a non-increasing function, $M_2(T)$ is a monotonically increasing function, the condition

$$\min_{T>0} \max(M_1(T), M_2(T)) = \varepsilon \tag{4.53}$$

means that there exists $T_* > 0$ such that

$$M_1(T_*) \leq \varepsilon, \quad M_2(T_*) = \varepsilon. \tag{4.54}$$

The latter in relations (4.54) means that $T_* = 1/\varepsilon$. Then the former in relations (4.54) means that

$$\max_{|t| \leq 1/\varepsilon} |f_X(t) - f_Y(t)| \leq 2\varepsilon. \tag{4.55}$$

Note that condition (4.53) is equivalent to the condition $\lambda(X, Y) = \varepsilon$. Thus, from $\lambda(X, Y) = \varepsilon$ we get (4.55). Analogously from $\lambda(X, Y) \leq \varepsilon$ we obtain the same (4.55).

It means that from condition (4.10) we have that

$$|f(t) - f^{k_j}(t/k_j^{1/\alpha})| \leq 2\varepsilon \qquad \text{for} \quad |t| \leq 1/\varepsilon, \quad j = 1, 2. \tag{4.56}$$

Since $f(t)$ is a continuous function such that $f(0) = 1$, and, besides, $f(t)$ is real as a characteristic function of a symmetric random variable, there exists p_0 such that $\min\{t\colon\ f(t) = 1/2\} = p_0(f) = p_0 > 0$. Only two cases are possible: $p_0 \leq 1/\varepsilon$ or $p_0 > 1/\varepsilon$. So, let $p = \min(p_0, 1/\varepsilon)$.

Instead of the characteristic function $f(t)$, we introduce the characteristic function $f_p(t) = f(pt)$, for which

$$\begin{aligned} p_0(f_p) &= \min\left\{|t|\colon\ f_p(t) = 1/2\right\} = \min\left\{|t|\colon\ f(pt) = 1/2\right\} \\ &= \frac{1}{p}\min\left\{|u|\colon\ f(u) = \frac{1}{2}\right\} = \frac{1}{p}\cdot p_0 \geq 1. \end{aligned}$$

If $f(t)$ satisfies (4.56) for $|t| \leq 1/\varepsilon$, then $f_p(t)$ satisfies (4.56) for $|t| \leq 1/(\varepsilon p)$.

Since $p \leq 1/\varepsilon$, i.e. $1/(\varepsilon p) \geq 1$, for the characteristic function $f_p(t)$ we have that

$$|f_p(t) - f_p^{k_j}(t/k_j^{1/\alpha})| \leq 2\varepsilon \qquad \forall t \in [-1, 1], \tag{4.57}$$

$$\min\{|t|\colon\ f_p(t) = 1/2\} \geq 1, \quad \text{i.e.} \quad |f_p(t)| \geq 1/2 \quad \forall t \in [1, 1]. \tag{4.58}$$

Applying Lemma 3.1 and having in mind that $f_p(t)$ is real, we get that

$$\max_{|t|\leq 1} |f_p(t) - \exp\{-|A_p||t|^\alpha\}| \leq C_2\varepsilon^{\triangle}, \tag{4.59}$$

where $A_p = k_1 \log f_p(k_1^{-1/\alpha})$.

Thus, it remains to consider the domain $1 < |t| \leq 1/(\varepsilon p)$ (of course, only in the case where $p < 1/\varepsilon$; if $p = 1/\varepsilon$, the proof is completed).

Note that the method for extending the estimate of type (4.59) from the interval $\mid t \mid\leq 1$ to a considerably wider interval was first applied by authors in [25] for a particular case $k_1 = 2$ and $k_2 = 3$.

We denote

$$r_j(t) = f_p(t) - f_p^{k_j}(t/k_j^{1/\alpha}), \tag{4.60}$$

$$h(t) = f_p(t) - \exp\{-|A_p||t|^{1/\alpha}\}. \tag{4.61}$$

According to (4.56), $|r_j(t)| \leq 2\varepsilon$ for $|t| \leq 1/(p\varepsilon)$. And according to (4.60), (4.61) we have that, for $|t| \leq 1/(p\varepsilon)$,

$$\begin{aligned} &h(t) + \exp\{-|A_p||t|^\alpha\} \\ &= \left(h(t/k_j^{1/\alpha}) + \exp\Big\{-|A_p|\frac{|t|^\alpha}{k_j}\Big\}\right)^{k_j} + r_j(t) \\ &= h^{k_j}\big(t/k_j^{1/\alpha}\big) + \sum_{i=1}^{k_j-1} \mathbf{C}_{k_j}^i \exp\Big\{-\frac{i|A_p|}{k_j}|t|^\alpha\Big\} h^{k_j-i}(t/k_j^{1/\alpha}) \\ &\quad + \exp\{-|A_p||t|^\alpha\} + r_j(t), \end{aligned}$$

where $\mathbf{C}_k^i$ is a binomial coefficient. So,

$$h(t) = h^{k_j}\big(t/k_j^{1/\alpha}\big) + \sum_{i=1}^{k_j-1} \mathbf{C}_{k_j}^i \exp\Big\{-\frac{i|A_p|}{k_j}|t|^\alpha\Big\} h^{k_j-i}\big(t/k_j^{1/\alpha}\big) + r_j(t) \tag{4.62}$$

for $|t| \leq 1/(p\varepsilon), \quad j = 1, 2.$

Having assumed that for some $t_0 \in [1, 1/(p\varepsilon))$

$$\sup_{|t|<t_0} |h(t)|^{k_j} \leq \varepsilon^{\triangle}. \tag{4.63}$$

We prove that the estimate of the same type is also true in the interval $k_j^{1/\alpha}$ times wider, i.e. we prove that

$$\sup_{|t|<k_j^{1/\alpha} t_0} |h(t)|^{k_j} \leq \varepsilon^{\triangle} \tag{4.64}$$

if $k_j^{1/\alpha} t_0 \leq 1/(p\varepsilon)$.

We note at first that $k_j \geq 2$. If $C_2\varepsilon^{\triangle} \leq 1$, then we have from (4.59) that

$$\sup_{|t|<1} |h(t)|^{k_j} \leq (\sup_{|t|<1} |h(t)|)^{k_j} \leq (C_2\varepsilon^{\triangle})^{k_j} \leq (C_2\varepsilon^{\triangle})^2$$

$$\leq (C_2^2\varepsilon^{\triangle})\varepsilon^{\triangle}. \tag{4.65}$$

Since, as mentioned at the beginning of this paper, we are interested only in small enough $\varepsilon > 0$, let us consider only those ε for which $C_2^2\varepsilon^{\triangle} \leq 1, \quad i.e.\ \varepsilon \leq C_2^{-2/\triangle}$.

Consequently, we obtain from (4.65) that $\sup_{|t|<1} |h(t)|^{k_j} \leq \varepsilon^{\triangle}$, i.e. in the interval $[1, 1/(p\varepsilon))$ there exists t_0 such that relation (4.63) holds. In case we succeed to obtain (4.64) from this, then, because of $k_j^{1/\alpha} > 1$, we would also get thereby that (4.63) is also valid for $t_0 = 1/(p\varepsilon)$.

Thus, if we have assumption (4.63), we shall prove (4.64).

Since according to the assumption (4.63), $|h(t)|^{k_j} \leq \varepsilon^{\triangle}$ in the interval $|t| < t_0$, for any natural m and $n, \quad n \geq 2$

$$|h(d_j^m t)|^n \leq \varepsilon^{\triangle} \tag{4.66}$$

for $|t| < t_0/d_j$, where $d_j = 1/k_j^{1/\alpha}$. Using (4.62), (4.63) and (4.66), in the interval $|t| < t_0/d_j$ we obtain:

$$\left|h(d_j^{m-1}t)\right| \leq \sum_{i=1}^{k_j-1} \mathbf{C}_{k_j}^i \exp\{-i|A_p||t|^\alpha/k_j^m\}|h(d_j^m t)|^{k_j-i} + \varepsilon^{\triangle} + 2\varepsilon.$$

Since $\varepsilon^{\triangle} < 1$, it is obvious that among all the members of the type

$$\exp\{-i|A_p||t|^\alpha/k_j^m\}|h(d_j^m t)|^{k_j-i}$$

under summation sign, the first member is the largest one, therefore

$$\begin{aligned}|h(d_j^{m-1}t)| &\leq \exp\{-|A_p||t|^\alpha/k_j^m\}|h(d_j^m t)|^{k_j-1}\sum_{i=1}^{k_j-1}\mathbf{C}_{k_j}^i + \varepsilon^\triangle + 2\varepsilon \\ &= (2^{k_j}-2)\exp\{-|A_p||t|^\alpha/k_j^m\}|h(d_j^m t)|^{k_j-1} + \varepsilon^\triangle + 2\varepsilon \\ &\leq (2^{k_j}-2)|h(d_j^m t)|\exp\{-|A_p||t|^\alpha/k_j^m\} + \varepsilon^\triangle + 2\varepsilon\,.\end{aligned}$$

So, if we make $s+1$ steps we shall get the next result in the interval $|t| < t_0/d_j$:

$$\begin{aligned}|h(t)| &\leq (2^{k_j}-2)|h(d_j t)|\exp\{-|A_p|k_j^{-1}|t|^\alpha\} + \varepsilon^\triangle + 2\varepsilon \\ &\leq (2^{k_j}-2)^2|h(d_j^2 t)|\exp\{-|A_p|(k_j^{-1}+k_j^{-2})|t|^\alpha)\} \\ &\quad +(2^{k_j}-2)(\varepsilon^\triangle+2\varepsilon)\exp\{-|A_p|k_j^{-1}|t|^\alpha\} + \varepsilon^\triangle + 2\varepsilon \leq \ldots \\ &\leq (2^{k_j}-2)^{s+1}|h(d_j^{s+1}t)|\exp\{-|A_p|k_j^{-1}|t|^\alpha\} \\ &+ (2^{k_j}-2)^{s+1}(\varepsilon^\triangle+2\varepsilon)\exp\{-|A_p|k_j^{-1}|t|^\alpha)\} + \varepsilon^\triangle + 2\varepsilon. \qquad (4.67)\end{aligned}$$

Let us define s as follows: $s = 1 + [\alpha \log_{k_j} t_0]$. Note that for $|t| < t_0/d_j$,

$$d_j^{s+1}|t| < 1. \qquad (4.68)$$

Having denoted

$$F(t) = (2^{k_j}-2)^2|t|^{\alpha/\log_{2^{k_j}-2} k_j}\exp\{-|A_p|k_j^{-1}|t|^\alpha\}$$

we see that, for $|t| \geq t_0$,

$$(2^{k_j}-2)^{s+1}\exp\{-|A_p|k_j^{-1}|t|^\alpha\} \leq F(t_0). \qquad (4.69)$$

Since $F(t)$ is even, it is easy to verify that the maximum $F(t)$ is attained at the points t^* and $-t^*$, where $t^* = k_j/(|A_p|\log_{2^{k_j}-2} k_j)^{1/\alpha}$. Hence and from relations (4.67), (4.69) we obtain for $t_0 \leq |t| < t_0/d_j$

$$|h(t)| \leq F(t^*)(|h(d_j^{s+1}t)| + \varepsilon^\triangle + 2\varepsilon) + \varepsilon^\triangle + 2\varepsilon.$$

This implies that there exists a constant $C_7 = C_7(k_j)$ such that for $t_0 \leq |t| < t_0 k_j^{1/\alpha}$

$$|h(t)| \leq C_7 |A_p|^{-1/C_8}(|h(d_j^{s+1}t)| + \varepsilon^{\triangle} + 2\varepsilon) + \varepsilon^{\triangle} + 2\varepsilon, \qquad (4.70)$$

where $C_8 = \log_{2^{k_j}-2} k_j$.

If $p = p_0$ (the case $p = 1/\varepsilon$ is trivial) then according to (4.58) $f_p(1) = 1/2$ and from (4.57) we have that $f_p^{k_j}(d_j) \leq 1/2 + \varepsilon$. Recalling the definition of A_p, we obtain for $\varepsilon \leq 1/4$, that

$$|A_p|^{-1} \leq |\log(1/2 + \varepsilon)|^{-1} \leq |\log(3/4)|^{-1} \leq 3.5. \qquad (4.71)$$

From (4.70) and (4.71) we conclude that for $t_0 \leq |t| < t_0/d_j$

$$|h(t)| \leq C_9 |h(d_j^{s+1}t)| + (C_9 + 1)(\varepsilon^{\triangle} + 2\varepsilon), \qquad (4.72)$$

where $C_9 = C_9(k_1, k_2)$ is a constant. By virtue of (4.68) and (4.59), from (4.72) we derive that for $t_0 \leq |t| < t_0/d_j$

$$|h(t)| \leq C_2 C_9 \varepsilon^{\triangle} + (C_9 + 1)(\varepsilon^{\triangle} + 2\varepsilon) \leq C_{10}\varepsilon^{\triangle}, \qquad (4.73)$$

where $C_{10} = C_2 C_9 + 3(C_9 + 1)$.

Consequently, if

$$\varepsilon^{1-\triangle/k_j} \leq C_{10}^{-1},$$

we obtain that $|h(t)| \leq \varepsilon^{\triangle/k_j}$ for $|t| < t_0/d_j$, i.e., assuming that (4.63) is true in the interval $|t| < t_0$, we have proved that (4.63) is true in the wider interval $|t| < t_0/d_j$, and simultaneously, as mentioned above, in the whole interval $|t| < 1/(p\varepsilon)$, if $\varepsilon \leq \varepsilon_0$, where $\varepsilon_0 = C_{10}^{-1/(1-\triangle/k_j)}$. Thus, by virtue of (4.72), (4.68) and (4.59), we get (4.73) in the whole interval $|t| < 1/(p\varepsilon)$:

$$|f_p(t) - \exp\{-|A_p||t|^{\alpha}\} \leq C_{10}\varepsilon^{\triangle},$$

i.e. for $|u| < 1/\varepsilon$ we have that $|f(u) - \exp\{-|A_p|p^{-\alpha}|u|^{\alpha}\} \leq C_{10}\varepsilon^{\triangle}$.

According to the definition of the λ-metric, it means that relation (4.11) is proved.

Chapter 5

Stability estimations of a characterization of the Normal distribution in weak metric

5.1 Characterization by the property of identically distributed linear statistics

The Chapter 5 is devoted to the estimation of the stability of characterization of the normal law by the property of identically distributed linear statistics

$$X = X_1, \quad S = \sum_{i=1}^{n} b_i X_i,$$

where $X_1, X_2, ..., X_n$ are independent identically distributed (i.i.d.) random variables and $b_1, b_2, ...b_n$ are real coefficients.

Such a characterization theorem is well known (see, for example, the monograph by Kagan, Linnik, Rao [5], Theorem 13.7.2). If X and S are identically distributed and $\sum b_j^2 = 1$, then $X_1, X_2, ..., X_n$ is a normal sample. It is important to emphasize that in the formulation of this characterization theorem any moment restrictions are absent.

Only Zinger, Klebanov, Yanushkevichius [30] succeeded to preserve the absence of moment restrictions in the investigation of the stability of this characterization but only in the case $n = 2, \;\; b_1 = b_2 = 1/\sqrt{2}$. In the general case all authors, which investigate this problem, require the moment or pseudo-moment restrictions (see, for example, the monograph by Yanushkevichius [20]).

Following V.M. Zolotarev [31], let us introduce metrics μ_k and ν_r in the space of random variables,

$$\mu_k(X,Y) = \int\limits_{-\infty}^{\infty} x^k d\left(F_X - F_Y\right), \;\; k = 0,1,2,...$$

$$\nu_r(X,Y) = \int\limits_{-\infty}^{\infty} |x|^r \left|d\left(F_X - F_Y\right)\right|, \;\; r > 0.$$

Conditions in these metrics are analogous to the corresponding moment and pseudo-moment conditions, therefore we can reformulate the result of R. Shimizu [16] (see also [22]) in the following manner:

Theorem 5.1 (R.Shimizu [16]). *Let $X, X_1, X_2, ..., X_n$ be i.i.d. random variables. Under the assumptions*

$$\mu_1(X,S) = \mu_2(X,S) = 0, \quad \nu_3(X,S) \le \varepsilon \tag{5.1}$$

and

$$b_1^2 + b_2^2 + ... + b_n^2 = 1, \;\; a = \max\left\{|b_1|, ..., |b_n|\right\} < 1, \tag{5.2}$$

the random variable X has finite mean θ and variance σ^2 and the following inequality holds:

$$\rho\left(X, \mathcal{Z}_{\theta,\sigma^2}\right) = \sup_x \left|P(X < x) - P(\mathcal{Z}_{\theta,\sigma^2} < x)\right| \leq$$
$$\leq 1.8(1-a)^{-1/4}\sigma^{-3/4}\varepsilon^{1/4}, \tag{5.3}$$

where $\mathcal{Z}_{\theta,\sigma^2}$ is a normal random variable with mean θ and variance σ^2.

Estimation (5.3) is non-informative in the cases where σ are very small. In addition, the observation error is estimated in the metric ν_3 in assumptions of Theorem 5.1, but the closeness with a normal random variable is estimated in another metric - a uniform metric ρ.

Our purpose is to avoid these imperfections. We attain the aim by choosing Lévy metric L instead of the metrics ν_3 and ρ.

5.2 Comparison of metrics

Recall that Lévy metric L is defined by the formula

$$L(X,Y) = \inf\{\varepsilon : P(X < x - \varepsilon) - \varepsilon \leq P(Y < x) \leq$$

$$\leq P(X < x + \varepsilon) + \varepsilon \; for \; all \; x \in R^1\}.$$

Since the conditions of our Theorem 5.2 are formulated in Lévy metric L, it is interesting to compare R. Shimizu conditions (5.1) with their analogue in Lévy metric L. Of course, we can replace ρ in (3.2) by L, since for any random variables X, Y always $L(X,Y) \leq \rho(X,Y)$. Unfortunately, the multiplier $\sigma^{-3/4}$ in R. Shimizu estimation (5.3) does not allow us to conclude that the characterization theorem under consideration is stable in ρ. Indeed, in the case where the sequence of random variables under consideration $\{X_k^*\}$ satisfies the condition $\lim_{k\to\infty} DX_k^* = 0$, estimation (5.3) becomes non-informative.

But, may be, the multiplier of such a kind is not necessary in (5.3)? The answer is negative in the case of uniform metric ρ, because it is simple to construct an example, from which we conclude that our characterization model is not stable in the uniform metric ρ. Indeed, let $X^*(\varepsilon)$, $X_1^*(\varepsilon)$, $X_2^*(\varepsilon)$, ..., $X_n^*(\varepsilon)$ be a sequence (by ε) of i.i.d. normal random variables with zero mean and dispersion ε^2. Then

$$\rho(X^*(\varepsilon), \sum b_i X_i^*(\varepsilon)) = 0 \le \varepsilon.$$

Since the dispersion $\varepsilon^2 \downarrow 0$, our random variables $X_i^*(\varepsilon)$ are degenerating as $\varepsilon \downarrow 0$ but, on the other hand,

$$\rho(X_i^*(\varepsilon), E) = 1/2 \text{ for all } i = 1, 2, ...n \text{ and } \varepsilon > 0.$$

Thus,

$$\lim_{\varepsilon\downarrow 0} \rho(X_i^*(\varepsilon), E) \ne 0, \tag{5.4}$$

where E is a degenerate in zero normal random variable. According to (3.4) for sufficiently small $\sigma = \varepsilon$ we indeed have no effect of stability in the uniform metric ρ. However, in the case of any weak metric (for example, in the case of Lévy metric L) we have another picture:

$$\lim_{\varepsilon\downarrow 0} L(X_i^*(\varepsilon), E) = 0.$$

We also note that in the case of Lévy metric it is not difficult to correct the situation by slightly changing the proof of R. Shimizu [16]. Indeed, let $f(t)$ be the characteristic function of X. Shimizu [16] has proved that, if conditions of Theorem 5.1 are satisfied, then

$$\int_{-T}^{T} \frac{|f(t) - \exp(i\theta t - \sigma^2 t^2/2)|}{t} dt \leq \frac{\varepsilon}{3}(1-a)^{-1} \int_{0}^{T} t^2 dt = \frac{\varepsilon}{9}(1-a)^{-1} T^3. \tag{5.5}$$

According to formula (1.5.18) in V.M. Zolotarev [31],

$$L\left(X, \mathcal{Z}_{\theta,\sigma^2}\right) \leq \frac{1}{\pi} \int_{0}^{T} \frac{|f(t) - \exp(i\theta t - \sigma^2 t^2/2)|}{t} dt + 5.66 \frac{\log(1+T)}{T} \tag{5.6}$$

So, choosing $T = \varepsilon^{-1/4}$, we have from (5.5) and (5.6) that, for sufficiently small $\varepsilon > 0$,

$$L\left(X, \mathcal{Z}_{\theta,\sigma^2}\right) \leq 6\varepsilon^{1/4} \log \frac{1}{\varepsilon}. \tag{5.7}$$

It means, that we successfully avoided the multiplier $\sigma^{-3/4}$ in formula (5.3).

Can we improve the order of stability in (5.7) and avoid a logarithmic multiplier? The answer is positive. Indeed, note that according to (1.5.44) in [31],

$$L^4(X,Y) \leq 32\zeta_3(X,Y), \tag{5.8}$$

where $\zeta_s(X,Y) = \sup\left\{|E(f(X) - f(Y))| : f \in \mathcal{F}_s\right\}, \; s = m + \alpha$, $m \geq 0$ is the integer, $0 < \alpha \leq 1$, and $\mathcal{F}_s$ is a set of all real bounded functions on R^1 with the derivatives of order m at all the points and

$$\left|f^{(m)}(x) - f^{(m)}(y)\right| \leq |x-y|^{\alpha}.$$

According to Zolotarev's theorem 1.4.2 in [31], the metric $\zeta_s(X,Y)$, $s \geq 0$ is an ideal metric of order s. On the other hand, since $\mu_1(X,S) = \mu_2(X,S) = 0$, according to formula (1.5.41) in [31],

$$\zeta_3(X,S) \leq \frac{\nu_3(X,S)}{\Gamma(4)} = \frac{\nu_3(X,S)}{6}. \tag{5.9}$$

From (5.9) and the relation $\nu_3(X,L) \leq \varepsilon$ in (5.1) we conclude that

$$\zeta_3(X,S) \leq \varepsilon/6. \tag{5.10}$$

As R. Shimizu [16] has noted, from the condition

$$\mu_1(X,S) = \mu_2(X,S) = 0$$

we derive that

$$\theta \sum_{j=1}^{n} b_j = \theta. \tag{5.11}$$

Let now $Z_1, Z_2, ..., Z_n$ be i.i.d. normal random variables with the mean θ and dispersion σ^2 and let $Z = \sum_{j=1}^{n} b_j Z_j$. Then, it follows from (5.11) that Z is also a normal random variable with the mean θ and dispersion σ^2, i.e. Z and $\mathcal{Z}_{\theta,\sigma^2}$ have the same normal distribution with the parameters θ and σ^2. Thus, since ζ_3 is an ideal metric of order 3, by virtue of (5.10) we obtain

$$\zeta_3(X,Z) \leq \zeta_3(X,S) + \zeta_3(S,Z) \leq \varepsilon/6 + \zeta_3\left(\sum_{j=1}^{n} b_j X_j, \sum_{j=1}^{n} b_j Z_j\right) \leq$$

$$\leq \varepsilon/6 + \sum_{j=1}^{n} |b_j|^3 \zeta_3(X_j, Z_j). \tag{5.12}$$

Let us denote $\delta = \zeta_3(X_j, Z_j)$. It is easy to see that $\delta = \zeta_3(X,Z)$ and under (5.2) and (5.12)

$$\delta \leq \varepsilon/6 + \delta \sum_{j=1}^{n} |b_j|^3 \leq \varepsilon/6 + \delta \max\{|b_1|, ..., |b_n|\} \sum_{j=1}^{n} |b_j|^2 = \varepsilon/6 + \delta a.$$

So, $\delta(1-a) \leq \varepsilon/6$ and

$$\zeta_3(X,Z) \leq \varepsilon/(6(1-a)).$$

By virtue of (5.8) we find that

$$L(X, \mathcal{Z}_{\theta,\sigma^2}) \leq 32^{1/4}\zeta_3^{1/4}(X, \mathcal{Z}_{\theta,\sigma^2}) \leq 2\left(3(1-a)\right)^{-1/4}\varepsilon^{1/4}. \qquad (5.13)$$

By comparing (5.7) with (5.13) we see that in (5.13) we have successfully avoided the logarithmic multiplier which is in (5.7) and, consequently, improved the order of stability.

5.3 Main results

It is well known that if $X, X_1, X_2, ..., X_n$ are i.i.d. random variables, X and S are identically distributed and assumption (5.2) is satisfied, then $X_1, X_2, ..., X_n$ are normal random variables. We investigate the stability of this characterization theorem in Lévy metric L. It means that, instead of the condition $\nu_3(X, S) \leq \varepsilon$ in (5.1), we have only $L(X, S) \leq \varepsilon$.

The indicator function I_ϑ is a function defined as

$$I_\vartheta = \begin{cases} 1, if\ \max\{\vartheta, 1/4\} = \vartheta, \\ 0, if\ \max\{\vartheta, 1/4\} \neq \vartheta. \end{cases}$$

Let us denote $\Xi\left(M, r\right) = \left\{X : E\left|X\right|^r \leq M\right\}$, where r is a constant from the interval $(2, 3]$.

Theorem 5.2 (R. Yanushkevichius, O. Yanushkevichiene [29]). *Let $X, X_1, X_2, ..., X_n$ be i.i.d. random variables from the class $\Xi\left(M, r\right)$. Under the assumptions*

$$\mu_1(X, S) = \mu_2(X, S) = 0, \quad L(X, S) \leq \varepsilon \qquad (5.14)$$

and (5.2) there exist a constant $C = C(M, r, n, b_1, ..., b_n)$ and normal random variable $\mathcal{Z}$ such that the following inequality holds:

$$L\left(X,\mathcal{Z}\right)\leq C\varepsilon^{\max\{\vartheta,1/4\}}(1+I_{\vartheta}\cdot\log\frac{1}{\varepsilon}), \tag{5.15}$$

where $\vartheta=\Delta r/(2r+1)$,

$$\Delta=1/(1-\log n/\left(\log(|b_1|^r+|b_2|^r+...+|b_n|^r)\right))>0. \tag{5.16}$$

Since our proof is based on the use of characteristic functions, it is also natural to use the metric defined in the class of characteristic functions. As in previous chapter, we have chosen the weak metric λ,

$$\lambda\big(X,Y\big)=\min\left\{\max\left\{\frac{1}{2}\max(|f_X(t)-\right.\right.$$
$$\left.\left.-f_Y(t)|:\ |t|\leq T),\frac{1}{T}\right\}:\ T>0\right\},$$

where $f_X\left(t\right)$ and $f_Y\left(t\right)$ denote the characteristic functions of the random variables X and Y, respectively.

Two-sided estimations of this metric are studied by V. Zolotarev and V. Senatov [32]. If $X\in\Xi\left(M,r\right)$, then from [32] we derive that

$$\lambda(X,S)\leq 12kL^{r/(2r+1)}(X,S)=12k\varepsilon^{r/(2r+1)}. \tag{5.17}$$

Let $f(t)$ be the characteristic function of a random variable X, i.e. $f\left(t\right)=f_X\left(t\right)$. Then, from (5.17) we get for $|t|\leq 1/(12k\varepsilon^{r/(2r+1)})$ that

$$f(t)=f(b_1t)f(b_2t)...f(b_nt)+h(t),\ |h(t)|\leq\varepsilon_1, \tag{5.18}$$

where

$$h(t)=\int_{-\infty}^{\infty}\exp(itx)d(\boldsymbol{P}(X<x)-\boldsymbol{P}(S<x)),$$
$$\varepsilon_1=12k\varepsilon^{r/(2r+1)}.$$

Since $f(t)$ is a characteristic function, $f(0)=1$ and $f(t)$ is continuous. So, if p is a real number from the interval $(0,1)$, then

$$u_* = u_*(f, p) = \inf \{|t| : |f(t)| = p\} > 0.$$

Since the case $u_* \geq 1/\varepsilon_1$ is trivial, let $u_* < 1/\varepsilon_1$.

We will introduce the characteristic function $f^*(t)$ instead of the characteristic function $f(t)$ into the consideration,

$$f^*(t) = f(tu_*). \tag{5.19}$$

If $f(t)$ satisfies (5.18) for $|t| \leq 1/\varepsilon_1$, then $f^*(t)$ satisfies (5.18) for $|t| \leq 1/(u_*\varepsilon_1)$, i.e.

$$f^*(t) = f^*(b_1t)f^*(b_2t)...f^*(b_nt) + h(u_*t), \tag{5.20}$$

where $|h(u_*t)| \leq \varepsilon_1$ for $|t| \leq 1/(u_*\varepsilon_1)$. Since $u_* < 1/\varepsilon_1$, (5.20) is valid for $|t| \leq 1$ as well.

By (5.19) $u_*(f^*, p) = 1$, and for this reason $|f^*(t)| \geq p$ for $|t| \leq 1$. Thus,

$$\log f^*(t) = \sum_{i=1}^{n} \log f^*(b_it) + R(t), \tag{5.21}$$

where

$$R(t) = \log\left(1 + \frac{h(z_*t)}{f^*(b_1t)f^*(b_2t)...f^*(b_nt)}\right), \tag{5.22}$$

$$|R(t)| = \left|\log\left(1 + \frac{h(z_*t)}{\prod\limits_{i=1}^{n} f^*(b_it)}\right)\right| \leq \frac{2\,|h(z_*t)|}{\prod\limits_{i=1}^{n} |f^*(b_it)|} \leq 2\varepsilon_1 p^{-n}. \tag{5.23}$$

From (5.20)–(5.23) and relations (3.1.17)–(3.1.27), (3.2.6)–(3.2.13) in Yanushkevichius [20] (see also relations (58)–(74) in Yanushkevichius and

Yanushkevichiene [28]), we conclude that

$$\left|f^*(t)-\exp\left(-\sigma^2t^2/2\right)\right| \le C\varepsilon^{\Delta} \text{ for } |t|\le 1, \tag{5.24}$$

$$\left|f^*(t)-\exp\left(-\sigma^2t^2/2\right)\right| \le$$

$$\le C(1+\sigma^B)\max_{|t|\le 1}\left|f^*(t)-\exp\left(-\sigma^2t^2/2\right)\right| \text{ for } |t|\le 1/(u_*\varepsilon_1), \tag{5.25}$$

where

$$B=-\log_a n,\ a=1/\max\{|b_j|:j=1,2,...,n\}.$$

Formula (5.25) is useful for us only in the case where we can estimate σ^B from above. To this end, we note that since $|f^*(1)|=p$, from (5.24) we derive

$$|1-\exp\left(-\sigma^2t^2/2\right)|\le C\varepsilon^{\Delta}.$$

Consequently, $\exp\left(-\sigma^2/2\right)\ge p/2$, i.e.,

$$\sigma\le\sqrt{2\log\frac{2}{p}}. \tag{5.26}$$

We have used the fact that ε_1 is a small positive number, because the proof of Theorem 5.2 is non-trivial only in this case. It can be appreciated as follows: there exists a small positive number ε_0, depending only on n, r and $b_1, b_2, ..., b_n$, such that inequality (5.15) is valid for all $\varepsilon_1\in(0,\varepsilon_0]$. In all the other cases Theorem 5.2 is trivial.

It follows from (5.24), (5.25) and (5.26) that for $|t|\le 1/\varepsilon_1$

$$\left|f(t)-\exp\left(-\sigma^2t^2/(2z_*)\right)\right|\le C(b_1,b_2,...,b_n,n,r)\varepsilon_1^{\Delta},$$

i.e.,

$$\lambda\left(X,\mathcal{Z}\right)\leq C\varepsilon^{\Delta r/(2r+1)}. \tag{5.27}$$

Since by V.M. Zolotarev and V.V. Senatov [32],

$$L\leq 8\left(1+\tfrac{1}{r}\log M+\log\tfrac{1}{\lambda}\right)\lambda$$

in the class $\Xi\left(M,r\right)=\left\{X:E\left|X\right|^{r}\leq M\right\}$, we conclude from (5.27) that inequality

$$L\left(X,\mathcal{Z}\right)\leq C\varepsilon^{\Delta r/(2r+1)}\log\frac{1}{\varepsilon} \tag{5.28}$$

is proved.

So, on the one hand we have estimate (5.28), on the other hand - the estimate (5.13). Combining these estimates we get that

$$L\left(X,\mathcal{Z}\right)\leq\min\{C\varepsilon^{\Delta r/(2r+1)}\log\frac{1}{\varepsilon},2\left(3(1-a)\right)^{-1/4}\varepsilon^{1/4}\}\leq$$

$$\leq C\varepsilon^{\max\{\Delta r/(2r+1),1/4\}}(1+I_{\Delta r/(2r+1)}\cdot\log\frac{1}{\varepsilon}),$$

i.e. inequality (5.15) is proved.

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*My family name according Lithuanian passport is **R. Januškevičius**. But according SIAM translation of my first scientific work from Russian to English in 1978 it was written as **R. V. Yanushkyavichius** (see, for example, "On estimates of the stability measure for decompositions of probability distributions into components", SIAM, 1978, Theory Probab. Appl., 23 (3), 507-520). Later my family name was written as **R. Yanushkevichius**.*

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