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Closure properties of randomly stopped sums

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VILNIAUS UNIVERSITETAS

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Atsitiktinių dydžių atsitiktinių sumų uždarumo savybės

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Contents

Notation

$\mathbb{N}$	set of the positive integers
$\mathbb{N}_0$	set of the non-negative integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of non-negative numbers
$\mathcal{H}$	class of heavy-tailed distributions
$\mathcal{S}$	class of subexponential distributions
$\mathcal{S}^*$	class of strongly subexponential distributions
$\mathcal{L}$	class of long-tailed distributions
$\mathcal{OL}$	class of $\mathcal{O}$-exponential distributions
$\mathcal{L}(\gamma)$	class of exponential distributions
$\mathcal{D}$	class of dominatedly varying distributions
$F_\xi(x) = P(\xi \leqslant x)$	distribution function of the random variable ξ
$\overline{F}_\xi = 1 - F_\xi$	tail of the distribution function F_ξ
$\lfloor x \rfloor$	integer part of the real number x
$\{x\}$	fractional part of the real number x
$\mathbb{1}_A$	indicator function of the set A

$d.f.$	abbreviation for "distribution function"
$r.v.$	abbreviation for "random variable"
$r.v.'s$	abbreviation for "random variables"
$i.i.d.$	abbreviation for "independent and identically distributed"
$f(x) = o(g(x))$	denotes that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$
$Q_X(\lambda) = \sup\limits_{x\in\mathbb{R}} \mathbb{P}(x \leqslant X \leqslant x+\lambda)$	Lévy concentration function of r.v. X
$\mathrm{supp(X)} = \{\mathrm{x} \in \mathbb{R} : \mathbb{P}(\mathrm{X} = \mathrm{x}) > 0\}$	support of the discrete random variable X

Chapter 1

Introduction

Research problem, topicality and novelty

The research objects of the thesis are the randomly stopped sum S_η, the randomly stopped maximum $\xi_{(\eta)}$ and the randomly stopped maximum of sums $S_{(\eta)}$:

$$S_\eta = \xi_1 + \cdots + \xi_\eta,$$
$$\xi_{(\eta)} = \max\{0, \xi_1, \ldots, \xi_\eta\},$$
$$S_{(\eta)} = \max\{S_0, S_1, \ldots, S_\eta\},$$

where $\{\xi_1, \xi_2, \ldots\}$ is a sequence of random variables and η is a counting random variable. *We say that η is a counting random variable if it is non-negative, integer-valued and non-degenerate at 0.*

The thesis is devoted to finding conditions for the independent random variables $\{\xi_1, \xi_2, \ldots\}$ under which the randomly stopped sum, the randomly stopped maximum and the randomly stopped maximum of sums belong to the special classes of heavy-tailed distributions. The motivation for this investigation comes mainly from insurance and finance, where questions related to extremal or rare events are traditionally considered (see, e.g., [5, 29, 49, 54]). For instance, data from motor third liability insurance as well as fire and catastrophe insurance (earthquakes, flooding etc.) clearly show the heavy tail behavior. In particular, Pareto, lognormal and loggamma distributions are extremely popular in actuarial mathematics.

Mathematical aspects of risk theory related to calculation of ruin probabilities are considered in a large number of works (see, e.g., [5, 11, 34, 35, 36, 49, 52, 54, 57] and references therein). From the mathematical point of view, the success of any insurance business depends on the asymptotic behavior of the distribution of S_η and $S_{(\eta)}$. If the distribution of individual claim sizes is light-tailed, then the corresponding ruin probability is also small for large values of the initial surplus and usually decreases with an exponential rate (see, e.g., [5, 32, 34, 36, 49, 54, 57] and references therein). If individual claim sizes belong to heavy-tailed distributions, then the ruin probability decreases much more slowly with increasing initial surplus (see, e.g., [31, 38, 39, 43, 44, 45, 48, 57, 59]). Thus, it is necessary to find out at the beginning of the investigation whether the distribution of individual claim sizes is light- or heavy-tailed.

One of the most significant research directions in risk theory is investigation of the ruin probability when the distribution of claim sizes is heavy-tailed. In this case, ruin typically occurs because of one large claim, and results are usually obtained for some special classes of heavy-tailed distributions. Results on asymptotic behavior of the ruin probability typically turn out to be different for different classes. Asymptotics of the ruin probabilities in the case of heavy-tailed claim sizes was investigated in [3, 6, 7, 8, 30, 42, 48], and also in [4, 39, 43, 59] for models with constant interest rate. Various bounds for the ruin probability are obtained in [23, 31, 38, 44, 45]. Optimal control problems are also solved for some special classes of heavy-tailed distributions in [55, 56, 57]. Therefore, to apply all these results, we need to know whether the distribution of claim sizes belongs to some special classes.

The closure problem is classical. Bingham, Goldie and Teugels [9] are one of the first researchers in this field. It is worth to mention that all the classical results related to the closure problem are obtained for identically distributed random variables $\{\xi_1, \xi_2, \ldots\}$. The main novelty of this theses is that not only identically distributed random variables are considered.

All results presented in the thesis are new and original. They are based on 5 scientific publications.

Aim and tasks

The main aim of the thesis is to find conditions for the independent random variables $\{\xi_1, \xi_2, \dots\}$ and the counting random variable η under which the distribution functions of S_η, $\xi_{(\eta)}$ and $S_{(\eta)}$ belong to some classes of functions.

To achieve the aim, the following tasks are raised:

- To establish conditions under which the randomly stopped sum S_η belongs to the class of dominatedly varying distributions.
- To find conditions under which the randomly stopped sum S_η belongs to the class of exponential distributions.
- To find conditions under which the randomly stopped sum S_η, the randomly stopped maximum $\xi_{(\eta)}$ and the randomly stopped maximum of sums $S_{(\eta)}$ belongs to the class of O-exponential distributions.

Methodology of investigation

Belonging to the classes of heavy-tailed distributions is usually associated with the tail behavior of the distribution function. Therefore, to estimate tail probabilities for sums of random variables and the maximum of sums, we use standard methods of probability theory in this thesis. The majority of estimates for the classes of heavy-tailed distributions are related to properties of special indices such as the Matuszewska index, the L-index, etc. To investigate the tails of randomly stopped sums, randomly stopped maximums and randomly stopped maximums of sums, the set of all possible values of the counting random variable is usually divided into a few subsets, where the tails are studied separately using different methods. The tails of sums of random variables are evaluated using classical methods when the values of the counting random variables are fixed. Asymptotic properties of distributions which are related to the special indices, namely the Matuszewska index and the L-index, are applied when the values of the counting random variables grow together with the tail bound. Concentration inequalities are used for average values of counting random variables.

Defended propositions

1. Conditions for the independent random variables $\{\xi_1, \xi_2, \ldots\}$ and the counting random variable η under which the distribution function of the random sum $S_\eta = \xi_1 + \xi_2 + \ldots + \xi_\eta$:

- preserves dominatedly varying tails;
- belongs to the class of exponential distributions.

2. Conditions under which the randomly stopped sum S_η, the randomly stopped maximum $\xi_{(\eta)}$ and the randomly stopped maximum of sums $S_{(\eta)}$ are distributed according to $\mathcal{O}$-exponential laws. In this case, identically and not necessarily identically distributed independent random variables $\{\xi_1, \xi_2, \ldots\}$ are considered.

Publications

- Danilenko, S., Šiaulys, J. (2015). Random Convolution of $\mathcal{O}$-exponential distributions. *Nonlinear Analysis: Modelling and Control*, 20(3): 447-454.
- Danilenko, S., Šiaulys, J. (2016). Randomly stopped sums of not identically distributed heavy tailed random variables. *Statistics and Probability Letters*, 113: 84-93.
- Danilenko, S., Paškauskaitė, S., Šiaulys, J. (2016). Random convolution of inhomogeneous distributions with $\mathcal{O}$-exponential tail. *Modern Stochastics: Theory and Applications*, 3(1): 79-94.
- Danilenko, S., Markevičiūtė, J.; Šiaulys, J. (2017). Randomly stopped sums with exponential-type distributions. *Nonlinear Analysis: Modelling and Control*, 22(6): 793-807.
- Danilenko, S., Šiaulys, J., Stepanauskas G. (2018). Closure properties of $\mathcal{O}$-exponential distributions. *Statistics and Probability Letters*, 140: 63-70.

Conferences

- Random convolution of $\mathcal{O}$-exponential distributions. *56th conference of Lithuanian Mathematical Society*, June 16-17, 2015, Kaunas.
- Sunkiauodegių skirstinių atsitiktinių sumų savybės. *57th conference of Lithuanian Mathematical Society*, June 20-21, 2016, Vilnius.
- Randomly stopped sum of distributions with dominatingly varying tails. *The X Tartu Conference on Multivariate Statistics*, June 28 - July 1, 2016, Tartu.
- Eksponentiškai pasiskirsčiusios atsitiktinės sumos. *58th conference of Lithuanian Mathematical Society*, June 21-22, 2017, Vilnius.
- Closure properties of $\mathcal{O}$-exponential distributions. *Modern Stochastics: Theory and Applications. IV*, May 24-25, 2018, Kyiv.
- $\mathcal{O}$-eksponentinių skirstinių uždarumo savybės. *59th conference of Lithuanian Mathematical Society*, June 18-19, 2018, Kaunas.

Structure of the thesis

In Chapters 1 – 2, the necessary notation is introduced and an overview of known results is given. In Section 2.1, we formulate the main notions and recall the definitions of the classes $\mathcal{L}$, $\mathcal{OL}$, $\mathcal{D}$ and $\mathcal{L}(\gamma)$, $\gamma \geqslant 0$, as well as some other related classes. In addition, we describe some interrelationships among the classes of heavy-tailed distributios. In Section 2.2, we give a number of typical examples of d.f.'s from all classes under consideration. In Section 2.3, we formulate a few known results that describe conditions under which the d.f. F_{S_η} belongs to some classes.

In Chapters 3 – 5, we present our main results.

In Chapter 3, we investigate conditions under which the random sum S_η belongs to the class of dominatedly varying distributions. To be more precise, we prove two assertions that describe conditions under which the randomly stopped sum belongs to the class $\mathcal{D}$ and consider two examples. In Theorem 3.1.1, no moment conditions for the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are required, whereas the conditions of Theorem 3.1.2 imply that the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ have finite means. Yang and Gao [66] give conditions under which $F_{S_\eta} \in \mathcal{D}$ when the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are identically distributed but follow some dependence structure. Yang and Gao [66] and Xu et al. [64] consider conditions under which $F_{S_\eta} \in \mathcal{L}$ when the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are independent and not identically distributed. Combining the results of [64, 66] with our results we can obtain conditions under which $F_{S_\eta} \in \mathcal{L} \cap \mathcal{D}$.

In Chapter 4, we consider conditions under which the random sum S_η belongs to the class of exponential distributions. We prove three theorems yielding conditions under which the d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ and consider some examples. Theorem 4.1.1 deals with the case of a finitely supported counting r.v. η, whereas Theorems 4.1.2 and 4.1.3 imply that the right tail of η is unbounded. We suppose that the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are non-negative in Theorems 4.1.1 and 4.1.3, whereas they can be real-valued in Theorem 4.1.2. The proofs of the main results are based on ideas from the papers [40, 63, 65]. Some similar results for the class $\mathcal{L} = \mathcal{L}(0)$ are obtained in [47, 64].

In Chapter 5, we study conditions under which the randomly stopped sum S_η, the randomly stopped maximum $\xi_{(\eta)}$ and the randomly stopped maximum of sums $S_{(\eta)}$ belong to the class of $\mathcal{O}$-exponential distributions. Moreover, we illustrate the results with examples. The class $\mathcal{OL}$ has never been investigated thoroughly before. We also note that the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ under consideration can be real-valued and non-identically distributed. In addition, we study the closure properties of distributions not only for the randomly stopped sum S_η but also for the randomly stopped maximum $\xi_{(\eta)}$ and the randomly stopped maximum of sums $S_{(\eta)}$.

Finally, the conclusions are formulated in Chapter 6.

Chapter 2

Classification of distribution functions

2.1 The main notions

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of real-valued random variables (r.v.'s), which are identically or non-identically distributed, and let η be a counting r.v., which is independent of the sequence $\{\xi_1, \xi_2, \ldots\}$. As usual, *a counting r.v. is a r.v. which is non-negative, integer-valued and non-degenerate at 0.*

We denote by $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$, $n \geqslant 1$, the partial sums, and we write $S_\eta = \xi_1 + \cdots + \xi_\eta$ for the randomly stopped sum of the r.v.'s $\{\xi_1, \xi_2, \ldots\}$. Similarly, set $\xi_{(0)} = 0$ and $\xi_{(n)} = \max\{0, \xi_1, \ldots, \xi_n\}$, $n \geqslant 1$, and let $\xi_{(\eta)} = \max\{0, \xi_1, \ldots, \xi_\eta\}$ be the randomly stopped maximum of the r.v.'s $\{\xi_1, \xi_2, \ldots\}$. Finally, set $S_{(n)} = \max\{S_0, S_1, \ldots, S_n\}$, $n \geqslant 0$, and let $S_{(\eta)} = \max\{S_0, S_1, \ldots, S_\eta\}$ be the randomly stopped maximum of sums $\{S_0, S_1, S_2, \ldots\}$.

The distribution functions (d.f.'s) of the r.v.'s S_η, $\xi_{(\eta)}$ and $S_{(\eta)}$ can be expressed as follows:

$$F_{S_\eta}(x) := \mathbb{P}(S_\eta \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(S_n \leqslant x)\mathbb{P}(\eta = n),$$

$$F_{\xi_{(\eta)}}(x) := \mathbb{P}(\xi_{(\eta)} \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(\xi_{(n)} \leqslant x)\mathbb{P}(\eta = n),$$

$$F_{S_{(\eta)}}(x) := \mathbb{P}(S_{(\eta)} \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(S_{(n)} \leqslant x)\mathbb{P}(\eta = n).$$

We denote by $\overline{F}$ the tail of a d.f. F, that is, $\overline{F}(x) = 1 - F(x)$ for all

$x \in \mathbb{R}$.

Note that the tails of these d.f.'s can be expressed similarly:

$$\overline{F}_{S_\eta}(x) = \sum_{n=0}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n),$$

$$\overline{F}_{\xi_{(\eta)}}(x) = \sum_{n=0}^{\infty} \mathbb{P}(\xi_{(n)} > x)\mathbb{P}(\eta = n),$$

$$\overline{F}_{S_{(\eta)}}(x) = \sum_{n=0}^{\infty} \mathbb{P}(S_{(n)} > x)\mathbb{P}(\eta = n).$$

In what follows, we are interested in the closure property of the distributions of the randomly stopped sum S_η, the randomly stopped maximum $\xi_{(\eta)}$ and the randomly stopped maximum of sums $S_{(\eta)}$. This property states that if $F_{\xi_\varkappa}$ belongs to some special class of distributions for some fixed $\varkappa \geqslant 1$, then the functions F_{S_η}, $F_{\xi_{(\eta)}}$ and $F_{S_{(\eta)}}$ belong to the same class.

In this section, we recall the definitions of the classes $\mathcal{OL}$, $\mathcal{D}$ and $\mathcal{L}(\gamma)$, $\gamma > 0$, which are investigated in this thesis, as well as some related classes $\mathcal{L}$, $\mathcal{S}$ and $\mathcal{S}^*$ because a lot of the methods that are used in proofs of various assertions are similar for all the classes. Furthermore, we describe the related classes for the more complete presentation.

Definition 2.1.1. *A d.f. F is said to be heavy-tailed ($F \in \mathcal{H}$) if for any fixed $\delta > 0$, we have*

$$\lim_{x\to\infty} \overline{F}(x)\mathrm{e}^{\delta x} = \infty.$$

The class of heavy-tailed random variables $\mathcal{H}$ has a very rich structure. The most important subclasses of $\mathcal{H}$ are defined below.

We start with subexponential and long-tailed distributions, which were first introduced and studied by Chistyakov [13] in the context of the branching process. In particular, he proved that the subexponential distribution class is contained in the class of long-tailed distributions. Later subexponential distributions have been used in a wide variety of applications in probability theory, for instance, in renewal theory and the theory of infinitely divisible distributions (see, e.g., [28, 29, 33, 51, 61]).

Definition 2.1.2. *A d.f. F supported on the interval $[0, \infty)$ is said to be subexponential ($F \in \mathcal{S}$) if*

$$\lim_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2.$$

Here and subsequently, $*$ denotes the convolution of d.f.'s.

The class of strongly subexponential distributions was introduced by Klüppelberg [41].

Definition 2.1.3. *A d.f. F supported on the interval $[0,\infty)$ is said to be strongly subexponential ($F \in \mathcal{S}^*$) if*

$$\mu := \int\limits_{[0,\infty)} x \mathrm{d}F(x) < \infty \quad \text{and} \quad \int\limits_0^x \overline{F}(x-y)\overline{F}(y)\mathrm{d}y \underset{x\to\infty}{\sim} 2\mu\overline{F}(x).$$

If a d.f. F is supported on $\mathbb{R}$, i.e. $F(0-) > 0$, then F is supposed to belong to either $\mathcal{S}$ or $\mathcal{S}^*$ when $F^+(x) = F(x 1\!\!1_{\{[0,\infty)\}}(x))$ belongs to the corresponding class.

Definition 2.1.4. *A d.f. F is said to be long-tailed ($F \in \mathcal{L}$) if for any fixed $a > 0$, we have*

$$\lim_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} = 1.$$

Shimura and Watanabe [58] introduced the class $\mathcal{OL}$, which is wider than the class $\mathcal{L}$ and similar to it, and investigated some subclasses of this class.

Definition 2.1.5. *A d.f. F is said to be $\mathcal{O}$-exponential ($F \in \mathcal{OL}$) if for any fixed $a \in \mathbb{R}$, we have*

$$0 < \liminf_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} < \infty.$$

The last definition implies that $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ if $F \in \mathcal{OL}$.

It is obvious that $F \in \mathcal{OL}$ if and only if

$$\limsup_{x\to\infty} \frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty, \tag{2.1.1}$$

or, equivalently,

$$\sup_{x\geqslant 0} \frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty.$$

The last condition shows that the class $\mathcal{OL}$ is quite wide. Now we describe the most popular subclasses of $\mathcal{OL}$ because we present some results on the random convolution of distributions from these subclasses later.

Definition 2.1.6. *A d.f. F is said to belong to the class of exponential distributions ($F \in \mathcal{L}(\gamma)$) with some $\gamma > 0$ if for any fixed $a > 0$, we have*

$$\lim_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} = \mathrm{e}^{-a\gamma}.$$

For $\gamma > 0$, the class $\mathcal{L}(\gamma)$ was introduced by Embrechts and Goldie [26]. If $\gamma = 0$, then it is clear that $\mathcal{L}(\gamma) = \mathcal{L}$.

Another famous class of heavy-tailed distributions is the class of dominatedly varying distributions $\mathcal{D}$ introduced by Feller [32].

Definition 2.1.7. *A d.f. F is said to be dominatingly varying ($F \in \mathcal{D}$) if for any fixed $a \in (0,1)$, we have*

$$\limsup_{x\to\infty} \frac{\overline{F}(xa)}{\overline{F}(x)} < \infty.$$

Now we summarize interrelationships among the most important classes of heavy-tailed distributions introduced above. Most of these interrelationships are well known.

The definitions given above together with [13, Lemma 2], [24, Lemma 9], [29, Lemma 1.3.5(a)] and [37, Lemma 1] imply that

$$\mathcal{L} \cap \mathcal{D} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}, \quad \mathcal{D} \subset \mathcal{H}.$$

Figure 2.1 shows the interrelationships among the classes of heavy-tailed distributions $\mathcal{D}$, $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{L}$ and $\mathcal{H}$.

Similarly, we can conclude that

$$\mathcal{L} \subset \mathcal{OL}, \quad \mathcal{D} \subset \mathcal{OL} \quad \text{and} \quad \bigcup_{\gamma>0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

This relationship is presented in Figure 2.2.

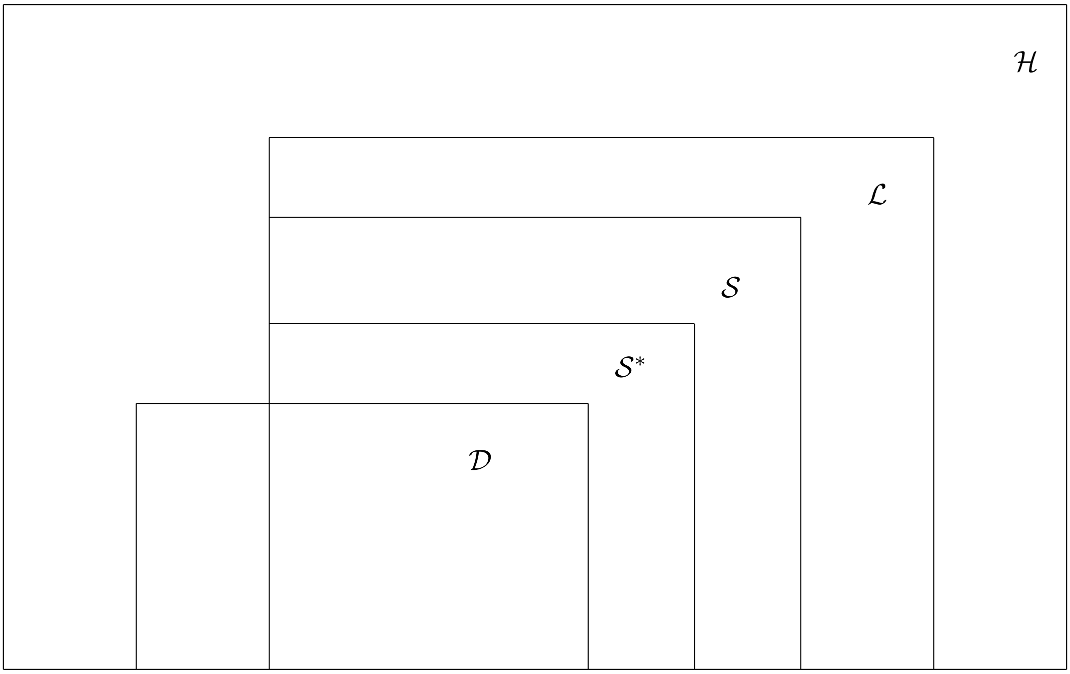

Figure 2.1: Classification of classes of heavy-tailed distributions

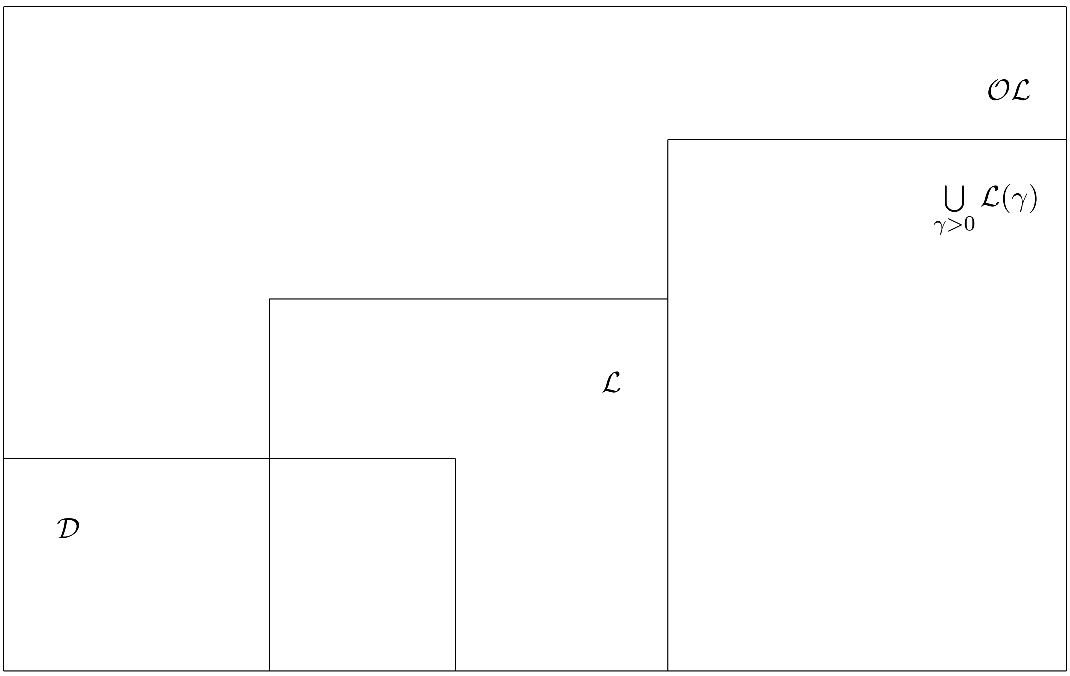

Figure 2.2: Classification of subclasses of the class $\mathcal{OL}$

2.2 Examples

In this section, we give a number of d.f.'s, which belong to the classes defined in Section 2.1.

EXAMPLE 2.2.1. *The Pareto distribution with d.f.*

$$F(x) = 1 - \left(1 + \frac{x}{b}\right)^{-a}, \ x \geqslant 0,$$

where $b > 0$ is the scale parameter and $a > 0$ is the shape parameter.

For the Pareto distribution, we have $F \in \mathcal{L} \cap \mathcal{D}$.

It is clear that $\overline{F}(x) \sim (x/b)^{-a}$ as $x \to \infty$. For this reason, the Pareto distribution is sometimes referred to as the power-law distribution. The Pareto distribution has finite moments of order $k < a$, whereas all moments of order $k \geqslant a$ are infinite.

EXAMPLE 2.2.2. *The Weibull distribution with d.f.*

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{\lambda}\right)^{\beta}\right\}, \ x \geqslant 0,$$

where $\lambda > 0$ is the scale parameter and $\beta > 0$ is the shape parameter.

If $0 < \beta < 1$, then $F \in \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}$. However, $F \notin \mathcal{D}$.

Note that in the case $\beta = 1$ we have the exponential distribution. All moments of the Weibull distribution are finite, but $\mathbb{E}\mathrm{e}^{\delta \mathrm{X}} = \infty$ for all $\delta > 0$ if $\beta < 1$.

EXAMPLE 2.2.3. *The distribution from the paper by Cline and Samorodnitsky [16] with*

$$\overline{F}(x) = \exp\Big\{ - \lfloor \log(1+x) \rfloor + \\ - \max\left\{(1+x)(\log(1+x) - \lfloor \log(1+x) \rfloor), 1\right\}\Big\}, \quad x \geqslant 0,$$

where $\lfloor z \rfloor$ denotes the integer part of z.

In this case we have $F \in \mathcal{D}$ and $F \notin \mathcal{L}$.

EXAMPLE 2.2.4. *The Burr distribution on $\mathbb{R}^+$ with the tail defined by*

$$\overline{F}(x) = \left(\frac{\kappa}{x^{\tau} + \kappa}\right)^{\alpha},$$

where $\alpha > 0$, $\kappa > 0$ and $\tau > 0$ are some parameters.

The Burr distribution has finite moments of order $\tau < \alpha\tau$, whereas all moments of order $\tau \geqslant \alpha\tau$ are infinite.

In this case we have $F \in \mathcal{L} \cap \mathcal{D}$ as in Example 2.2.1.

Example 2.2.5. *By Proposition 2.6 from [2], an absolutely continuous d.f. F belongs to the class $\mathcal{L}(\gamma)$ if and only if*

$$\overline{F}(x) = \exp\left\{-\int_{-\infty}^{x} (a(u)+b(u))du\right\}$$

for $x \in \mathbb{R}$, where measurable functions a and b satisfy the following conditions:

(i) $a(u) + b(u) \geqslant 0, u \in \mathbb{R}$;

(ii) $\lim_{u\to\infty} a(u) = \gamma$;

(iii) $\lim_{x\to\infty} \int_{-\infty}^{x} a(u)du = \infty$;

(iv) $\lim_{x\to\infty} \int_{-\infty}^{x} b(u)du$ *exists.*

If we choose

$$a(u) = \left(2 - \frac{1}{u}\right) \mathbb{1}_{[1,\infty)}(u)$$

and

$$b(u) = \left(\frac{1}{1+u^2}\right) \mathbb{1}_{[1,\infty)}(u),$$

then we get the d.f. F with tail

$$\overline{F}(x) = x \exp\left\{2 + \frac{\pi}{4} - 2x - \arctan x\right\}, x \geqslant 1,$$

which belongs to the class $\mathcal{L}(2)$ because

$$\begin{aligned}
\lim_{u\to\infty} a(u) &= \lim_{u\to\infty}\left(2 - \frac{1}{u}\right) = 2,\\
\int_{-\infty}^{x} a(u)du &= \int_{-\infty}^{x}\left(2 - \frac{1}{u}\right)du = 2x - 2 + \ln x \underset{x\to\infty}{\longrightarrow} \infty,\\
\lim_{x\to\infty}\int_{-\infty}^{x} b(u)du &= \lim_{x\to\infty}\int_{-\infty}^{x}\left(\frac{1}{1+u^2}\right)du\\
&= \lim_{x\to\infty}(\mathrm{arctg(x)} - \mathrm{arctg(1)}) = \frac{\pi}{4}.
\end{aligned}$$

Example 2.2.6. *By Proposition 2.6 from [2], an absolutely continuous d.f. F belongs to the class $\mathcal{OL}$ if its tail $\overline{F}$ has the representation*

$$\overline{F}(x) = \exp\left\{-\int_{-\infty}^{x}(a(u)+b(u))du\right\} \qquad (2.2.1)$$

for $x \in \mathbb{R}$, where some measurable functions a and b satisfy the following conditions:

(i) $a(u)+b(u) \geqslant 0, u \in \mathbb{R}$;

(ii) $\limsup\limits_{u\to\infty} |a(u)| < \infty$;

(iii) $\liminf\limits_{x\to\infty} \int_{-\infty}^{x} a(u)du = \infty$;

(iv) $\limsup\limits_{x\to\infty} \left|\int_{-\infty}^{x} b(u)du\right| < \infty$.

If we choose

$$a(u) = (1+\sin u)\, \mathbb{1}_{[1,\infty)}(u)$$

and

$$b(u) = (\cos u)\, \mathbb{1}_{[1,\infty)}(u),$$

then we get the d.f. F with tail

$$\overline{F}(x) = x\exp\{1+\sin 1 - \cos 1 - x + \cos x - \sin x\}, x \geqslant 1,$$

which belongs to the class $\mathcal{OL}$ because

$$\limsup_{u\to\infty}|a(u)| = \limsup_{u\to\infty}|1+\sin u| = 2,$$

$$\liminf_{x\to\infty}\int_{-\infty}^{x} a(u)du = \liminf_{x\to\infty}\int_{-\infty}^{x}(1+\sin u)du = \infty,$$

$$\limsup_{x\to\infty}\left|\int_{-\infty}^{x} b(u)du\right| = \limsup_{x\to\infty}\left|\int_{-\infty}^{x}\cos u du\right| \leqslant 2.$$

Example 2.2.7. *Let the r.v. ξ have the geometric distribution with parameter $p \in (0;1)$, i.e.*

$$\mathbb{P}(\xi = k) = (1-p)^k p,\ \ k = 0, 1, 2, ...$$

This distribution belongs to the class $\mathcal{OL}$ because

$$\begin{aligned}
\overline{F}_\xi(x) &= \sum_{k>x} \mathbb{P}(\xi = k) = \sum_{k=\lfloor x\rfloor+1}^{\infty} (1-p)^k p \\
&= p\left((1-p)^{\lfloor x\rfloor+1} + (1-p)^{\lfloor x\rfloor+2} + ...\right) \\
&= p(1-p)^{\lfloor x\rfloor+1}\,(1 + (1-p) + ...) \\
&= (1-p)^{\lfloor x\rfloor+1}, \quad x \geqslant 1,
\end{aligned}$$

and, consequently,

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)} &= \limsup_{x\to\infty} \frac{(1-p)^{\lfloor x-1\rfloor+1}}{(1-p)^{\lfloor x\rfloor+1}} \\
&= \limsup_{x\to\infty} \frac{(1-p)^{x-1-\{x-1\}}}{(1-p)^{x-\{x\}}} \\
&= \frac{1}{1-p} \limsup_{x\to\infty} (1-p)^{\{x\}-\{x-1\}} < \infty.
\end{aligned}$$

EXAMPLE 2.2.8. *Let the r.v. ξ be distributed according to the Peter and Paul law with parameter $\frac{1}{3}$, i.e.*

$$\overline{F}_\xi(x) = 2 \sum_{\substack{2^l > x \\ l \geqslant 1}} \frac{1}{3^l}, \quad x \geqslant 1.$$

Since $\overline{F}_\xi$ is a piecewise constant function, ξ is a discrete random variable.

For all $x \geqslant 1$, we have

$$\begin{aligned}
\overline{F}_\xi(x) &= 2 \sum_{2^l > x} \frac{1}{3^l} = 2 \sum_{l > \frac{\log x}{\log 2}} \frac{1}{3^l} = 2 \sum_{l=\lfloor \frac{\log x}{\log 2}\rfloor+1}^{\infty} \frac{1}{3^l} \\
&= 2\left(\frac{1}{3^{\lfloor \frac{\log x}{\log 2}\rfloor+1}} + \frac{1}{3^{\lfloor \frac{\log x}{\log 2}\rfloor+2}} + ...\right) \\
&= 3^{-\lfloor \frac{\log x}{\log 2}\rfloor} 2 \left(\frac{1}{3} + \frac{1}{3^2} + ...\right) \\
&= 3^{-\lfloor \frac{\log x}{\log 2}\rfloor}.
\end{aligned}$$

This distribution belongs to the class $\mathcal{D}$ because

$$\begin{aligned}\limsup_{x\to\infty}\frac{\overline{F}_\xi(ax)}{\overline{F}_\xi(x)} &= \limsup_{x\to\infty}\frac{3^{\lfloor\frac{\log x}{\log 2}\rfloor}}{3^{\lfloor\frac{\log ax}{\log 2}\rfloor}}\\ &= \limsup_{x\to\infty}\frac{3^{\frac{\log x}{\log 2}-\{\frac{\log x}{\log 2}\}}}{3^{\frac{\log ax}{\log 2}-\{\frac{\log ax}{\log 2}\}}}\\ &\leqslant 3^{-\frac{\log a}{\log 2}}3 = 3^{1+\frac{1}{\log 2}\log\frac{1}{a}} < \infty\end{aligned}$$

for any fixed $a \in (0,1)$.

2.3 Known results

There are a lot of results giving either sufficient or necessary and sufficient conditions in order that the d.f. of the random sum F_{S_η} belong to some classes of heavy-tailed distributions. It is usually assumed that the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are not only independent but also identically distributed. Now we formulate a few known results, which guarantee that the d.f. F_{S_η} belongs to some classes. The first assertion describes the closeness of the class $\mathcal{S}$.

Theorem 2.3.1. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent copies of a non-negative r.v. ξ with subexponential d.f. F_ξ. In addition, let η be counting r.v., which is independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbb{E}(1+\delta)^\eta < \infty$ for some $\delta > 0$, then $F_{S_\eta} \in \mathcal{S}$.*

The proof of Theorem 2.3.1 can be found in several papers (see, e.g., [27, Theorem 4.2], [14, Theorem 2.13], [29, Theorems 1.3.9 and A3.20], [33, Corollary 3.13 and Theorem 3.37]). In a more general case where F_ξ belongs to the so-called convolution-equivalent class $\mathcal{S}(\alpha)$, $\alpha \geqslant 0$, a similar result is obtained in [62]. In the case of strongly subexponential d.f.'s, the following result, which involves weaker restrictions on the r.v. η, can be derived from Theorem 1 of Denisov et al. [25]. To be more precise, using that assertion we can get the following theorem.

Theorem 2.3.2. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent copies of a non-negative r.v. ξ with strongly subexponential d.f. F_ξ and finite mean $\mathbb{E}\xi$. Moreover, let η be a counting r.v., which is independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbb{P}(\eta > x/c) \underset{x\to\infty}{=} o(\overline{F}_\xi(x))$ for some $c > \mathbb{E}\xi$, then $F_{S_\eta} \in \mathcal{S}^*$.*

Similar properties for the class of dominatedly varying distributions are proved in [46] (see Theorems 4 and 5 and Corollary 1). Below we formulate Theorem 4 from that paper. We recall only that a d.f. F belongs to the class $\mathcal{D}$ if and only if the upper Matuszewska index $J_F^+ < \infty$, where, by definition,

$$J_F^+ = -\lim_{y\to\infty} \frac{1}{\log y} \log\left(\liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}\right).$$

Theorem 2.3.3. *Let $\{\xi_1, \xi_2, \ldots\}$ be i.i.d. non-negative r.v.'s with d.f. $F_\xi \in \mathcal{D}$. In adddition, let η be a counting r.v., which is independent of $\{\xi_1, \xi_2, \ldots\}$, such that $\mathbb{E}\eta^{p+1} < \infty$ for some $p > J_{F_\xi}^+$. Then the d.f. F_{S_η} of the randomly stopped sum S_η belongs to the class $\mathcal{D}$ as well.*

The closeness of the class $\mathcal{L}$ under random convolution is considered in [1, 14, 15, 46]. Below we formulate the assertion of Theorem 6 from [46].

Theorem 2.3.4. *Suppose that $\{\xi_1, \xi_2, \ldots\}$ are i.i.d. non-negative r.v.'s with d.f. $F_\xi \in \mathcal{L}$. Moreover, let η be a counting r.v., which is independent of $\{\xi_1, \xi_2, \ldots\}$, with d.f. F_η. If $\overline{F}_\eta(\delta x) = o\Big(\sqrt{x}\ \overline{F}_\xi(x)\Big)$ for any $\delta \in (0,1)$, then $F_{S_\eta} \in \mathcal{L}$.*

The conditions under which $F_{S_\eta} \in \mathcal{D}$ are considered in [66] in the case where the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are identically distributed but follow some dependence structure, whereas the conditions under which $F_{S_\eta} \in \mathcal{L}$ are considered in [66, 64] in the case where the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are independent and non-identically distributed.

Cai and Tang [12] discuss the max-sum equivalence and the convolution closure of heavy-tailed distributions. They prove that the class $\mathcal{D}$ is closed under convolution and establish the max-sum equivalence for the class $\mathcal{D} \cap \mathcal{L}$ (see [12, Proposition 2.1 and Theorem 2.1]).

Theorem 2.3.5. *If $F_1 \in \mathcal{D} \cap L$ and $F_2 \in \mathcal{D} \cap L$, then $F_1 * F_2 \in \mathcal{D} \cap L$ and*

$$\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) + \overline{F_2}(x).$$

In [14, 15], Cline establishes that the d.f F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ if the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are identically distributed with d.f. $F_\xi \in \mathcal{L}(\gamma)$ and η is an arbitrary counting r.v. Albin [1] constructs a counterexample and shows that Cline's result is false in general. In his paper [1], Albin states that the d.f. F_{S_η} remains in the class $\mathcal{L}(\gamma)$ if the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are identically distributed with d.f. $F_\xi \in \mathcal{L}(\gamma)$ and $\mathbb{E}\, e^{\delta\eta} < \infty$ for all $\delta > 0$. In order to prove his assertion, Albin uses the implication

$$\begin{aligned} & \sup_{x \geqslant c} \frac{\overline{F}(x-t)}{\overline{F}(x)} \leqslant (1+\varepsilon)\, e^{\gamma t} \\ \Rightarrow \quad & \sup_{x \geqslant n(c-t)+t} \frac{\overline{F^{*n}}(x-t)}{\overline{F^{*n}}(x)} \leqslant (1+\varepsilon)\, e^{\gamma t}, \quad n \in \mathbb{N}, \end{aligned} \tag{2.3.1}$$

for some $c \in \mathbb{R}$, where $\varepsilon > 0$, $t \in \mathbb{R}$ and F is a d.f from the class $\mathcal{L}(\gamma)$, $\gamma \geqslant 0$. Unfortunately, if parameter $\gamma > 0$, then the implication above holds only for $t > 0$. Watanabe and Yamamuro show that (2.3.1) is incorrect in the case of $\gamma > 0$ and $t < 0$ (see [63, Remark 6.1]). If $\gamma = 0$ and $t > 0$, the implication above is sufficient to prove Albin's statement under some weaker restrictions on the counting r.v. η (see [46, Theorem 6]). Thus, Albin's assertion related to conditions under which $F_{S_\eta} \in \mathcal{L}(\gamma)$ remains only a hypothesis in the case $\gamma > 0$. Watanabe and Yamamuro do not prove Albin's hypothesis in the general case, but they give the following assertion related to the Poisson r.v. (see [63, Proposition 6.1]).

Theorem 2.3.6. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent identically distributed r.v.'s with d.f. F_ξ. If $F_\xi \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$, then F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for any counting r.v. η distributed according to the Poisson law.*

Theorem 2.3.6 is generalized in [65], where the following assertion is proved (see [65, Theorem 2.3]).

Theorem 2.3.7. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent non-negative r.v.'s with d.f. F_ξ such that $F_\xi^{*N} \in \mathcal{L}(\gamma)$ for some integer $N \geqslant 1$ and $\gamma \geqslant 0$. Then $F_{S_\eta} \in \mathcal{L}(\gamma)$ if either conditions (i) and (ii) or conditions (i) and (iii) hold, where*

(i) *for any $\varepsilon \in (0,1)$, there is an integer $M = M(\varepsilon)$ such that*

$$\sum_{k=M}^{\infty} \mathbb{P}(\eta = k+1)\overline{F_\xi^{*k}}(x) \leqslant \varepsilon \overline{F}_{S_\eta}(x)$$

for all $x \geqslant 0$*;*

(ii) $\overline{F}_\xi(x) = o\big(\overline{F^2_\xi}(x)\big)$;

(iii) *for all* $t > 0$ *and* $1 \leqslant i \leqslant N-1$,

$$\liminf_{x\to\infty} \frac{\overline{F^{*i}_\xi}(x-t)}{\overline{F^{*i}_\xi}(x)} \geqslant \mathrm{e}^{\gamma t}.$$

Motivated by the results presented above, we also consider conditions under which the d.f. F_{S_η} belongs to the classes $\mathcal{OL}$, $\mathcal{D}$ and $\mathcal{L}(\gamma)$, $\gamma > 0$. We investigate the randomly stopped sum S_η, the randomly stopped maximum ξ_η and the randomly stopped maximum of sums $S_{(\eta)}$ for independent but not necessarily identically distributed r.v.'s. We suppose that some of the d.f.'s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ are in a certain class, and we find conditions on $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ and η such that the distributions of S_η, ξ_η and $S_{(\eta)}$ remain in the same class. We present various collections of such conditions.

Chapter 3

Randomly stopped sums of dominatedly varying distributions

3.1 Main results

In this section, we formulate and prove two assertions, which describe conditions under which the randomly stopped sum S_η belongs to the class $\mathcal{D}$. In Theorem 3.1.1, no moment conditions on the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are required, whereas the conditions of Theorem 3.1.2 imply that the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ have finite expectations.

Theorem 3.1.1. [21] *Let $\{\xi_1, \xi_2, \ldots\}$ be independent non-negative r.v.'s, and let η be a counting r.v., which is independent of $\{\xi_1, \xi_2, \ldots\}$. Then $F_{S_\eta} \in \mathcal{D}$ if the following three conditions hold:*

$$\begin{aligned}
&\text{(a)} \quad F_{\xi_\kappa} \in \mathcal{D} \quad \textit{for some} \quad \kappa \in \operatorname{supp}(\eta);\\
&\text{(b)} \quad \limsup_{x\to\infty} \sup_{n\geqslant\kappa} \frac{1}{n\overline{F}_{\xi_\kappa}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty;\\
&\text{(c)} \quad \mathbb{E}\eta^{p+1} < \infty \quad \textit{for some} \quad p > J^+_{F_{\xi_\kappa}}.
\end{aligned}$$

Theorem 3.1.2. [21] *Suppose that the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are independent and non-negative, and the counting r.v. η is independent of $\{\xi_1, \xi_2, \ldots\}$. In addition, let condition (b) of Theorem 3.1.1 hold for some $\kappa \in \operatorname{supp}(\eta)$ together with the following requirements:*

$$F_{\xi_\kappa} \in \mathcal{D}, \quad \max\{\mathbb{E}\xi_\kappa, \mathbb{E}\eta\} < \infty,$$

$$\limsup_{u\to\infty}\sup_{n\geqslant\kappa}\frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ \mathbb{E}\xi_k\geqslant u}}\mathbb{E}\xi_k=0,\quad \liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\xi_k>0.$$

Then $F_{S_\eta}\in\mathcal{D}$ *if and only if* $\min\{F_{\xi_\kappa},F_\eta\}\in\mathcal{D}$.

It follows from inequality (3.3.3) below that condition (b) of Theorem 3.1.1 and the condition $\mathbb{E}\xi_\kappa<\infty$ imply that $\mathbb{E}\xi_k$ are finite for all k. Therefore, all the conditions of Theorem 3.1.2 are meaningful.

Now we give two examples of the r.v.'s $\{\xi_1,\xi_2,\ldots\}$ and η where the random sum F_{S_η} belongs to the class of dominatedly varying distributions. In Example 3.1.1, we see that only one distribution belongs to the class $\mathcal{D}$, and all other distributions have light tails. Selecting the counting r.v. η in a special way we can achieve that the randomly stopped sum belongs to the class $\mathcal{D}$. This result follows from Theorem 3.1.1.

Example 3.1.1. *Let* $\{\xi_1,\xi_2,\ldots\}$ *be independent r.v.'s such that* ξ_1 *is distributed according to the Peter and Paul law with parameter* $\frac{1}{2}$, *i.e.*

$$\overline{F}_{\xi_1}(x)=\sum_{l\geqslant1,\,2^l>x}\frac{1}{2^l}=2^{-\left\lfloor\frac{\log x}{\log 2}\right\rfloor},\quad x\geqslant1,$$

whereas ξ_k, $k\geqslant2$, *are exponentially distributed with parameter* k. *In addition, let* η *be a counting r.v., which is independent of* $\{\xi_1,\xi_2,\ldots\}$ *and has the following distribution:*

$$\mathbb{P}(\eta=m)=\frac{1}{\zeta(5)}\frac{1}{(m+1)^5},\quad m\in\mathbb{N}_0,$$

where ζ *denotes the Riemann zeta function, i.e.*

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad Re\ s>1.$$

Theorem 3.1.1 implies that the d.f. of the randomly stopped sum S_η has a dominatedly varying tail.

Indeed, for any fixed $y \in (0,1)$, we have

$$\begin{aligned}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_1}(xy)}{\overline{F}_{\xi_1}(x)} &= \limsup_{x\to\infty}\frac{2^{-\lfloor \log xy/\log 2\rfloor}}{2^{-\lfloor \log x/\log 2\rfloor}}\\ &= \limsup_{x\to\infty}\frac{2^{-\log xy/\log 2+\{\log xy/\log 2\}}}{2^{-\log x/\log 2+\{\log x/\log 2\}}}\\ &= \limsup_{x\to\infty} 2^{-\frac{\log x}{\log 2}-\frac{\log y}{\log 2}+\frac{\log x}{\log 2}}2^{-\{\frac{\log xy}{\log 2}\}-\{\frac{\log x}{\log 2}\}}\\ &= 2^{-\frac{\log y}{\log 2}}\limsup_{x\to\infty}2^{\{\frac{\log xy}{\log 2}\}-\{\frac{\log x}{\log 2}\}} \leqslant 2^{1-\frac{\log y}{\log 2}} < \infty.\end{aligned}$$

Hence, $F_{\xi_1} \in \mathcal{D}$.

Moreover, we have

$$\begin{aligned}J^+_{F_{\xi_1}} &= -\lim_{y\to\infty}\frac{1}{\log y}\log\left(\liminf_{x\to\infty}\frac{\overline{F}_{\xi_1}(xy)}{\overline{F}_{\xi_1}(x)}\right)\\ &= -\lim_{y\to\infty}\frac{1}{\log y}\log\left(\liminf_{x\to\infty}\frac{2^{-\frac{\log xy}{\log 2}+\{\frac{\log xy}{\log 2}\}}}{2^{-\frac{\log x}{\log 2}+\{\frac{\log x}{\log 2}\}}}\right)\\ &= -\lim_{y\to\infty}\frac{1}{\log y}\log\left(\liminf_{x\to\infty}2^{-\frac{\log y}{\log 2}+\{\frac{\log xy}{\log 2}\}-\{\frac{\log x}{\log 2}\}}\right)\\ &= -\lim_{y\to\infty}\frac{1}{\log y}\log\left(2^{-\frac{\log y}{\log 2}}\liminf_{x\to\infty}2^{\{\frac{\log xy}{\log 2}\}-\{\frac{\log x}{\log 2}\}}\right).\end{aligned}$$

Next, since

$$\frac{1}{2} < \frac{2^{\{\frac{\log xy}{\log 2}\}}}{2^{\{\frac{\log x}{\log 2}\}}} < 2,$$

we obtain

$$\begin{aligned}J^+_{F_{\xi_1}} &= -\lim_{y\to\infty}\frac{1}{\log y}\log\left(2^{-\frac{\log y}{\log 2}}2^{-1}\right)\\ &= -\lim_{y\to\infty}\frac{1}{\log y}\left(-\frac{\log y}{\log 2}-1\right)\log 2\\ &= \lim_{y\to\infty}\frac{\log 2}{\log y}\left(\frac{\log y}{\log 2}+1\right)\\ &= \lim_{y\to\infty}\left(1+\frac{\log 2}{\log y}\right) = 1.\end{aligned}$$

Finally, we have

$$\mathbb{E}\eta^3 = \frac{1}{\zeta(5)}\sum_{m=0}^{\infty}\frac{m^3}{(m+1)^5} < \frac{\zeta(2)}{\zeta(5)}$$

and

$$\begin{aligned}\limsup_{x\to\infty}\sup_{n\geqslant 1}\frac{1}{n\overline{F}_{\xi_1}(x)}\sum_{i=1}^{n}\overline{F}_{\xi_i}(x) &= \limsup_{x\to\infty}\sup_{n\geqslant 1}\frac{1}{n}\left(1+2^{\left\lfloor\frac{\log x}{\log 2}\right\rfloor}\sum_{i=2}^{n}\mathrm{e}^{-ix}\right)\\ &\leqslant \limsup_{x\to\infty}\left(1+2^{\left\lfloor\frac{\log x}{\log 2}\right\rfloor}/(\mathrm{e}^{2x}-\mathrm{e}^{x})\right)=1.\end{aligned}$$

In Example 3.1.2, the set of the r.v.'s $\{\xi_1,\xi_2,\ldots\}$ is divided into two sets. The r.v.'s from the first set belong to the class $\mathcal{D}$, whereas r.v.'s from the second set do not belong to this class. Selecting the counting r.v. η in a special way, we achieve that the random sum of such random variables belongs to the class $\mathcal{D}$. This fact follows from Theorem 3.1.2.

EXAMPLE 3.1.2. *Let $\{\xi_1,\xi_2,\ldots\}$ be independent r.v.'s such that ξ_k are distributed according to the Pareto law for all odd k and ξ_k are distributed according to the Weibull law for all even k. To be more precise, let*

$$\begin{aligned}&\overline{F}_{\xi_k}(x)=(1+x)^{-2},\ \ x\geqslant 0,\ \ k\in\{1,3,5,\ldots\},\\ &\overline{F}_{\xi_k}(x)=\mathrm{e}^{-\sqrt{x}},\ \ x\geqslant 0,\ \ k\in\{2,4,6,\ldots\}.\end{aligned}$$

In addition, let η be a counting r.v, which is independent of $\{\xi_1,\xi_2,\ldots\}$ and distributed according to the Poisson law.

Theorem 3.1.2 implies that d.f. of the randomly stopped sum S_η belongs to the class $\mathcal{D}$.

Indeed, since

$$\begin{aligned}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_1}(xy)}{\overline{F}_{\xi_1}(x)} &= \limsup_{x\to\infty}\frac{\frac{1}{(1+xy)^2}}{\frac{1}{(1+x)^2}}=\limsup_{x\to\infty}\frac{(1+x)^2}{(1+xy)^2}\\ &= \limsup_{x\to\infty}\frac{\left(\frac{1}{x}+1\right)^2}{\left(\frac{1}{x}+y\right)^2}=\frac{1}{y^2}<\infty\end{aligned}$$

for any fixed $y\in(0,1)$, we conclude that $F_{\xi_1}\in\mathcal{D}$.

Next, we have

$$\begin{aligned}
\limsup_{x\to\infty}\frac{\overline{\min(F_{\xi_1},F_\eta)}(xy)}{\overline{\min(F_{\xi_1},F_\eta)}(x)} &= \limsup_{x\to\infty}\frac{(1-\min(F_{\xi_1},F_\eta))(xy)}{(1-\min(F_{\xi_1},F_\eta))(x)}\\
&= \limsup_{x\to\infty}\frac{(1-\min(1-\overline{F}_{\xi_1},1-\overline{F}_\eta))(xy)}{(1-\min(1-\overline{F}_{\xi_1},1-\overline{F}_\eta))(x)}\\
&= \limsup_{x\to\infty}\frac{1-(1-\max(\overline{F}_{\xi_1},\overline{F}_\eta))(xy)}{1-(1-\max(\overline{F}_{\xi_1},\overline{F}_\eta))(x)}\\
&= \limsup_{x\to\infty}\frac{\max(\overline{F}_{\xi_1},\overline{F}_\eta)(xy)}{\max(\overline{F}_{\xi_1},\overline{F}_\eta)(x)}\\
&= \limsup_{x\to\infty}\frac{\max\left(\frac{1}{(1+xy)^2},\sum\limits_{k>xy}\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}\right)}{\max\left(\frac{1}{(1+x)^2},\sum\limits_{k>x}\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}\right)}.
\end{aligned}$$

Since

$$\sum_{k>x}\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}<\frac{1}{(1+x)^2}$$

for x large enough, we obtain

$$\limsup_{x\to\infty}\frac{\overline{\min(F_{\xi_1},F_\eta)}(xy)}{\overline{\min(F_{\xi_1},F_\eta)}(x)}=\limsup_{x\to\infty}\frac{\frac{1}{(1+xy)^2}}{\frac{1}{(1+x)^2}}=\frac{1}{y^2}<\infty.$$

Thus, we deduce that $\min\{F_{\xi_1},F_\eta\}\in\mathcal{D}$.

Furthermore, we have

$$\mathbb{E}\xi_k=\int_0^\infty\overline{F}_{\xi_k}(x)dx=\int_0^\infty\frac{dx}{(1+x)^2}=1$$

for $k\in\{1,3,5,\ldots\}$ and

$$\mathbb{E}\xi_k=\int_0^\infty\overline{F}_{\xi_k}(x)dx=\int_0^\infty\mathrm{e}^{-\sqrt{x}}dx=2$$

for $k\in\{2,4,6,\ldots\}$.

Hence, we get

$$\inf_{k\in\mathbb{N}}\mathbb{E}\xi_k=1\quad\text{and}\quad\sup_{k\in\mathbb{N}}\mathbb{E}\xi_k=2.$$

For all $x \geqslant 0$, we have:
if $n = 1$, then

$$\frac{1}{n\overline{F}_{\xi_1}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) = 1 \leqslant 1 + \mathrm{e}^{-\sqrt{x}}(1+x)^2;$$

if n is even, then

$$\frac{1}{n\overline{F}_{\xi_1}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) = \frac{1}{2} + \frac{1}{2}\mathrm{e}^{-\sqrt{x}}(1+x)^2 \leqslant 1 + \mathrm{e}^{-\sqrt{x}}(1+x)^2;$$

if n is odd ($n = 2k+1$), then

$$\frac{1}{n\overline{F}_{\xi_1}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) = \frac{k+1}{2k+1} + \frac{k}{2k+1}\mathrm{e}^{-\sqrt{x}}(1+x)^2 \leqslant 1 + \mathrm{e}^{-\sqrt{x}}(1+x)^2.$$

Thus,

$$\sup_{n\geqslant 1} \frac{1}{n\overline{F}_{\xi_1}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) \leqslant 1 + \mathrm{e}^{-\sqrt{x}}(1+x)^2.$$

Therefore, we get

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n\overline{F}_{\xi_1}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) \leqslant 1.$$

On the other hand, we have

$$\sup_{n\geqslant 1} \frac{1}{n\overline{F}_{\xi_1}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) \geqslant \frac{1}{\overline{F}_{\xi_1}(x)} \overline{F}_{\xi_1}(x) = 1,$$

Finally, we conclude that

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n\overline{F}_{\xi_1}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) = 1.$$

3.2 Auxiliary results for Theorems 3.1.1-3.1.2

This section deals with a number of auxiliary results. The assertion of Lemma 3.2.1 below is well known and can be derived, for instance, from [9, Proposition 2.2.1].

Lemma 3.2.1. *For a d.f. $F \in \mathcal{D}$ and any $p > J_F^+$, there are constants $c_1 > 0$ and $c_2 > 0$ such that the inequality*

$$\frac{\overline{F}(u)}{\overline{F}(v)} \leqslant c_1 \left(\frac{v}{u}\right)^p$$

holds for $v \geqslant u \geqslant c_2$. In addition, we have $u^{-p} = o\big(\overline{F}(u)\big)$ for any $p > J_F^+$.

Lemma 3.2.2 is an inhomogeneous case of Theorem 3 from [17].

Lemma 3.2.2. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent non-negative r.v.'s with d.f.'s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, respectively, and let $F_{\xi_\kappa} \in \mathcal{D}$ for some $\kappa \geqslant 1$. In addition, we suppose that condition (b) of Theorem 3.1.1 holds. Then for any $p > J^+_{F_{\xi_\kappa}}$, there is a constant $c_3 > 0$ such that*

$$\frac{\overline{F_{\xi_1} * \ldots * F_{\xi_n}}(x)}{\overline{F}_{\xi_\kappa}(x)} \leqslant c_3\, n^{p+1}$$

for all $n \geqslant \kappa$ and $x \geqslant 0$.

Proof of Lemma 3.2.2. Suppose that $n \geqslant \kappa$ and $x \geqslant 0$. First of all, we observe that

$$\mathbb{P}(S_n > x) \leqslant \mathbb{P}\left(\bigcup_{i=1}^{n} \left\{\xi_i > \frac{x}{n}\right\}\right) \leqslant \sum_{i=1}^{n} \overline{F}_{\xi_i}\left(\frac{x}{n}\right). \tag{3.2.1}$$

By condition (b) of Theorem 3.1.1, we deduce that there are positive constants c_4 and c_5 such that

$$\sum_{i=1}^{n} \overline{F}_{\xi_i}(x) \leqslant c_5 n \overline{F}_{\xi_\kappa}(x), \quad x \geqslant c_4. \tag{3.2.2}$$

Therefore, inequality (3.2.1) implies that

$$\mathbb{P}(S_n > x) \leqslant c_5 n \overline{F}_{\xi_\kappa}\left(\frac{x}{n}\right) \quad \text{if} \quad x \geqslant c_4 n. \tag{3.2.3}$$

Since $F_{\xi_\kappa} \in \mathcal{D}$, by Lemma 3.2.1, there are positive constants c_6 and c_7 such that

$$\frac{\overline{F}_{\xi_\kappa}(u)}{\overline{F}_{\xi_\kappa}(v)} \leqslant c_6 \left(\frac{v}{u}\right)^p \tag{3.2.4}$$

for $v \geqslant u \geqslant c_7$.

From inequalities (3.2.3) and (3.2.4) it follows that

$$\frac{\mathbb{P}(S_n > x)}{\overline{F}_{\xi_\kappa}(x)} \leqslant c_5 n \frac{\overline{F}_{\xi_\kappa}(x/n)}{\overline{F}_{\xi_\kappa}(x)} \leqslant c_5 c_6\, n^{p+1} \tag{3.2.5}$$

for all $x \geqslant c_8 n$, where $c_8 = \max\{c_4, c_7\}$.

If $0 \leqslant x \leqslant c_8 n$, then

$$\frac{\mathbb{P}(S_n > x)}{\overline{F}_{\xi_\kappa}(x)} \leqslant \frac{1}{\overline{F}_{\xi_\kappa}(c_8 n)} = \frac{\overline{F}_{\xi_\kappa}(c_8)}{\overline{F}_{\xi_\kappa}(c_8 n)} \frac{1}{\overline{F}_{\xi_\kappa}(c_8)} \leqslant c_6 n^p \frac{1}{\overline{F}_{\xi_\kappa}(c_8)} \tag{3.2.6}$$

by (3.2.4) because $c_8 \geqslant c_7$.

From (3.2.5) and (3.2.6) we conclude that

$$\frac{\overline{F_{\xi_1} * \ldots * F_{\xi_n}}(x)}{\overline{F}_{\xi_\kappa}(x)} = \frac{\mathbb{P}(S_n > x)}{\overline{F}_{\xi_\kappa}(x)} \leqslant \max\left\{c_5 c_6, \frac{c_6}{\overline{F}_{\xi_\kappa}(c_8)}\right\} n^{p+1}$$

for all $x \geqslant 0$, which proves the lemma. □

The following lemma is an inhomogeneous case of Corollary 3.1 from [60].

Lemma 3.2.3. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent real-valued r.v.'s with d.f.'s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, respectively, and let $F_{\xi_\kappa} \in \mathcal{D}$ for some $\kappa \geqslant 1$. In addition, we suppose that*

$$\lim_{u\to\infty} \sup_{n\geqslant\kappa} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left(|\xi_k| \mathbb{1}_{\{\xi_k \leqslant -u\}}\right) = 0, \tag{3.2.7}$$

condition (b) of Theorem 3.1.1 holds, and $\mathbb{E}\xi_k = \mathbb{E}\xi_k^+ - \mathbb{E}\xi_k^- = 0$ for $k \in \mathbb{N}$. Then for any $\gamma > 0$, there is a constant $c_9 = c_9(\gamma) > 0$ such that

$$\mathbb{P}(S_n > x) \leqslant c_9 n \overline{F}_{\xi_\kappa}(x)$$

for all $x \geqslant \gamma n$ and $n \geqslant \kappa$.

Proof of Lemma 3.2.3. For all $x \geqslant 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{P}(S_n > x) &= \mathbb{P}\left(S_n > x, \bigcup_{k=1}^{n} \left\{\xi_k > \frac{x}{2}\right\}\right) + \mathbb{P}\left(S_n > x, \bigcap_{k=1}^{n} \left\{\xi_k \leqslant \frac{x}{2}\right\}\right) \\ &\leqslant \sum_{k=1}^{n} \overline{F}_{\xi_k}\left(\frac{x}{2}\right) + \mathbb{P}\left(\sum_{k=1}^{n} \widehat{\xi}_k > x\right), \end{aligned} \tag{3.2.8}$$

where $\widehat{\xi}_k = \xi_k \mathbb{1}_{\left\{\xi_k \leqslant \frac{x}{2}\right\}} + \frac{x}{2} \mathbb{1}_{\left\{\xi_k > \frac{x}{2}\right\}}$.

Let $y = y(x,n)$ and $a = a(x,n)$ be two functions defined as follows: $a(x,n) = \max\left\{- \log\left(n\overline{F}_{\xi_\kappa}\left(\frac{x}{2}\right)\right), 1\right\}$, whereas $y(x,n)$ is an arbitrary function such that $y(x,n) > 0$ and $\lim\limits_{n\to\infty} \sup\limits_{x \geqslant \gamma n} y(x,n) = 0$.

By Markov's inequality, for all $x > 0$ and $y = y(x,n)$, we have

$$\begin{aligned} \mathbb{P}\left(\sum_{k=1}^{n} \widehat{\xi}_k > x\right) &\leqslant \mathrm{e}^{-yx}\, \mathbb{E} \exp\left\{y \sum_{k=1}^{n} \widehat{\xi}_k\right\} \\ &= \mathrm{e}^{-yx} \prod_{k=1}^{n} \left(1 + \mathbb{E}\left(\mathrm{e}^{y\widehat{\xi}_k} - 1\right)\right). \end{aligned}$$

Therefore, using inequality $1 + v \leqslant \mathrm{e}^v$, $v \in \mathbb{R}$, we get

$$\mathcal{J} := \frac{\mathbb{P}\left(\sum\limits_{k=1}^{n} \widehat{\xi}_k > x\right)}{n\overline{F}_{\xi_\kappa}\left(\frac{x}{2}\right)} \leqslant \exp\Big\{ - yx + a + \sum_{k=1}^{n} \mathbb{E}\left(\mathrm{e}^{y\widehat{\xi}_k} - 1\right)\Big\}. \qquad (3.2.9)$$

For every fixed $k \in \mathbb{N}$, we split the expectation $\mathbb{E}\left(\mathrm{e}^{y\widehat{\xi}_k} - 1\right)$ into four parts as follows:

$$\begin{aligned} &\mathbb{E}\left(\mathrm{e}^{y\widehat{\xi}_k} - 1\right) \\ = &\left(\int\limits_{(-\infty,0]} + \int\limits_{(0,x/(2a^2)]} + \int\limits_{(x/(2a^2),x/2]}\right)(\mathrm{e}^{yu} - 1)\mathrm{d}F_{\xi_k}(u) \\ + &\left(\mathrm{e}^{yx/2} - 1\right)\overline{F}_{\xi_k}\left(\frac{x}{2}\right) \\ := &\mathcal{J}_{1k} + \mathcal{J}_{2k} + \mathcal{J}_{3k} + \mathcal{J}_{4k}. \end{aligned} \qquad (3.2.10)$$

The inequalities $|\mathrm{e}^v - 1| \leqslant |v|$ and $|\mathrm{e}^v - 1 - v| \leqslant v^2/2$, which are true for all $v \leqslant 0$

$$\begin{aligned} J_{1k} &= \mathbb{E}\left(\left(\mathrm{e}^{y\xi_k} - 1\right)\mathbb{1}_{\{\xi_k \leqslant 0\}}\right) \\ &= y\mathbb{E}\left(\xi_k \mathbb{1}_{\{\xi_k \leqslant 0\}}\right) + \mathbb{E}\left(\left(\mathrm{e}^{y\xi_k} - 1 - y\xi_k\right)\mathbb{1}_{\{\xi_k \leqslant 0\}}\right) \\ &= -y\mathbb{E}\xi_k^- + \mathbb{E}\left(\left(\mathrm{e}^{y\xi_k} - 1\right)\mathbb{1}_{\{\xi_k \leqslant -1/\sqrt[4]{y}\}}\right) - y\mathbb{E}\left(\xi_k \mathbb{1}_{\{\xi_k \leqslant -1/\sqrt[4]{y}\}}\right) \\ &+ \mathbb{E}\left(\left(\mathrm{e}^{y\xi_k} - 1 - y\xi_k\right)\mathbb{1}_{\{-1/\sqrt[4]{y} < \xi_k \leqslant 0\}}\right) \\ &\leqslant -y\mathbb{E}\xi_k^- + y\left(2\,\mathbb{E}\left(|\xi_k|\mathbb{1}_{\{\xi_k \leqslant -1/\sqrt[4]{y}\}}\right) + \sqrt{y}/2\right), \end{aligned} \qquad (3.2.11)$$

whereas the inequality $\mathrm{e}^v - 1 \leqslant v\mathrm{e}^v$, which holds for all $v \geqslant 0$, gives

$$\mathcal{J}_{2k} \leqslant y\mathrm{e}^{xy/(2a^2)} \int\limits_{[0,\,x/(2a^2)]} u\,\mathrm{d}F_{\xi_k}(u) \leqslant y\,\mathrm{e}^{xy/(2a^2)}\mathbb{E}\xi_k^+.$$

In addition, it is easily seen that

$$\mathcal{J}_{3k} \leqslant \mathrm{e}^{xy/2}\overline{F}_{\xi_k}(x/(2a^2)) \;\text{ and }\; \mathcal{J}_{4k} \leqslant \mathrm{e}^{xy/2}\overline{F}_{\xi_k}(x/2) \leqslant \mathrm{e}^{xy/2}\overline{F}_{\xi_k}(x/(2a^2)).$$

Using the bounds obtained together with relations (3.2.9) and (3.2.10) we get

$$\begin{aligned} \mathcal{J} \;&\leqslant\; \exp\Big\{ - yx + a - y\sum_{k=1}^{n}\mathbb{E}\xi_k^- + yn\varepsilon(y) \\ &+\; \mathrm{e}^{xy/(2a^2)}y\sum_{k=1}^{n}\mathbb{E}\xi_k^+ + 2\mathrm{e}^{xy/2}\sum_{k=1}^{n}\overline{F}_{\xi_k}\Big(\frac{x}{2a^2}\Big)\Big\}, \end{aligned} \tag{3.2.12}$$

where $n \geqslant \kappa$ and $\varepsilon(y) = y^{1/2}/2 + \sup\limits_{n\geqslant\kappa} \frac{1}{n}\sum\limits_{k=1}^{n} 2\,\mathbb{E}|\xi_k|\mathbb{1}_{\{\xi_k \leqslant -1/\sqrt[4]{y}\}}$.

The condition $\mathbb{E}\xi_\kappa^+ < \infty$ implies that

$$\lim_{n\to\infty}\inf_{x\geqslant\gamma n} a(x,n) \geqslant \lim_{n\to\infty}\log\left(\frac{1}{n\overline{F}_{\xi_\kappa}(\gamma n/2)}\right) = \infty. \tag{3.2.13}$$

Similarly, the condition $F_{\xi_\kappa} \in \mathcal{D}$ and Lemma 3.2.1 give

$$\begin{aligned} \lim_{n\to\infty}\inf_{x\geqslant\gamma n}\frac{x}{2a^2} &= \lim_{n\to\infty}\inf_{x\geqslant\gamma n}\frac{x}{2\log^2\big(1/(n\overline{F}_{\xi_\kappa}(x/2))\big)} \\ \geqslant \lim_{n\to\infty}\inf_{x\geqslant\gamma n}\frac{x}{2\log^2\big(1/(\overline{F}_{\xi_\kappa}(x))\big)} &= \lim_{n\to\infty}\inf_{x\geqslant\gamma n}\frac{x}{2\log^2\big(x^p/(x^p\overline{F}_{\xi_\kappa}(x))\big)} \\ \geqslant \lim_{n\to\infty}\inf_{x\geqslant\gamma n}\frac{x}{2p^2\log^2 x} &= \infty. \end{aligned} \tag{3.2.14}$$

Hence, by condition (b) of Theorem 3.1.1 and Lemma 3.2.1, we obtain

$$\sum_{k=1}^{n}\overline{F}_{\xi_k}\Big(\frac{x}{2a^2}\Big) \leqslant c_{10}n\overline{F}_{\xi_\kappa}\Big(\frac{x}{2a^2}\Big) \leqslant c_{11}n\,a^{2p}\overline{F}_{\xi_\kappa}\Big(\frac{x}{2}\Big),$$

where $x \geqslant \gamma n$, n is large enough and $p > J^+_{F_{\xi_\kappa}}$.

Substituting the last bound into (3.2.12) we conclude that the inequality

$$\begin{aligned} \mathcal{J} \;&\leqslant\; \exp\Big\{ - yx + a - y\sum_{k=1}^{n}\mathbb{E}\xi_k^- + yn\varepsilon(y) \\ &+\; \mathrm{e}^{xy/(2a^2)}y\sum_{k=1}^{n}\mathbb{E}\xi_k^+ + 2c_{11}\mathrm{e}^{xy/2}na^{2p}\overline{F}_{\xi_\kappa}\Big(\frac{x}{2}\Big)\Big\} \end{aligned} \tag{3.2.15}$$

holds if $x \geqslant \gamma n$ and n is large enough.

Let now $\widehat{y} = \max\left\{\frac{a-2p\log a}{x/2}, \frac{2}{x}\right\}$. It is clear that $\widehat{y} = \widehat{y}(x,n) > 0$ for all $x > 0$ and $n \in \mathbb{N}$. Moreover, $\lim\limits_{n\to\infty} \inf\limits_{x\geqslant\gamma n} \widehat{y} = 0$ by (3.2.14).

For $y = \widehat{y}$, inequality (3.2.15) yields

$$\mathcal{J} \leqslant \exp\Big\{ - \widehat{y}x + a + \widehat{y}n\varepsilon(\widehat{y}) + (\mathrm{e}^{1/a} - 1)\,\widehat{y}\sum_{k=1}^{n} \mathbb{E}\xi_k^+ + 2c_{11}\Big\} \qquad (3.2.16)$$

if $x \geqslant \gamma n$ and n is large enough.

By (3.2.2), we have

$$\begin{aligned}\sum_{k=1}^{n} \mathbb{E}\xi_k^+ &= \sum_{k=1}^{n}\left(\int_0^{c_4} + \int_{c_4}^{\infty}\right)\overline{F}_{\xi_k}(u)\mathrm{d}u \\ &\leqslant c_4 n + c_5 n \mathbb{E}\xi_\kappa^+ := c_{12} n. \qquad (3.2.17)\end{aligned}$$

Combining (3.2.16) with (3.2.2) we obtain

$$\begin{aligned}\mathcal{J} &\leqslant \mathrm{e}^{2c_{11}} \exp\left\{\widehat{y}x\Big(- 1 + \frac{\varepsilon(\widehat{y})}{\gamma} + \frac{(\mathrm{e}^{1/a} - 1)c_{12}}{\gamma}\Big) + a\right\} \\ &\leqslant \mathrm{e}^{2c_{11}} \exp\{\widehat{y}x(-3/4) + a\} \\ &= \mathrm{e}^{2c_{11}} \exp\{-a/2 + 3p\log a\}\end{aligned}$$

for all $x \geqslant \gamma n$ and n large enough.

Therefore, taking into account (3.2.9) and (3.2.13) we get

$$\limsup_{n\to\infty} \sup_{x\geqslant\gamma n} \frac{\mathbb{P}\left(\sum\limits_{k=1}^{n} \widehat{\xi}_k > x\right)}{n\overline{F}_{\xi_\kappa}\left(\frac{x}{2}\right)} = 0.$$

Hence, we deduce that

$$\limsup_{n\to\infty} \sup_{x\geqslant\gamma n} \frac{\mathbb{P}\left(S_n > x\right)}{n\overline{F}_{\xi_\kappa}(x)} \leqslant \limsup_{n\to\infty} \sup_{x\geqslant\gamma n} \frac{\sum\limits_{k=1}^{n} \overline{F}_{\xi_k}\left(\frac{x}{2}\right)}{n\overline{F}_{\xi_\kappa}\left(\frac{x}{2}\right)} \frac{\overline{F}_{\xi_\kappa}\left(\frac{x}{2}\right)}{\overline{F}_{\xi_\kappa}(x)} < \infty$$

because of inequality (3.2.8) and the conditions of the lemma.

From the last inequality it follows that

$$\sup_{x\geqslant\gamma n} \frac{\mathbb{P}\left(S_n > x\right)}{n\overline{F}_{\xi_\kappa}(x)} \leqslant c_{13} \qquad (3.2.18)$$

for some $c_{13} > 0$ and all $n \geqslant N$, where $N \geqslant \kappa$ is a positive integer number.

If $\kappa \leqslant n \leqslant N$ and $x \geqslant 0$, then by Lemma 3.2.2, there is a constant $c_{14} > 0$ such that

$$\mathbb{P}(S_n > x) \leqslant \mathbb{P}\left(\sum_{k=1}^{n} \xi_k^+ > x\right) \leqslant c_{14} n^{p+1} \overline{F}_{\xi_\kappa^+}(x) \leqslant c_{14} N^p n \overline{F}_{\xi_\kappa}(x). \quad (3.2.19)$$

The assertion of the lemma now follows immediately from (3.2.18) and (3.2.19). □

Lemma 3.2.4. *Let $\xi_1, \xi_2, ...$ be independent non-negative r.v.'s such that*

$$\lim_{u\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\Big(\xi_k \mathbb{I}_{\{\xi_k \geqslant u\}}\Big) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\xi_k > 0.$$

Then $\lim\limits_{x\to\infty} \sup\limits_{n>dx} \mathbb{P}(S_n \leqslant x) = 0$ *for some* $d > 1$.

Proof of Lemma 3.2.4. Let x and y be arbitrary positive numbers. Since $\xi_1, \xi_2, ...$ are non-negative, by Markov's inequality, we obtain

$$\begin{aligned} \mathbb{P}(\xi_1 + \xi_2 + ... + \xi_n \leqslant x) &= \mathbb{P}(\mathrm{e}^{-y(\xi_1+\xi_2+...+\xi_n)} \geqslant \mathrm{e}^{-yx}) \\ &\leqslant \mathrm{e}^{yx} \prod_{k=1}^{n} \Big(1 + \mathbb{E}(\mathrm{e}^{-y\xi_k} - 1)\Big). \end{aligned}$$

Next, applying arguments similar to those in (3.2.11), for all $k = 1, ..., n$ and $y > 0$, we get

$$\mathbb{E}(\mathrm{e}^{-y\xi_k} - 1) \leqslant -y\mathbb{E}\xi_k + y\left(2\mathbb{E}\Big(\xi_k \mathbb{I}_{\{\xi_k > 1/\sqrt[4]{y}\}}\Big) + \sqrt{y}/2\right).$$

Therefore,

$$\mathbb{P}(\xi_1 + \xi_2 + ... + \xi_n \leqslant x) \leqslant \exp\left\{y\Big(x - \sum_{k=1}^{n} \mathbb{E}\xi_k + n\widehat{\varepsilon}(y)\Big)\right\},$$

where $\widehat{\varepsilon}(y) = 2\sup\limits_{n\in\mathbb{N}} \frac{1}{n} \sum\limits_{k=1}^{n} \mathbb{E}(\xi_k \mathbb{I}_{\{\xi_k > 1/\sqrt[4]{y}\}}) + \sqrt{y}/2$.

By the conditions of the lemma, we have

$$\lim_{y\to 0} \widehat{\varepsilon}(y) = 0 \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\xi_k \geqslant d_1$$

for some $d_1 > 0$ and n large enough.

Hence, for this n and all $x > 0$ and $y > 0$, we obtain

$$\mathbb{P}(\xi_1 + \xi_2 + ... + \xi_n \leqslant x) \leqslant \exp\{y(x - nd_1 + n\widehat{\varepsilon}(y))\}.$$

Since $\widehat{\varepsilon}\,(y) \downarrow 0$ as $y \downarrow 0$, there is $y^* > 0$ such that

$$\mathbb{P}(\xi_1 + \xi_2 + ... + \xi_n \leqslant x) \leqslant \exp\{y^*(x - nd_1/2)\}.$$

The last inequality implies that

$$\mathbb{P}(\xi_1 + \xi_2 + ... + \xi_n \leqslant x) \leqslant \exp\{-y^* x\}$$

if $n \geqslant 4x/d_1$ and x is large enough.

The assertion of the lemma follows immediately from the last inequality. □

3.3 Proofs of Theorems 3.1.1-3.1.2

PROOF OF THEOREM 3.1.1. To prove the theorem, it is sufficient to show that

$$\limsup_{x\to\infty} \frac{\mathbb{P}\left(S_\eta > \frac{x}{2}\right)}{\mathbb{P}(S_\eta > x)} < \infty. \tag{3.3.1}$$

By Lemma 3.2.2, for any $x > 0$, we have

$$\begin{aligned}
\mathbb{P}\left(S_\eta > \frac{x}{2}\right) &= \left(\sum_{n=1}^{\kappa} + \sum_{n=\kappa+1}^{\infty}\right) \mathbb{P}\left(S_n > \frac{x}{2}\right) \mathbb{P}(\eta = n) \\
&\leqslant \mathbb{P}\left(S_\kappa > \frac{x}{2}\right) + \sum_{n=\kappa+1}^{\infty} \mathbb{P}\left(S_n > \frac{x}{2}\right) \mathbb{P}(\eta = n) \\
&\leqslant c_{15}\kappa^{p+1}\overline{F}_{\xi_\kappa}\left(\frac{x}{2}\right) + c_{15}\overline{F}_{\xi_\kappa}\left(\frac{x}{2}\right) \sum_{n=\kappa+1}^{\infty} n^{p+1}\mathbb{P}(\eta = n) \\
&\leqslant c_{15}\left(\kappa^{p+1} + \mathbb{E}\eta^{p+1}\right)\overline{F}_{\xi_\kappa}\left(\frac{x}{2}\right),
\end{aligned}$$

where c_{15} is a positive constant.

In the case of non-negative r.v.'s we know that

$$\begin{aligned}
\overline{F}_{S_\eta}(x) &= \sum_{n=1}^{\infty} \mathbb{P}\left(S_n > x\right)\mathbb{P}(\eta = n) \geqslant \mathbb{P}\left(S_k > x\right)\mathbb{P}(\eta = k) \\
&\geqslant \mathbb{P}\left(\xi_k > x\right)\mathbb{P}(\eta = k) = \overline{F}_{\xi_k}(x)\mathbb{P}(\eta = k)
\end{aligned} \tag{3.3.2}$$

for all $x \geqslant 0$ and $k \geqslant 1$.

Thus, from (3.3.2) we conclude that

$$\mathbb{P}(S_\eta > x) \geqslant \overline{F}_{\xi_\kappa}(x)\mathbb{P}(\eta = \kappa).$$

In addition, $\overline{F}_{\xi_\kappa}(x)\mathbb{P}(\eta=\kappa)>0$ because $F_{\xi_\kappa}\in\mathcal{D}$ and $\kappa\in\operatorname{supp}(\eta)$.

The last two inequalities imply that

$$\limsup_{x\to\infty}\frac{\mathbb{P}\left(S_\eta>\frac{x}{2}\right)}{\mathbb{P}(S_\eta>x)}\leqslant\frac{c_{15}\left(\kappa^{p+1}+\mathbb{E}\eta^{p+1}\right)}{\mathbb{P}(\eta=\kappa)}\limsup_{x\to\infty}\frac{\overline{F}_{\xi_\kappa}\left(\frac{x}{2}\right)}{\overline{F}_{\xi_\kappa}(x)}.$$

Inequality (3.3.1) now follows immediately from conditions (a) and (c) of the theorem, which completes the proof. □

Proof of Theorem 3.1.2. First, we suppose that $\min\{F_{\xi_\kappa},F_\eta\}\in\mathcal{D}$, or, equivalently,

$$\limsup_{x\to\infty}\frac{\max\left\{\overline{F}_{\xi_\kappa}(yx),\overline{F}_\eta(yx)\right\}}{\max\left\{\overline{F}_{\xi_\kappa}(x),\overline{F}_\eta(x)\right\}}<\infty$$

for all $y\in(0,1)$.

We now prove that the d.f. F_{S_η} belongs to the class $\mathcal{D}$.

Applying arguments similar to those in (3.2.17) and taking into account the conditions of theorem we get

$$\mathbb{E}S_n=\sum_{k=1}^{n}\mathbb{E}\xi_k\leqslant c_{16}n \tag{3.3.3}$$

for some constant $c_{16}>0$ and all $n\in\mathbb{N}$.

For all $x>4\kappa\,c_{16}$, we have

$$\begin{aligned}\overline{F}_{S_\eta}(x) &= \mathbb{P}(S_\eta>x,\eta\leqslant\kappa)+\mathbb{P}\left(S_\eta>x,\kappa<\eta\leqslant\frac{x}{4c_{16}}\right)\\ &+ \mathbb{P}\left(S_\eta>x,\eta>\frac{x}{4c_{16}}\right):=\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3.\end{aligned} \tag{3.3.4}$$

If x is large enough, then the conditions of the theorem imply that

$$\begin{aligned}\mathcal{J}_1 &\leqslant \mathbb{P}(S_\kappa>x)\leqslant\sum_{i=1}^{\kappa}\overline{F}_{\xi_i}\left(\frac{x}{\kappa}\right)\\ &\leqslant c_{17}\kappa\overline{F}_{\xi_\kappa}\left(\frac{x}{\kappa}\right)\leqslant c_{17}\kappa\max\left\{\overline{F}_{\xi_\kappa}\left(\frac{x}{\kappa}\right),\overline{F}_\eta\left(\frac{x}{\kappa}\right)\right\}\end{aligned}$$

for some constant $c_{17}>0$.

Next, for $\mathcal{J}_2$, we have

$$\begin{aligned}\mathcal{J}_2 &= \sum_{\kappa<n\leqslant x/(4c_{16})} \mathbb{P}\left(\sum_{i=1}^{n}(\xi_i - \mathbb{E}\xi_i) > x - \sum_{k=1}^{n}\mathbb{E}\xi_k\right)\mathbb{P}(\eta = n)\\ &\leqslant \sum_{\kappa<n\leqslant x/(4c_{16})} \mathbb{P}\left(\sum_{i=1}^{n}(\xi_i - \mathbb{E}\xi_i) > \frac{3}{4}x\right)\mathbb{P}(\eta = n). \qquad (3.3.5)\end{aligned}$$

The r.v.'s $\xi_1 - \mathbb{E}\xi_1$, $\xi_2 - \mathbb{E}\xi_2$, ... satisfy conditions of Lemma 3.2.3. To be more precise, $\mathbb{E}(\xi_k - \mathbb{E}\xi_k) = 0$ and $F_{\xi_\kappa - \mathbb{E}\xi_\kappa} \in \mathcal{D}$ for all $k \in \mathbb{N}$. In addition, the conditions of the theorem yield

$$\begin{aligned}&\limsup_{u\to\infty}\sup_{n\geqslant\kappa}\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left(|\xi_k - \mathbb{E}\xi_k|\,\mathbb{1}_{\{\xi_k - \mathbb{E}\xi_k \leqslant -u\}}\right)\\ =\ &\limsup_{u\to\infty}\sup_{n\geqslant\kappa}\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left((\mathbb{E}\xi_k - \xi_k)\,\mathbb{1}_{\{\xi_k - \mathbb{E}\xi_k \leqslant -u\}}\right)\\ \leqslant\ &\limsup_{u\to\infty}\sup_{n\geqslant\kappa}\frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ \mathbb{E}\xi_k\geqslant u}}\mathbb{E}\xi_k = 0. \qquad (3.3.6)\end{aligned}$$

Applying the assertion of Lemma 3.2.3 to (3.3.5) we get

$$\begin{aligned}\mathcal{J}_2 &\leqslant c_{18}\sum_{\kappa<n\leqslant x/(4c_{16})} n\overline{F}_{\xi_\kappa}\left(\frac{3x}{4} + \mathbb{E}\xi_\kappa\right)\mathbb{P}(\eta = n)\\ &\leqslant c_{18}\,\mathbb{E}\eta\max\left\{\overline{F}_{\xi_\kappa}\left(\frac{3x}{4}\right), \overline{F}_\eta\left(\frac{3x}{4}\right)\right\}.\end{aligned}$$

It is easily seen that $\mathcal{J}_3 \leqslant \overline{F}_\eta\,(x/(4c_{16}))$. Consequently, the bounds for $\mathcal{J}_1$ and $\mathcal{J}_2$ together with (3.3.4) give

$$\overline{F}_{S_\eta}(x) \leqslant (c_{17}\kappa + c_{18}\mathbb{E}\eta + 1)\max\left\{\overline{F}_{\xi_\kappa}(ax), \overline{F}_\eta(ax)\right\}, \qquad (3.3.7)$$

where $a = \min\{1/\kappa,\ 3/4,\ 1/(4c_{16})\}$.

On the other hand, Lemma 3.2.4 implies that

$$\begin{aligned}\mathbb{P}(S_\eta > x) &\geqslant \sum_{n>bx}\mathbb{P}(S_n > x)\mathbb{P}(\eta = n) \geqslant \overline{F}_\eta(bx)\left(1 - \sup_{n>bx}\mathbb{P}(S_n \leqslant x)\right)\\ &\geqslant \overline{F}_\eta(bx)/2 \qquad (3.3.8)\end{aligned}$$

for some $b > 1$ and x large enough because

$$\limsup_{u\to\infty}\sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left(\xi_\kappa\,\mathbb{1}_{\{\xi_k\geqslant u\}}\right)$$

$$
\begin{aligned}
&= \limsup_{u\to\infty} \max\left\{\max_{n<\kappa}\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left(\xi_k \mathbb{1}_{\{\xi_k\geqslant u\}}\right),\ \sup_{n\geqslant\kappa}\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left(\xi_k \mathbb{1}_{\{\xi_k\geqslant u\}}\right)\right\}\\
&= \limsup_{u\to\infty}\sup_{n\geqslant\kappa}\frac{1}{n}\sum_{k=1}^{n}\left(u\overline{F}_{\xi_k}(u)+\int_u^\infty \overline{F}_{\xi_k}(v)\mathrm{d}v\right)\\
&\leqslant \limsup_{u\to\infty} u\overline{F}_{\xi_\kappa}(u)\sup_{n\geqslant\kappa}\frac{1}{n\overline{F}_{\xi_\kappa}(u)}\sum_{k=1}^{n}\overline{F}_{\xi_k}(u)\\
&+ \limsup_{u\to\infty}\int_u^\infty \overline{F}_{\xi_\kappa}(v)\sup_{n\geqslant\kappa}\frac{1}{n\overline{F}_{\xi_\kappa}(v)}\sum_{k=1}^{n}\overline{F}_{\xi_k}(v)\mathrm{d}v\\
&\leqslant c_{19}\left(\limsup_{u\to\infty} u\overline{F}_{\xi_\kappa}(u)+\limsup_{u\to\infty}\int_u^\infty \overline{F}_{\xi_\kappa}(v)\mathrm{d}v\right)=0
\end{aligned}
$$

by the conditions of the theorem and bound (3.3.3).

From (3.3.2) and (3.3.8) we get

$$
\begin{aligned}
\overline{F}_{S_\eta}(x) &\geqslant \max\left\{\overline{F}_{\xi_\kappa}(x)\mathbb{P}(\eta=\kappa),1/2\ \overline{F}_\eta(bx)\right\}\\
&\geqslant \min\left\{\mathbb{P}(\eta=\kappa),1/2\right\}\max\left\{\overline{F}_{\xi_\kappa}(bx),\overline{F}_\eta(bx)\right\}. \qquad (3.3.9)
\end{aligned}
$$

Therefore, using (3.3.9) together with (3.3.7) we obtain

$$
\begin{aligned}
&\limsup_{x\to\infty}\frac{\overline{F}_{S_\eta}(x/2)}{\overline{F}_{S_\eta}(x)}\\
&\leqslant \frac{(c_{17}\kappa+c_{18}\mathbb{E}\eta+1)}{\min\left\{\mathbb{P}(\eta=\kappa),1/2\right\}}\limsup_{x\to\infty}\frac{\max\left\{\overline{F}_{\xi_\kappa}(ax/2),\overline{F}_\eta(ax/2)\right\}}{\max\left\{\overline{F}_{\xi_\kappa}(bx),\overline{F}_\eta(bx)\right\}}
\end{aligned}
$$

for some $0<a<1$ and $b>1$.

Thus, $F_{S_\eta}\in\mathcal{D}$, which proves the sufficiency of the conditions of the theorem.

Let now the d.f. F_{S_η} belongs to the class $\mathcal{D}$. By (3.3.9), we obtain

$$
\max\left\{\overline{F}_{\xi_\kappa}(x/2),\overline{F}_\eta(x/2)\right\}\leqslant\frac{\mathbb{P}(S_\eta>x/(2b))}{\min\left\{\mathbb{P}(\eta=\kappa),1/2\right\}}
$$

for some $b>1$ and x large enough.

Moreover, inequality (3.3.7) yields

$$
\max\left\{\overline{F}_{\xi_\kappa}(x),\overline{F}_\eta(x)\right\}\geqslant\frac{\overline{F}_{S_\eta}(x/a)}{(c_{17}\kappa+c_{18}\mathbb{E}\eta+1)}
$$

for some $0 < a < 1$ and x large enough.

Combining the last two inequalities we get

$$\limsup_{x\to\infty} \frac{\max\left\{\overline{F}_{\xi_\kappa}(x/2), \overline{F}_\eta(x/2)\right\}}{\max\left\{\overline{F}_{\xi_\kappa}(x), \overline{F}_\eta(x)\right\}} \leqslant c_{20} \limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(ax/(2b))}{\overline{F}_{S_\eta}(x)} < \infty,$$

where $c_{20} > 0$.

The necessity of the conditions of the theorem follows immediately from the last inequality, and the proof is complete. □

Chapter 4

Randomly stopped sums for exponential-type distributions

4.1 Main results

Now we formulate three theorems, which describe conditions under which the d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. In Theorem 4.1.1, the case of a finitely supported counting r.v. η is considered, whereas conditions of Theorems 4.1.2 and 4.1.3 imply that the right tail of η is unbounded. We suppose that the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are non-negative in Theorems 4.1.1 and 4.1.3, whereas they can be real-valued in Theorem 4.1.2.

If $\gamma > 0$, then the results presented in this chapter are new. If $\gamma = 0$, then all the assertions below can be derived from the theorems proved in [47]. For the sake of completeness, we include the case $\gamma = 0$ in our assumptions. Moreover, we apply the same methods to prove our results for $\gamma > 0$ and $\gamma = 0$.

Theorem 4.1.1. [18] *Let $n \geqslant 1$ and $\{\xi_1, \xi_2, \ldots, \xi_n\}$ be a collection of independent non-negative r.v.'s with d.f.'s $\{F_{\xi_1}, F_{\xi_2}, \ldots, F_{\xi_n}\}$, and let η be a counting r.v., which is independent of $\{\xi_1, \xi_2, \ldots, \xi_n\}$ and has a finite support* $\mathrm{supp}(\eta) \subseteq \{0, 1, \ldots, n\}$. *Then the d.f. of the randomly stopped sum F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if $F_{\xi_\nu} \in \mathcal{L}(\gamma)$ for some $1 \leqslant \nu \leqslant \min\Big\{\mathrm{supp}(\eta) \setminus \{0\}\Big\}$ and either $F_{\xi_k} \in \mathcal{L}(\gamma)$ or $\overline{F}_{\xi_k}(x) = o\Big(\overline{F}_{\xi_\nu}(x)\Big)$ for each $k \in \Big\{1, 2, \ldots, \max\{\mathrm{supp}(\eta)\}\Big\}$.*

Theorem 4.1.2. [18] *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.'s with d.f.'s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ such that*

$$\sup_{k\geqslant 1}\left|\frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y}\right| \underset{x\to\infty}{\longrightarrow} 0 \tag{4.1.1}$$

for some $\gamma \geqslant 0$ and any fixed $y \geqslant 0$. In addition, let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ and such that

$$\frac{\mathbb{P}(\eta = k+1)}{\mathbb{P}(\eta = k)} \underset{k\to\infty}{\longrightarrow} 0. \tag{4.1.2}$$

Then $F_{S_\eta} \in \mathcal{L}(\gamma)$.

Here we note that condition (4.1.1) is equivalent to the two-sided bound

$$\mathrm{e}^{-\gamma y} \leqslant \liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} \leqslant \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} \leqslant \mathrm{e}^{-\gamma y},$$

which holds for some $\gamma \geqslant 0$ and any fixed $y \geqslant 0$.

Moreover, we observe that condition (4.1.2) implies that $\mathbb{P}(\eta = k) > 0$ for all k large enough.

Theorem 4.1.3. [18] *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent non-negative r.v.'s with d.f.'s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. The d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if there are $\varkappa \geqslant 1$ and $1 \leqslant \nu \leqslant \varkappa$ such that*

(i) $\nu \leqslant \min\left\{\mathrm{supp}(\eta) \setminus \{0\}\right\}$;

(ii) $F_{\xi_\nu} \in \mathcal{L}(\gamma)$;

(iii) *for all* $1 \leqslant k \leqslant \varkappa$, $F_{\xi_k} \in \mathcal{L}(\gamma)$ *or* $\overline{F}_{\xi_k}(x) = o\left(\overline{F}_{\xi_\nu}(x)\right)$;

(iv) *for all* $y \geqslant 0$,

$$\sup_{k\geqslant \varkappa+1}\left|\frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y}\right| \underset{x\to\infty}{\longrightarrow} 0;$$

(v) $\mathbb{P}(\eta = k+1)/\mathbb{P}(\eta = k) \underset{k\to\infty}{\longrightarrow} 0.$

Next, we give two examples, which illustrate some applications of our theorems. In both of these examples, we construct randomly stopped sums belonging to the class of exponential distributions.

Example 4.1.1. *Suppose we have a three-seasonal sequence of independent Erlang r.v.'s with d.f.'s from the class $\mathcal{L}(2)$, i.e. let*

$$F_{\xi_k}(x) = \begin{cases} \Big(1-\mathrm{e}^{-2x}(1+2x)\Big)\mathbb{1}_{[0,\infty)}(x) & if \quad k \equiv 1 \bmod 3, \\ \Big(1-\mathrm{e}^{-2x}(1+2x+2x^2)\Big)\mathbb{1}_{[0,\infty)}(x) & if \quad k \equiv 2 \bmod 3, \\ \Big(1-\mathrm{e}^{-2x}(1+2x+2x^2+4x^3/3)\Big)\mathbb{1}_{[0,\infty)}(x) & if \quad k \equiv 0 \bmod 3. \end{cases}$$

In addition, we assume that the counting r.v. η is independent of $\{\xi_1, \xi_2, \ldots\}$ and distributed according to the Poisson law with an arbitrary positive parameter λ.

In this example, it is clear that

$$\begin{aligned} &\sup_{k\geqslant 1}\left|\frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-2y}\right| \\ = \quad &\max\left\{\left|\frac{\overline{F}_{\xi_1}(x+y)}{\overline{F}_{\xi_1}(x)} - \mathrm{e}^{-2y}\right|, \left|\frac{\overline{F}_{\xi_2}(x+y)}{\overline{F}_{\xi_2}(x)} - \mathrm{e}^{-2y}\right|, \right. \\ &\qquad\qquad \left.\left|\frac{\overline{F}_{\xi_3}(x+y)}{\overline{F}_{\xi_3}(x)} - \mathrm{e}^{-2y}\right|\right\} \underset{x\to\infty}{\to} 0 \end{aligned}$$

and

$$\mathbb{P}(\eta = k+1)/\mathbb{P}(\eta = k) = \frac{\lambda}{k+1} \underset{k\to\infty}{\to} 0.$$

It can be easily seen that all the conditions of Theorem 4.1.2 are satisfied. Consequently, $F_{S_\eta} \in \mathcal{L}(2)$.

Example 4.1.2. *Suppose that $\{\xi_1, \xi_2, \ldots\}$ is a sequence of non-negative r.v.'s such that*

$$\begin{aligned} &\overline{F}_{\xi_1}(x) = \mathrm{e}^{-x}, \ x \geqslant 0, \\ &\overline{F}_{\xi_k}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y, \ x \geqslant 0, \ k \in \{2, 3, \ldots, 10\}, \\ &\overline{F}_{\xi_k}(x) = \mathrm{e}^{-x}\Big(1 + x/(k-10)\Big), \ x \geqslant 0, \ k \in \{11, 12, \ldots\}. \end{aligned}$$

In addition, let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ and distributed according to the following law:

$$\mathbb{P}(\eta = k) = \frac{1}{\hat{c}}\,\mathrm{e}^{-k^2}, \ k \in \{0, 1, 2, \ldots\},$$

where

$$\hat{c} = \sum_{k=0}^{\infty} \mathrm{e}^{-k^2} \approx 1.3863.$$

The sequence $\{\xi_1, \xi_2, \ldots\}$ and the counting r.v. η satisfy conditions of Theorem 4.1.3 with $\gamma = 1$, $\nu = 1$ and $\varkappa = 10$ because:

$$\begin{aligned}
&F_{\xi_1} \in \mathcal{L}(1),\\
&\mathrm{supp}(\eta) \setminus \{0\} = \mathbb{N},\\
&\overline{F}_{\xi_k}(x) = o\Big(\overline{F}_{\xi_1}(x)\Big) \text{ if } k \in \{2, 3, \ldots, 10\},\\
&\mathbb{P}(\eta = k+1)/\mathbb{P}(\eta = k) = \mathrm{e}^{-2k-1},\ k \in \mathbb{N},
\end{aligned}$$

and

$$\left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-y} \right| = \frac{y\mathrm{e}^{-y}}{x+k-10}$$

for all $k \geqslant 11$, $x > 0$ and $y \geqslant 0$.

Consequently, the d.f. F_{S_η} belongs to the class $\mathcal{L}(1)$ by this theorem.

4.2 Auxiliary results for Theorems 4.1.1-4.1.3

In this section, we give all auxiliary assertions, which we use in the proofs of our main results. The first lemma is proved by Embrechts and Goldie (see [26, Theorem 3]).

Lemma 4.2.1. *Let F and G be two d.f.'s, and let F belong to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Then convolution $F * G$ belongs to the class $\mathcal{L}(\gamma)$ if one of the following conditions holds:*

(i) *the d.f. G belongs to the class $\mathcal{L}(\gamma)$;*

(ii) $\overline{G}(x) = o\Big(\overline{F}(x)\Big)$.

The next lemma is the inhomogeneous case of the upper bound given in the proof of Proposition 6.1 from [63].

Lemma 4.2.2. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent real-valued r.v.'s with d.f.'s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ such that*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+a)}{\overline{F}_{\xi_k}(x)} \leqslant \mathrm{e}^{-\gamma a} \tag{4.2.1}$$

for some $\gamma \geqslant 0$ *and* $a > 0$. *Then for any* $\varepsilon \in (0,1)$, *there is* $b = b(a,\varepsilon) > 0$ *such that*

$$\overline{F}_{S_{n+1}}(x+a) \leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma a}\,\overline{F}_{S_{n+1}}(x) + \overline{F}_{S_n}(x-b)$$

for all $x \in \mathbb{R}$ *and* $n \geqslant 1$.

Proof of Lemma 4.2.2. For all $x \in \mathbb{R}$ and $b > 0$, we have

$$\begin{aligned}
\overline{F}_{S_{n+1}}(x) &= \mathbb{P}(S_{n+1} > x) = \mathbb{P}(S_n + \xi_{n+1} > x)\\
&= \int\limits_{(-\infty,\,x-b]} \overline{F}_{\xi_{n+1}}(x-y)\mathrm{d}F_{S_n}(y) + \int\limits_{(x-b,\,\infty)} \overline{F}_{\xi_{n+1}}(x-y)\mathrm{d}F_{S_n}(y)\\
&:= \mathcal{J}_1(x,b) + \mathcal{J}_2(x,b). \qquad (4.2.2)
\end{aligned}$$

Condition (4.2.1) implies that

$$\sup_{n\geqslant 1} \frac{\overline{F}_{\xi_{n+1}}(x-y+a)}{\overline{F}_{\xi_{n+1}}(x-y)} \leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma a}$$

for any fixed $\varepsilon \in (0,1)$ if $y \leqslant x+a-b$, i.e. $x-y \geqslant b-a$, and b is large enough. For such values of b, we get

$$\begin{aligned}
\mathcal{J}_1(x+a,b) &= \int\limits_{(-\infty,\,x+a-b]} \overline{F}_{\xi_{n+1}}(x+a-y)\,\mathrm{d}F_{S_n}(y)\\
&= \int\limits_{(-\infty,\,x+a-b]} \frac{\overline{F}_{\xi_{n+1}}(x+a-y)}{\overline{F}_{\xi_{n+1}}(x-y)}\overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y)\\
&\leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma a} \int\limits_{(-\infty,\,x-b]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y)\\
&+ (1+\varepsilon)\,\mathrm{e}^{-\gamma a} \int\limits_{(x-b,\,x+a-b]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y)\\
&\leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma a}\Big(\mathcal{J}_1(x,b) + \mathcal{J}_2(x,b)\Big)\\
&= (1+\varepsilon)\,\mathrm{e}^{-\gamma a}\overline{F}_{S_{n+1}}(x).
\end{aligned}$$

Furthermore, it is obvious that

$$\begin{aligned}
\mathcal{J}_2(x+a,b) &\leqslant \int\limits_{(x+a-b,\,\infty)} \mathrm{d}F_{S_n}(y)\\
&\leqslant \overline{F}_{S_n}(x-b).
\end{aligned}$$

Therefore, for any $\varepsilon \in (0,1)$ and $b = b(a,\varepsilon)$ large enough, we obtain

$$\begin{aligned}\overline{F}_{S_{n+1}}(x+a) &= \mathcal{J}_1(x+a,b) + \mathcal{J}_2(x+a,b) \\ &\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma a}\,\overline{F}_{S_{n+1}}(x) + \overline{F}_{S_n}(x-b),\end{aligned}$$

which proves the lemma. □

The next lemma gives a lower bound for $\overline{F}_{S_{n+1}}(x+a)$ in the case of non-identical d.f.'s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$.

Lemma 4.2.3. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent real-valued r.v.'s with d.f.'s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ such that*

$$\liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+a)}{\overline{F}_{\xi_k}(x)} \geqslant \mathrm{e}^{-\gamma a} \tag{4.2.3}$$

for some $\gamma \geqslant 0$ and $a > 0$. Then for any $\varepsilon \in (0, 1/2)$, there is $\hat{b} = \hat{b}(a,\varepsilon) > 0$ such that

$$\overline{F}_{S_{n+1}}(x+a) \geqslant (1-\varepsilon)\,\mathrm{e}^{-\gamma a}\,\overline{F}_{S_{n+1}}(x) - \overline{F}_{S_n}(x-\hat{b})$$

for all $x \in \mathbb{R}$ and $n \geqslant 1$.

Proof of Lemma 4.2.3. By representation (4.2.2), we obtain

$$\overline{F}_{S_{n+1}}(x) = \mathcal{J}_1(x,\hat{b}) + \mathcal{J}_2(x,\hat{b})$$

for all $x \in \mathbb{R}$ and $\hat{b} > 0$.

Next, by (4.2.3), for any fixed $\varepsilon \in (0, 1/2)$, we have

$$\inf_{n\geqslant 1} \frac{\overline{F}_{\xi_{n+1}}(x-y+a)}{\overline{F}_{\xi_{n+1}}(x-y)} \geqslant (1-\varepsilon)\,\mathrm{e}^{-\gamma a}$$

for all $y \leqslant x + a - \hat{b}$ and $\hat{b} = \hat{b}(a,\varepsilon)$ large enough.

Applying arguments similar to those in the proof of Lemma 4.2.2 we get

$$\begin{aligned}\mathcal{J}_1(x+a,\hat{b}) &= \int\limits_{(-\infty,\,x+a-\hat{b}]} \frac{\overline{F}_{\xi_{n+1}}(x+a-y)}{\overline{F}_{\xi_{n+1}}(x-y)} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\ &\geqslant (1-\varepsilon)\,\mathrm{e}^{-\gamma a} \int\limits_{(-\infty,\,x+a-\hat{b}]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\ &\geqslant (1-\varepsilon)\,\mathrm{e}^{-\gamma a} \int\limits_{(-\infty,\,x-\hat{b}]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\ &= (1-\varepsilon)\,\mathrm{e}^{-\gamma a}\mathcal{J}_1(x,\hat{b})\end{aligned}$$

for such values of $\hat{b}$. Therefore,

$$\begin{aligned}\overline{F}_{S_{n+1}}(x+a) &\geqslant \mathcal{J}_1(x+a,\hat{b})\\ &\geqslant (1-\varepsilon)\,\mathrm{e}^{-\gamma a}\mathcal{J}_1(x,\hat{b})\\ &= (1-\varepsilon)\,\mathrm{e}^{-\gamma a}(\mathcal{J}_1(x,\hat{b})+\mathcal{J}_2(x,\hat{b}))-(1-\varepsilon)\,\mathrm{e}^{-\gamma a}\mathcal{J}_2(x,\hat{b})\\ &\geqslant (1-\varepsilon)\,\mathrm{e}^{-\gamma a}\,\overline{F}_{S_{n+1}}(x)-\overline{F}_{S_n}(x-\hat{b}),\end{aligned}$$

and the assertion of the lemma follows. □

The last auxiliary assertion is a mild generalization of one of Braverman's lemmas (see [10, Lemma 1]).

Lemma 4.2.4. *Let $\{\xi_1,\xi_2,\ldots\}$ be independent r.v.'s, and let $F_{\xi_1}\in\mathcal{L}(\gamma)$ for some $\gamma\geqslant 0$. Then for any $a>0$ there is a constant c_a such that*

$$\mathbb{P}(S_n>x-a)\leqslant c_a\mathbb{P}(S_n>x)$$

for all $n\in\mathbb{N}$ and $x\in\mathbb{R}$.

Proof of Lemma 4.2.4. The definition of the class $\mathcal{L}(\gamma)$ implies that

$$\begin{aligned}\overline{F}_{S_1}(x-a) &= \overline{F}_{\xi_1}(x-a)\\ &\leqslant 2\mathrm{e}^{\gamma a}\overline{F}_{\xi_1}(x)\end{aligned}$$

if $x>x_a$ and x_a is large enough.

If $x\leqslant x_a$, then it is evident that

$$\frac{\overline{F}_{S_1}(x-a)}{\overline{F}_{S_1}(x)} = \frac{\overline{F}_{\xi_1}(x-a)}{\overline{F}_{\xi_1}(x)}\leqslant\frac{1}{\overline{F}_{\xi_1}(x_a)}.$$

Consequently,

$$\sup_{x\in\mathbb{R}}\frac{\overline{F}_{S_1}(x-a)}{\overline{F}_{S_1}(x)}\leqslant\max\left\{2\mathrm{e}^{\gamma a},\frac{1}{\overline{F}_{\xi_1}(x_a)}\right\}:=c_a.$$

If $n \geqslant 2$, then for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
\overline{F}_{S_n}(x-a) &= \mathbb{P}(\xi_1+\xi_2+\ldots+\xi_n > x-a)\\
&:= \mathbb{P}(\xi_1+S_{2,n} > x-a)\\
&= \int\limits_{-\infty}^{\infty} \mathbb{P}(\xi_1 > x-a-y)\,\mathrm{d}\mathbb{P}(S_{2,n} \leqslant y)\\
&= \int\limits_{-\infty}^{\infty} \frac{\overline{F}_{\xi_1}(x-y-a)}{\overline{F}_{\xi_1}(x-y)}\overline{F}_{\xi_1}(x-y)\,\mathrm{d}\mathbb{P}(S_{2,n} \leqslant y)\\
&\leqslant \sup_{z\in\mathbb{R}} \frac{\overline{F}_{\xi_1}(z-a)}{\overline{F}_{\xi_1}(z)} \int\limits_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y)\,\mathrm{d}\mathbb{P}(S_{2,n} \leqslant y)\\
&\leqslant c_a \int\limits_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y)\,\mathrm{d}\mathbb{P}(S_{2,n} \leqslant y)\\
&\leqslant c_a\mathbb{P}(\xi_1+S_{2,n} > x)\\
&= c_a\mathbb{P}(S_n > x),
\end{aligned}$$

which completes the proof of the lemma. □

4.3 Proofs of Theorems 4.1.1-4.1.3

In the proofs below, we mainly use approaches from [19, 40, 63].

Proof of Theorem 4.1.1. For all $x > 0$, we have

$$\overline{F}_{S_\eta}(x) = \sum_{k\in\mathrm{supp}(\eta)} \mathbb{P}(\eta = k)\overline{F}_{S_k}(x).$$

Since the support $\mathrm{supp}(\eta)$ is finite, for any $y > 0$, we get

$$\min_{k\in\mathrm{supp}(\eta)} \left\{\frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)}\right\} \leqslant \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant \max_{k\in\mathrm{supp}(\eta)} \left\{\frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)}\right\}$$

by the two-sided bound

$$\begin{aligned}
\min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_m}{b_m}\right\} &\leqslant \frac{a_1+a_2+\ldots+a_m}{b_1+b_2+\ldots+b_m}\\
&\leqslant \max\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_m}{b_m}\right\}, \qquad (4.3.1)
\end{aligned}$$

which holds for $a_i \geqslant 0$, $b_i > 0$ and $m \in \mathbb{N}$, where $i \in \{1, 2, \ldots, m\}$.

If $k \in \operatorname{supp}(\eta)$, then

$$S_k = \sum_{i \in \mathcal{K}} \xi_i + \sum_{i \notin \mathcal{K}} \xi_i,$$

where $\mathcal{K} = \{1 \leqslant i \leqslant k : F_{\xi_i} \in \mathcal{L}(\gamma)\}$.

Since $F_{\xi_\nu} \in \mathcal{L}(\gamma)$ for $1 \leqslant \nu \leqslant \min\{\operatorname{supp}(\eta) \setminus \{0\}\}$, the set of indices $\mathcal{K}$ is not empty. Lemma 4.2.1 implies that the d.f. $F_{\mathcal{K}}$ of the sum $\sum_{i \in \mathcal{K}} \xi_i$ belongs to the class $\mathcal{L}(\gamma)$.

Next, if $i^* \notin \mathcal{K}$, then $\overline{F}_{\xi_{i^*}}(x) = o\big(\overline{F}_{\xi_\nu}(x)\big)$ by the conditions of the theorem. Therefore,

$$\frac{\overline{F}_{\xi_{i^*}}(x)}{\overline{F}_{\mathcal{K}}(x)} = \frac{\mathbb{P}(\xi_{i^*} > x)}{\mathbb{P}\Big(\sum_{i \in \mathcal{K}} \xi_i > x\Big)} \leqslant \frac{\overline{F}_{\xi_{i^*}}(x)}{\overline{F}_{\xi_\nu}(x)} \underset{x \to \infty}{\longrightarrow} 0,$$

and consequently, $F_{\mathcal{K}} * F_{\xi_{i^*}}$ belongs to the class $\mathcal{L}(\gamma)$ by part (ii) of Lemma 4.2.1. Continuing our considerations we conclude that the d.f.

$$F_{S_k} = F_{\mathcal{K}} * \Big\{ \underset{i \notin \mathcal{K}}{\circledast} F_{\xi_i} \Big\}$$

belongs to the class $\mathcal{L}(\gamma)$ as well for any index $k \in \operatorname{supp}(\eta)$. Here $\underset{i \notin \mathcal{K}}{\circledast} F_{\xi_i}$ denotes the d.f. of the sum $\sum_{i \notin \mathcal{K}} \xi_i$.

Consequently, applying the two-sided bound (4.3.1) yields the inequalities

$$\limsup_{x \to \infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant \max_{k \in \operatorname{supp}(\eta)} \left\{ \limsup_{x \to \infty} \frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)} \right\} = \mathrm{e}^{-\gamma y}$$

and

$$\liminf_{x \to \infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \geqslant \min_{k \in \operatorname{supp}(\eta)} \left\{ \liminf_{x \to \infty} \frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)} \right\} = \mathrm{e}^{-\gamma y}$$

for all $y > 0$, and the proof of the theorem is complete. □

Proof of Theorem 4.1.2. In order to show that $F_{S_\eta} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$, it is sufficient to establish the inequalities

$$\limsup_{x \to \infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant \mathrm{e}^{-\gamma y} \tag{4.3.2}$$

and

$$\liminf_{x \to \infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \geqslant \mathrm{e}^{-\gamma y} \tag{4.3.3}$$

for all $y > 0$.

(I) First, we show inequality (4.3.2). To this end, we suppose that $y > 0$ is an arbitrary number and choose $\varepsilon \in (0,1)$. By condition (4.1.2), we have

$$\mathbb{P}(\eta = n+1) \leqslant \varepsilon\, \mathbb{P}(\eta = n) \tag{4.3.4}$$

for all $n \geqslant N = N(\varepsilon) \geqslant 2$. For such values of N, we get

$$\begin{aligned} \overline{F}_{S_\eta}(x+y) &= \sum_{n=1}^{N} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x+y) \\ &+ \sum_{n=N+1}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x+y). \end{aligned} \tag{4.3.5}$$

Applying Lemma 4.2.2 we obtain

$$\begin{aligned} \sum_{n=N+1}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x+y) &\leqslant \sum_{n=N+1}^{\infty} (1+\varepsilon)\, \mathrm{e}^{-\gamma y}\mathbb{P}(\eta = n)\overline{F}_{S_n}(x) \\ &+ \sum_{n=N+1}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_{n-1}}(x-b) \end{aligned}$$

for some $b = b(y,\varepsilon) > 0$. This relation together with inequality (4.3.4) shows that

$$\begin{aligned} \overline{F}_{S_\eta}(x+y) &\leqslant \sum_{n=1}^{N} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x+y) \\ &+ (1+\varepsilon)\,\mathrm{e}^{-\gamma y} \sum_{n=N+1}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x) \\ &+ \varepsilon \sum_{n=N+1}^{\infty} \mathbb{P}(\eta = n-1)\overline{F}_{S_{n-1}}(x-b). \end{aligned} \tag{4.3.6}$$

Condition (4.1.1) implies that

$$\mathrm{e}^{-\gamma u} \leqslant \liminf_{x\to\infty} \frac{\overline{F}_{\xi_k}(x+u)}{\overline{F}_{\xi_k}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(x+u)}{\overline{F}_{\xi_k}(x)} \leqslant \mathrm{e}^{-\gamma u}$$

for all fixed k and u. From this it follows that $\overline{F}_{\xi_k} \in \mathcal{L}(\gamma)$ for all k. Hence, by Lemma 4.2.1, we conclude that $\overline{F}_{S_n} \in \mathcal{L}(\gamma)$ for all fixed $n \in \mathbb{N}$. Therefore,

$$\max_{1\leqslant n\leqslant N} \frac{\overline{F}_{S_n}(x+y)}{\overline{F}_{S_n}(x)} \leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma y} \tag{4.3.7}$$

for all $x \geqslant \hat{x} = \hat{x}(N, y, \varepsilon)$. Thus, for some pair $N \in \mathbb{N}$ and $b > 0$, chosen in a special way, and all $x \geqslant \hat{x}$, we have

$$\begin{aligned}
\overline{F}_{S_\eta}(x+y) &\leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma y} \sum_{n=1}^{N} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x) \\
&+ (1+\varepsilon)\,\mathrm{e}^{-\gamma y} \sum_{n=N+1}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x) \\
&+ \varepsilon \sum_{n=N}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x-b) \\
&= (1+\varepsilon)\,\mathrm{e}^{-\gamma y}\overline{F}_{S_\eta}(x) \\
&+ \varepsilon \sum_{n=N}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x-b).
\end{aligned}$$

By Lemma 4.2.4, $\overline{F}_{S_n}(x-b) \leqslant c_{21}\overline{F}_{S_n}(x)$ for some constant $c_{21} = c_{21}(b(y,\varepsilon)) > 0$. Therefore,

$$\begin{aligned}
\overline{F}_{S_\eta}(x+y) &\leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma y}\overline{F}_{S_\eta}(x) + \varepsilon c_{21} \sum_{n=N}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x) \\
&\leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma y}\overline{F}_{S_\eta}(x) + \varepsilon c_{21}\ \overline{F}_{S_\eta}(x)
\end{aligned}$$

for all x large enough.

The last inequality implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma y} + \varepsilon\ c_{21}.$$

Since $\varepsilon \in (0,1)$ and $y > 0$ are arbitrary, the desired inequality (4.3.2) holds for all $y > 0$.

(II) In this part, we show inequality (4.3.3). We fix $y > 0$ and choose $\varepsilon \in (0, 1/2)$. Let N be a natural number for which inequality (4.3.4) holds. By Lemma 4.2.3, we have

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x+y) &\geqslant \sum_{n=N+1}^{\infty} (1-\varepsilon)\,\mathrm{e}^{-\gamma y}\mathbb{P}(\eta = n)\overline{F}_{S_n}(x) \\
&- \sum_{n=N+1}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_{n-1}}(x-\hat{b}) \qquad (4.3.8)
\end{aligned}$$

for some $\hat{b} = \hat{b}(y,\varepsilon) > 0$. Substituting (4.3.4) and (4.3.8) into equality (4.3.5) we get

$$\begin{aligned}\overline{F}_{S_\eta}(x+y) &\geqslant \sum_{n=1}^{N} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x+y)\\ &+ (1-\varepsilon)\,\mathrm{e}^{-\gamma y} \sum_{n=N+1}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x)\\ &- \varepsilon \sum_{n=N+1}^{\infty} \mathbb{P}(\eta = n-1)\overline{F}_{S_{n-1}}(x-\hat{b}).\end{aligned} \tag{4.3.9}$$

Since $\overline{F}_{\xi_k} \in \mathcal{L}(\gamma)$ for all fixed k, applying Lemma 4.2.1 we obtain $F_{S_n} \in \mathcal{L}(\gamma)$ for each fixed $n \in \mathbb{N}$. Applying arguments similar to those for deriving (4.3.7) we get

$$\min_{1\leqslant n\leqslant N} \frac{\overline{F}_{S_n}(x+y)}{\overline{F}_{S_n}(x)} \geqslant (1-\varepsilon)\,\mathrm{e}^{-\gamma y} \tag{4.3.10}$$

for all $x \geqslant \tilde{x} = \tilde{x}(N,y,\varepsilon)$.

Therefore, by (4.3.9) and (4.3.10), for all $x \geqslant \tilde{x}$, we have

$$\begin{aligned}\overline{F}_{S_\eta}(x+y) &\geqslant (1-\varepsilon)\,\mathrm{e}^{-\gamma y}\overline{F}_{S_\eta}(x)\\ &- \varepsilon \sum_{n=N}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x-\hat{b}).\end{aligned}$$

Next, from Lemma 4.2.4 we deduce that $\overline{F}_{S_n}(x-\hat{b}) \leqslant c_{22}\overline{F}_{S_n}(x)$ for some constant $c_{22} = c_{22}(\hat{b}(y,\varepsilon)) > 0$. Therefore,

$$\overline{F}_{S_\eta}(x+y) \geqslant (1-\varepsilon)\,\mathrm{e}^{-\gamma y}\overline{F}_{S_\eta}(x) - \varepsilon\, c_{22}\overline{F}_{S_\eta}(x)$$

for all $x \geqslant \tilde{x}$. This last inequality implies that

$$\liminf_{x\to\infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \geqslant (1-\varepsilon)\,\mathrm{e}^{-\gamma y} - \varepsilon\, c_{22}$$

for any $\varepsilon \in (0,1/2)$. Letting $\varepsilon \to 0$ in the last inequality we get the desired inequality (4.3.3), which completes the proof of the theorem. □

Proof of Theorem 4.1.3. If $\varkappa = 1$, then the assertion of the theorem follows immediately from Theorem 4.1.2. So we suppose that $\varkappa \geqslant 2$ and break down our proof into two parts.

(I) If $\mathbb{P}(\eta \leqslant \varkappa) = 0$, then the r.v. η has an infinite support

$$\operatorname{supp}(\eta) \subset \{\varkappa + 1, \varkappa + 2, \ldots\}.$$

Conditions (i)-(iii) of the theorem imply that $F_{S_\varkappa} \in \mathcal{L}(\gamma)$ by Theorem 4.1.1. Since $F_{\xi_{\varkappa+1}} \in \mathcal{L}(\gamma)$ by condition (iv), the convolution $F_{S_{\varkappa+1}} = F_{S_\varkappa} * F_{\xi_{\varkappa+1}}$ belongs to the class $\mathcal{L}(\gamma)$ as well by Lemma 4.2.1. From this and condition (iv) we conclude that

$$\sup_{k\geqslant 1}\left|\frac{\overline{F}_{\hat{\xi}_k}(x+y)}{\overline{F}_{\hat{\xi}_k}(x)} - \mathrm{e}^{-\gamma y}\right| \underset{x\to\infty}{\to} 0$$

for any fixed $y \geqslant 0$, where $\hat{\xi}_1 = S_{\varkappa+1}$, $\hat{\xi}_2 = \xi_{\varkappa+2}$, $\hat{\xi}_3 = \xi_{\varkappa+3}$, ...

Let $\hat{\eta}$ be the counting r.v. defined by the equality $\mathbb{P}(\hat{\eta} = k) = \mathbb{P}(\eta = \varkappa + k)$, where $k = 1, 2, \ldots$, and let $\hat{S}_n = \hat{\xi}_1 + \hat{\xi}_2 + \ldots + \hat{\xi}_n$ for all $n \geqslant 1$.

The r.v.'s $\{\hat{\xi}_1, \hat{\xi}_2, \ldots\}$ and $\hat{\eta}$ satisfy conditions of Theorem 4.1.2. Hence, $F_{\hat{S}_{\hat{\eta}}} \in \mathcal{L}(\gamma)$. It is easily seen that

$$\begin{aligned}
\overline{F}_{\hat{S}_{\hat{\eta}}}(x) &= \mathbb{P}(\hat{\eta} = 1)\mathbb{P}(\hat{S}_1 > x) + \sum_{k=2}^{\infty}\mathbb{P}(\hat{\eta} = k)\mathbb{P}(\hat{S}_k > x)\\
&= \mathbb{P}(\eta = \varkappa + 1)\mathbb{P}(S_{\varkappa+1} > x) + \sum_{k=2}^{\infty}\mathbb{P}(\eta = \varkappa + k)\mathbb{P}(S_{\varkappa+k} > x)\\
&= \overline{F}_{S_\eta}(x)
\end{aligned}$$

for any $x \geq 0$. Consequently, $F_{S_\eta} \in \mathcal{L}(\gamma)$ as well in the case under consideration.

(II) Let now $\mathbb{P}(\eta \leqslant \varkappa) > 0$. Since $\mathbb{P}(\eta \geqslant \varkappa + 1) > 0$ by condition (v), we have

$$\overline{F}_{S_\eta}(x) = \mathbb{P}(\eta \leqslant \varkappa)\overline{F}_{S_{\tilde{\eta}}}(x) + \mathbb{P}(\eta \geqslant \varkappa + 1)\overline{F}_{S_{\hat{\eta}}}(x) \tag{4.3.11}$$

for any $x \geq 0$, where $\tilde{\eta}$ and $\hat{\eta}$ are two counting r.v.'s, which are independent of $\{\xi_1, \xi_2, \ldots\}$ and have the following distributions:

$$\begin{aligned}
\mathbb{P}(\tilde{\eta} = k) &= \frac{\mathbb{P}(\eta = k)}{\mathbb{P}(\eta \leqslant \varkappa)},\ k \in \{0, 1, \ldots, \varkappa\},\\
\mathbb{P}(\hat{\eta} = k) &= \frac{\mathbb{P}(\eta = k)}{\mathbb{P}(\eta \geqslant \varkappa + 1)},\ k \in \{\varkappa + 1, \varkappa + 2, \ldots\}.
\end{aligned}$$

Theorem 4.1.1 implies that $F_{S_{\tilde{\eta}}} \in \mathcal{L}(\gamma)$ because of the finiteness of the support $\operatorname{supp}(\tilde{\eta})$. Applying arguments similar to those in part (I) we deduce that $F_{S_{\hat{\eta}}} \in \mathcal{L}(\gamma)$ as well. Now the assertion of the theorem follows immediately from equality (4.3.11), which completes the proof. □

Chapter 5

Closure properties of $\mathcal{O}$-exponential distributions

5.1 Main results

In this chapter, we deal with the class $\mathcal{OL}$, which is wider than the class $\mathcal{L}(\gamma)$. All assertions in this section are presented in the chronological order. First, we formulate two assertions for the case of i.i.d. summands. Such two assertions were obtained in paper [20]. We suppose that the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are i.i.d. with d.f. from the class $\mathcal{OL}$ and show that the d.f. of the sum $\xi_1 + \xi_2 + \ldots + \xi_\eta$ remains in the class $\mathcal{OL}$ if the r.v. η satisfies the conditions similar to those in [1, Theorem 3.1].

Theorem 5.1.1. [20] *Let $\{\xi_1, \xi_2, \ldots\}$ be independent copies of a non-negative r.v. ξ with d.f. F_ξ. In addition, let η be a non-negative, integer-valued, non-degenerate at 0 and independent of $\{\xi_1, \xi_2, \ldots\}$ r.v. with d.f. F_η. If $F_\xi \in \mathcal{OL}$ and $\overline{F}_\eta(\delta x) = O\Big(\sqrt{x}\ \overline{F}_\xi(x)\Big)$ for all $\delta \in (0,1)$, then $F_{S_\eta} \in \mathcal{OL}$.*

The assertion of this theorem can be derived from Theorem 5.1.5 or Corollary 7.2.2, but in Section 5.3 we give the original proof of this statement presented in [20]. Note that it is similar to the proof of Theorem 6 in [46].

The following corollary actually shows that Albin's conditions for the counting r.v. η are sufficient for the d.f. F_{S_η} to remain in the class $\mathcal{OL}$. The proof of the corollary below is also given in Section 5.3.

Corollary 5.1.1. [20] *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent non-negative r.v.'s with a common d.f. $F_\xi \in \mathcal{OL}$. Then the following assertions hold:*

(i) The d.f. $\mathbb{P}(\xi_1 + ... + \xi_n \leqslant x)$ belongs to the class $\mathcal{OL}$ for all fixed $n \in \mathbb{N}$.

(ii) If, in addition, η is a non-negative, integer-valued, non-degenerate at 0 and independent of $\{\xi_1, \xi_2, \ldots\}$ r.v. Such that $\mathbb{E}\mathrm{e}^{\varepsilon\eta} < \infty$ for each $\varepsilon > 0$, then $F_{S_\eta} \in \mathcal{OL}$.

Further in this section, we consider independent but not necessarily identically distributed r.v.'s. If $\{\xi_1, \xi_2, \ldots\}$ are possibly non-identically distributed, then different collections of conditions on r.v.'s $\{\xi_1, \xi_2, \ldots\}$ and η imply that $F_{S_\eta} \in \mathcal{OL}$.

We formulate our next main results for not necessarily identically distributed heavy-tailed r.v.'s. The first assertion corresponds to the situation where the tails of the d.f.'s F_{ξ_k} are uniformly comparable with themselves at the points x and $x - 1$ for all $x \in [0, \infty)$ and large indices k.

Theorem 5.1.2. [19] *Let $\{\xi_1, \xi_2, \ldots\}$ be independent non-negative r.v.'s with d.f.'s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then $F_{S_\eta} \in \mathcal{OL}$ if the following three conditions hold:*

- *For some $\kappa \in \mathrm{supp}(\eta) \setminus \{0\}$, the d.f. $F_{\xi_\kappa} \in \mathcal{OL}$;*
- *For any $k \in \mathrm{supp}(\eta)$, $k \leqslant \kappa$, either $\lim\limits_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0$ or $F_{\xi_k} \in \mathcal{OL}$;*
- $\sup\limits_{x\geqslant 0} \sup\limits_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} < \infty$.

Since any d.f. from the class $\mathcal{OL}$ is comparable with itself, the next assertion follows immediately from Theorem 5.1.2.

Corollary 5.1.2. [19] *Let $\{\xi_1, \xi_2, \ldots\}$ be independent non-negative r.v.'s with a common d.f. $F_\xi \in \mathcal{OL}$. Then the d.f. of the random sum F_{S_η} is $\mathcal{O}$-exponential for any counting r.v. η.*

It is easily seen that this corollary generalizes the assertion of Theorem 5.1.1. Nevertheless, we presented both results because of different methods of proofs, which are given in Section 5.3.

Our next assertion deals with the case where the counting r.v. η has a finite support.

Theorem 5.1.3. [19] *Let* $\{\xi_1, \xi_2, \ldots, \xi_D\}$*,* $D \in \mathbb{N}$*, be independent non-negative r.v.'s with d.f.'s* $\{F_{\xi_1}, F_{\xi_2}, \ldots F_{\xi_D}\}$*, and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots, \xi_D\}$*. Then* $F_{S_\eta} \in \mathcal{OL}$ *if the following three conditions hold:*

- $\mathbb{P}(\eta \leqslant D) = 1$*;*
- *For some* $\kappa \in \operatorname{supp}(\eta) \setminus \{0\}$*, the d.f.* $F_{\xi_\kappa} \in \mathcal{OL}$*;*
- *For any* $k \in \{1, 2, \ldots, D\}$*, either* $\lim\limits_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0$ *or* $F_{\xi_k} \in \mathcal{OL}$*.*

The next assertion deals with the case where the tails of the d.f.'s F_{ξ_k} are comparable at x and $x-1$ asymptotically and uniformly with respect to large indices k. In this case, conditions on the counting r.v. η are more restrictive.

Theorem 5.1.4. [19] *Let* $\{\xi_1, \xi_2, \ldots\}$ *be independent non-negative random variables with d.f.'s* $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$*, and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$ *and having a d.f.* F_η*. Then* $F_{S_\eta} \in \mathcal{OL}$ *if the following five conditions hold:*

- *For some* $\kappa \in \operatorname{supp}(\eta) \setminus \{0\}$*, the d.f.* $F_{\xi_\kappa} \in \mathcal{OL}$*;*
- *For any* $k \in \operatorname{supp}(\eta)$*,* $k \leqslant \kappa$*, either* $\lim\limits_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0$ *or* $F_{\xi_k} \in \mathcal{OL}$*;*
- $\limsup\limits_{x\to\infty} \sup\limits_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} < \infty$*;*
- $\limsup\limits_{k\to\infty} \frac{1}{k} \sum\limits_{l=1}^{k} \sup\limits_{x\geqslant 0} \left(\overline{F}_{\xi_{\kappa+l}}(x-1) - \overline{F}_{\xi_{\kappa+l}}(x)\right) < 1$*;*
- *For any* $\delta \in (0,1)$*, it holds* $\overline{F}_\eta(\delta x) = O\left(\sqrt{x}\,\overline{F}_{\xi_\kappa}(x)\right)$*.*

Our last theorem on the randomly stopped sum S_η gives sufficient conditions under which the d.f. of the randomly stopped sum F_{S_η} belongs to the class $\mathcal{OL}$. We note that the assertion below is more general and easily verifiable.

Theorem 5.1.5. [22] *Let* $\{\xi_1, \xi_2, \ldots\}$ *be independent real-valued r.v.'s, and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$*. Then* $F_{S_\eta} \in \mathcal{OL}$ *if* $F_{\xi_\varkappa} \in \mathcal{OL}$ *for* $\varkappa = \min\{n \in \operatorname{supp}(\eta), n \geqslant 1\}$*.*

For the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ and the counting r.v. η, we can construct not only randomly stopped sum S_η, but also the randomly stopped maximum $\xi_{(\eta)}$ and the randomly stopped maximum of sums $S_{(\eta)}$. The following two theorems describe conditions under which the d.f. $F_{\xi_{(\eta)}}$ belongs to the class of $\mathcal{O}$-exponential distributions. In Theorem 5.1.6, the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ satisfy minor requirements, but conditions on the counting r.v. η are more restrictive. In Theorem 5.1.7, the situation is completely different: $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ are uniformly $\mathcal{O}$-exponential d.f.'s of independent r.v.'s, but the counting r.v. η may be arbitrary.

Theorem 5.1.6. [22] *Let* $\{\xi_1, \xi_2, \ldots\}$ *be* (*arbitrarily dependent*) *real-valued r.v.'s, and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *Suppose that there are* $\varkappa \in \operatorname{supp}(\eta)$ *and a positive sequence* $\{\varphi(n)\}_{n=1}^{\infty}$ *such that* $F_{\xi_\varkappa} \in \mathcal{OL}$, $\mathbb{E}\varphi(\eta)\mathbb{1}_{\{\eta \geqslant 1\}} < \infty$ *and*

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{\varphi(n)\overline{F}_{\xi_\varkappa}(x)} \sum_{k=1}^{n} \overline{F}_{\xi_k}(x) < \infty. \tag{5.1.1}$$

Then $F_{\xi_{(\eta)}} \in \mathcal{OL}$.

Theorem 5.1.7. [22] *Let* $\{\xi_1, \xi_2, \ldots\}$ *be independent real-valued r.v.'s, and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *If*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x-1)}{\overline{F}_{\xi_k}(x)} < \infty,$$

then $F_{\xi_{(\eta)}} \in \mathcal{OL}$.

The last two assertions give sufficient conditions under which the d.f. of the randomly stopped maximum of sums belongs to the class $\mathcal{OL}$. Theorem 5.1.8 deals with the case where the r.v. η has an infinite support, whereas Theorem 5.1.9 corresponds to the case of finitely supported counting r.v. η.

Theorem 5.1.8. [22] *Let* $\{\xi_1, \xi_2, \ldots\}$ *be independent real-valued r.v.'s, and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *Then the d.f. of the randomly stopped maximum of random sums* $F_{S_{(\eta)}}$ *belongs to the class* $\mathcal{OL}$ *if* $F_{\xi_1} \in \mathcal{OL}$ *and the r.v.* η *satisfies the following two conditions:*

$$\limsup_{n\to\infty} \frac{\mathbb{P}(\eta > n)}{\min\limits_{1\leqslant k\leqslant n} \mathbb{P}(\eta = k)} < \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{\mathbb{P}(\eta \geqslant n)}{\mathbb{P}(\eta = n)} < \infty. \tag{5.1.2}$$

Theorem 5.1.9. [22] *Let* $\{\xi_1, \ldots, \xi_m\}$ *be independent real-valued r.v.'s with d.f.'s* $\{F_{\xi_1}, \ldots, F_{\xi_m}\}$. *In addition, let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots, \xi_m\}$ *and having a finite support* $\operatorname{supp}(\eta)$ *such that* $\mu := \max\{n \in \operatorname{supp}(\eta)\} \leqslant m$. *Then the d.f. of* $S_{(\eta)}$ *belongs to the class* $\mathcal{OL}$ *if* $F_{\xi_1} \in \mathcal{OL}$.

The following corollary deals with the homogeneous case and follows immediately from Theorems 5.1.5, 5.1.7 and 5.1.8.

Corollary 5.1.3. [22] *Let $\{\xi_1, \xi_2, \ldots\}$ be i.i.d. real-valued r.v.'s with d.f. F_ξ, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If F_ξ is $\mathcal{O}$-exponential, then the d.f.'s F_{S_η} and $F_{\xi_{(\eta)}}$ are $\mathcal{O}$-exponential. If, in addition, the r.v. η satisfies condition* (5.1.2)*, then $F_{S_{(\eta)}}$ is also $\mathcal{O}$-exponential.*

Further, in this section, we give some examples of the random sums S_η with $\mathcal{O}$-exponential d.f.'s F_{S_η}. These examples illustrate the application of Theorems 5.1.2-5.1.5.

Example 5.1.1. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.'s. We suppose that the r.v. ξ_k is distributed according to the Pareto law with parameters k and α, i.e.*

$$\overline{F}_{\xi_k}(x) = \left(\frac{k}{k+x}\right)^{\alpha}, \quad x \geqslant 0,$$

where $k \in \{1, 2, \ldots, D\}$, $D \geqslant 1$ and $\alpha > 0$. In addition, we assume that the r.v. ξ_{D+k}, $k \in \mathbb{N}$, is distributed according to the exponential law with parameter λ/k, i.e.

$$\overline{F}_{\xi_{D+k}}(x) = \mathrm{e}^{-\lambda x/k}, \quad x \geqslant 0.$$

It follows from Theorem 5.1.2 that the d.f. of the random sum S_η is $\mathcal{O}$-exponential for any counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$ and satisfying the condition $\mathbb{P}(\eta = \kappa) > 0$ for some $\kappa \in \{1, 2, \ldots, D\}$ because:

- $F_{\xi_k} \in \mathcal{L} \subset \mathcal{OL}$ for all $k \leqslant \kappa$;
- $\displaystyle \sup_{x \geqslant 0} \sup_{k \geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)}$

$$= \max\left\{ \sup_{0 \leqslant x \leqslant 1} \sup_{k \geqslant 1} \frac{1}{\overline{F}_{\xi_{\kappa+k}}(x)}, \ \sup_{x>1} \sup_{k \geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \right\}$$

$$= \max\left\{ \sup_{0 \leqslant x \leqslant 1} \max\left\{ \max_{1 \leqslant k \leqslant D-\kappa} \left(\frac{\kappa+k+x}{\kappa+k}\right)^{\alpha}, \ \sup_{k \geqslant 1} \mathrm{e}^{\lambda x/k} \right\}, \right.$$

$$\left. \sup_{x>1} \max\left\{ \max_{1 \leqslant k \leqslant D-\kappa} \left(\frac{\kappa+k+x}{\kappa+k+x-1}\right)^{\alpha}, \ \sup_{k \geqslant 1} \mathrm{e}^{\lambda/k} \right\} \right\}$$

$$\leqslant \max\left\{2^{\alpha}, \mathrm{e}^{\lambda}\right\}.$$

Example 5.1.2. *Let* η *be the r.v. uniformly distributed on the set* $\{1, 2, \dots, D\}$, *i.e.*

$$\mathbb{P}(\eta = k) = \frac{1}{D}, \quad k \in \{1, 2, \dots, D\},$$

for some $D \geqslant 2$. *Moreover, let* $\{\xi_1, \xi_2, \dots, \xi_D\}$ *be independent r.v.'s, where* ξ_1 *is exponentially distributed and* $\xi_2, \dots, \xi_D$ *are uniformly distributed.*

If the r.v. η is independent of the r.v.'s $\{\xi_1, \xi_2, \dots, \xi_D\}$, then Theorem 5.1.3 or Theorem 5.1.5 imply that the d.f. of the random sum S_η is $\mathcal{O}$-exponential.

Example 5.1.3. *Let* $\{\xi_1, \xi_2, \dots\}$ *be independent r.v.'s, where* $\{\xi_1, \xi_2, \dots, \xi_{\kappa-1}\}$ *are finitely supported for some* $\kappa \geqslant 2$ *and* ξ_κ *is distributed according to the Weibul law, i.e.*

$$\overline{F}_{\xi_\kappa}(x) = e^{-\sqrt{x}}, \quad x \geqslant 0.$$

In addition, we suppose that the r.v. $\xi_{\kappa+k}$ *has a d.f. with the tail*

$$\overline{F}_{\xi_{\kappa+k}}(x) = \begin{cases} 1 & \text{if } x < 0, \\ \frac{1}{k} & \text{if } 0 \leqslant x < k, \\ \frac{1}{k}\mathrm{e}^{-(x-k)} & \text{if } x \geqslant k, \end{cases}$$

for all $k = m^2$, $m \geqslant 2$, *whereas for all remaining indices* $k \notin \{m^2, m \in \mathbb{N} \setminus \{1\}\}$, *the r.v.* $\xi_{\kappa+k}$ *is exponentially distributed, i.e*

$$\overline{F}_{\xi_{\kappa+k}}(x) = \mathrm{e}^{-x}, \quad x \geqslant 0.$$

If the counting r.v. η is independent of $\{\xi_1, \xi_2, \dots\}$ and distributed according to the Poisson law with parameter λ, then it follows from Theorem 5.1.4 that the random sum S_η is $\mathcal{O}$-exponentially distributed because:

- $F_{\xi_\kappa} \in \mathcal{L} \subset \mathcal{OL}$;
- $\displaystyle\lim_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0 \quad \text{if} \quad k = 1, 2, \ldots, \kappa - 1;$
- $$\sup_{x\geqslant 1}\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)}$$
$$= \sup_{x\geqslant 1} \max\left\{ \sup_{k\geqslant 1,\, k=m^2,\, m\geqslant 2} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)},\ \sup_{k\geqslant 1,\, k\neq m^2} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \right\}$$
$$= \sup_{x\geqslant 1} \max\left\{ \sup_{k\geqslant 1,\, k=m^2,\, m\geqslant 2} \left\{ \mathbb{1}_{[1,k)}(x) + \mathrm{e}^{x-k}\mathbb{1}_{[k,k+1)}(x) + \mathrm{e}\mathbb{1}_{[k+1,\infty)}(x) \right\}, \right.$$
$$\left. \sup_{k\geqslant 1,\, k\neq m^2} \mathrm{e} \right\} = \mathrm{e};$$
- $$\limsup_{k\to\infty} \frac{1}{k} \sum_{l=1}^{k} \sup_{x\geqslant 0} \left(\overline{F}_{\xi_{\kappa+l}}(x-1) - \overline{F}_{\xi_{\kappa+l}}(x) \right)$$
$$= \limsup_{k\to\infty} \frac{1}{k} \left(\sum_{l=1,\, l=m^2}^{k} \left(1 - \frac{1}{l}\right) + \left(1 - \frac{1}{\mathrm{e}}\right) \sum_{l=1,\, l\neq m^2}^{k} 1 \right)$$
$$\leqslant \left(1 - \frac{1}{\mathrm{e}}\right);$$
- $\displaystyle\overline{F}_{\eta}(x) < \left(\frac{\mathrm{e}\lambda}{x}\right)^x, \quad x > \lambda.$

Here the last inequality is the well-known Chernoff bound for the Poisson law (see, for instance, [50, p. 97]).

It is easily seen that the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ from the last example satisfy conditions of Theorem 5.1.4, whereas the third condition of Theorem 5.1.2 does not hold because

$$\sup_{x\geqslant 0}\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \geqslant \sup_{0\leqslant x<1}\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \geqslant \sup_{0\leqslant x<1}\ \sup_{k=m^2, m\geqslant 2} k = \infty$$

in this case.

At the end of this section we present two examples, which illustrate the results of Theorems 5.1.5–5.1.9.

Example 5.1.4. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.'s with shifted exponential distributions defined by*

$$\overline{F}_{\xi_k}(x) = \mathrm{e}^{-(x+k-1)}, \quad x \geqslant -k+1, \quad k \in \mathbb{N}.$$

It is obvious that $F_{\xi_1} \in \mathcal{OL}$ and $\overline{F}_{\xi_k}(x-1)/\overline{F}_{\xi_k}(x) = \mathrm{e}$ for all $x \geqslant 1$ and $k \in \mathbb{N}$. Hence, Theorems 5.1.5 and 5.1.7 imply that the d.f.'s F_{S_η} and $F_{\xi_{(\eta)}}$ belong to the class $\mathcal{OL}$ for an arbitrary counting r.v. η, whereas from the assertion of Theorem 5.1.8 we conclude that the d.f. $F_{S_{(\eta)}}$ belongs to the class $\mathcal{OL}$ for any counting r.v. η satisfying conditions (5.1.2).

Example 5.1.5. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.'s such that ξ_n, $n \neq 3$, are standard normal random variables, whereas the d.f. of ξ_3 has the tail*

$$\overline{F}_{\xi_3}(x) = \mathbb{I}_{(-\infty,-\pi)}(x) + \exp\{-x + \pi - \sin x\}\mathbb{I}_{[-\pi,\infty)}(x),$$

which can be obtained by taking $a(x) = \mathbb{I}_{[-\pi,\infty)}(x)$ and $b(x) = \cos x\, \mathbb{I}_{[-\pi,\infty)}(x)$ in (2.2.1).

In this case, we see that $F_{\xi_3} \in \mathcal{OL}$ and

$$\sup_{n\geqslant 1} \frac{1}{n\,\overline{F}_{\xi_3}(x)} \sum_{k=1}^{n} \overline{F}_{\xi_k}(x) < \frac{1}{3} + 0.002, \quad x \geqslant 3.$$

By Theorem 5.1.6, the d.f. $F_{\xi_{(\eta)}}$ belongs to the class $\mathcal{OL}$ for any counting r.v. η with finite mean $\mathbb{E}\eta$. If the counting r.v. has a support with $\min\{n \in \mathrm{supp}(\eta), n \geqslant 1\} = 3$, then by Theorem 5.1.5, the d.f. F_{S_η} belongs to the class $\mathcal{OL}$. In this case, the conditions of Theorems 5.1.8 and 5.1.9 are not satisfied because $F_{\xi_1} \notin \mathcal{OL}$, and a simple example of the counting r.v. η with the distribution $\mathbb{P}(\eta = 1) = \mathbb{P}(\eta = 2) = 1/2$ shows that the d.f. $F_{S_{(\eta)}}$ cannot be $\mathcal{O}$-exponential.

5.2 Auxiliary results for Theorems 5.1.1-5.1.9

To prove our main results, we give some auxiliary assertions. The first lemma is a well-known classical bound for the concentration function of a sum of i.i.d. r.v.'s. Its proof can be found, for instance, in [53, Theorem 2.22].

Lemma 5.2.1. *Let $X_1, X_2, \ldots$ be a sequence of independent r.v.'s with a common non-degenerate d.f., and let $Z_n = X_1 + X_2 + \ldots + X_n$. Then there is a constant c_{23}, which is independent of λ and n, such that*

$$Q_{Z_n}(\lambda) = \sup_{x\in\mathbb{R}} \mathbb{P}\left(x \leqslant X_1 + X_2 + \ldots + X_n \leqslant x + \lambda\right) \leqslant c_{23}\,(\lambda + 1)\, n^{-1/2}$$

for all $\lambda \geqslant 0$ *and* $n \in \mathbb{N}$.

The second lemma is proved by Shimura and Watanabe (see [58, Proposition 2.2]). The lemma describes an important property of d.f.'s from the class $\mathcal{OL}$.

Lemma 5.2.2. *Let* F *be a d.f. from the class* $\mathcal{OL}$. *Then there is some* $\Delta > 0$ *such that*

$$\lim_{x\to\infty} \mathrm{e}^{\Delta x}\,\overline{F}(x) = \infty.$$

The next lemma is crucial for the proof of Theorem 5.1.1. Some its elements can be found in [26] (see the proof of Theorem 3(b)). Inequality (2.3.1), which is a special case of the assertion below, is proved in [1, Lemma 2.1]. Leipus and Šiaulys [46] generalize Albin's inequality (2.3.1) to any d.f. with unbounded support. An analytical proof of Lemma 5.2.3 is given in [46] (see the proof of Lemma 4). Taking into account the importance of the assertion we give another proof of Lemma 5.2.3 in this section, which is completely probabilistic.

Lemma 5.2.3. *Let the d.f.* F *be such that* $\overline{F}(x) > 0$ *for all* $x \in \mathbb{R}$. *Moreover, suppose that*

$$\sup_{x\geqslant d_2} \frac{\overline{F}(x-t)}{\overline{F}(x)} \leqslant d_1$$

for some constants $t > 0$, $d_1 > 0$ *and* $d_2 > t$. *Then for all* $n = 1, 2, \ldots$, *we have*

$$\sup_{x\geqslant n(d_2-t)+t} \frac{\overline{F^{*n}}(x-t)}{\overline{F^{*n}}(x)} \leqslant d_1.$$

Proof of Lemma 5.2.3. Let X be a r.v. with d.f. F. Then the conditions of Lemma 5.2.3 give

$$\sup_{x\geqslant d_2} \frac{\mathbb{P}(X > x-t)}{\mathbb{P}(X > x)} \leqslant d_1 \tag{5.2.1}$$

for some $t > 0$, $d_1 > 0$ and $d_2 > t$, and we need to prove that

$$\sup_{x\geqslant n(d_2-t)+t} \frac{\mathbb{P}(S_n^X > x-t)}{\mathbb{P}(S_n^X > x)} \leqslant d_1 \tag{5.2.2}$$

for all $n \in \mathbb{N}$, where $S_n^X = X_1 + ... + X_n$ and $X_1, X_2, \dots$ are independent copies of X.

The proof is proceeded by induction on n. By condition (5.2.1), inequality (5.2.2) holds for $n = 1$. We suppose now that $N \geqslant 1$. For any $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $t > 0$, we obtain

$$\begin{aligned}
\mathbb{P}(S_{N+1}^X > x) &= \mathbb{P}(S_N^X + X_{N+1} > x,\, X_{N+1} \leqslant x - z) \\
&\quad + \mathbb{P}(S_N^X + X_{N+1} > x,\, S_N^X \leqslant z) \\
&\quad + \mathbb{P}(X_{N+1} > x - z)\, \mathbb{P}(S_N^X > z) \\
&\geqslant \mathbb{P}(S_N^X > x - X_{N+1},\, x - X_{N+1} \geqslant z) \\
&\quad + \mathbb{P}(X_{N+1} > x - S_N^X,\, x - S_N^X \geqslant x - z + t) \\
&\quad + \mathbb{P}(X_{N+1} > x - z)\, \mathbb{P}(S_N^X > z)\,. \qquad (5.2.3)
\end{aligned}$$

If we replace x by $x - t$ and z by $z - t$, then we get

$$\begin{aligned}
\mathbb{P}(S_{N+1}^X > x - t) &= \mathbb{P}(S_N^X + X_{N+1} > x - t,\, X_{N+1} \leqslant x - z) \\
&\quad + \mathbb{P}(S_N^X + X_{N+1} > x - t,\, S_N^X \leqslant z - t) \\
&\quad + \mathbb{P}(X_{N+1} > x - z)\, \mathbb{P}(S_N^X > z - t) \\
&= \mathbb{P}(S_N^X > x - X_{N+1} - t,\, x - X_{N+1} \geqslant z) \\
&\quad + \mathbb{P}(X_{N+1} > x - S_N^X - t,\, x - S_N^X \geqslant x - z + t) \\
&\quad + \mathbb{P}(X_{N+1} > x - z)\, \mathbb{P}(S_N^X > z - t)\,. \qquad (5.2.4)
\end{aligned}$$

If

$$\mathbb{P}\Big(S_N^X > x - X_{N+1} - t,\, x - X_{N+1} \geqslant z\Big) > 0$$

and

$$\mathbb{P}\Big(X_{N+1} > x - S_N^X - t,\, x - S_N^X \geqslant x - z + t\Big) > 0,$$

then (5.2.3), (5.2.4) and (4.3.1) imply that

$$\begin{aligned}
\frac{\mathbb{P}(S_{N+1}^X > x - t)}{\mathbb{P}(S_{N+1}^X > x)} &\leqslant \max\left\{ \frac{\mathbb{P}(S_N^X > x - X_{N+1} - t,\, x - X_{N+1} \geqslant z)}{\mathbb{P}(S_N^X > x - X_{N+1},\, x - X_{N+1} \geqslant z)}, \right. \\
&\left. \frac{\mathbb{P}(X_{N+1} > x - S_N^X - t,\, x - S_N^X > x - z + t)}{\mathbb{P}(X_{N+1} > x - S_N^X,\, x - S_N^X > x - z + t)}, \frac{\mathbb{P}(S_N^X > z - t)}{\mathbb{P}(S_N^X > z)} \right\} \\
&\leqslant \max\left\{ \sup_{y \geqslant z} \frac{\mathbb{P}(S_N^X > y - t)}{\mathbb{P}(S_N^X > y)}, \sup_{y \geqslant x - z + t} \frac{\mathbb{P}(X > y - t)}{\mathbb{P}(X > y)} \right\} \qquad (5.2.5)
\end{aligned}$$

If

$$\mathbb{P}\Big(S_N^X > x - X_{N+1} - t,\, x - X_{N+1} \geqslant z\Big) = 0$$

and

$$\mathbb{P}\Big(X_{N+1} > x - S_N^X - t,\, x - S_N^X \geqslant x - z + t\Big) > 0,$$

then expressions (5.2.3), (5.2.4) and (4.3.1) imply that

$$\frac{\mathbb{P}(S_{N+1}^X > x - t)}{\mathbb{P}(S_{N+1}^X > x)} \leqslant \max\left\{\frac{\mathbb{P}(X_{N+1} > x - S_N^X - t,\, x - S_N^X > x - z + t)}{\mathbb{P}(X_{N+1} > x - S_N^X,\, x - S_N^X > x - z + t)},\right.$$
$$\left.\frac{\mathbb{P}(S_N^X > z - t)}{\mathbb{P}(S_N^X > z)}\right\}.$$

Hence, inequality (5.2.5) holds again.

The cases

$$\left\{\begin{array}{l}\mathbb{P}\Big(S_N^X > x - X_{N+1} - t,\, x - X_{N+1} \geqslant z\Big) > 0, \\ \mathbb{P}\Big(X_{N+1} > x - S_N^X - t,\, x - S_N^X \geqslant x - z + t\Big) = 0\end{array}\right\}$$

and

$$\left\{\begin{array}{l}\mathbb{P}\Big(S_N^X > x - X_{N+1} - t,\, x - X_{N+1} \geqslant z\Big) = 0, \\ \mathbb{P}\Big(X_{N+1} > x - S_N^X - t,\, x - S_N^X \geqslant x - z + t\Big) = 0\end{array}\right\}$$

can be considered analogously.

Therefore, inequality (5.2.5) holds under conditions of the lemma for any $x \in \mathbb{R}$, $z \in \mathbb{R}$, $t > 0$ and $N \geqslant 1$.

We now assume that (5.2.2) is satisfied for $n = N$, and we show that (5.2.2) holds for $n = N + 1$.

Condition (5.2.1) and bound (5.2.5) with $z = \frac{Nx}{N+1} + \frac{t}{N+1}$ imply that

$$\frac{\mathbb{P}(S_{N+1}^X > x - t)}{\mathbb{P}(S_{N+1}^X > x)}$$
$$\leqslant \max\left\{\sup_{y \geqslant \frac{Nx}{N+1} + \frac{t}{N+1}} \frac{\mathbb{P}(S_N^X > y - t)}{\mathbb{P}(S_N^X > y)},\, \sup_{y \geqslant \frac{x}{N+1} + \frac{Nt}{N+1}} \frac{\mathbb{P}(X > y - t)}{\mathbb{P}(X > y)}\right\} \leqslant d_1$$

if $x \geqslant (N+1)(d_2 - t) + t$, because in this case we have

$$\frac{Nx}{N+1} + \frac{t}{N+1} \geqslant N(d_2 - t) + t,$$

and

$$\frac{x}{N+1} + \frac{Nt}{N+1} \geqslant d_2.$$

Thus, bound (5.2.2) holds for $n = N + 1$. Therefore, (5.2.2) is true for all n by induction. □

In fact, the next assertion is proved by deriving inequality (5.2.5). Another analytic proof can be found in [26] (see equation (2.12)).

Lemma 5.2.4. *Let F and G be two d.f.'s such that $\overline{F}(x) > 0$ and $\overline{G}(x) > 0$ for all $x \in \mathbb{R}$. Then*

$$\frac{\overline{F * G}(x-t)}{\overline{F * G}(x)} \leqslant \max\left\{\sup_{y \geqslant v} \frac{\overline{F}(y-t)}{\overline{F}(y)}, \sup_{y \geqslant x-v+t} \frac{\overline{G}(y-t)}{\overline{G}(y)}\right\}$$

for all $x \in \mathbb{R}$, $v \in \mathbb{R}$ and $t > 0$.

The following assertion is the well-known Kolmogorov-Rogozin inequality for concentration functions. We recall that the Lévy concentration function, or simply the concentration function, of a r.v. X we denote by $Q_X(\lambda)$. This function is well-defined for all $\lambda \in [0, \infty)$. The proof of the lemma below can be found in [53, Theorem 2.15]. It is evident that this lemma is a generalization of Lemma 5.2.1.

Lemma 5.2.5. *Let $X_1, X_2, \ldots, X_n$ be independent r.v.'s, and let $Z_n = \sum_{k=1}^{n} X_k$. Then for any $n \in \mathbb{N}$, we have*

$$Q_{Z_n}(\lambda) \leqslant A\lambda \left\{\sum_{k=1}^{n} \lambda_k^2 \left(1 - Q_{X_k}(\lambda_k)\right)\right\}^{-1/2},$$

where A is an absolute constant and $0 < \lambda_k \leqslant \lambda$ for all $k \in \{1, 2, \ldots, n\}$.

The following assertion describes sufficient conditions under which the d.f. of two independent r.v.'s belongs to the class $\mathcal{OL}$.

Lemma 5.2.6. *Let X_1 and X_2 be independent r.v.'s with d.f.'s F_{X_1} and F_{X_2}, respectively. Then the d.f $F_{X_1} * F_{X_2}$ of the sum $X_1 + X_2$ is $\mathcal{O}$-exponential if $F_{X_1} \in \mathcal{OL}$ and one of the following two conditions holds:*

- $$\lim_{x \to \infty} \frac{\overline{F}_{X_2}(x)}{\overline{F}_{X_1}(x)} = 0; \qquad (5.2.6)$$
- $F_{X_2} \in \mathcal{OL}$.

Proof of Lemma 5.2.6. We split the proof into three parts.

I. First, we suppose that $\mathbb{P}(X_2 \leqslant D) = 1$ for some $D > 0$. It is evident that condition (5.2.6) holds in this case.

For all $x \in \mathbb{R}$, we have

$$\begin{aligned}\overline{F_{X_1} * F_{X_2}}(x) &= \mathbb{P}(X_1 + X_2 > x)\\ &= \int\limits_{(-\infty,D]} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y).\end{aligned}$$

Hence, for all $x \in \mathbb{R}$, we get

$$\begin{aligned}\frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} &= \frac{\int\limits_{(-\infty,D]} \overline{F}_{X_1}(x-1-y)\frac{\overline{F}_{X_1}(x-y)}{\overline{F}_{X_1}(x-y)}\mathrm{d}F_{X_2}(y)}{\int\limits_{(-\infty,D]} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y)}\\ &\leqslant \frac{\int\limits_{(-\infty,D]} \sup\limits_{y\leqslant D} \frac{\overline{F}_{X_1}(x-1-y)}{\overline{F}_{X_1}(x-y)}\overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y)}{\int\limits_{(-\infty,D]} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y)}\\ &= \sup_{z\geqslant x-D} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)},\end{aligned}$$

which implies that

$$\begin{aligned}\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} &\leqslant \limsup_{x\to\infty} \sup_{z\geqslant x-D} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}\\ &= \limsup_{y\to\infty} \frac{\overline{F}_{X_1}(y-1)}{\overline{F}_{X_1}(y)}\\ &< \infty\end{aligned}$$

because $F_{X_1} \in \mathcal{OL}$. Thus, $F_{X_1} * F_{X_2} \in \mathcal{OL}$ as well.

II. Now let us consider the case where condition (5.2.6) holds but $\overline{F}_{X_2}(x) > 0$ for all $x \in \mathbb{R}$.

For all $x \in \mathbb{R}$, we have

$$\overline{F_{X_1} * F_{X_2}}(x) = \int\limits_{-\infty}^{\infty} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y).$$

Therefore,

$$\overline{F_{X_1} * F_{X_2}}(x-1) = \left(\int\limits_{(-\infty, x-M]} + \int\limits_{(x-M,\infty)} \right) \overline{F}_{X_1}(x-1-y)\mathrm{d}F_{X_2}(y)$$

$$\leqslant \int\limits_{(-\infty, x-M]} \overline{F}_{X_1}(x-1-y)\frac{\overline{F}_{X_1}(x-y)}{\overline{F}_{X_1}(x-y)}\mathrm{d}F_{X_2}(y) + \overline{F}_{X_2}(x-M)$$

$$\leqslant \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)} \int\limits_{(-\infty, x-M]} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y) + \overline{F}_{X_2}(x-M)$$

for all M and x such that $0 < M < x - 1$.

In addition, for such values of M and x, we obtain

$$\overline{F_{X_1} * F_{X_2}}(x) \geqslant \int\limits_{(-\infty, x-M]} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y),$$

$$\overline{F_{X_1} * F_{X_2}}(x) \geqslant \int\limits_{(M,\infty)} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y) \geqslant \overline{F}_{X_1}(x-M)\overline{F}_{X_2}(M).$$

The bounds obtained above imply that

$$\frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)} + \frac{\overline{F}_{X_2}(x-M)}{\overline{F}_{X_1}(x-M)\overline{F}_{X_2}(M)}$$

for all x and M such that $0 < M < x - 1$.
Consequently,

$$\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)} + \frac{1}{\overline{F}_{X_2}(M)} \limsup_{x\to\infty} \frac{\overline{F}_{X_2}(x-M)}{\overline{F}_{X_1}(x-M)} = \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}$$

for any $M > 0$, which yields

$$\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \limsup_{M\to\infty} \frac{\overline{F}_{X_1}(M-1)}{\overline{F}_{X_1}(M)} < \infty$$

because F_{X_1} is $\mathcal{O}$-exponential. Thus, $F_{X_1} * F_{X_2} \in \mathcal{OL}$ by (2.1.1).

III. It remains to prove the assertion in the case where both d.f.'s F_{X_1} and F_{X_2} are $\mathcal{O}$-exponential. By Lemma 5.2.4, we have

$$\frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \max\left\{\sup_{z \geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}, \sup_{z \geqslant x-M+1} \frac{\overline{F}_{X_2}(z-1)}{\overline{F}_{X_2}(z)}\right\}$$

for all x and M such that $0 < M < x-1$.

Therefore, for any $M > 0$, we get

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \\
&\qquad \leqslant \max\left\{\sup_{z \geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}, \limsup_{x\to\infty} \sup_{z \geqslant x-M+1} \frac{\overline{F}_{X_2}(z-1)}{\overline{F}_{X_2}(z)}\right\} \\
&\qquad = \max\left\{\sup_{z \geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}, \limsup_{y\to\infty} \frac{\overline{F}_{X_2}(y-1)}{\overline{F}_{X_2}(y)}\right\}.
\end{aligned}$$

Letting $M \to \infty$ we obtain

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \\
&\qquad \leqslant \max\left\{\limsup_{M\to\infty} \frac{\overline{F}_{X_1}(M-1)}{\overline{F}_{X_1}(M)}, \limsup_{y\to\infty} \frac{\overline{F}_{X_2}(y-1)}{\overline{F}_{X_2}(y)}\right\} < \infty
\end{aligned}$$

because F_{X_1} and F_{X_2} belong to the class $\mathcal{OL}$. Consequently, $F_{X_1} * F_{X_2} \in \mathcal{OL}$ by definition (2.1.1), which completes the proof of the lemma. □

Lemma 5.2.7. *Let* $\{X_1, X_2, ..., X_n\}$ *be independent non-negative r.v.'s with d.f.'s* $\{F_{X_1}, F_{X_2}, \ldots, F_{X_n}\}$. *In addition, we suppose that* $F_{X_1} \in \mathcal{OL}$ *and for each* $k \in \{2, 3, ..., n\}$, *either* $\lim_{x\to\infty} \overline{F}_{X_k}(x)/\overline{F}_{X_1}(x) = 0$ *or* $F_{X_k} \in \mathcal{OL}$. *Then the d.f.* $F_{X_1} * F_{X_2} * ... * F_{X_n}$ *belongs to the class* $\mathcal{OL}$.

Proof of Lemma 5.2.7. We prove the lemma by induction on n. If $n = 2$, then the assertion of the lemma follows from Lemma 5.2.6. Suppose that it holds if $n = m$, i.e. $F_{X_1} * F_{X_2} * ... * F_{X_m} \in \mathcal{OL}$, and we now show that the assertion is true for $n = m + 1$.

The conditions of the lemma imply that either $F_{X_{m+1}} \in \mathcal{OL}$ or

$$\begin{aligned}
\lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\overline{F_{X_1} * F_{X_2} * ... * F_{X_m}}(x)} &= \lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\mathbb{P}(X_1 + ... + X_m > x)} \\
&\leqslant \lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\mathbb{P}(X_1 > x)} \\
&= \lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\overline{F}_{X_1}(x)} = 0.
\end{aligned}$$

Thus, applying Lemma 5.2.6 again we get

$$F_{X_1} * F_{X_2} * ... * F_{X_{m+1}} = (F_{X_1} * F_{X_2} * ... * F_{X_m}) * F_{X_{m+1}} \in \mathcal{OL}.$$

From this we see that the assertion of the lemma holds for $n = m + 1$, and consequently, for all $n \in \mathbb{N}$ by induction. This completes the proof. □

The next auxiliary assertion gives a bound for the tail distribution of the maximum. We observe that estimate of Lemma 5.2.8 below is similar to the bound for the tail distribution of the sum given in [26] (see inequality (2.12)).

Lemma 5.2.8. *Let X and Y be two r.v.'s with d.f.'s F and G, respectively, and let H be the d.f. of* $\max\{X, Y\}$.

(i) *If X and Y are independent, then*

$$\frac{\overline{H}(x-t)}{\overline{H}(x)} \leqslant \max\left\{ \frac{\overline{F}(x-t)}{\overline{F}(x)}, \frac{\overline{G}(x-t)}{\overline{G}(x)} \right\}$$

for all $t > 0$ if $\overline{F}(x) > 0$ and $\overline{G}(x) > 0$.

(ii) *If X and Y are arbitrarily dependent, then*

$$\frac{\overline{H}(x-t)}{\overline{H}(x)} \leqslant \frac{\overline{F}(x-t)}{\overline{F}(x)} + \frac{\overline{G}(x-t)}{\overline{G}(x)}$$

for all $t \in \mathbb{R}$ if $\overline{F}(x) > 0$ and $\overline{G}(x) > 0$.

Proof of Lemma 5.2.8. (i) In this case, we have $H(x) = F(x)G(x)$. Therefore, for all $x \in \mathbb{R}$, we get

$$\overline{H}(x) = 1 - H(x) = \overline{G}(x) + G(x) - F(x)G(x) = \overline{G}(x) + G(x)\overline{F}(x).$$

If $t > 0$ and $G(x) > 0$, then

$$\begin{aligned}\frac{\overline{H}(x-t)}{\overline{H}(x)} &\leqslant \frac{\overline{F}(x-t)G(x)+\overline{G}(x-t)}{\overline{F}(x)G(x)+\overline{G}(x)} \\ &\leqslant \max\left\{\frac{\overline{F}(x-t)}{\overline{F}(x)}, \frac{\overline{G}(x-t)}{\overline{G}(x)}\right\}\end{aligned}$$

by inequality (4.3.1).

If $G(x) = 0$, then for any $t > 0$, we have $\overline{H}(x-t) = \overline{H}(x) = \overline{G}(x-t) = \overline{G}(x) = 1$, and the inequality in part (i) is obvious.

(ii) If the r.v.'s X and Y are arbitrarily dependent, then for all $x \in \mathbb{R}$, we have

$$\begin{aligned}&\overline{H}(x) = \mathbb{P}(\{X > x\} \cup \{Y > x\}) \leqslant \overline{F}(x) + \overline{G}(x), \\ &\overline{H}(x) \geqslant \overline{F}(x), \quad \overline{H}(x) \geqslant \overline{G}(x),\end{aligned}$$

and the desired inequality follows immediately. □

The next lemma is a direct consequence of the bounds given above.

Lemma 5.2.9. *Let $X_1, X_2, \ldots, X_n$, $n \geqslant 1$, be (arbitrarily dependent) r.v.'s. If their d.f.'s belong to the class $\mathcal{OL}$, then the d.f. of* $\max\{X_1, X_2, ..., X_n\}$ *belongs to the class $\mathcal{OL}$ as well.*

Proof of Lemma 5.2.9. Since

$$\max\{X_1, X_2, \ldots, X_n\} = \max\{\max\{X_1, \ldots, X_{n-1}\}, X_n\}$$

for all $n \geqslant 3$, the assertion follows immediately from Lemma 5.2.8. □

The last lemma describes conditions under which the convolution of several d.f.'s belongs to the class $\mathcal{OL}$. In fact, the assertion generalizes Lemma 3 from [19].

Lemma 5.2.10. *Let $\{F_1, F_2, \ldots, F_n\}$, $n \geqslant 2$, be a collection of d.f.'s of real-valued r.v.'s. If $F_1 \in \mathcal{OL}$, then the d.f. $G_n := F_1 * F_2 * \ldots * F_n$ belongs to the class $\mathcal{OL}$ as well and*

$$\sup_{x\in\mathbb{R}} \frac{\overline{G}_n(x-1)}{\overline{G}_n(x)} \leqslant \sup_{x\in\mathbb{R}} \frac{\overline{F}_1(x-1)}{\overline{F}_1(x)}.$$

Proof of Lemma 5.2.10. Since $F_1 \in \mathcal{OL}$, by (2.1.1), there is a constant $c > 0$ such that

$$\sup_{x\in\mathbb{R}} \frac{\overline{F}_1(x-1)}{\overline{F}_1(x)} = c.$$

Thus, we get

$$\begin{aligned}\overline{F_1 * F_2}(x-1) &= \int_{\mathbb{R}} \overline{F}_1(x-1-y)\,\mathrm{d}F_2(y)\\ &\leqslant c\int_{\mathbb{R}} \overline{F}_1(x-y)\,\mathrm{d}F_2(y) = c\,\overline{F_1 * F_2}(x),\end{aligned}$$

and for $n = 2$, the assertion of the lemma follows because, in addition, we have

$$\begin{aligned}\overline{F_1 * F_2}(x) &\geqslant \int_{(-x/2,\,x/2]} \overline{F}_1(x-y)\,\mathrm{d}F_2(y)\\ &\geqslant \overline{F}_1\Big(\frac{3x}{2}\Big)\Big(F_2\Big(\frac{x}{2}\Big) - F_2\Big(-\frac{x}{2}\Big)\Big) > 0\end{aligned}$$

for x large enough.

If $n \geqslant 3$, then the assertion of the lemma holds because

$$F_1 * F_2 * \ldots * F_n = (F_1 * F_2 * \ldots * F_{n-1}) * F_n,$$

and the proof is complete. □

5.3 Proofs of Theorems 5.1.1-5.1.9

Proof of Theorem 5.1.1. First, we show that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x-a)}{\overline{F}_{S_\eta}(x)} = \limsup_{x\to\infty} \frac{\mathbb{P}(S_\eta > x-a)}{\mathbb{P}(S_\eta > x)} < \infty \tag{5.3.1}$$

for all $a \in \mathbb{R}$.

If $a \leqslant 0$, then $\mathbb{P}(S_\eta > x-a) \leqslant \mathbb{P}(S_\eta > x)$ for all $x \in \mathbb{R}$, and (5.3.1) is obvious.

We now suppose that $a > 0$. Since $F_\xi \in \mathcal{OL}$, we deduce that

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(x-a)}{\overline{F}_\xi(x)} = c_{24} \tag{5.3.2}$$

for some finite constant $c_{24} > 0$, which may depend on a. Thus, there is some $K = K_a > a+1$ such that

$$\sup_{x \geqslant K} \frac{\overline{F}_\xi(x-a)}{\overline{F}_\xi(x)} \leqslant 2\, c_{24}. \tag{5.3.3}$$

Applying Lemma 5.2.3 we obtain

$$\sup_{x \geqslant n\,(K-a)+a} \frac{\mathbb{P}(S_n > x-a)}{\mathbb{P}(S_n > x)} = \sup_{x \geqslant n\,(K-a)+a} \frac{\overline{F_\xi^{*n}}(x-a)}{\overline{F_\xi^{*n}}(x)} \leqslant 2\, c_{24}. \tag{5.3.4}$$

Here and subsequently, $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for all $n \in \mathbb{N}$.

For any positive $x > 0$, we have

$$\begin{aligned} \mathbb{P}(S_\eta > x) &= \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n) \geqslant \sum_{n=1}^{\infty} \mathbb{P}(\xi_1 > x)\mathbb{P}(\eta = n) \\ &= \overline{F}_\xi(x)\mathbb{P}(\eta \geqslant 1). \end{aligned} \tag{5.3.5}$$

If $x \geqslant K$, then applying (5.3.4) yields

$$\begin{aligned} & \mathbb{P}(S_\eta > x-a) \\ = \; & \mathbb{P}\Big(S_\eta > x-a,\, \eta \leqslant \frac{x-a}{K-a}\Big) + \mathbb{P}\Big(S_\eta > x-a,\, \eta > \frac{x-a}{K-a}\Big) \\ = \; & \sum_{n \leqslant \frac{x-a}{K-a}} \mathbb{P}(S_n > x-a)\,\mathbb{P}(\eta = n) + \sum_{n > \frac{x-a}{K-a}} \mathbb{P}(S_n > x-a)\,\mathbb{P}(\eta = n) \\ \leqslant \; & 2\, c_{24} \sum_{n \leqslant \frac{x-a}{K-a}} \mathbb{P}(S_n > x)\,\mathbb{P}(\eta = n) + \sum_{n > \frac{x-a}{K-a}} \mathbb{P}(S_n > x-a)\mathbb{P}(\eta = n) \\ + \; & \sum_{n > \frac{x-a}{K-a}} \mathbb{P}(S_n > x)\,\mathbb{P}(\eta = n) - \sum_{n > \frac{x-a}{K-a}} \mathbb{P}(S_n > x)\,\mathbb{P}(\eta = n) \\ \leqslant \; & c_{25} \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n) \\ + \; & \sum_{n > \frac{x-a}{K-a}} \mathbb{P}(x - a < S_n \leqslant x)\,\mathbb{P}(\eta = n) \end{aligned} \tag{5.3.6}$$

with $c_{25} = \max\{2c_{24}, 1\}$.

By Lemma 5.2.1, we obtain

$$\sup_{x \in \mathbb{R}} \mathbb{P}(x - a < S_n \leqslant x) \leqslant c_{26}(a+1)\frac{1}{\sqrt{n}},$$

where the constant c_{26} is independent of a and n. Thus, inequality (5.3.6) implies that

$$\begin{aligned}
\mathbb{P}(S_\eta > x-a) &\leqslant c_{25}\mathbb{P}(S_\eta > x) + c_{26}(a+1)\sum_{n>\frac{x-a}{K-a}} \frac{\mathbb{P}(\eta = n)}{\sqrt{n}} \\
&\leqslant c_{25}\mathbb{P}(S_\eta > x) \\
&+ c_{26}\sqrt{\frac{K-a}{x-a}}\,(a+1)\,\mathbb{P}\Big(\eta > \frac{x-a}{K-a}\Big) \qquad (5.3.7)
\end{aligned}$$

provided that $x \geqslant K$.

From inequalities (5.3.5) and (5.3.7) we conclude that

$$\frac{\mathbb{P}(S_\eta > x-a)}{\mathbb{P}(S_\eta > x)} \leqslant c_{25} + \frac{c_{26}\,\sqrt{K-a}\,(a+1)}{\sqrt{x-a}\,\mathbb{P}(\eta \geqslant 1)\,\overline{F}_\xi(x)}\,\overline{F}_\eta\Big(\frac{x-a}{K-a}\Big)$$

for all $x \geqslant K$. Consequently, we have

$$\begin{aligned}
&\limsup_{x\to\infty}\frac{\mathbb{P}(S_\eta > x-a)}{\mathbb{P}(S_\eta > x)} \\
\leqslant\;& c_{25} + c_{26}\frac{(a+1)\,\sqrt{K-a}}{\mathbb{P}(\eta \geqslant 1)}\limsup_{x\to\infty}\frac{\overline{F}_\eta\big((x-a)/(K-a)\big)}{\sqrt{x-a}\;\overline{F}_\xi(x-a)} \\
&\limsup_{x\to\infty}\frac{\overline{F}_\xi(x-a)}{\overline{F}_\xi(x)} \\
=\;& c_{25} + c_{24}\,c_{26}\frac{(a+1)\sqrt{K-a}}{\mathbb{P}(\eta \geqslant 1)}\;\limsup_{x\to\infty}\frac{\overline{F}_\eta(\frac{x}{K-a})}{\sqrt{x}\;\overline{F}_\xi(x)} < \infty
\end{aligned}$$

by equality (5.3.2) and the condition $\overline{F}_\eta(\delta x) = O\big(\sqrt{x}\;\overline{F}_\xi(x)\big)$, which holds for any $\delta \in (0,1)$. Therefore, (5.3.1) is true for all $a \in \mathbb{R}$.

It remains to prove that

$$\liminf_{x\to\infty}\frac{\overline{F}_{S_\eta}(x-a)}{\overline{F}_{S_\eta}(x)} = \liminf_{x\to\infty}\frac{\mathbb{P}(S_\eta > x-a)}{\mathbb{P}(S_\eta > x)} > 0,$$

where a is any real number. It is easily seen that this follows from (5.3.1) because

$$\mathbb{P}(S_\eta > x) \geqslant \overline{F}_\xi(x)\mathbb{P}(\eta \geqslant 1) > 0$$

for all $x > 0$, and hence, we have

$$\liminf_{x\to\infty}\frac{\mathbb{P}(S_\eta > x-a)}{\mathbb{P}(S_\eta > x)} = \frac{1}{\limsup\limits_{x\to\infty}\frac{\mathbb{P}(S_\eta > x+a)}{\mathbb{P}(S_\eta > x)}} > 0.$$

The last inequality together with (5.3.1) implies that the d.f. F_{S_η} belongs to the class $\mathcal{OL}$, which completes the proof. □

Proof of Corollary 5.1.1. Part (i) of the corollary is evident. So we prove only part (ii). Let $\delta \in (0,1)$. By Markov's inequality, we have

$$\overline{F}_\eta(\delta x) = \mathbb{P}(\eta > \delta x) = \mathbb{P}(\mathrm{e}^{y\eta} > \mathrm{e}^{y\delta x}) \leqslant \mathrm{e}^{-\delta y x}\mathbb{E}\mathrm{e}^{y\eta} \tag{5.3.8}$$

for all $y > 0$. Since the d.f. F_ξ belongs to the class $\mathcal{OL}$, by Lemma 5.2.2, we get

$$\mathrm{e}^{\Delta x}\overline{F}_\xi(x) \underset{x\to\infty}{\longrightarrow} \infty$$

for some $\Delta > 0$.

Choosing $y = \Delta/\delta > 0$ in (5.3.8) we obtain

$$\frac{\overline{F}_\eta(\delta x)}{\sqrt{x}\,\overline{F}_\xi(x)} \leqslant \frac{\mathbb{E}\mathrm{e}^{y\eta}}{\mathrm{e}^{\delta y x}\sqrt{x}\,\overline{F}_\xi(x)} = \frac{1}{\sqrt{x}}\,\frac{1}{\mathrm{e}^{\Delta x}\overline{F}_\xi(x)}\mathbb{E}\,\mathrm{e}^{(\Delta/\delta)\eta} \underset{x\to\infty}{\longrightarrow} 0$$

because $\mathbb{E}\mathrm{e}^{\varepsilon\eta}$ is finite for all $\varepsilon > 0$ by the conditions of Corollary 5.1.1. The assertion of the corollary now follows from Theorem 5.1.1.

□

Proof of Theorem 5.1.2. From the conditions of the theorem and Lemma 5.2.7 we conclude that the d.f. $F_{S_\kappa}(x) = \mathbb{P}(S_\kappa \leqslant x)$ belongs to the class $\mathcal{OL}$, i.e. we have

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\kappa}(x-1)}{\overline{F}_{S_\kappa}(x)} < \infty, \tag{5.3.9}$$

or, equivalently,

$$\sup_{x\geqslant 0} \frac{\overline{F}_{S_\kappa}(x-1)}{\overline{F}_{S_\kappa}(x)} \leqslant c_{27} \tag{5.3.10}$$

for some constant $c_{27} > 0$.

Next, we see that

$$\frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} = \mathcal{J}_1(x) + \mathcal{J}_2(x) \tag{5.3.11}$$

for all $x \geqslant 0$, where

$$\mathcal{J}_1(x) = \frac{\mathbb{P}(S_\eta > x-1, \eta \leqslant \kappa)}{\mathbb{P}(S_\eta > x)}$$

and

$$\mathcal{J}_2(x) = \frac{\mathbb{P}(S_\eta > x-1, \eta > \kappa)}{\mathbb{P}(S_\eta > x)}.$$

Since $\kappa \in \operatorname{supp}(\eta)$, we obtain

$$\begin{aligned}
\mathcal{J}_1(x) &= \frac{\sum\limits_{n=0}^{\kappa} \mathbb{P}(S_n > x-1)\mathbb{P}(\eta = n)}{\sum\limits_{n=0}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n)} \\
&\leqslant \frac{1}{\mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa)} \sum_{n=0}^{\kappa} \mathbb{P}(S_\kappa > x-1)\mathbb{P}(\eta = n) \\
&= \frac{\mathbb{P}(S_\kappa > x-1)}{\mathbb{P}(S_\kappa > x)} \frac{\mathbb{P}(\eta \leqslant \kappa)}{\mathbb{P}(\eta = \kappa)}.
\end{aligned}$$

Hence, it follows from (5.3.9) that

$$\limsup_{x\to\infty} \mathcal{J}_1(x) < \infty. \tag{5.3.12}$$

By Lemma 5.2.4, we have

$$\frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\left\{ \sup_{z\geqslant M} \frac{\mathbb{P}(S_\kappa > z-1)}{\mathbb{P}(S_\kappa > z)}, \sup_{z\geqslant x-M+1} \frac{\overline{F}_{\xi_{\kappa+1}}(z-1)}{\overline{F}_{\xi_{\kappa+1}}(z)} \right\} \tag{5.3.13}$$

for all $x \in \mathbb{R}$ and $M \in \mathbb{R}$.

The third condition of the theorem implies that

$$\sup_{x\geqslant 0} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \leqslant c_{28} \tag{5.3.14}$$

for all $k \in \mathbb{N}$ and some $c_{28} > 0$.

If we choose $M = x/2$ in (5.3.13), then using (5.3.10) we get

$$\sup_{x\geqslant 0} \frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\{c_{27}, c_{28}\} := c_{29}. \tag{5.3.15}$$

Applying Lemma 5.2.4 again we obtain

$$\frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant \max\left\{ \sup_{z\geqslant M} \frac{\mathbb{P}(S_{\kappa+1} > z-1)}{\mathbb{P}(S_{\kappa+1} > z)}, \sup_{z\geqslant x-M+1} \frac{\overline{F}_{\xi_{\kappa+2}}(z-1)}{\overline{F}_{\xi_{\kappa+2}}(z)} \right\}.$$

If we choose $M = x/2$, then from inequalities (5.3.14) and (5.3.15) we deduce that

$$\sup_{x \geqslant 0} \frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant c_{29}.$$

Continuing the process we get

$$\sup_{x \geqslant 0} \frac{\mathbb{P}(S_{\kappa+k} > x-1)}{\mathbb{P}(S_{\kappa+k} > x)} \leqslant c_{29}$$

for all $k \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \mathcal{J}_2(x) &= \frac{1}{\mathbb{P}(S_\eta > x)} \sum_{k=1}^{\infty} \mathbb{P}(S_{\kappa+k} > x-1)\mathbb{P}(\eta = \kappa + k) \\ &\leqslant \frac{c_{29}}{\mathbb{P}(S_\eta > x)} \sum_{k=1}^{\infty} \mathbb{P}(S_{\kappa+k} > x)\mathbb{P}(\eta = \kappa + k) \\ &\leqslant \frac{c_{29}\mathbb{P}(S_\eta > x)}{\mathbb{P}(S_\eta > x)} = c_{29} \end{aligned} \tag{5.3.16}$$

for all $x \geqslant 0$.

From relations (5.3.11), (5.3.12) and (5.3.16) it follows that

$$\limsup_{x \to \infty} \frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} < \infty.$$

Therefore, the d.f. F_{S_η} belongs to the class $\mathcal{OL}$ by (2.1.1), which completes the proof of the theorem. □

Proof of Theorem 5.1.3. The assertion of the theorem can be either derived from Theorem 5.1.2 or proved directly. We now present a direct proof of Theorem 5.1.3.

It is evident that $S_k = \xi_\kappa + \sum\limits_{n=1,\ n \neq \kappa}^{k} \xi_n$ for all $k \geqslant \kappa$. Hence, by Lemma 5.2.7, $F_{S_k} \in \mathcal{OL}$ for all $\kappa \leqslant k \leqslant D$.

If $x \geqslant 1$, then we have

$$\frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} = \frac{\sum\limits_{\substack{n=1\\ \eta\in\text{supp }\eta}}^{D} \mathbb{P}(S_n > x-1)\mathbb{P}(\eta = n)}{\sum\limits_{\substack{n=1\\ \eta\in\text{supp }\eta}}^{D} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n)}$$

$$\leqslant \frac{\mathbb{P}(S_\kappa > x-1)\mathbb{P}(\eta \leqslant \kappa) + \sum\limits_{\substack{n=\kappa+1\\ n\in\text{supp }\eta}}^{D} \mathbb{P}(S_n > x-1)\mathbb{P}(\eta = n)}{\mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa) + \sum\limits_{\substack{n=\kappa+1\\ \eta\in\text{supp }\eta}}^{D} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n)}$$

$$\leqslant \max\left\{\frac{\mathbb{P}(S_\kappa > x-1)\mathbb{P}(\eta \leqslant \kappa)}{\mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa)}, \max_{\substack{\kappa+1\leqslant n\leqslant D\\ n\in\text{supp }\eta}} \frac{\mathbb{P}(S_n > x-1)}{\mathbb{P}(S_n > x)}\right\}. \tag{5.3.17}$$

Note that here in the last step, we use inequality (4.3.1).

Since $F_{S_n} \in \mathcal{OL}$ for all $n \geqslant \kappa$, from (5.3.17) we conclude that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} < \infty, \tag{5.3.18}$$

and the assertion of the theorem 5.1.3 follows. □

Proof of Theorem 5.1.4. As usual, it is sufficient to prove relation (5.3.18).

For all $x \geqslant 0$, we have

$$\begin{aligned}\mathbb{P}(S_\eta > x) &= \sum_{n=1}^{\infty} \mathbf{P}(S_n > x)\mathbb{P}(\eta = n)\\ &\geqslant \mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa)\\ &\geqslant \overline{F}_{\xi_\kappa}(x)\mathbb{P}(\eta = \kappa)).\end{aligned} \tag{5.3.19}$$

Similarly, for all $K \geqslant 2$ and $x \geqslant 2K$, we obtain

$$\begin{aligned}\mathbb{P}(S_\eta > x-1) &= \sum_{n=1}^{\kappa} \mathbf{P}(S_n > x-1)\mathbb{P}(\eta = n)\\ &+ \sum_{1\leqslant k\leqslant (x-1)/(K-1)} \mathbf{P}(S_{\kappa+k} > x-1)\mathbb{P}(\eta = \kappa+k)\\ &+ \sum_{k>(x-1)/(K-1)} \mathbf{P}(x-1 < S_{\kappa+k} \leqslant x)\mathbb{P}(\eta = \kappa+k)\\ &+ \sum_{k>(x-1)/(K-1)} \mathbf{P}(S_{\kappa+k} > x)\mathbb{P}(\eta = \kappa+k)\\ &:= \mathcal{K}_1(x) + \mathcal{K}_2(x) + \mathcal{K}_3(x) + \mathcal{K}_4(x).\end{aligned} \tag{5.3.20}$$

Since the d.f. F_{S_κ} belongs to the class $\mathcal{OL}$ by Lemma 5.2.7, applying (5.3.12) we get

$$\limsup_{x\to\infty} \frac{\mathcal{K}_1(x)}{\mathbb{P}(S_\eta > x)} = \limsup_{x\to\infty} \mathcal{J}_1(x) < \infty. \tag{5.3.21}$$

Now we consider the summand $\mathcal{K}_2(x)$. Since F_{S_κ} is $\mathcal{O}$-exponential, we have

$$\sup_{x\geqslant 0} \frac{\mathbb{P}(S_\kappa > x-1)}{\mathbb{P}(S_\kappa > x)} \leqslant c_{30}$$

for some constant $c_{30} > 0$.

Next, the third condition of Theorem 5.1.4 gives

$$\sup_{x\geqslant c_{31}} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \leqslant c_{32}$$

for some constants $c_{31} > 2$ and $c_{32} > 0$ and all $k \in \mathbb{N}$.

By Lemma 5.2.4 (with $v = c_{31}$), we have

$$\frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\left\{ \sup_{z\geqslant x-c_{31}+1} \frac{\mathbb{P}(S_\kappa > z-1)}{\mathbb{P}(S_\kappa > z)}, \sup_{z\geqslant c_{31}} \frac{\overline{F}_{\xi_{\kappa+1}}(z-1)}{\overline{F}_{\xi_{\kappa+1}}(z)} \right\}.$$

Consequently, we obtain

$$\sup_{x\geqslant c_{31}} \frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\{c_{30}, c_{32}\} := c_{33}.$$

Applying Lemma 5.2.4 again for the sum $S_{\kappa+2} = S_{\kappa+1} + \xi_{\kappa+2}$ (with $v = x/2 + 1/2$) we get

$$\frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant \max\left\{ \sup_{z\geqslant \frac{x}{2}+\frac{1}{2}} \frac{\mathbb{P}(S_{\kappa+1} > z-1)}{\mathbb{P}(S_{\kappa+1} > z)}, \sup_{z\geqslant \frac{x}{2}+\frac{1}{2}} \frac{\overline{F}_{\xi_{\kappa+2}}(z-1)}{\overline{F}_{\xi_{\kappa+2}}(z)} \right\}.$$

If $x \geqslant 2(c_{31}-1)+1$, then $x/2 + 1/2 \geqslant c_{31}$. Therefore, by the last inequality we obtain

$$\sup_{x\geqslant 2(c_{31}-1)+1} \frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant c_{33}.$$

Applying Lemma 5.2.4 once again (with $v = x/3 + 2/3$) we get

$$\frac{\mathbb{P}(S_{\kappa+3} > x-1)}{\mathbb{P}(S_{\kappa+3} > x)} \leqslant \max\left\{ \sup_{z\geqslant \frac{2x}{3}+\frac{1}{3}} \frac{\mathbb{P}(S_{\kappa+2} > z-1)}{\mathbb{P}(S_{\kappa+2} > z)}, \sup_{z\geqslant \frac{x}{3}+\frac{2}{3}} \frac{\overline{F}_{\xi_{\kappa+3}}(z-1)}{\overline{F}_{\xi_{\kappa+3}}(z)} \right\}.$$

If $x \geqslant 3(c_5 31-1)+1$, then $2x/3+1/3 \geqslant 2(c_{31}-1)+1$ and $x/3+2/3 \geqslant c_{31}$. Hence, the last estimate yields

$$\sup_{x \geqslant 3(c_{31}-1)+1} \frac{\mathbb{P}(S_{\kappa+3} > x-1)}{\mathbb{P}(S_{\kappa+3} > x)} \leqslant c_{33}.$$

Continuing the process we conclude that

$$\sup_{x \geqslant k(c_{31}-1)+1} \frac{\mathbb{P}(S_{\kappa+k} > x-1)}{\mathbb{P}(S_{\kappa+k} > x)} \leqslant c_{33} \tag{5.3.22}$$

for all $k \in \mathbb{N}$.

We can suppose that $K = c_{31}$ in representation (5.3.20). In this case, it follows from inequality (5.3.22) that

$$\begin{aligned} &\limsup_{x\to\infty} \frac{\mathcal{K}_2(x)}{\mathbb{P}(S_\eta > x)} \\ &\quad \leqslant \limsup_{x\to\infty} \frac{c_{33}}{\mathbb{P}(S_\eta > x)} \sum_{1 \leqslant k \leqslant \frac{x-1}{c_{31}-1}} \mathbb{P}(S_{\kappa+k} > x)\mathbb{P}(\eta = \kappa + k) \\ &\quad \leqslant c_{33}. \end{aligned} \tag{5.3.23}$$

It is easily seen that

$$\limsup_{x\to\infty} \frac{\mathcal{K}_4(x)}{\mathbb{P}(S_\eta > x)} \leqslant 1, \tag{5.3.24}$$

so it remains to get a bound for the summand $\mathcal{K}_3(x)$.

Applying Lemma 5.2.5 we obtain

$$\mathcal{K}_3(x) \leqslant A \sum_{k > \frac{x-1}{c_{31}-1}} \mathbb{P}(\eta = \kappa + k) \left(\sum_{l=1}^{k} \left(1 - \sup_{x\in\mathbb{R}} \mathbb{P}(x - 1 \leqslant \xi_{\kappa+l} \leqslant x) \right) \right)^{-1/2}$$

with some absolute positive constant A. By the fourth condition of the theorem, we have

$$\frac{1}{k} \sum_{l=1}^{k} \sup_{x\in\mathbb{R}} \left(\overline{F}_{\xi_{\kappa+l}}(x-1) - \overline{F}_{\xi_{\kappa+l}}(x) \right) \leqslant 1 - \Delta$$

for some $0 < \Delta < 1$ and all k large enough. Hence, for such values of k, we get

$$\sum_{l=1}^{k} \left(1 - \sup_{x\in\mathbb{R}} \mathbb{P}(x - 1 \leqslant \xi_{\kappa+l} \leqslant x) \right) \geqslant k\Delta.$$

From the last inequality it follows that

$$\begin{aligned}\mathcal{K}_3(x) &\leqslant \frac{A}{\sqrt{\Delta}} \sum_{k>\frac{x-1}{c_{31}-1}} \frac{1}{\sqrt{k}} \mathbb{P}(\eta = \kappa + k) \\ &\leqslant \frac{A}{\sqrt{\Delta}} \sqrt{\frac{c_{31}-1}{x-1}} \mathbb{P}\left(\eta > \kappa + \frac{x-1}{c_{31}-1}\right)\end{aligned}$$

for all x large enough. Therefore,

$$\begin{aligned}&\limsup_{x\to\infty} \frac{\mathcal{K}_3(x)}{\mathbb{P}(S_\eta > x)} \\ &\leqslant \frac{A}{\sqrt{\Delta}} \frac{\sqrt{c_{31}-1}}{\mathbb{P}(\eta = \kappa)} \limsup_{x\to\infty} \frac{\overline{F}_\eta\left(\frac{x-1}{c_{31}-1}\right)}{\sqrt{x-1}\,\overline{F}_{\xi_\kappa}(x-1)} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_\kappa}(x-1)}{\overline{F}_{\xi_\kappa}(x)} \\ &< \infty \qquad (5.3.25)\end{aligned}$$

by (5.3.19) and the last condition of the theorem. Representation (5.3.20) and bounds (5.3.21), (5.3.23), (5.3.24) and (5.3.25) yield the desired inequality (5.3.18), which completes the proof of the theorem. □

Proof of Theorem 5.1.5. The conditions of the theorem and Lemma 5.2.10 imply that

$$\sup_{x\in\mathbb{R}} \frac{\overline{F}_{S_n}(x-1)}{\overline{F}_{S_n}(x)} \leqslant \sup_{x\in\mathbb{R}} \frac{\overline{F}_{\xi_\varkappa}(x-1)}{\overline{F}_{\xi_\varkappa}(x)} := c_{34}$$

for all $n \geqslant \varkappa$ and for some constant $c_{34} > 0$. If $x > 1$, then

$$\begin{aligned}\overline{F}_{S_\eta}(x-1) &= \sum_{n=\varkappa}^{\infty} \overline{F}_{S_n}(x-1)\mathbb{P}(\eta = n) \\ &\leqslant c_{34} \sum_{n=\varkappa}^{\infty} \overline{F}_{S_n}(x)\mathbb{P}(\eta = n) \\ &= c_{34}\overline{F}_{S_\eta}(x),\end{aligned}$$

and the assertion of the theorem follows. □

Proof of Theorem 5.1.6. For all $x > 0$, we have

$$\begin{aligned}\overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcup_{k=1}^{n} \xi_k > x\right) \mathbb{P}(\eta = n) \\ &\leqslant \sum_{n=1}^{\infty} \sum_{k=1}^{n} \overline{F}_{\xi_k}(x)\mathbb{P}(\eta = n).\end{aligned}$$

If x is large enough ($x \geqslant x_1$), then inequality (5.1.1) implies that

$$\sum_{k=1}^{n} \overline{F}_{\xi_k}(x) \leqslant c_{35}\varphi(n)\overline{F}_{\xi_\varkappa}(x)$$

for all $n \geqslant 1$ and for some constant $c_{35} > 0$. Hence, if $x \geqslant x_1 + 1$, then

$$\begin{aligned} \overline{F}_{\xi_{(\eta)}}(x-1) &\leqslant c_{35} \sum_{n=1}^{\infty} \varphi(n)\overline{F}_{\xi_\varkappa}(x-1)\mathbb{P}(\eta = n) \\ &= c_{35}\overline{F}_{\xi_\varkappa}(x-1)\mathbb{E}\varphi(\eta)\mathbb{1}_{\{\eta\geqslant 1\}}. \end{aligned} \tag{5.3.26}$$

To get a lower bound, we notice that

$$\begin{aligned} \overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcup_{k=1}^{n} \{\xi_k > x\}\right) \mathbb{P}(\eta = n) \\ &\geqslant \mathbb{P}\left(\bigcup_{k=1}^{\varkappa} \{\xi_k > x\}\right) \mathbb{P}(\eta = \varkappa) \\ &\geqslant \mathbb{P}(\xi_\varkappa > x)\mathbb{P}(\eta = \varkappa) \\ &= \overline{F}_{\xi_\varkappa}(x)\mathbb{P}(\eta = \varkappa). \end{aligned} \tag{5.3.27}$$

From bounds (5.3.26) and (5.3.27) we get

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi_{(\eta)}}(x-1)}{\overline{F}_{\xi_{(\eta)}}(x)} \leqslant \frac{c_{35}\,\mathbb{E}\varphi(\eta)\mathbb{1}_{\{\eta\geqslant 1\}}}{\mathbb{P}(\eta = \varkappa)} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_\varkappa}(x-1)}{\overline{F}_{\xi_\varkappa}(x)} < \infty$$

because $F_{\xi_\varkappa} \in \mathcal{OL}$, $\varkappa \in \operatorname{supp}(\eta)$ and $\mathbb{E}\varphi(\eta)\mathbb{1}_{\{\eta\geqslant 1\}} < \infty$. The last inequality implies that $F_{\xi_{(\eta)}} \in \mathcal{OL}$, and the theorem is proved. □

Proof of Theorem 5.1.7. It is obvious that

$$\overline{F}_{\xi_{(\eta)}}(x-1) = \sum_{n=1}^{\infty} \mathbb{P}(\xi_{(n)} > x-1)\mathbb{P}(\eta = n) \tag{5.3.28}$$

for all $x > 1$. The conditions of the theorem implies that $\overline{F}_{\xi_k}(x) > 0$ for all $x \in \mathbb{R}$ and $k \geqslant 1$. Hence, by part (i) of Lemma 5.2.8, we have

$$\frac{\mathbb{P}(\xi_{(n)} > x-1)}{\mathbb{P}(\xi_{(n)} > x)} \leqslant \max_{1\leqslant k\leqslant n} \frac{\overline{F}_{\xi_k}(x-1)}{\overline{F}_{\xi_k}(x)} \leqslant \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x-1)}{\overline{F}_{\xi_k}(x)}$$

for all $n \geqslant 1$. Consequently, we obtain

$$\begin{aligned} \overline{F}_{\xi_{(\eta)}}(x-1) &\leqslant \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x-1)}{\overline{F}_{\xi_k}(x)} \sum_{n=1}^{\infty} \mathbb{P}(\xi_{(n)} > x)\mathbb{P}(\eta = n) \\ &= \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x-1)}{\overline{F}_{\xi_k}(x)}\, \overline{F}_{\xi_{(\eta)}}(x) \end{aligned}$$

for all $x > 1$, and the assertion of the theorem follows. □

Proof of Theorem 5.1.8. By Lemma 5.2.10, the d.f.'s F_{S_n} belong to the class $\mathcal{OL}$ for all fixed n and

$$\sup_{x\in\mathbb{R}} \frac{\overline{F}_{S_n}(x-1)}{\overline{F}_{S_n}(x)} \leqslant c_{36} := \sup_{x\in\mathbb{R}} \frac{\overline{F}_{\xi_1}(x-1)}{\overline{F}_{\xi_1}(x)}. \tag{5.3.29}$$

Lemma 5.2.9 implies that $F_{S_{(n)}} \in \mathcal{OL}$ for all $1 \leqslant n \leqslant K$, where $K \geqslant 1$ is arbitrary. Therefore, we obtain

$$\max_{1\leqslant n\leqslant K} \left\{ \sup_{x\in\mathbb{R}} \frac{\overline{F}_{S_{(n)}}(x-1)}{\overline{F}_{S_{(n)}}(x)} \right\} \leqslant c_{37} \tag{5.3.30}$$

for some constant $c_{37} = c_{37}(K) \geqslant 1$.

For all $x > 1$, we have

$$\begin{aligned}
\overline{F}_{S_{(\eta)}}(x-1) \leqslant & \sum_{1\leqslant n\leqslant K} \mathbb{P}(S_{(n)} > x-1)\mathbb{P}(\eta = n) \\
& + \sum_{n>K} \sum_{k=1}^{n} \mathbb{P}(S_k > x-1)\mathbb{P}(\eta = n) \\
& = \sum_{1\leqslant n\leqslant K} \overline{F}_{S_{(n)}}(x-1)\mathbb{P}(\eta = n) + \mathbb{P}(\eta > K) \sum_{n=1}^{K} \overline{F}_{S_n}(x-1) \\
& + \sum_{n>K} \overline{F}_{S_n}(x-1)\mathbb{P}(\eta \geqslant n).
\end{aligned}$$

Inequalities (5.3.29) and (5.3.30) and conditions (5.1.2) of the theorem give

$$\begin{aligned}
\overline{F}_{S_{(\eta)}}(x-1) \; & \leqslant \; c_{37} \sum_{1\leqslant n\leqslant K} \overline{F}_{S_{(n)}}(x)\mathbb{P}(\eta = n) \\
& + \; c_{36} \left(\mathbb{P}(\eta > K) \sum_{n=1}^{K} \overline{F}_{S_n}(x) + \sum_{n>K} \overline{F}_{S_n}(x)\mathbb{P}(\eta \geqslant n) \right) \\
& \leqslant \; c_{37} \overline{F}_{S_{(\eta)}}(x) \\
& + \; c_{38} \left(\sum_{n=1}^{K} \overline{F}_{S_n}(x)\mathbb{P}(\eta = n) + \sum_{n>K} \overline{F}_{S_n}(x)\mathbb{P}(\eta = n) \right) \\
& \leqslant \; (c_{37} + c_{38})\overline{F}_{S_{(\eta)}}(x)
\end{aligned}$$

for all K large enough with some constant $c_{38} = c_{38}(K) > 0$. The last inequality implies that the d.f. $F_{S_{(\eta)}}$ is O-exponential. □

Proof of Theorem 5.1.9. It is clear that

$$\overline{F}_{S_{(\eta)}}(x) = \sum_{n\in\operatorname{supp}(\eta)\setminus\{0\}} \mathbb{P}(S_{(n)} > x)\mathbb{P}(\eta = n).$$

Since $\operatorname{supp}(\eta)$ is finite, from inequality (4.3.1) we conclude that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_{(\eta)}}(x-1)}{\overline{F}_{S_{(\eta)}}(x)} \leqslant \max_{n\in\operatorname{supp}(\eta)\setminus\{0\}} \left\{ \limsup_{x\to\infty} \frac{\mathbb{P}(S_{(n)} > x-1)}{\mathbb{P}(S_{(n)} > x)} \right\}.$$

For any $n \in \operatorname{supp}(\eta)$, $n \geqslant 1$, the d.f. of S_n is $\mathcal{O}$-exponential by Lemma 5.2.10. Therefore, by Lemma 5.2.9, the d.f. of $S_{(n)} = \max\{S_1, \ldots, S_n\}$ is $\mathcal{O}$-exponential for any $n \in \operatorname{supp}(\eta) \setminus \{0\}$. The assertion of the theorem follows from the last inequality. $\square$

Chapter 6

Conclusions

Here we summarize the main results obtained in this thesis:

- We give two assertions, which describe conditions under which the randomly stopped sum belongs to the class of dominatedly varying distributions. We consider the case where no moment conditions on the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are required, as well as the case where these r.v.'s have finite means of some order. We obtain conditions under which $F_{S_\eta} \in \mathcal{D}$ when the r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are independent but not necessarily identically distributed.

- We find conditions under which the d.f. F_{S_η} belongs to the class of exponential distributions. One of the main theorems deals with the case of a finitely supported counting r.v. η, whereas other assertions imply that the right tail of η is unbounded. We consider both non-negative and real-valued r.v.'s $\{\xi_1, \xi_2, \ldots\}$. We obtain conditions under which $F_{S_\eta} \in \mathcal{L}(\gamma)$ for some $\gamma > 0$ when $\{\xi_1, \xi_2, \ldots\}$ are possibly non-identically distributed r.v.'s.

- We find conditions under which the randomly stopped sum S_η, the randomly stopped maximum $\xi_(\eta)$ and the randomly stopped maximum of sums $S_{(\eta)}$belong to the class of $\mathcal{O}$-exponential distributions. We study the case where the sequence $\{\xi_1, \xi_2, \ldots\}$ consists of not necessarily identically distributed r.v.'s.

7 skyrius

Santrauka

7.1 Įžanga

Mokslinė problema, aktualumas ir naujumas

Disertacijoje tiriama atsitiktinai sustabdyta suma $S_\eta = \xi_1 + \cdots + \xi_n$, atsitiktinai sustabdytas maksimumas $\xi_{(\eta)} = \max\{0, \xi_1, \ldots, \xi_\eta\}$ ir atsitiktinai sustabdytas sumų maksimumas $S_{(\eta)} = \max\{S_0, S_1, \ldots, S_\eta\}$, kai $\{\xi_1, \xi_2, \ldots\}$ yra nepriklausomi atsitiktiniai dydžiai, o η skaičiuojantis atsitiktinis dydis, nepriklausantis nuo atsitiktinių dydžių $\{\xi_1, \xi_2, \ldots\}$. Sakykime, kad η yra skaičiuojantis atsitiktinis dydis, jei η yra neneigiamas, sveikareikšmis ir neišigimęs taške 0. Disertacijoje ieškome sąlygų, kurioms esant nepriklausomų atsitiktinių dydžių S_η, $\xi_{(\eta)}$, $S_{(\eta)}$ pasiskirstymo funkcijos F_{S_η}, $F_{\xi_{(\eta)}}$, $F_{S_{(\eta)}}$ priklauso tam tikroms sunkiauodegių skirstinių klasėms. Šio tyrimo motyvacija susijusi su draudimine ir finansine veikla, kur tradiciškai nagrinėjamos problemos, atsirandančios dėl ekstremalių ar retų įvykių. Matematiniu požiūriu, bet kokio draudimo verslo sėkmė priklauso nuo asimptotinio S_η ar $S_{(\eta)}$ pasiskirstymo funkcijų elgesio. Jei šių dydžių pasiskirstymo funkcijos uodega yra lengva, tai bankroto tikimybė yra maža didelėms pradinio kapitalo vertėms. Įprastai tokiomis sąlygomis ši tikimybė gęsta eksponentiniu greičiu (žr. pvz., [5, 32, 34, 36, 49, 54, 57]) pradiniam kapitalui neaprėžtai augant. Jei dydžių S_η ar $S_{(\eta)}$ skirstiniai priklauso sunkiauodegių klasei, tada modelio aprašančio draudiminę veiklą bankroto tikimybė didėjant pradiniam kapitalui gali gęsti tik laipsniškai (žr. pvz., [31, 38, 39, 43, 44, 45, 48, 57, 59]). Taigi tyrimo pradžioje būtina išsiaiškinti kokio pavidalo yra minėtų atsitiktinių dydžių pasiskirstymo funkcijų uodegų elgesys.

Viena iš svarbiausių rizikos teorijos ar draudimo matematikos tyrimų krypčių yra bankroto tikimybės nagrinėjimas, kai ieškinių dydžių pasiskirstymo funkcijos turi sunkias uodegas. Šiuo atveju bankrotas įprastai įvyksta dėl vieno didelio ieškinio, o teoriniai rezultatai gaunami tik specifinėms sunkiauodegių skirstinių klasėms, nes asimptotinis bankroto tikimybės elgesys ženkliai skiriasi skirtingėms sunkiauodegių skirstinių poklasėms. Todėl visada svarbu žinoti ar suminių ieškinių S_η, $S_{(\eta)}$ pasiskirstymai priklauso tam tikroms klasėms.

Antra vertus, darbe gauti rezultatai susiję su klasikine uždarumo problema. Bingham, Goldie ir Teugels (žr. [9]) vieni iš pirmųjų šios srities mokslininkų. Verta paminėti, kad visi klasikiniai rezultatai, susiję su uždarumo problema, nagrinėja vienodai pasiskirsčiusius atsitiktinius dydžius $\{\xi_1, \xi_2, \ldots\}$. Tuo tarpu šioje disertacijoje yra nagrinėjami ne tik vienodai pasiskirstę atsitiktiniai dydžiai. Parodoma, kad atsitiktinių dydžių S_η, $\xi_{(\eta)}$, $S_{(\eta)}$ pasiskirstymo funkcijos pasilieka tam tikroje reguliarumo klasėje, jeigu tik keli ar, atskirais atvejais, visi skirstiniai yra toje klasėje.

Visi disertacijoje pateikti rezultatai yra nauji ir originalūs. Disertacijos rezultatai publikuoti 5 moksliniuose straipsniuose.

Tikslas ir uždaviniai

Pagrindinis disertacijos tikslas - rasti sąlygas nepriklausomiems atsitiktiniams dydžiams $\{\xi_1, \xi_2, \ldots\}$ ir skaičiuojančiam atsitiktiniam dydžiui η, kurioms esant S_η, $\xi_{(\eta)}$ ir $S_{(\eta)}$ paskirstymo funkcijos priklauso tam tikroms funkcijų klasėms.

Tikslas pasiekiamas išsprendus tokius uždavinius:

- Nustatyti sąlygas, kurioms esant atsitiktinės sustabdytos sumos S_η pasiskirstymo funkcija priklauso dominuojamai kintančių skirstinių klasei ($\mathcal{D}$).

- Nustatyti sąlygas, kurioms esant atsitiktinės sustabdytos sumos S_η pasiskirstymo funkcija priklauso eksponentinių skirstinių klasei ($\mathcal{L}(\gamma)$, $\gamma \geqslant 0$).

- Nustatyti sąlygas, kurioms esant atsitiktinės sustabdytos sumos S_η, atsitiktinio sustabdyto maksimumo $\xi_{(\eta)}$ ir atsitiktinio sustabdyto sumų maksimumo $S_{(\eta)}$ pasiskirstymo funkcijos priklauso $\mathcal{O}$-eksponentinių skirstinių klasei ($\mathcal{OL}$).

Tyrimo metodika

Kuriai iš skirstinių klasių priklauso tam tikras skirstinys paprastai susijęs su pasiskirstymo funkcijos uodegos elgesiu. Todėl, norint įvertinti atsitiktinės sustabdytos sumos ir atsitiktinio sustabdyto sumų maksimumo uodegos pasiskirstymą, šioje disertacijoje naudojami tipiniai tikimybių teorijos ir matematinės analizės metodai. Dauguma sunkiauodegių paskirstymo funkcijų darinių įvertinimų yra susiję su Matuszewska indekso ir L-indekso savybėmis.

7.2 Pagrindiniai rezultatai

7.2.1 Apibrėžimai

Darbe nagrinėjame kelias sunkiauodegių skirstinių klases. Pateiksime apibrėžimus nagrinėjamų disertacijoje pagrindinių klasių $\mathcal{OL}$, $\mathcal{D}$ ir $\mathcal{L}(\gamma)$, $\gamma \geqslant 0$, ir susijusių klasių $\mathcal{L}$, $\mathcal{S}$ ir $\mathcal{S}*$. Viena iš priežasčių yra ta, kad daugybė metodų naudojamų įvairiuose teiginiuose yra panašūs visoms klasėms, kita - noras pateikti išsamesnę sunkiauodegių skirstinių klasių informaciją.

7.2.1 APIBRĖŽIMAS. *Sakoma, kad pasiskirstymo funkcija F turi sunkią uodegą (priklauso klasei $\mathcal{H}$), jeigu bet kuriam $\delta > 0$*

$$\lim_{x\to\infty} e^{\delta x}\ \overline{F}(x) = \infty.$$

Čia, ir toliau: $\overline{F}(x) = 1 - F(x)$, $x \in \mathbb{R}$ - pasiskirstymo funkcijos F uodega.

Svarbiausių sunkiauodegių skirstinių klasių apibrėžimus pateiksime žemiau. Pradėsime nuo subeksponentinių ir ilgauodegių skirstinių klasių, kurias pirmasis aprašė Chistyakov [13]. Pavyzdžiui, jis įrodė, kad subeksponetinė skirstinių klasė yra ilgauodegių skirstinių klasės dalis.

7.2.2 APIBRĖŽIMAS. *Sakoma, kad neneigiamo atsitiktinio dydžio pasiskirstymo funkcija F yra subeksponentinė (priklauso klasei $\mathcal{S}$), jeigu*

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2.$$

7.2.3 APIBRĖŽIMAS. *Sakoma, kad pasiskirstymo funkcija F priklauso klasei $\mathcal{L}$, jeigu bet kuriam teigiamam y:*

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1.$$

Shimura ir Watanabe [58] aprašė klasę $\mathcal{OL}$, kuri yra platesnė už klasę $\mathcal{L}$, bet kažkiek panaši į ją savo struktūra.

7.2.4 APIBRĖŽIMAS. *Sakoma, kad pasiskirstymo funkcija F yra $\mathcal{O}$-eksponentinė (priklauso klasei $\mathcal{OL}$), jeigu bet kuriam fiksuotam $a \in \mathbb{R}$*

$$0 < \liminf_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} < \infty.$$

Faktiškai, pasiskirstymo funkcija F priklauso klasei $\mathcal{OL}$ tada ir tik tada, kai

$$\sup_{x\geqslant 0} \frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty.$$

Paskutinė sąlyga rodo, kad klasė $\mathcal{OL}$ yra gana plati. Dabar mes apibūdinsime populiariausias $\mathcal{OL}$ poklases.

7.2.5 APIBRĖŽIMAS. *Sakoma, kad pasiskirstymo funkcija F priklauso klasei $\mathcal{L}(\gamma)$, $\gamma \geqslant 0$, jeigu bet kuriam fiksuotam $y > 0$*

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-y\gamma}.$$

Kai $\gamma > 0$, klasė $\mathcal{L}(\gamma)$ buvo įvesta Embrechts ir Goldie [26].

Kai $\gamma = 0$, akivaizdu, kad $\mathcal{L}(\gamma) = \mathcal{L}$.

Kita žinoma sunkiauodegių skirstinių klasė - dominuojamai kintančių skirstinių klasė $\mathcal{D}$, pirmą kartą nusakyta Feller [32].

7.2.6 APIBRĖŽIMAS. *Sakoma, kad pasiskirstymo funkcija F priklauso klasei $\mathcal{D}$, jeigu bet kuriam $y \in (0,1)$*

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

7.2.7 APIBRĖŽIMAS. *Bet kuriai pasiskirstymo funkcijai F viršutinis Matuszewska indeksas nusakomas lygybe*

$$J_F^+ = -\lim_{y\to\infty} \frac{1}{\log y} \log\left(\liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}\right).$$

Dabar mes nusakysime tarpusavio ryšius tarp svarbiausių sunkiauodegių skirstinių klasių, dauguma tų ryšių yra gerai žinomi. Aukščiau pateikti apibrėžimai kartu su [13, Lemma 2], [24, Lemma 9], [29, Lemma 1.3.5 (a)] ir [37, Lemma 1] reiškia, kad

$$\mathcal{L}\cap\mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}, \quad \mathcal{D} \subset \mathcal{H}.$$

Be to

$$\mathcal{L} \subset \mathcal{OL}, \quad \mathcal{D} \subset \mathcal{OL} \quad \text{and} \quad \bigcup_{\gamma>0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

7.2.2 Atsitiktinės sumos dominuojamai kintančių skirstinių klasėje

Mes formuluojame dvi teoremas apibūdinančias sąlygas, kurioms esant atsitiktinė sustabdyta suma S_η priklauso klasei $\mathcal{D}$. 7.2.1 teoremoje nereikalaujama jokių sąlygų nepriklausomų atsitiktinių dydžių $\{\xi_1, \xi_2, \ldots\}$ momentams, tuo tarpu sąlygos 7.2.2 teoremoje rodo, kad atsitiktiniai dydžiai $\{\xi_1, \xi_2, \ldots\}$ turi turėti baigtinius vidurkius.

7.2.1 teorema. [21] *Sakykime $\{\xi_1, \xi_2, \ldots\}$ yra nepriklausomi neneigiami atsitiktiniai dydžiai, o η yra skaičiuojantis atsitiktinis dydis nepriklausantis nuo atsitiktinių dydžių $\{\xi_1, \xi_2, \ldots\}$. Tada $F_{S_\eta} \in \mathcal{D}$, jei tenkinamos sąlygos:*

(a) $F_{\xi_\kappa} \in \mathcal{D}$, *kažkokiam* $\kappa \in \operatorname{supp}(\eta)$;

(b) $\displaystyle\limsup_{x\to\infty} \sup_{n\geqslant\kappa} \frac{1}{n\overline{F}_{\xi_\kappa}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$;

(c) $\mathbb{E}\eta^{p+1} < \infty$, *kažkokiam* $p > J^+_{F_{\xi_\kappa}}$.

7.2.2 teorema. [21] *Tegul* $\{\xi_1, \xi_2, \dots\}$ *nepriklausomi neneigiami atsitiktiniai dydžiai, o* η *yra skaičiuojantis atsitiktinis dydis nepriklausantis nuo* $\{\xi_1, \xi_2, \dots\}$. *Tegul, be to kažkokiam* $\kappa \in \operatorname{supp}(\eta)$ *tenkinama 7.2.1 teoremos (b) sąlyga kartu su reikalavimais:*

$$F_{\xi_\kappa} \in \mathcal{D}, \quad \max\{\mathbb{E}\xi_\kappa, \mathbb{E}\eta\} < \infty,$$

$$\limsup_{u\to\infty} \sup_{n\geqslant\kappa} \frac{1}{n} \sum_{\substack{1\leqslant k\leqslant n\\ \mathbb{E}\xi_k\geqslant u}} \mathbb{E}\xi_k = 0, \quad \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\xi_k > 0.$$

Tada $F_{S_\eta} \in \mathcal{D}$, *tada ir tik tada kai* $\min\{F_{\xi_\kappa}, F_\eta\} \in \mathcal{D}$.

7.2.3 Eksponentinių skirstinių atsitiktinių sumų savybės

Šiame skyrelyje formuluojame teoremas apibūdinančias sąlygas, kurioms esant atsitiktinė sustabdyta suma S_η priklauso klasei $\mathcal{L}(\gamma)$ kažkokiam $\gamma \geqslant 0$. 7.2.3 teoremoje skaičiuojantis atsitiktinis dydis η turi baigtinę atramą, tuo tarpu 7.2.4 ir 7.2.5 teoremose skaičiuojančio atsitiktinio dydžio η dešinė uodega yra neapribuojama. Atsitiktiniai dydžiai $\{\xi_1, \xi_2, \dots\}$ yra neneigiami 7.2.3 ir 7.2.5 teoremose, tuo tarpu 7.2.4 teoremoje šie dydžiai gali įgyti realias reikšmes.

Jeigu $\gamma > 0$, tai visi pateikiami rezultatai yra nauji. Jeigu $\gamma = 0$, tuomet visi toliau pateikti teiginiai gali būti išvesti iš įrodytų teoremų darbe [47]. Siekiant išsamumo į mūsų prielaidas įtraukiame atvejį $\gamma = 0$. Be to, mes taikome tuos pačius metodus, įrodydami savo rezultatus kai $\gamma > 0$ ir $\gamma = 0$.

7.2.3 teorema. [18] *Tarkime* $\{\xi_1, \xi_2, \dots \xi_n\}$, $n \geqslant 1$ *yra nepriklausomi neneigiami atsitiktiniai dydžiai ir* η *yra skaičiuojantis atsitiktinis dydis, nepriklausantis nuo jų su baigtine atrama* $\operatorname{supp}(\eta) \subset \{0, 1, \dots, n\}$. *Tada* $F_{S_\eta} \in \mathcal{L}(\gamma)$ *kažkokiam* $\gamma \geqslant 0$, *jei* $F_{\xi_\nu} \in \mathcal{L}(\gamma)$ *kuriam nors* $1 \leqslant \nu \leqslant \min\Big\{\operatorname{supp}(\eta) \setminus \{0\}\Big\}$ *ir tenkinama sąlyga* $F_{\xi_k} \in \mathcal{L}(\gamma)$ *arba* $\overline{F}_{\xi_k}(x) = o\Big(\overline{F}_{\xi_\nu}(x)\Big)$, *visiems* $k \in \Big\{1, 2, \dots, \max\{\operatorname{supp}(\eta)\}\Big\}$.

7.2.4 teorema. [18] *Tegul* $\{\xi_1, \xi_2, \dots\}$ *nepriklausomi, realias reikšmes įgyjantys, atsitiktiniai dydžiai su pasiskirstymo funkcijomis* $\{F_{\xi_1}, F_{\xi_2}, \dots\}$ *tokiomis, kad*

$$\sup_{k\geqslant 1} \left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y} \right| \underset{x\to\infty}{\longrightarrow} 0 \tag{7.2.1}$$

kažkokiam $\gamma \geqslant 0$ *ir bet kuriam fiksuotam* $y \geqslant 0$. *Papildomai, tarkime* η *yra skaičiuojantis atsitiktinis dydis nepriklausomas nuo atsitiktinių dydžių* $\{\xi_1, \xi_2, \ldots\}$ *toks, kad*

$$\frac{\mathbb{P}(\eta = k+1)}{\mathbb{P}(\eta = k)} \underset{k\to\infty}{\longrightarrow} 0. \tag{7.2.2}$$

Tada $F_{S_\eta} \in \mathcal{L}(\gamma)$.

Atkreipime dėmesį, kad sąlyga (7.2.1) ekvivalenti dvipusiam apribojimui

$$\mathrm{e}^{-\gamma y} \leqslant \liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} \leqslant \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} \leqslant \mathrm{e}^{-\gamma y},$$

kuris galioja kažkokiam $\gamma \geqslant 0$ ir bet kuriam fiksuotam $y \geqslant 0$.

Be to, mes pastebime, kad iš sąlygos (7.2.2) išplaukia, kad $\mathbb{P}(\eta = k) > 0$ visiems pakankamai dideliems k.

7.2.5 teorema. [18] *Tegul* $\{\xi_1, \xi_2, \ldots\}$ *yra nepriklausomi neneigiami atsitiktiniai dydžiai su pasiskirstymo funkcijomis* $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, *ir tegul* η *yra skaičiuojantis atsitiktinis dydis nepriklausomas nuo* $\{\xi_1, \xi_2, \ldots\}$. *Atsitiktinės sustabdytos sumos pasiskirstymo funkcija* F_{S_η} *priklauso klasei* $\mathcal{L}(\gamma)$, *su* $\gamma \geqslant 0$, *jei kažkokiems* $\varkappa \geqslant 1$ *ir* $1 \leqslant \nu \leqslant \varkappa$ *tenkinamos sąlygos:*

(i) $\nu \leqslant \min\big\{\mathrm{supp}(\eta) \setminus \{0\}\big\}$;

(ii) $F_{\xi_\nu} \in \mathcal{L}(\gamma)$;

(iii) $F_{\xi_k} \in \mathcal{L}(\gamma)$ *ar* $\overline{F}_{\xi_k}(x) = o\big(\overline{F}_{\xi_\nu}(x)\big)$, *kai* $1 \leqslant k \leqslant \varkappa$;

(iv) $\sup\limits_{k\geqslant \varkappa+1} \left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y} \right| \underset{x\to\infty}{\longrightarrow} 0$, *kai* $y \geqslant 0$;

(v) $\mathbb{P}(\eta = k+1)/\mathbb{P}(\eta = k) \underset{k\to\infty}{\longrightarrow} 0$.

7.2.4 $\mathcal{O}$-eksponentinių skirstinių uždarumo savybės

Šiame skyrelyje nagrinėsime kokiais atvejais atsitiktinės sustabdytos sumos S_η, atsitiktinio sustabdyto maksimumo $\xi_{(\eta)}$ ir atsitiktinio sustabdyto sumų maksimumo $S_{(\eta)}$ pasiskirstymo funkcijos priklauso $\mathcal{OL}$ klasei su prielaida, kad atsitiktiniai dydžiai $\{\xi_1, \xi_2, \ldots\}$ ir η tenkina tam tikras sąlygas.

7.2.6 teoremoje pateikiame pakankamas sąlygas, kurioms esant atsitiktinės sustabdytos sumos S_η pasiskirstymo funkcija priklauso klasei $\mathcal{OL}$.

7.2.6 teorema. [22] *Tegul* $\{\xi_1, \xi_2, \ldots\}$ *yra nepriklausomi realieji (real-valued) atsitiktiniai dydžiai ir* η *yra skaičiuojantis atsitiktinis dydis, nepriklausantis nuo* $\{\xi_1, \xi_2, \ldots\}$*. Tada* $F_{S_\eta} \in \mathcal{OL}$*, jeigu* $F_{\xi_\varkappa} \in \mathcal{OL}$ *indeksui* $\varkappa = \min\{n \in \operatorname{supp}(\eta), n \geqslant 1\}$.

Žemiau pateiktos išvados suformuluotos anksčiau [19], bet jos gali būti gautos ir iš 7.2.6 teoremos. Suformuluoti rezultatai pateikti nebūtinai vienodai pasiskirsčiusiems atsitiktiniams dydžiams.

Pirmoji išvada apibūdina situacija, kai pasiskirstymo funkcijos uodegos $\overline{F}_{\xi_k}$ reikšmės esant dideliems k indeksams yra palyginamos taškuose x ir $x-1$ visiems $x \in [0, \infty)$.

7.2.1 išvada. [19] *Tegul* $\{\xi_1, \xi_2, \ldots\}$ *yra nepriklausomi neneigiami atsitiktiniai dydžiai su pasiskirstymo funkcijomis* $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$*, ir tegul* η *yra skaičiuojantis atsitiktinis dydis, nepriklausantis nuo* $\{\xi_1, \xi_2, \ldots\}$*. Tada* $F_{S_\eta} \in \mathcal{OL}$ *, jeigu galioja tokios trys sąlygos:*

- *Kažkokiam* $\kappa \in \operatorname{supp}(\eta) \setminus \{0\} = \{n \in \mathbb{N} : \mathbb{P}(\eta = n) > 0\}$*, pasiskirstymo funkcija* $F_{\xi_\kappa} \in \mathcal{OL}$*;*
- *Kiekvienam* $k \in \operatorname{supp}(\eta)$*,* $k \leqslant \kappa$*, arba* $\lim\limits_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0$*, arba* $F_{\xi_k} \in \mathcal{OL}$*;*
- $\sup\limits_{x\geqslant 0} \sup\limits_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} < \infty.$

Kadangi kiekvienos pasiskirstymo funkcijos iš $\mathcal{OL}$ klasės reikšmės taškuose x ir $x-1$ yra palyginamos, tai kitas teiginys tiesiogiai išplaukia iš 7.2.1 išvados.

7.2.2 išvada. [19] *Tegul* $\{\xi_1, \xi_2, \ldots\}$ *yra nepriklausomi neneigiami atsitiktiniai dydžiai su bendra pasiskirstymo funkcija* $F_\xi \in \mathcal{OL}$*. Tada atsitiktinės sumos pasiskirstymo funkcija* F_{S_η} *yra yra* $\mathcal{O}$*-eksponentinė bet kuriam skaičiuojančiam atsitiktiniui dydžiui* η*.*

Kitas mūsų rezultatas nagrinėjamas esant skaičiuojančiam atsitiktiniam dydžiui η su baigtine atrama.

7.2.3 išvada. [19] *Tegul* $\{\xi_1, \xi_2, \ldots, \xi_D\}$*,* $D \in \mathbb{N}$ *yra nepriklausomi neneigiami atsitiktiniai dydžiai su pasiskirstymo funkcijomis* $\{F_{\xi_1}, F_{\xi_2}, \ldots F_{\xi_D}\}$ *ir tegul* η *yra skaičiuojantis atsitiktinis dydis, nepriklausantis nuo* $\{\xi_1, \xi_2, \ldots, \xi_D\}$*. Tada* $F_{S_\eta} \in \mathcal{OL}$*, kai tenkinamos tokios trys sąlygos:*

- $\mathbb{P}(\eta \leqslant D) = 1$;
- *Kažkokiam* $\kappa \in \operatorname{supp}(\eta) \setminus \{0\}$, *pasiskirstymo funkcija* $F_{\xi_\kappa} \in \mathcal{OL}$;
- *Kiekvienam* $k \in \{1, 2, \ldots, D\}$, *arba* $\lim\limits_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0$, *arba* $F_{\xi_k} \in \mathcal{OL}$.

Kitos dvi teoremos nusako situacijas, kurioms esant atsitiktinio sustabdyto maksimumo pasiskirstymo funkcija $F_{\xi_{(\eta)}}$ priklauso klasei $\mathcal{OL}$. 7.2.7 teoremoje atsitiktiniai dydžiai $\{\xi_1, \xi_2, \ldots\}$ tenkina nedidelius reikalavimus, bet skaičuojantis atsitiktinis dydis η yra labiau apribotas. 7.2.8 teoremoje situacija kitokia - nepriklausomų atsitiktinių dydžių pasiskirstymo funkcijos $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ yra sulyginamos taškuose x ir $x-1$ bei priklauso $\mathcal{OL}$ klasei, bet skaičiuojantis atsitiktinis dydis η gali būti bet koks.

7.2.7 teorema. [22] *Tegul* $\{\xi_1, \xi_2, \ldots\}$ *yra galimai priklausomi, realias reikšmes įgyjantys, atsitiktiniai dydžiai, ir tegul* η *yra skaičiuojantis atsitiktinis dydis, nepriklausantis nuo* $\{\xi_1, \xi_2, \ldots\}$. *Tarkime, kad egzistuoja* $\varkappa \in \operatorname{supp}(\eta)$ *ir teigiama seka* $\{\varphi(n)\}_{n=1}^{\infty}$ *tokia, kad* $F_{\xi_\varkappa} \in \mathcal{OL}$, $\mathbb{E}\varphi(\eta)\mathbb{1}_{\{\eta \geqslant 1\}} < \infty$ *ir*

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{\varphi(n)\overline{F}_{\xi_\varkappa}(x)} \sum_{k=1}^{n} \overline{F}_{\xi_k}(x) < \infty.$$

Tada $F_{\xi_{(\eta)}} \in \mathcal{OL}$.

7.2.8 teorema. [22] *Tegul* $\{\xi_1, \xi_2, \ldots\}$ *yra nepriklausomi, realias reikšmes įgyjantys, atsitiktiniai dydžiai, ir tegul* η *yra skaičiuojantis atsitiktinis dydis, nepriklausantis nuo* $\{\xi_1, \xi_2, \ldots\}$. *Jeigu*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x-1)}{\overline{F}_{\xi_k}(x)} < \infty,$$

tada $F_{\xi_{(\eta)}} \in \mathcal{OL}$.

Paskutinės dvi teoremos apibūdina situacijas, kurioms esant atsitiktinio sustabdyto sumų maksimumo pasiskirstymo funkcija $F_{S_{(\eta)}}$ priklauso klasei $\mathcal{OL}$. 7.2.9 teoremoje nagrinėjamas atvejis su skaičiuojančiu atsitiktiniu dydžiu η, kuris neturi baigtinės atramos, o 7.2.10 teoremoje priešingai η - turi baigtinę atramą.

7.2.9 teorema. [22] *Tegul* $\{\xi_1, \xi_2, \ldots\}$ *yra nepriklausomi, realias reikšmes įgyjantys, atsitiktiniai dydžiai, ir tegul* η *yra skaičiuojantis atsitiktinis dydis, nepriklausantis nuo* $\{\xi_1, \xi_2, \ldots\}$. *Atsitiktinių sumų atsitiktinio maksimumo* *pasiskirstymo funkcija $F_{S_{(\eta)}}$ priklauso klasei $\mathcal{OL}$, jeigu $F_{\xi_1} \in \mathcal{OL}$ ir atsitiktinis dydis η tenkina šias dvi sąlygas:*

$$\limsup_{n\to\infty} \frac{\mathbb{P}(\eta > n)}{\min\limits_{1\leqslant k\leqslant n} \mathbb{P}(\eta = k)} < \infty \quad and \quad \limsup_{n\to\infty} \frac{\mathbb{P}(\eta \geqslant n)}{\mathbb{P}(\eta = n)} < \infty.$$

7.2.10 teorema. [22] *Tegul $\{\xi_1, \ldots, \xi_m\}$ yra nepriklausomi, realias reikšmes įgyjantys, atsitiktiniai dydžiai su pasiskirstymo funkcijomis $\{F_{\xi_1}, \ldots, F_{\xi_m}\}$. Tegul η skaičiuojantis atsitiktinis dydis nepriklausomas nuo $\{\xi_1, \xi_2, \ldots, \xi_m\}$ ir turintis baigtinę atramą $\operatorname{supp}(\eta)$ tokią, kad $\mu := \max\{n \in \operatorname{supp}(\eta)\} \leqslant m$. Tada $F_{S_{(\eta)}}$ priklauso klasei $\mathcal{OL}$, jeigu $F_{\xi_1} \in \mathcal{OL}$.*

Iš šio skyrelio teiginių išplaukia sekantis tvirtinimas apie vienodai pasiskirsčiusių atsitiktinius dydžius.

7.2.4 išvada. *Tegul $\{\xi_1, \xi_2, \ldots\}$ yra nepriklausomi vienodai pasiskirstę, realias reikšmes įgynantys, atsitiktiniai dydžiai su bendra pasiskirstymo funkcija F_ξ, ir tegul η skaičiuojantis atsitiktinis dydis nepriklausomas nuo $\{\xi_1, \xi_2, \ldots\}$. Jeigu F_ξ yra $\mathcal{O}$-eksponentinė, tai pasiskirstymo funkcija F_{S_η} ir $F_{\xi_{(\eta)}}$ yra $\mathcal{O}$-eksponentinės. Jeigu papildomai, atsitiktinis dydis η tenkina 7.2.9 teoremos sąlygas, tai $F_{S_{(\eta)}}$ taip pat yra $\mathcal{O}$-eksponentinė.*

7.3 Išvados

Čia apibendriname pagrindinius šios disertacijos rezultatus:

1. Pateikėme du teiginius, apibūdinančius sąlygas, kuriom esant atsitiktinės sustabdytos sumos pasiskirstymo funkcija priklauso dominuojamai kintančių skirstinių klasei. Išnagrinėtas atvejis, kai nėra reikalaujamos jokios sąlygos atsitiktinių dydžių $\{\xi_1, \xi_2, \ldots\}$ momentams, ir atvejis, kai šie atsitiktiniai dydžiai turi baigtinius vidurkius. Abiem atvejais gautos sąlygos, kurioms esant $F_{S_\eta} \in \mathcal{D}$, kai atsitiktiniai dydžiai $\{\xi_1, \xi_2, \ldots\}$ yra nepriklausomi, bet nebūtinai vienodai paskirstę.

2. Nustatėme sąlygas, kuriom esant pasiskirstymo funkcija F_{S_η} priklauso eksponentinių skirstinių klasei. Vienoje iš pagrindinių teoremų skaičiuojantis atsitiktinis dydis η turi baigtinę atramą, tuo tarpu kitose teoremose skaičiuojančio atsitiktinio dydžio η uodega iš dešinės yra

neapribuojama. Išnagrinėjome tiek neneigiamus, tiek realias reikšmes įgyjančius atsitiktinius dydžius $\{\xi_1, \xi_2, \ldots\}$. Gavome sąlygas, kuriomis esant $F_{S_\eta} \in \mathcal{L}(\gamma)$, kažkokiam $\gamma \geqslant 0$, kai $\{\xi_1, \xi_2, \ldots\}$ nebūtinai vienodai pasiskirstę atsitiktiniai dydžiai.

3. Radome sąlygas, kuriom esant atsitiktinės sustabdytos sumos S_η, atsitiktinio sustabdyto maksimumo $\xi_{(\eta)}$ ir atsitiktinio sustabdyto sumų maksimumo $S_{(\eta)}$ pasiskirstymo funkcijos priklauso $\mathcal{O}$-eksponentinių skirstinių klasei. Išnagrinėjome atvejus, kai atsitiktiniai dydžiai $\{\xi_1, \xi_2, \ldots\}$ nebūtinai vienodai pasiskirstę atsitiktiniai dydžiai.

7.4 Rezultatų sklaida

Disertacijos rezultatai publikuoti šiuose moksliniuose straipsniuose:

- Danilenko, S., Šiaulys, J. (2015). Random Convolution of $\mathcal{O}$-exponential distributions. *Nonlinear Analysis: Modelling and Control*, 20(3): 447-454.
- Danilenko, S., Šiaulys, J. (2016). Randomly stopped sums of not identically distributed heavy tailed random variables. *Statistics and Probability Letters*, 113: 84-93.
- Danilenko, S., Paškauskaitė, S., Šiaulys, J. (2016). Random convolution of inhomogeneous distributions with $\mathcal{O}$-exponential tail. *Modern Stochastics: Theory and Applications*, 3(1): 79-94.
- Danilenko, S., Markevičiūtė, J.; Šiaulys, J. (2017). Randomly stopped sums with exponential-type distributions. *Nonlinear Analysis: Modelling and Control*, 22(6): 793-807.
- Danilenko, S., Šiaulys, J., Stepanauskas G. (2018). Closure properties of $\mathcal{O}$-exponential distributions. *Statistics and Probability Letters*, 140: 63-70.

Konferencijos, kuriose pristatyti pranešimai disertacijos tema:

- Random convolution of $\mathcal{O}$-exponential distributions. *Lietuvos matematikų draugijos 56-oji konferencija*, 2015 m. birželio 16-17 d., Kaunas.

- Sunkiauodegių skirstinių atsitiktinių sumų savybės. *Lietuvos matematikų draugijos 57-oji konferencija*, 2016 m. birželio 20-21 d., Vilnius.

- Randomly stopped sum of distributions with dominatingly varying tails. *The X Tartu Conference on Multivariate Statistics*, 2016 m. birželio 28 - liepos 1 d., Tartu.

- Eksponentiškai pasiskirsčiusios atsitiktinės sumos. *Lietuvos matematikų draugijos 58-oji konferencija*, 2017 m. birželio 21-22 d., Vilnius.

- Closure properties of $\mathcal{O}$-exponential distributions. *Modern Stochastics: Theory and Applications. IV*, 2018 m. gegužės 24-25, Kyiv.

- $\mathcal{O}$-eksponentinių skirstinių uždarumo savybės. *Lietuvos matematikų draugijos 59-oji konferencija*, 2018 m. birželio 18-19 d., Kaunas.

8 Chapter

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Random convolution of $\mathcal{O}$-exponential distributions*

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Abstract. Assume that $\xi_1, \xi_2, \dots$ are independent and identically distributed non-negative random variables having the $\mathcal{O}$-exponential distribution. Suppose that η is a nonnegative non-degenerate at zero integer-valued random variable independent of $\xi_1, \xi_2, \dots$. In this paper, we consider the conditions for η under which the distribution of random sum $\xi_1 + \xi_2 + \dots + \xi_\eta$ remains in the class of $\mathcal{O}$-exponential distributions.

Keywords: long tail, random sum, closure property, $\mathcal{O}$-exponential distribution.

1 Introduction

Let $\xi_1, \xi_2, \dots$ be independent copies of a random variable (r.v.) ξ with distribution function (d.f.) F_ξ. Let η be a nonnegative non-degenerate at zero integer-valued r.v. independent of $\{\xi_1, \xi_2, \dots\}$. We suppose that F_ξ is $\mathcal{O}$-exponential and we find minimal conditions under which the d.f.

$$\begin{aligned} F_{S_\eta}(x) &:= \mathbf{P}(\xi_1 + \xi_2 + \dots + \xi_\eta \leqslant x) \\ &= \sum_{n=0}^{\infty} \mathbf{P}(\eta = n) \mathbf{P}(\xi_1 + \xi_2 + \dots + \xi_n \leqslant x) \\ &= \sum_{n=0}^{\infty} \mathbf{P}(\eta = n) F_\xi^{*n}(x) \end{aligned}$$

belongs to the class of $\mathcal{O}$-exponential distributions as well. Here and elsewhere in this paper, F^{*n} denotes the n-fold convolution of d.f. F. Theorem 1 below is the main result of this paper. Before the exact formulation of this theorem, we recall the definition of $\mathcal{O}$-exponential and some related d.f.'s classes. In all definitions below, we assume that $\overline{F}(x) = 1 - F(x) > 0$ for all $x \in \mathbb{R}$.

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Definition 1. For $\gamma > 0$, by $\mathcal{L}(\gamma)$ we denote the class of exponential d.f.s, i.e. $F \in \mathcal{L}(\gamma)$ if for any fixed real y,

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\gamma y}.$$

In the case $\gamma = 0$, class $\mathcal{L}(0)$ is called the long-tailed distribution class and is denoted by $\mathcal{L}$.

Definition 2. A d.f. F belongs to the dominated varying-tailed class ($F \in \mathcal{D}$) if for any fixed $y \in (0,1)$,

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

Definition 3. A d.f. F is $\mathcal{O}$-exponential ($F \in \mathcal{OL}$) if for any fixed $y \in \mathbb{R}$,

$$0 < \liminf_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} < \infty.$$

It is easy to see that the following inclusions hold:

$$\mathcal{D} \subset \mathcal{OL}, \qquad \mathcal{L} \subset \mathcal{OL}, \qquad \bigcup_{\gamma\geqslant 0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

In [2, 3], Cline claimed that d.f. F_{S_η} remains in the class $\mathcal{L}(\gamma)$ if $F_\xi \in \mathcal{L}(\gamma)$ and η is any nonnegative non-degenerate at zero integer-valued r.v. Albin [1] observed that Cline's result is false in general. He obtained that d.f. F_{S_η} remains in the class $\mathcal{L}(\gamma)$ if F_ξ belongs to the class $\mathcal{L}(\gamma)$ and $\mathbf{E}\mathrm{e}^{\delta\eta} < \infty$ for each $\delta > 0$. In order to prove this claim, author used the upper estimate

$$\frac{\overline{F^{*n}}(x-t)}{\overline{F^{*n}}(x)} \leqslant (1+\varepsilon)\mathrm{e}^{\gamma t}, \tag{1}$$

provided that $\varepsilon > 0$, $t \in \mathbb{R}$, $F \in \mathcal{L}(\gamma)$, $x \geqslant n(c_1 - t) + t$ and $c_1 = c_1(\varepsilon, t)$ is sufficiently large such that

$$\frac{\overline{F}(x-t)}{\overline{F}(x)} \leqslant (1+\varepsilon)\mathrm{e}^{\gamma \mathrm{t}}$$

for $x \geqslant c_1$ (see [1, Lemma 1]). Unfortunately, the obtained estimate holds for positive t only. If t is negative, then the above estimate is incorrect in general. This fact was shown by Watanabe and Yamamuro (see [8, Remark 6.1]). Thus, the Cline proposition that $\mathbf{P}(\xi_1 + \xi_2 + \cdots + \xi_\eta \leqslant x)$ belongs to the class $\mathcal{L}(\gamma)$ remains not proved.

In this paper, we investigate a wider class, $\mathcal{OL}$, instead of the class $\mathcal{L}(\gamma)$. We show that the d.f. of the sum $\xi_1 + \xi_2 + \cdots + \xi_\eta$ remains in the class $\mathcal{OL}$, if r.v. η satisfies the conditions similar to that in [1]. The following theorem is the main statement in this paper.

Theorem 1. *Let $\xi_1, \xi_2, \ldots$ be independent copies of a nonnegative r.v. ξ with d.f. F_ξ. Let η be a nonnegative, non-degenerate at zero, integer-valued and independent of $\{\xi_1, \xi_2, \ldots\}$ r.v. with d.f. F_η. If F_ξ belongs to the class $\mathcal{OL}$ and $\overline{F_\eta}(\delta x) = O(\sqrt{x}\overline{F_\xi}(x))$ for each $\delta \in (0,1)$, then $F_{S_\eta} \in \mathcal{OL}$.*

A detailed proof of Theorem 1 is presented in Section 3. Note that the proof is similar to that of Theorem 6 in [5].

The following assertion actually shows that Albin's conditions for the counting r.v. η are sufficient for d.f. F_{S_η} to remain in the class $\mathcal{OL}$. The proof of the following corollary is also presented in Section 3.

Corollary 1. *Let $\xi_1, \xi_2, \ldots$ be a sequence of independent nonnegative r.v.s with common d.f. $F_\xi \in \mathcal{OL}$.*

(i) *D.f. $\mathbf{P}(\xi_1 + \cdots + \xi_n \leqslant x)$ belongs to the class $\mathcal{OL}$ for each fixed $n \in \mathbb{N}$.*
(ii) *Let η be a r.v. which is nonnegative, non-degenerate at zero, integer-valued and independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbf{E}\mathrm{e}^{\varepsilon\eta} < \infty$ for each $\varepsilon > 0$, then $F_{S_\eta} \in \mathcal{OL}$.*

2 Auxiliary lemmas

Before proving our main results, we give three auxiliary lemmas. The first lemma is well known classical estimate for the concentration function of a sum of independent and identically distributed r.v.s. The proof of Lemma 1 can be found in [6] (see Theorem 2.22), for instance.

Lemma 1. *Let $X_1, X_2, \ldots,$ be a sequence of independent r.v.s with a common non-degenerate d.f. Then there exists a constant c_2, independent of λ and n, such that*

$$\sup_{x\in\mathbb{R}} \mathbf{P}(x \leqslant X_1 + X_2 + \cdots + X_n \leqslant x + \lambda) \leqslant c_2(\lambda + 1)n^{-1/2}$$

for all $\lambda \geqslant 0$ and all $n \in \mathbb{N}$.

The second auxiliary lemma is due to Shimura and Watanabe (see [7, Prop. 2.2]). The lemma describes an important property of a d.f. from the class $\mathcal{OL}$.

Lemma 2. *Let F be a d.f. from the class $\mathcal{OL}$. Then there exists positive Δ such that*

$$\lim_{x\to\infty} \mathrm{e}^{\Delta x}\overline{F}(x) = \infty.$$

The last auxiliary lemma is crucial in the proof of Theorem 1. The elements of the statement below can be found in [4] (see the proof of Theorem 3(b)). Inequality (1), which is a particular case of the statement below, is proved in [1] (see Lemma 2.1). Leipus and Šiaulys [5] generalized Albin's inequality (1) for an arbitrary d.f. with unbounded support. The analytical proof of Lemma 3 is given in [5] (see proof of Lemma 4). In this paper, we present another, completely probabilistic proof of the lemma below having in mind the importance of the statement.

Lemma 3. *Let d.f. F be such that $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$. Suppose that*

$$\sup_{x\geqslant d_2} \frac{\overline{F}(x-t)}{\overline{F}(x)} \leqslant d_1$$

for some positive constants t, d_1 *and* $d_2 > t$. *Then, for all* $n = 1, 2, \dots$, *we have*:

$$\sup_{x \geqslant n(d_2 - t) + t} \frac{\overline{F^{*n}}(x - t)}{\overline{F^{*n}}(x)} \leqslant d_1.$$

Proof of Lemma 3. Let X be a r.v. with d.f. F. Then the condition of Lemma 3 says that

$$\sup_{x \geqslant d_2} \frac{\mathbf{P}(X > x - t)}{\mathbf{P}(X > x)} \leqslant d_1 \tag{2}$$

for some positive $t, d_1, d_2 > t$, and we need to prove that

$$\sup_{x \geqslant (nd_2 - t) + t} \frac{\mathbf{P}(S_n^X > x - t)}{\mathbf{P}(S_n^X > x)} \leqslant d_1 \tag{3}$$

for all $n \in \mathbb{N}$, where $S_n^X = X_1 + \dots + X_n$, and $X_1, X_2, \dots$ are independent copies of X.

The proof is proceeded by induction on n. According to condition (2), inequality (3) holds for $n = 1$. Suppose now that $N \geqslant 1$. For arbitrary real x, z and $t > 0$, we obtain

$$\begin{aligned}
\mathbf{P}\big(S_{N+1}^X > x\big) &= \mathbf{P}\big(S_N^X + X_{N+1} > x,\, X_{N+1} \leqslant x - z\big) \\
&\quad + \mathbf{P}\big(S_N^X + X_{N+1} > x,\, S_N^X \leqslant z\big) \\
&\quad + \mathbf{P}(X_{N+1} > x - z)\,\mathbf{P}\big(S_N^X > z\big) \\
&\geqslant \mathbf{P}\big(S_N^X > x - X_{N+1},\, x - X_{N+1} \geqslant z\big) \\
&\quad + \mathbf{P}\big(X_{N+1} > x - S_N^X,\, x - S_N^X \geqslant x - z + t\big) \\
&\quad + \mathbf{P}(X_{N+1} > x - z)\mathbf{P}\big(S_N^X > z\big).
\end{aligned} \tag{4}$$

If we replace x by $x - t$ and z by $z - t$ then we get

$$\begin{aligned}
\mathbf{P}\big(S_{N+1}^X > x - t\big) &= \mathbf{P}\big(S_N^X + X_{N+1} > x - t,\, X_{N+1} \leqslant x - z\big) \\
&\quad + \mathbf{P}\big(S_N^X + X_{N+1} > x - t,\, S_N^X \leqslant z - t\big) \\
&\quad + \mathbf{P}(X_{N+1} > x - z)\mathbf{P}\big(S_N^X > z - t\big) \\
&= \mathbf{P}\big(S_N^X > x - X_{N+1} - t,\, x - X_{N+1} \geqslant z\big) \\
&\quad + \mathbf{P}\big(X_{N+1} > x - S_N^X - t,\, x - S_N^X \geqslant x - z + t\big) \\
&\quad + \mathbf{P}(X_{N+1} > x - z)\mathbf{P}\big(S_N^X > z - t\big).
\end{aligned} \tag{5}$$

R.v.s $X_1, X_2, \dots$ are independent. Therefore,

$$\begin{aligned}
&\mathbf{P}\big(S_N^X > x - X_{N+1} - t,\, x - X_{N+1} \geqslant z\big) \\
&\quad = \mathbf{E}\big(\mathbf{E}\big(\mathbf{1}_{\{S_N^X > x - X_{N+1} - t\}} \mathbf{1}_{\{x - X_{N+1} \geqslant z\}} \;\big|\; x - X_{N+1} = y\big)\big) \\
&\quad = \mathbf{E}\big(\mathbf{1}_{\{y \geqslant z\}} \mathbf{E}\big(\mathbf{1}_{\{S_N^X > y - t\}} \;\big|\; x - X_{N+1} = y\big)\big) \\
&\quad = \mathbf{E}\big(\mathbf{1}_{\{y \geqslant z\}} \mathbf{P}\big(S_N^X > y - t\big)\big)
\end{aligned}$$

$$\leqslant \sup_{y \geqslant z} \frac{\mathbf{P}(S_N^X > y - t)}{\mathbf{P}(S_N^X > y)} \mathbf{E}\big(\mathbf{1}_{\{y \geqslant z\}} \mathbf{P}\big(S_N^X > y\big)\big)$$
$$= \sup_{y \geqslant z} \frac{\mathbf{P}(S_N^X > y - t)}{\mathbf{P}(S_N^X > y)} \mathbf{P}\big(S_N^X > x - X_{N+1},\, x - X_{N+1} \geqslant z\big), \tag{6}$$

where $\mathbf{1}_A$ denotes the indicator function of an event A. Similarly,

$$\mathbf{P}\big(X_{N+1} > x - S_N^X - t,\, x - S_N^X \geqslant x - z + t\big)$$
$$\leqslant \sup_{y \geqslant x - z + t} \frac{\mathbf{P}(X_{N+1} > y - t)}{\mathbf{P}(X_{N+1} > y)} \mathbf{P}\big(X_{N+1} > x - S_N^X,\, x - S_N^X \geqslant x - z + t\big). \tag{7}$$

Using estimates (4)–(7), we obtain

$$\frac{\mathbf{P}(S_{N+1}^X > x - t)}{\mathbf{P}(S_{N+1}^X > x)} \leqslant \max\left\{ \sup_{y \geqslant z} \frac{\mathbf{P}(S_N^X > y - t)}{\mathbf{P}(S_N^X > y)},\ \sup_{y \geqslant x - z + t} \frac{\mathbf{P}(X > y - t)}{\mathbf{P}(X > y)} \right\} \tag{8}$$

if $x, z \in \mathbb{R}$, $t > 0$ and $N \geqslant 1$.

Suppose now that (3) is satisfied for $n = N$. We will show that (3) holds for $n = N + 1$.

Condition (2) and estimate (8) imply, taking $z = z_N = Nx/(N+1) + t/(N+1)$ and $w_N = x - z_N + t = x/(N+1) + Nt/(N+1)$, that

$$\frac{\mathbf{P}(S_{N+1}^X > x - t)}{\mathbf{P}(S_{N+1}^X > x)} \leqslant \max\left\{ \sup_{y \geqslant z_N} \frac{\mathbf{P}(S_N^X > y - t)}{\mathbf{P}(S_N^X > y)},\ \sup_{y \geqslant w_N} \frac{\mathbf{P}(X > y - t)}{\mathbf{P}(X > y)} \right\} \leqslant d_1$$

if $x \geqslant (N+1)(d_2 - t) + t$, because, in this case,

$$z_N \geqslant N(d_2 - t) + t \quad \text{and} \quad w_N \geqslant d_2.$$

So, estimate (3) holds for $n = N + 1$ and the validity of (3) for all n follows by induction. □

3 Proofs of main results

In this section, we present detailed proofs of our main results.

Proof of Theorem 1. First, we show that

$$\limsup_{x \to \infty} \frac{\overline{F_{S_\eta}}(x - a)}{\overline{F_{S_\eta}}(x)} = \limsup_{x \to \infty} \frac{\mathbf{P}(S_\eta > x - a)}{\mathbf{P}(S_\eta > x)} < \infty \tag{9}$$

for each $a \in \mathbb{R}$.

If $a \leqslant 0$, then $\mathbf{P}(S_\eta > x - a) \leqslant \mathbf{P}(S_\eta > x)$ for all $x \in \mathbb{R}$, and estimate (9) is obvious.

Suppose now that $a > 0$. Since $F_\xi \in \mathcal{OL}$, we derive that

$$\limsup_{x\to\infty} \frac{\overline{F_\xi}(x-a)}{\overline{F_\xi}(x)} = c_3 \tag{10}$$

for some finite positive quantity c_3 maybe depending on a. So, there exists some $K = K_a > a+1$ such that

$$\sup_{x\geqslant K} \frac{\overline{F_\xi}(x-a)}{\overline{F_\xi}(x)} \leqslant 2c_3. \tag{11}$$

Applying Lemma 3, we obtain that

$$\sup_{x\geqslant n(K-a)+a} \frac{\mathbf{P}(S_n > x-a)}{\mathbf{P}(S_n > x)} = \sup_{x\geqslant n\,(K-a)+a} \frac{\overline{F_\xi^{*n}}(x-a)}{\overline{F_\xi^{*n}}(x)} \leqslant 2c_3, \tag{12}$$

where and below $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ if $n \in \mathbb{N}$.

For an arbitrarily chosen positive x, we have

$$\begin{aligned}\mathbf{P}(S_\eta > x) &= \sum_{n=1}^{\infty} \mathbf{P}(S_n > x)\mathbf{P}(\eta = n) \geqslant \sum_{n=1}^{\infty} \mathbf{P}(\xi_1 > x)\mathbf{P}(\eta = n)\\ &= \overline{F}_\xi(x)\mathbf{P}(\eta \geqslant 1).\end{aligned} \tag{13}$$

If $x \geqslant K$, then, using (12), we get:

$$\begin{aligned}\mathbf{P}(S_\eta > x-a) &= \mathbf{P}\biggl(S_\eta > x-a,\, \eta \leqslant \frac{x-a}{K-a}\biggr) + \mathbf{P}\biggl(S_\eta > x-a,\, \eta > \frac{x-a}{K-a}\biggr)\\ &= \sum_{n\leqslant(x-a)/(K-a)} \mathbf{P}(S_n > x-a)\,\mathbf{P}(\eta = n)\\ &\quad + \sum_{n>(x-a)/(K-a)} \mathbf{P}(S_n > x-a)\mathbf{P}(\eta = n)\\ &\leqslant 2c_3 \sum_{n\leqslant(x-a)/(K-a)} \mathbf{P}(S_n > x)\mathbf{P}(\eta = n)\\ &\quad + \sum_{n>(x-a)/(K-a)} \mathbf{P}(S_n > x-a)\mathbf{P}(\eta = n)\\ &\quad + \sum_{n>(x-a)/(K-a)} \mathbf{P}(S_n > x)\,\mathbf{P}(\eta = n)\\ &\quad - \sum_{n>(x-a)/(K-a)} \mathbf{P}(S_n > x)\mathbf{P}(\eta = n)\end{aligned}$$

$$\leqslant c_4 \sum_{n=1}^{\infty} \mathbf{P}(S_n > x)\mathbf{P}(\eta = n) + \sum_{n>(x-a)/(K-a)} \mathbf{P}(x - a < S_n \leqslant x)\, \mathbf{P}(\eta = n) \tag{14}$$

with $c_4 = \max\{2c_3, 1\}$.

According to Lemma 1, we obtain

$$\sup_{x\in\mathbb{R}} \mathbf{P}(x - a < S_n \leqslant x) \leqslant c_5(a + 1)\frac{1}{\sqrt{n}},$$

where the constant c_5 is independent of a and n. Thus, inequality (14) implies

$$\begin{aligned}\mathbf{P}(S_\eta > x - a) &\leqslant c_4\mathbf{P}(S_\eta > x) + c_5(a + 1) \sum_{n>(x-a)/(K-a)} \frac{\mathbf{P}(\eta = n)}{\sqrt{n}} \\ &\leqslant c_4\mathbf{P}(S_\eta > x) + c_5\sqrt{\frac{K-a}{x-a}}(a + 1)\mathbf{P}\left(\eta > \frac{x-a}{K-a}\right)\end{aligned} \tag{15}$$

provided that $x \geqslant K$.

Inequalities (13) and (15) imply that, for $x \geqslant K$, it holds

$$\frac{\mathbf{P}(S_\eta > x - a)}{\mathbf{P}(S_\eta > x)} \leqslant c_4 + \frac{c_5\sqrt{K-a}(a+1)}{\sqrt{x-a}\,\mathbf{P}(\eta \geqslant 1)\overline{F_\xi}(x)}\overline{F_\eta}\left(\frac{x-a}{K-a}\right).$$

Consequently,

$$\begin{aligned}&\limsup_{x\to\infty} \frac{\mathbf{P}(S_\eta > x - a)}{\mathbf{P}(S_\eta > x)} \\ &\quad\leqslant c_4 + c_5\frac{(a+1)\sqrt{K-a}}{\mathbf{P}(\eta \geqslant 1)} \limsup_{x\to\infty} \frac{\overline{F_\eta}((x-a)/(K-a))}{\sqrt{x-a}\,\overline{F_\xi}(x-a)} \limsup_{x\to\infty} \frac{\overline{F_\xi}(x-a)}{\overline{F_\xi}(x)} \\ &\quad= c_4 + c_3c_5\frac{(a+1)\sqrt{K-a}}{\mathbf{P}(\eta \geqslant 1)} \limsup_{x\to\infty} \frac{\overline{F_\eta}(x/(K-a))}{\sqrt{x}\,\overline{F_\xi}(x)} < \infty\end{aligned}$$

due to equality (10) and requirement $\overline{F_\eta}(\delta x) = O(\sqrt{x}\,\overline{F_\xi}(x))$ which holds for arbitrary $\delta \in (0, 1)$. Therefore, relation (9) is satisfied for for all $a \in \mathbb{R}$.

It remains to prove that

$$\liminf_{x\to\infty} \frac{\overline{F_{S_\eta}}(x-a)}{\overline{F_{S_\eta}}(x)} = \liminf_{x\to\infty} \frac{\mathbf{P}(S_\eta > x - a)}{\mathbf{P}(S_\eta > x)} > 0,$$

where a is an arbitrarily chosen real number. But this relation follows from the proved estimate (9), because

$$\mathbf{P}(S_\eta > x) \geqslant \overline{F_\xi}(x)\mathbf{P}(\eta \geqslant 1) > 0$$

for each positive number x, and so

$$\liminf_{x\to\infty}\frac{\mathbf{P}(S_\eta > x-a)}{\mathbf{P}(S_\eta > x)} = \left(\limsup_{x\to\infty}\frac{\mathbf{P}(S_\eta > x+a)}{\mathbf{P}(S_\eta > x)}\right)^{-1} > 0.$$

The last inequality, together with estimate (9), implies that d.f. F_{S_η} belongs to the class $\mathcal{OL}$. Theorem 1 is proved. □

Proof of Corollary 1. Part (i) of Corollary 1 is evident. So we only prove part (ii). Let $\delta \in (0,1)$. According to the Markov inequality, we have

$$\overline{F_\eta}(\delta x) = \mathbf{P}(\eta > \delta x) = \mathbf{P}\big(\mathrm{e}^{y\eta} > \mathrm{e}^{y\delta x}\big) \leqslant \mathrm{e}^{-\delta yx}\mathbf{E}\mathrm{e}^{y\eta} \tag{16}$$

for each $y > 0$. The d.f. F_ξ belongs to the class $\mathcal{OL}$. Therefore, Lemma 2 implies that $\mathrm{e}^{\Delta x}\overline{F_\xi}(x) \to \infty$ as $x \to \infty$. for some positive Δ.

Choosing $y = \Delta/\delta > 0$ in (16), we obtain:

$$\frac{\overline{F_\eta}(\delta x)}{\sqrt{x}\,\overline{F_\xi}(x)} \leqslant \frac{\mathbf{E}\mathrm{e}^{y\eta}}{\mathrm{e}^{\delta yx}\sqrt{x}\,\overline{F_\xi}(x)} = \frac{1}{\sqrt{x}}\frac{1}{\mathrm{e}^{\Delta x}\overline{F_\xi}(x)}\mathbf{E}\mathrm{e}^{(\Delta/\delta)\eta} \underset{x\to\infty}{\to} 0$$

because $\mathbf{E}\mathrm{e}^{\varepsilon\eta}$ is finite for an arbitrarily positive ε according to the main condition of Corollary 1. The statement of Corollary 1 follows now from Theorem 1. □

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Random convolution of inhomogeneous distributions with $\mathcal{O}$-exponential tail

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Abstract Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (not necessarily identically distributed), and η be a counting random variable independent of this sequence. We obtain sufficient conditions on $\{\xi_1, \xi_2, \ldots\}$ and η under which the distribution function of the random sum $S_\eta = \xi_1 + \xi_2 + \cdots + \xi_\eta$ belongs to the class of $\mathcal{O}$-exponential distributions.

Keywords Heavy tail, exponential tail, $\mathcal{O}$-exponential tail, random sum, random convolution, inhomogeneous distributions, closure property

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1 Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v., that is, an integer-valued, nonnegative, and nondegenerate at zero r.v. In addition, suppose that the r.v. η and r.v.s $\{\xi_1, \xi_2, \ldots\}$ are independent. Let $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, $n \in \mathbb{N}$, be the partial sums, and let

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$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the random sum of $\{\xi_1, \xi_2, \ldots\}$.

We are interested in conditions under which the d.f. of S_η

$$F_{S_\eta}(x) = \mathbb{P}(S_\eta \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n \leqslant x)$$

belongs to the class of $\mathcal{O}$-*exponential* distributions.

According to Albin and Sunden [1] or Shimura and Watanabe [15], *a d.f. F belongs to the class of $\mathcal{O}$-exponential distributions $\mathcal{OL}$ if*

$$0 < \liminf_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} < \infty$$

for all $a \in \mathbb{R}$, where $\overline{F}(x) = 1 - F(x)$, $x \in \mathbb{R}$, is the tail of a d.f. F.

Note that if $F \in \mathcal{OL}$, then $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$.

It is obvious that a d.f. F belongs to the class $\mathcal{OL}$ if and only if

$$\limsup_{x\to\infty} \frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty \tag{1}$$

or, equivalently, if and only if

$$\sup_{x\geqslant 0} \frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty.$$

The last condition shows that class $\mathcal{OL}$ is quite wide. We further describe some more popular subclasses of $\mathcal{OL}$ for which we will present some results on the random convolution of distributions from these subclasses.

A d.f. F is said to belong to the class $\mathcal{L}$ of long-tailed d.f.s if for every fixed $a > 0$, we have

$$\lim_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} = 1.$$

A d.f. F is said to belong to the class $\mathcal{L}(\gamma)$ of exponential distributions with some $\gamma > 0$ if for any fixed $a > 0$, we have

$$\lim_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} = \mathrm{e}^{-a\gamma}.$$

A d.f. F belongs to the class $\mathcal{D}$ (or *has a dominatingly varying tail*) *if for every fixed $a \in (0, 1)$, we have*

$$\limsup_{x\to\infty} \frac{\overline{F}(xa)}{\overline{F}(x)} < \infty.$$

A d.f. F supported on the interval $[0, \infty)$ *belongs to the class $\mathcal{S}$* (or *is subexponential*) *if*

$$\lim_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2,$$

where, as usual, $*$ denotes the convolution of d.f.s.

A d.f. F supported on the interval $[0,\infty)$ belongs to the class $\mathcal{S}^*$ (or is strongly subexponential) if

$$\mu := \int_{[0,\infty)} x\mathrm{d}F(x) < \infty \quad \text{and} \quad \int_0^x \overline{F}(x-y)\overline{F}(y)\mathrm{d}y \underset{x\to\infty}{\sim} 2\mu\overline{F}(x).$$

If a d.f. F is supported on $\mathbb{R}$, then F belongs to some of the classes $\mathcal{S}$ or $\mathcal{S}^*$ if $F^+(x) = F(x)\mathbb{1}_{\{[0,\infty)\}}(x)$ belongs to the corresponding class.

The presented definitions, together with Lemma 2 of Chistyakov [2], Lemma 9 of Denisov et al. [5], Lemma 1.3.5(a) of Embrechts et al. [9], and Lemma 1 of Kaas and Tang [11], imply that

$$\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{OL},\ \mathcal{D} \subset \mathcal{OL},\quad \bigcup_{\gamma>0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

Now we present a few known results on when the d.f. F_{S_η} belongs to some class. The first result about subexponential distributions was proved by Embrechts and Goldie (Theorem 4.2 in [8]) and Cline (Theorem 2.13 in [3]).

Theorem 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent copies of a nonnegative r.v. ξ with subexponential d.f. F_ξ. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbb{E}(1+\delta)^\eta < \infty$ for some $\delta > 0$, then $F_{S_\eta} \in \mathcal{S}$.*

In the case of strongly subexponential d.f.s, the following result, which involves weaker restrictions on the r.v. η, can be derived from Theorem 1 of Denisov et al. [6] and Corollary 2.36 of Foss et al. [10].

Theorem 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent copies of a nonnegative r.v. ξ with strongly subexponential d.f. F_ξ and finite mean $\mathbb{E}\xi$. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbb{P}(\eta > x/c) \underset{x\to\infty}{=} o(\overline{F}_\xi(x))$ for some $c > \mathbb{E}\xi$, then $F_{S_\eta} \in \mathcal{S}^*$.*

Similar results for classes $\mathcal{D}$, $\mathcal{L}$, and $\mathcal{OL}$ can be found in the papers of Leipus and Šiaulys [12] and Danilenko and Šiaulys [4]. We further present Theorem 6 from [12].

Theorem 3. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s with common d.f. $F_\xi \in \mathcal{L}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ having d.f. F_η. If $\overline{F}_\eta(\delta x) \underset{x\to\infty}{=} o(\sqrt{x}\,\overline{F}_\xi(x))$ for each $\delta \in (0,1)$, then $F_{S_\eta} \in \mathcal{L}$.*

In all presented results, r.v.s $\{\xi_1, \xi_2, \ldots\}$ are identically distributed. In this work, we consider independent, but not necessarily identically distributed, r.v.s. As was noted, we restrict our consideration on the class $\mathcal{OL}$. In fact, in this paper, we generalize the results of [4]. If $\{\xi_1, \xi_2, \ldots\}$ may be not identically distributed, then various collections of conditions on r.v.s $\{\xi_1, \xi_2, \ldots\}$ and η imply that $F_{S_\eta} \in \mathcal{OL}$. The rest of the paper is organized as follows. In Section 2, we formulate our main results. In Section 3, we present all auxiliary assertions, and the detailed proofs of the main results are presented in Section 4. Finally, a few examples of $\mathcal{O}$-exponential random sums are described in Section 5.

2 Main results

In this section, we formulate our main results. The first result describes the situation where the tails of d.f.s F_{ξ_k} for large indices k are uniformly comparable with itself at the points x and $x-1$ for all $x \in [0,\infty)$.

Theorem 4. *Let* $\{\xi_1,\xi_2,\ldots\}$ *be independent nonnegative random variables with d.f.s* $\{F_{\xi_1},F_{\xi_2},\ldots\}$*, and let* η *be a counting r.v. independent of* $\{\xi_1,\xi_2,\ldots\}$*. Then* $F_{S_\eta} \in \mathcal{OL}$ *if the following three conditions are satisfied.*

- *For some* $\kappa \in \operatorname{supp}(\eta)\setminus\{0\} = \{n\in\mathbb{N} : \mathbb{P}(\eta=n)>0\}$*,* $F_{\xi_\kappa}\in\mathcal{OL}$*.*
- *For each* $k\in\operatorname{supp}(\eta)$*,* $k\leqslant\kappa$*, either* $\lim\limits_{x\to\infty}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)}=0$ *or* $F_{\xi_k}\in\mathcal{OL}$*.*
- $\sup\limits_{x\geqslant 0}\sup\limits_{k\geqslant 1}\frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)}<\infty$.

Since each d.f. from the class $\mathcal{OL}$ is comparable with itself, the next assertion follows immediately from Theorem 4.

Corollary 1. *Let* $\{\xi_1,\xi_2,\ldots\}$ *be independent nonnegative random variables with common d.f.* $F_\xi\in\mathcal{OL}$*. Then the d.f. of random sum* F_{S_η} *is* $\mathcal{O}$*-exponential for an arbitrary counting r.v.* η*.*

Our second main assertion is dealt with counting r.v.s having finite support.

Theorem 5. *Let* $\{\xi_1,\xi_2,\ldots,\xi_D\}$*,* $D\in\mathbb{N}$*, be independent nonnegative random variables with d.f.s* $\{F_{\xi_1},F_{\xi_2},\ldots F_{\xi_D}\}$*, and let* η *be a counting r.v. independent of* $\{\xi_1,\xi_2,\ldots,\xi_D\}$*. Then* $F_{S_\eta}\in\mathcal{OL}$ *under the following three conditions.*

- $\mathbb{P}(\eta\leqslant D)=1$.
- *For some* $\kappa\in\operatorname{supp}(\eta)\setminus\{0\}$*,* $F_{\xi_\kappa}\in\mathcal{OL}$*.*
- *For each* $k\in\{1,2,\ldots,D\}$*, either* $\lim\limits_{x\to\infty}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)}=0$ *or* $F_{\xi_k}\in\mathcal{OL}$*.*

Our last main assertion describes the case where the tails of d.f.s F_{ξ_k} are comparable at x and $x-1$ asymptotically and uniformly with respect to large indices k. In this case, conditions are more restrictive for a counting r.v.

Theorem 6. *Let* $\{\xi_1,\xi_2,\ldots\}$ *be independent nonnegative random variables with d.f.s* $\{F_{\xi_1},F_{\xi_2},\ldots\}$*, and let* η *be a counting r.v. d.f.* F_η *independent of* $\{\xi_1,\xi_2,\ldots\}$*. Then* $F_{S_\eta}\in\mathcal{OL}$ *if the following five conditions are satisfied.*

- *For some* $\kappa\in\operatorname{supp}(\eta)\setminus\{0\}$*,* $F_{\xi_\kappa}\in\mathcal{OL}$*.*
- *For each* $k\in\operatorname{supp}(\eta)$*,* $k\leqslant\kappa$*, either* $\lim\limits_{x\to\infty}\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)}=0$ *or* $F_{\xi_k}\in\mathcal{OL}$*.*
- $\limsup\limits_{x\to\infty}\sup\limits_{k\geqslant 1}\frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)}<\infty$.
- $\limsup\limits_{k\to\infty}\frac{1}{k}\sum_{l=1}^{k}\sup\limits_{x\geqslant 0}(\overline{F}_{\xi_{\kappa+l}}(x-1)-\overline{F}_{\xi_{\kappa+l}}(x))<1$.
- *For each* $\delta\in(0,1)$*,* $\overline{F}_\eta(\delta x)=O(\sqrt{x}\,\overline{F}_{\xi_\kappa}(x))$*.*

3 Auxiliary lemmas

In this section, we present all assertions that we use in the proofs of our main results. We present some of auxiliary results with proofs. The first assertion can be found in [7] (see Eq. (2.12)).

Lemma 1. *Let F and G be two d.f.s satisfying $\overline{F}(x) > 0$, $\overline{G}(x) > 0$, $x \in \mathbb{R}$. Then*

$$\frac{\overline{F * G}(x-t)}{\overline{F * G}(x)} \leqslant \max\left\{\sup_{y \geqslant v} \frac{\overline{F}(y-t)}{\overline{F}(y)}, \sup_{y \geqslant x-v+t} \frac{\overline{G}(y-t)}{\overline{G}(y)}\right\}$$

for all $x \in \mathbb{R}$, $v \in \mathbb{R}$, and $t > 0$.

The following assertion is the well-known Kolmogorov–Rogozin inequality for concentration functions. Recall that the Lévy concentration function or simply concentration function of a r.v. X is the function

$$Q_X(\lambda) = \sup_{x \in \mathbb{R}} \mathbb{P}(x \leqslant X \leqslant x + \lambda), \quad \lambda \in [0, \infty).$$

The proof of the next lemma can be found in [14] (Theorem 2.15).

Lemma 2. *Let $X_1, X_2, \ldots, X_n$ be independent r.v.s, and let $Z_n = \sum_{k=1}^n X_k$. Then, for all $n \in \mathbb{N}$,*

$$Q_{Z_n}(\lambda) \leqslant A\lambda \left\{\sum_{k=1}^n \lambda_k^2 \big(1 - Q_{X_k}(\lambda_k)\big)\right\}^{-1/2},$$

where A is an absolute constant, and $0 < \lambda_k \leqslant \lambda$ for each $k \in \{1, 2, \ldots, n\}$.

The following assertion describes sufficient conditions under which the d.f. of two independent r.v.s belongs to the class $\mathcal{OL}$.

Lemma 3. *Let X_1 and X_2 be independent r.v.s with d.f.s F_{X_1} and F_{X_2}, respectively. Then the d.f. $F_{X_1} * F_{X_2}$ of the sum $X_1 + X_2$ is $\mathcal{O}$-exponential if $F_{X_1} \in \mathcal{OL}$ and one of the following two conditions holds:*

- $$\lim_{x \to \infty} \frac{\overline{F}_{X_2}(x)}{\overline{F}_{X_1}(x)} = 0, \tag{2}$$
- $F_{X_2} \in \mathcal{OL}$.

Proof. We split the proof into three parts.

I. First, suppose that $\mathbb{P}(X_2 \leqslant D) = 1$ for some $D > 0$. In this case, condition (2) holds evidently.

For each real x, we have

$$\overline{F_{X_1} * F_{X_2}}(x) = \mathbb{P}(X_1 + X_2 > x) = \int\limits_{(-\infty, D]} \overline{F}_{X_1}(x-y) \mathrm{d}F_{X_2}(y).$$

Hence, for such x,

$$\begin{aligned}
\frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} &= \frac{\int_{(-\infty, D]} \overline{F}_{X_1}(x-1-y) \frac{\overline{F}_{X_1}(x-y)}{\overline{F}_{X_1}(x-y)} \mathrm{d}F_{X_2}(y)}{\int_{(-\infty, D]} \overline{F}_{X_1}(x-y) \mathrm{d}F_{X_2}(y)} \\
&\leqslant \frac{\int_{(-\infty, D]} \sup_{y \leqslant D} \frac{\overline{F}_{X_1}(x-1-y)}{\overline{F}_{X_1}(x-y)} \overline{F}_{X_1}(x-y) \mathrm{d}F_{X_2}(y)}{\int_{(-\infty, D]} \overline{F}_{X_1}(x-y) \mathrm{d}F_{X_2}(y)} \\
&= \sup_{z \geqslant x-D} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}.
\end{aligned}$$

This estimate implies that

$$\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \limsup_{x\to\infty} \sup_{z\geqslant x-D} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}$$
$$= \limsup_{y\to\infty} \frac{\overline{F}_{X_1}(y-1)}{\overline{F}_{X_1}(y)}$$
$$< \infty$$

because $F_{X_1} \in \mathcal{OL}$. So, $F_{X_1} * F_{X_2} \in \mathcal{OL}$ as well.

II. Now let us consider the case where condition (2) holds but $\overline{F}_{X_2}(x) > 0$ for all $x \in \mathbb{R}$. For each real x, we have

$$\overline{F_{X_1} * F_{X_2}}(x) = \int_{-\infty}^{\infty} \overline{F_{X_1}}(x-y)\mathrm{d}F_{X_2}(y).$$

Therefore,

$$\overline{F_{X_1} * F_{X_2}}(x-1) = \Bigg(\int_{(-\infty,\, x-M]} + \int_{(x-M,\infty)} \Bigg)\overline{F}_{X_1}(x-1-y)\mathrm{d}F_{X_2}(y)$$
$$\leqslant \int_{(-\infty,\, x-M]} \overline{F}_{X_1}(x-1-y)\frac{\overline{F}_{X_1}(x-y)}{\overline{F}_{X_1}(x-y)}\mathrm{d}F_{X_2}(y) + \overline{F}_{X_2}(x-M)$$
$$\leqslant \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)} \int_{(-\infty,\, x-M]} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y) + \overline{F}_{X_2}(x-M)$$

for all M, x such that $0 < M < x-1$. In addition, for such M and x, we obtain

$$\overline{F_{X_1} * F_{X_2}}(x) \geqslant \int_{(-\infty,\, x-M]} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y),$$
$$\overline{F_{X_1} * F_{X_2}}(x) \geqslant \int_{(M,\infty)} \overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y)$$
$$\geqslant \overline{F}_{X_1}(x-M)\overline{F}_{X_2}(M).$$

The obtained estimates imply that

$$\frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)} + \frac{\overline{F}_{X_2}(x-M)}{\overline{F}_{X_1}(x-M)\overline{F}_{X_2}(M)}$$

for all x and M such that $0 < M < x-1$. Consequently,

$$\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)} + \frac{1}{\overline{F}_{X_2}(M)} \limsup_{x\to\infty} \frac{\overline{F}_{X_2}(x-M)}{\overline{F}_{X_1}(x-M)}$$

$$= \sup_{z \geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}$$

for all positive M. Therefore,

$$\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \limsup_{M\to\infty} \frac{\overline{F}_{X_1}(M-1)}{\overline{F}_{X_1}(M)} < \infty$$

because F_{X_1} is $\mathcal{O}$-exponential. Consequently, $F_{X_1} * F_{X_2} \in \mathcal{OL}$ by (1).

III. It remains to prove the assertion when both d.f.s F_{X_1} and F_{X_2} are $\mathcal{O}$-exponential. By Lemma 1 we have

$$\frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \max\left\{ \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}, \sup_{z\geqslant x-M+1} \frac{\overline{F}_{X_2}(z-1)}{\overline{F}_{X_2}(z)} \right\}$$

for all x and M such that $0 < M < x-1$. Therefore, for every positive M,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \\
&\quad\leqslant \max\left\{ \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}, \limsup_{x\to\infty} \sup_{z\geqslant x-M+1} \frac{\overline{F}_{X_2}(z-1)}{\overline{F}_{X_2}(z)} \right\} \\
&\quad= \max\left\{ \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}, \limsup_{y\to\infty} \frac{\overline{F}_{X_2}(y-1)}{\overline{F}_{X_2}(y)} \right\}.
\end{aligned}$$

Letting M tend to infinity, we get that

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \\
&\quad\leqslant \max\left\{ \limsup_{M\to\infty} \frac{\overline{F}_{X_1}(M-1)}{\overline{F}_{X_1}(M)}, \limsup_{y\to\infty} \frac{\overline{F}_{X_2}(y-1)}{\overline{F}_{X_2}(y)} \right\} < \infty
\end{aligned}$$

because F_{X_1} and F_{X_2} belong to class $\mathcal{OL}$. Consequently, $F_{X_1} * F_{X_2} \in \mathcal{OL}$ due to requirement (1). Lemma 3 is proved. □

Lemma 4. *Let $\{X_1, X_2, \ldots, X_n\}$ be independent nonnegative r.v.s with d.f.s $\{F_{X_1}, F_{X_2}, \ldots, F_{X_n}\}$. Let $F_{X_1} \in \mathcal{OL}$ and suppose that, for each $k \in \{2, 3, \ldots, n\}$, either $\lim_{x\to\infty} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} = 0$ or $F_{X_k} \in \mathcal{OL}$. Then the d.f. $F_{X_1} * F_{X_2} * \cdots * F_{X_n}$ belongs to the class $\mathcal{OL}$.*

Proof. We use induction on n. If $n = 2$, then the statement follows from Lemma 3. Suppose that the statement holds if $n = m$, that is, $F_{X_1} * F_{X_2} * \cdots * F_{X_m} \in \mathcal{OL}$, and we will show that the statement is correct for $n = m+1$.

Conditions of the lemma imply that $F_{X_{m+1}} \in \mathcal{OL}$ or

$$\lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\overline{F_{X_1} * F_{X_2} * \cdots * F_{X_m}}(x)} = \lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\mathbb{P}(X_1 + \cdots + X_m > x)}$$

$$\leqslant \lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\mathbb{P}(X_1 > x)} = \lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\overline{F}_{X_1}(x)} = 0.$$

So, using Lemma 3 again, we get

$$F_{X_1} * F_{X_2} * \cdots * F_{X_{m+1}} = (F_{X_1} * F_{X_2} * \cdots * F_{X_m}) * F_{X_{m+1}} \in \mathcal{OL}.$$

We see that the statement of the lemma holds for $n = m + 1$ and, consequently, by induction, for all $n \in \mathbb{N}$. The lemma is proved. □

4 Proofs of the main results

In this section, we present proofs of our main results.

Proof of Theorem 4. Conditions of Theorem and Lemma 4 imply that the d.f. $F_{S_\kappa}(x) = \mathbb{P}(S_\kappa \leqslant x)$ belongs to the class $\mathcal{OL}$. So, we have

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\kappa}(x-1)}{\overline{F}_{S_\kappa}(x)} < \infty \tag{3}$$

or, equivalently,

$$\sup_{x\geqslant 0} \frac{\overline{F}_{S_\kappa}(x-1)}{\overline{F}_{S_\kappa}(x)} \leqslant c_1 \tag{4}$$

for some positive constant c_1.

We observe that, for all $x \geqslant 0$,

$$\frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} = \mathcal{J}_1(x) + \mathcal{J}_2(x), \tag{5}$$

where

$$\mathcal{J}_1(x) = \frac{\mathbb{P}(S_\eta > x-1, \eta \leqslant \kappa)}{\mathbb{P}(S_\eta > x)},$$
$$\mathcal{J}_2(x) = \frac{\mathbb{P}(S_\eta > x-1, \eta > \kappa)}{\mathbb{P}(S_\eta > x)}.$$

Since $\kappa \in \operatorname{supp}(\eta)$, we obtain

$$\begin{aligned}
\mathcal{J}_1(x) &= \frac{\sum_{n=0}^{\kappa} \mathbb{P}(S_n > x-1)\mathbb{P}(\eta = n)}{\sum_{n=0}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n)} \\
&\leqslant \frac{1}{\mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa)} \sum_{n=0}^{\kappa} \mathbb{P}(S_\kappa > x-1)\mathbb{P}(\eta = n) \\
&= \frac{\mathbb{P}(S_\kappa > x-1)}{\mathbb{P}(S_\kappa > x)} \frac{\mathbb{P}(\eta \leqslant \kappa)}{\mathbb{P}(\eta = \kappa)}.
\end{aligned}$$

Hence, it follows from (3) that

$$\limsup_{x\to\infty} \mathcal{J}_1(x) < \infty. \tag{6}$$

By Lemma 1 we have

$$\frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\left\{\sup_{z\geqslant M} \frac{\mathbb{P}(S_\kappa > z-1)}{\mathbb{P}(S_\kappa > z)}, \sup_{z\geqslant x-M+1} \frac{\overline{F}_{\xi_{\kappa+1}}(z-1)}{\overline{F}_{\xi_{\kappa+1}}(z)}\right\} \tag{7}$$

for all real x and M.

The third condition of the theorem implies that

$$\sup_{x\geqslant 0} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \leqslant c_2 \tag{8}$$

for all $k \in \mathbb{N}$ and some positive c_2.

If we choose $M = x/2$ in estimate (7), then, using (4), we get

$$\sup_{x\geqslant 0} \frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\{c_1, c_2\} := c_3. \tag{9}$$

Applying Lemma 1 again, we obtain

$$\frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant \max\left\{\sup_{z\geqslant M} \frac{\mathbb{P}(S_{\kappa+1} > z-1)}{\mathbb{P}(S_{\kappa+1} > z)}, \sup_{z\geqslant x-M+1} \frac{\overline{F}_{\xi_{\kappa+2}}(z-1)}{\overline{F}_{\xi_{\kappa+2}}(z)}\right\}.$$

By choosing $M = x/2$ we get from inequalities (8) and (9) that

$$\sup_{x\geqslant 0} \frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant c_3.$$

Continuing the process, we find

$$\sup_{x\geqslant 0} \frac{\mathbb{P}(S_{\kappa+k} > x-1)}{\mathbb{P}(S_{\kappa+k} > x)} \leqslant c_3$$

for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned}\mathcal{J}_2(x) &= \frac{1}{\mathbb{P}(S_\eta > x)} \sum_{k=1}^{\infty} \mathbb{P}(S_{\kappa+k} > x-1)\mathbb{P}(\eta = \kappa+k) \\ &\leqslant \frac{c_3}{\mathbb{P}(S_\eta > x)} \sum_{k=1}^{\infty} \mathbb{P}(S_{\kappa+k} > x)\mathbb{P}(\eta = \kappa+k) \\ &\leqslant \frac{c_3\mathbb{P}(S_\eta > x)}{\mathbb{P}(S_\eta > x)} = c_3 \end{aligned} \tag{10}$$

for all $x \geqslant 0$.

The obtained relations (5), (6), and (10) imply that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} < \infty.$$

Therefore, the d.f. F_{S_η} belongs to the class $\mathcal{OL}$ due to requirement (1). Theorem 4 is proved. □

Proof of Theorem 5. The statement of the theorem can be derived from Theorem 4 or proved directly. We present the direct proof of Theorem 5.

It is evident that $S_k = \xi_\kappa + \sum_{n=1,\ n\neq\kappa}^{k} \xi_n$ for each $k \geqslant \kappa$. Hence, by Lemma 4, $F_{S_k} \in \mathcal{OL}$ for all $\kappa \leqslant k \leqslant D$.

If $x \geqslant 1$, then we have

$$\begin{aligned}
\frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} &= \frac{\sum_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{D} \mathbb{P}(S_n > x-1)\mathbb{P}(\eta = n)}{\sum_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{D} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n)}\\
&\leqslant \frac{\mathbb{P}(S_\kappa > x-1)\mathbb{P}(\eta \leqslant \kappa) + \sum_{\substack{n=\kappa+1\\ n\in\mathrm{supp}(\eta)}}^{D} \mathbb{P}(S_n > x-1)\mathbb{P}(\eta = n)}{\mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa) + \sum_{\substack{n=\kappa+1\\ n\in\mathrm{supp}(\eta)}}^{D} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n)}\\
&\leqslant \max\left\{\frac{\mathbb{P}(S_\kappa > x-1)\mathbb{P}(\eta \leqslant \kappa)}{\mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa)},\ \max_{\substack{\kappa+1\leqslant n\leqslant D\\ n\in\mathrm{supp}(\eta)}} \frac{\mathbb{P}(S_n > x-1)}{\mathbb{P}(S_n > x)}\right\}, \quad (11)
\end{aligned}$$

where in the last step we use the inequality

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leqslant \max\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right\},$$

provided that $n \geqslant 1$ and $a_i, b_i > 0$ for $i \in \{1, 2, \ldots, n\}$.

Since $F_{S_n} \in \mathcal{OL}$ for all $n \geqslant \kappa$, we get from (11) that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} < \infty, \quad (12)$$

and the statement of Theorem 5 follows. □

Proof of Theorem 6. As usual, it suffices to prove relation (12). If $x \geqslant 0$, then we have

$$\begin{aligned}
\mathbb{P}(S_\eta > x) &= \sum_{n=1}^{\infty} \mathbf{P}(S_n > x)\mathbb{P}(\eta = n)\\
&\geqslant \mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa)\\
&\geqslant \overline{F}_{\xi_\kappa}(x)\mathbb{P}(\eta = \kappa). \quad (13)
\end{aligned}$$

Similarly, for $K \geqslant 2$ and $x \geqslant 2K$,

$$\mathbb{P}(S_\eta > x-1) = \sum_{n=1}^{\kappa} \mathbf{P}(S_n > x-1)\mathbb{P}(\eta = n)$$

$$
\begin{aligned}
&+ \sum_{1 \leqslant k \leqslant (x-1)/(K-1)} \mathbf{P}(S_{\kappa+k} > x-1)\mathbb{P}(\eta = \kappa + k) \\
&+ \sum_{k > (x-1)/(K-1)} \mathbf{P}(x-1 < S_{\kappa+k} \leqslant x)\mathbb{P}(\eta = \kappa + k) \\
&+ \sum_{k > (x-1)/(K-1)} \mathbf{P}(S_{\kappa+k} > x)\mathbb{P}(\eta = \kappa + k) \\
&:= \mathcal{K}_1(x) + \mathcal{K}_2(x) + \mathcal{K}_3(x) + \mathcal{K}_4(x). \qquad (14)
\end{aligned}
$$

The distribution function F_{S_κ} belongs to the class $\mathcal{OL}$ due to Lemma 4. So, by estimate (6) we have

$$\limsup_{x\to\infty} \frac{\mathcal{K}_1(x)}{\mathbb{P}(S_\eta > x)} = \limsup_{x\to\infty} \mathcal{J}_1(x) < \infty. \qquad (15)$$

Now we consider the sum $\mathcal{K}_2(x)$. Since F_{S_κ} is $\mathcal{O}$-exponential, we have

$$\sup_{x\geqslant 0} \frac{\mathbb{P}(S_\kappa > x-1)}{\mathbb{P}(S_\kappa > x)} \leqslant c_4$$

with some positive constant c_4. On the other hand, the third condition of Theorem 6 implies that

$$\sup_{x\geqslant c_5} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \leqslant c_6$$

for some constants $c_5 > 2$, $c_6 > 0$ and all $k \in \mathbb{N}$.

By Lemma 1 (with $v = c_5$) we have

$$\frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\left\{ \sup_{z\geqslant x-c_5+1} \frac{\mathbb{P}(S_\kappa > z-1)}{\mathbb{P}(S_\kappa > z)}, \sup_{z\geqslant c_5} \frac{\overline{F}_{\xi_{\kappa+1}}(z-1)}{\overline{F}_{\xi_{\kappa+1}}(z)} \right\}.$$

Consequently,

$$\sup_{x\geqslant c_5} \frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\{c_4, c_6\} := c_7.$$

Applying Lemma 1 again for the sum $S_{\kappa+2} = S_{\kappa+1} + \xi_{\kappa+2}$ (with $v = x/2 + 1/2$), we get

$$\frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant \max\left\{ \sup_{z\geqslant \frac{x}{2}+\frac{1}{2}} \frac{\mathbb{P}(S_{\kappa+1} > z-1)}{\mathbb{P}(S_{\kappa+1} > z)}, \sup_{z\geqslant \frac{x}{2}+\frac{1}{2}} \frac{\overline{F}_{\xi_{\kappa+2}}(z-1)}{\overline{F}_{\xi_{\kappa+2}}(z)} \right\}.$$

If $x \geqslant 2(c_5-1)+1$, then $x/2+1/2 \geqslant c_5$. Therefore, by the last inequality we obtain that

$$\sup_{x\geqslant 2(c_5-1)+1} \frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant c_7.$$

Applying Lemma 1 once again (with $v = x/3 + 2/3$), we get

$$\frac{\mathbb{P}(S_{\kappa+3} > x-1)}{\mathbb{P}(S_{\kappa+3} > x)} \leqslant \max\left\{ \sup_{z\geqslant \frac{2x}{3}+\frac{1}{3}} \frac{\mathbb{P}(S_{\kappa+2} > z-1)}{\mathbb{P}(S_{\kappa+2} > z)}, \sup_{z\geqslant \frac{x}{3}+\frac{2}{3}} \frac{\overline{F}_{\xi_{\kappa+3}}(z-1)}{\overline{F}_{\xi_{\kappa+3}}(z)} \right\}.$$

If $x \geqslant 3(c_5 - 1) + 1$, then $2x/3 + 1/3 \geqslant 2(c_5 - 1) + 1$ and $x/3 + 2/3 \geqslant c_5$. So, the last estimate implies

$$\sup_{x \geqslant 3(c_5-1)+1} \frac{\mathbb{P}(S_{\kappa+3} > x - 1)}{\mathbb{P}(S_{\kappa+3} > x)} \leqslant c_7.$$

Continuing the process, we can get that

$$\sup_{x \geqslant k(c_5-1)+1} \frac{\mathbb{P}(S_{\kappa+k} > x - 1)}{\mathbb{P}(S_{\kappa+k} > x)} \leqslant c_7 \tag{16}$$

for all $k \in \mathbb{N}$.

We can suppose that $K = c_5$ in representation (14). In such a case, it follows from inequality (16) that

$$\begin{aligned} \limsup_{x \to \infty} \frac{\mathcal{K}_2(x)}{\mathbb{P}(S_\eta > x)} &\leqslant \limsup_{x \to \infty} \frac{c_7}{\mathbb{P}(S_\eta > x)} \sum_{1 \leqslant k \leqslant \frac{x-1}{c_5-1}} \mathbb{P}(S_{\kappa+k} > x)\mathbb{P}(\eta = \kappa + k) \\ &\leqslant c_7. \end{aligned} \tag{17}$$

Since, obviously,

$$\limsup_{x \to \infty} \frac{\mathcal{K}_4(x)}{\mathbb{P}(S_\eta > x)} \leqslant 1, \tag{18}$$

it remains to estimate sum $\mathcal{K}_3(x)$. Using Lemma 2, we obtain

$$\mathcal{K}_3(x) \leqslant A \sum_{k > \frac{x-1}{c_5-1}} \mathbb{P}(\eta = \kappa + k) \left(\sum_{l=1}^{k} \left(1 - \sup_{x \in \mathbb{R}} \mathbb{P}(x - 1 \leqslant \xi_{\kappa+l} \leqslant x) \right) \right)^{-1/2}$$

with some absolute positive constant A. By the fourth condition of the theorem,

$$\frac{1}{k} \sum_{l=1}^{k} \sup_{x \in \mathbb{R}} \left(\overline{F}_{\xi_{\kappa+l}}(x - 1) - \overline{F}_{\xi_{\kappa+l}}(x) \right) \leqslant 1 - \Delta$$

for some $0 < \Delta < 1$ and all sufficiently large k. So, for such k,

$$\sum_{l=1}^{k} \left(1 - \sup_{x \in \mathbb{R}} \mathbb{P}(x - 1 \leqslant \xi_{\kappa+l} \leqslant x) \right) \geqslant k\Delta.$$

From the last estimate it follows that

$$\begin{aligned} \mathcal{K}_3(x) &\leqslant \frac{A}{\sqrt{\Delta}} \sum_{k > \frac{x-1}{c_5-1}} \frac{1}{\sqrt{k}} \mathbb{P}(\eta = \kappa + k) \\ &\leqslant \frac{A}{\sqrt{\Delta}} \sqrt{\frac{c_5 - 1}{x - 1}} \mathbb{P}\left(\eta > \kappa + \frac{x - 1}{c_5 - 1} \right) \end{aligned}$$

for sufficiently large x. Therefore,

$$
\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathcal{K}_3(x)}{\mathbb{P}(S_\eta > x)} \\
&\leqslant \frac{A}{\sqrt{\Delta}} \frac{\sqrt{c_5 - 1}}{\mathbb{P}(\eta = \kappa)} \limsup_{x\to\infty} \frac{\overline{F}_\eta(\frac{x-1}{c_5-1})}{\sqrt{x-1}\,\overline{F}_{\xi_\kappa}(x-1)} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_\kappa}(x-1)}{\overline{F}_{\xi_\kappa}(x)} \\
&< \infty \qquad (19)
\end{aligned}
$$

by estimate (13) and the last condition of the theorem. Representation (14) and estimates (15), (17), (18), and (19) imply the desired inequality (12). Theorem 6 is proved. □

5 Examples of $\mathcal{O}$-exponential random sums

In this section, we present three examples of random sums S_η for which the d.f.s F_{S_η} are $\mathcal{O}$-exponential.

Example 1. Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s. We suppose that the r.v. ξ_k for $k \in \{1, 2, \ldots, D\}$ is distributed according to the Pareto law with parameters k and α, that is,

$$\overline{F}_{\xi_k}(x) = \left(\frac{k}{k+x}\right)^\alpha, \quad x \geqslant 0,$$

where $k \in \{1, 2, \ldots, D\}$, $D \geqslant 1$, and $\alpha > 0$. In addition, we suppose that the r.v. ξ_{D+k} for each $k \in \mathbb{N}$ is distributed according to the exponential law with parameter λ/k, that is,

$$\overline{F}_{\xi_{D+k}}(x) = \mathrm{e}^{-\lambda x/k}, \quad x \geqslant 0.$$

It follows from Theorem 4 that the d.f. of the random sum S_η is $\mathcal{O}$-exponential for each counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$ under the condition $\mathbb{P}(\eta = \kappa) > 0$ for some $\kappa \in \{1, 2, \ldots, D\}$ because:

- $F_{\xi_k} \in \mathcal{L} \subset \mathcal{OL}$ for each $k \leqslant \kappa$,
- $$
\begin{aligned}
&\sup_{x\geqslant 0}\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \\
&= \max\left\{ \sup_{0\leqslant x\leqslant 1}\sup_{k\geqslant 1} \frac{1}{\overline{F}_{\xi_{\kappa+k}}(x)},\ \sup_{x>1}\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \right\} \\
&= \max\left\{ \sup_{0\leqslant x\leqslant 1} \max\left\{ \max_{1\leqslant k\leqslant D-\kappa} \left(\frac{\kappa+k+x}{\kappa+k}\right)^\alpha,\ \sup_{k\geqslant 1} \mathrm{e}^{\lambda x/k} \right\}, \right. \\
&\qquad \left. \sup_{x>1} \max\left\{ \max_{1\leqslant k\leqslant D-\kappa} \left(\frac{\kappa+k+x}{\kappa+k+x-1}\right)^\alpha,\ \sup_{k\geqslant 1} \mathrm{e}^{\lambda/k} \right\} \right\} \\
&\leqslant \max\left\{2^\alpha,\ \mathrm{e}^\lambda\right\}.
\end{aligned}
$$

Example 2. Let a r.v. η be uniformly distributed on $\{1, 2, \ldots, D\}$, that is,

$$\mathbb{P}(\eta = k) = \frac{1}{D}, \quad k \in \{1, 2, \ldots, D\},$$

for some $D \geqslant 2$. Let $\{\xi_1, \xi_2, \ldots, \xi_D\}$ be independent r.v.s, where ξ_1 is exponentially distributed, and $\xi_2, \ldots, \xi_D$ are uniformly distributed.

If the r.v. η is independent of the r.v.s $\{\xi_1, \xi_2, \ldots, \xi_D\}$, then Theorem 5 implies that the d.f. of the random sum S_η is $\mathcal{O}$-exponential.

Example 3. Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s, where $\{\xi_1, \xi_2, \ldots, \xi_{\kappa-1}\}$ are finitely supported, $\kappa \geqslant 2$, and ξ_κ is distributed according to the Weibull law, that is,

$$\overline{F}_{\xi_\kappa}(x) = \mathrm{e}^{-\sqrt{x}}, \quad x \geqslant 0.$$

In addition, we suppose that the r.v. $\xi_{\kappa+k}$ for each $k = m^2$, $m \geqslant 2$, has the d.f. with tail

$$\overline{F}_{\xi_{\kappa+k}}(x) = \begin{cases} 1 & \text{if } x < 0, \\ \frac{1}{k} & \text{if } 0 \leqslant x < k, \\ \frac{1}{k}\mathrm{e}^{-(x-k)} & \text{if } x \geqslant k, \end{cases}$$

whereas for each remaining index $k \notin \{m^2, m \in \mathbb{N} \setminus \{1\}\}$, the r.v. $\xi_{\kappa+k}$ has the exponential distribution, that is,

$$\overline{F}_{\xi_{\kappa+k}}(x) = \mathrm{e}^{-x}, \quad x \geqslant 0.$$

If the counting r.v. η is independent of $\{\xi_1, \xi_2, \ldots\}$ and is distributed according to the Poisson law with parameter λ, then it follows from Theorem 6 that the random sum S_η is $\mathcal{O}$-exponentially distributed because:

- $F_{\xi_\kappa} \in \mathcal{L} \subset \mathcal{OL}$;
- $\lim\limits_{x\to\infty} \dfrac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0 \quad \text{if } k = 1, 2, \ldots, \kappa - 1;$
- $$\sup_{x\geqslant 1}\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)}$$ $$= \sup_{x\geqslant 1}\max\left\{ \sup_{k\geqslant 1,\, k=m^2,\, m\geqslant 2} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)}, \sup_{k\geqslant 1,\, k\neq m^2} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \right\}$$ $$= \sup_{x\geqslant 1}\max\left\{ \sup_{k\geqslant 1,\, k=m^2,\, m\geqslant 2} \left\{\mathbb{1}_{[1,k)}(x) + \mathrm{e}^{x-k}\mathbb{1}_{[k,k+1)}(x) + \mathrm{e}\mathbb{1}_{[k+1,\infty)}(x)\right\}, \sup_{k\geqslant 1,\, k\neq m^2} \mathrm{e} \right\} = \mathrm{e};$$
- $$\limsup_{k\to\infty} \frac{1}{k}\sum_{l=1}^{k} \sup_{x\geqslant 0} \left(\overline{F}_{\xi_{\kappa+l}}(x-1) - \overline{F}_{\xi_{\kappa+l}}(x)\right)$$ $$= \limsup_{k\to\infty} \frac{1}{k}\left(\sum_{l=1,\, l=m^2}^{k} \left(1 - \frac{1}{l}\right) + \left(1 - \frac{1}{\mathrm{e}}\right) \sum_{l=1,\, l\neq m^2}^{k} 1\right)$$

$$\leqslant \left(1 - \frac{1}{\mathrm{e}}\right);$$

- $\overline{F}_\eta(x) < \left(\frac{\mathrm{e}\lambda}{x}\right)^x, \quad x > \lambda.$

Here the last estimate is the well-known Chernof bound for the Poisson law (see, e.g., p. 97 in [13]).

As we can see, the r.v.s $\{\xi_1, \xi_2, \dots\}$ from the last example satisfy the conditions of Theorem 6, whereas the third condition of Theorem 4 does not hold because, in this case,

$$\sup_{x \geqslant 0} \sup_{k \geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \geqslant \sup_{0 \leqslant x < 1} \sup_{k \geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \geqslant \sup_{0 \leqslant x < 1} \sup_{k = m^2, m \geqslant 2} k = \infty.$$

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Randomly stopped sums with exponential-type distributions

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Abstract. Assume that $\{\xi_1, \xi_2, \ldots\}$ are independent and possibly nonidentically distributed random variables. Suppose that η is a nonnegative, nondegenerate at zero and integer-valued random variable, which is independent of $\{\xi_1, \xi_2, \ldots\}$. In this paper, we consider conditions for η and $\{\xi_1, \xi_2, \ldots\}$ under which the distribution of the random sum $\xi_1 + \xi_2 + \cdots + \xi_\eta$ belongs to the class of exponential distributions.

Keywords: class of exponential distributions, random sum, closure property.

1 Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v., i.e. an integer-valued, nonnegative and nondegenerate at zero r.v. In addition, suppose that r.v. η and the sequence $\{\xi_1, \xi_2, \ldots\}$ are independent.

Let $S_0 = 0$, $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \in \mathbb{N}$, and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the randomly stopped sum of r.v.s $\xi_1, \xi_2, \ldots$. We denote the d.f. of S_η by F_{S_η} together with its tail $\overline{F}_{S_\eta}$. It is obvious that

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}(S_n > x)$$

for any positive x.

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In this paper, we consider possibly nonidentically distributed r.v.s $\xi_1, \xi_2, \ldots$. We find conditions under which d.f. F_{S_η} belongs to the class of exponential distributions. If $F_{\xi_1}, F_{\xi_2}, \ldots$ are different, then the various collections of conditions on d.f.s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and the counting r.v. η imply exponentiality of d.f. F_{S_η}. Before discussing the properties of F_{S_η}, we recall some d.f. classes related to exponentiality.

- *For* $\gamma > 0$, *by* $\mathcal{L}(\gamma)$ *we denote the class of exponential d.f.s. It is said that* $F \in \mathcal{L}(\gamma)$ *if for any* $y > 0$,
$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\gamma y}.$$
- *For* $\gamma = 0$, *the class* $\mathcal{L}(0)$ *is called the long-tailed distribution class and is denoted by* $\mathcal{L}$.
- *A d.f.* F *is* $\mathcal{O}$*-exponential* ($F \in \mathcal{OL}$) *if for any* $y > 0$,
$$\liminf_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} > 0$$
or, equivalently,
$$\limsup_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty.$$

According to Proposition 2.6 by Albin and Sundén in [2], an absolutely continuous d.f. F belongs to the class $\mathcal{L}(\gamma)$ if and only if

$$F(x) = 1 - \exp\left\{-\int_{-\infty}^{x} \bigl(\alpha(u) + \beta(u)\bigr)\,\mathrm{d}u\right\} \quad \text{for } x \in \mathbb{R},$$

for some measurable functions α and β with $\alpha(u) + \beta(u) \geqslant 0$, for all $u \in \mathbb{R}$ such that

$$\lim_{u\to\infty} \alpha(u) = \gamma, \qquad \lim_{x\to\infty} \int_{-\infty}^{x} \alpha(u)\,\mathrm{d}u = \infty \quad \text{and} \quad \lim_{x\to\infty} \int_{-\infty}^{x} \beta(u)\,\mathrm{d}u$$

exists. We note that each exponential distribution, each Erlang's distribution and each gamma distribution belong to the class $\mathcal{L}(\gamma)$ with some $\gamma > 0$.

It is easy to see that the following two inclusions hold:

$$\mathcal{L} \subset \mathcal{OL}, \qquad \bigcup_{\gamma>0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

In [4, 5], Cline claimed that d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if r.v.s $\{\xi_1, \xi_2, \ldots\}$ are identically distributed with d.f. $F_\xi \in \mathcal{L}(\gamma)$ and η is any counting r.v. Albin [1] constructed a counterexample and showed that Cline's result is false in general. In his paper [1], Albin stated that d.f. F_{S_η} remains in the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if r.v.s $\{\xi_1, \xi_2, \ldots\}$ are identically distributed with common d.f. F_ξ belonging to the class $\mathcal{L}(\gamma)$

and $\mathbf{E}\mathrm{e}^{\delta\eta} < \infty$ for each $\delta > 0$. In order to prove his statement, Albin used the following implication for $c \in \mathbb{R}$:

$$\sup_{x \geqslant c} \frac{\overline{F}(x-t)}{\overline{F}(x)} \leqslant (1+\varepsilon)\,\mathrm{e}^{\gamma t}$$
$$\Longrightarrow \quad \sup_{x \geqslant n(c-t)+t} \frac{\overline{F^{*n}}(x-t)}{\overline{F^{*n}}(x)} \leqslant (1+\varepsilon)\,\mathrm{e}^{\gamma t}, \quad n \in \mathbb{N},$$

provided that $\varepsilon > 0$, $t \in \mathbb{R}$ and F is a d.f. from the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Here and later F^{*n} denotes n-fold convolution of d.f. F with itself. Unfortunately, if $\gamma > 0$, then the obtained relation holds for positive t only. Watanabe and Yamamuro (see [13, Remark 6.1]) showed that the above implication is incorrect in the case of positive γ and negative t. When $\gamma = 0$, the above implication for positive t is sufficient to prove the Albin's statement under the weaker restrictions on the counting r.v. η (see [11, Thm. 6]). The Albin's statement on conditions for which $F_{S_\eta} \in \mathcal{L}(\gamma)$ has remained only as a hypothesis in the case $\gamma > 0$. Watanabe and Yamamuro [13] do not prove the Albin's hypothesis in this case, they presented the following statement (see [13, Prop. 6.1]).

Theorem 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent identically distributed r.v.s with a common d.f. F_ξ. If $F_\xi \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$, then F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for each counting r.v. η distributed according to the Poisson law.*

The above result was generalized in [15], where the following statement was proved (see [15, Thm. 2.3]).

Theorem 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent nonnegative r.v.s with a common d.f. F_ξ such that $F_\xi^{*\kappa} \in \mathcal{L}(\gamma)$ for some integer $\kappa \geqslant 1$ and some $\gamma \geqslant 0$. In addition, let counting r.v. η be independent of $\{\xi_1, \xi_2, \ldots\}$ and $\mathbf{P}(\eta \geqslant \kappa) > 0$. Then $F_{S_\eta} \in \mathcal{L}(\gamma)$ if any pair* (i), (ii) *or* (i), (iii) *of conditions holds, where*

(i) *for any $\varepsilon \in (0,1)$, there is an integer $M = M(\varepsilon)$ such that for $x \geqslant 0$,*

$$\sum_{k=M}^{\infty} \mathbf{P}(\eta = k+1)\overline{F_\xi^{*k}}(x) \leqslant \varepsilon \overline{F}_{S_\eta}(x),$$

(ii) $\overline{F}_\xi(x) = o(\overline{F_\xi^{*2}}(x))$,

(iii) *for all $t > 0$ and $1 \leqslant i \leqslant \kappa - 1$,*

$$\liminf_{x\to\infty} \frac{\overline{F_\xi^{*i}}(x-t)}{\overline{F_\xi^{*i}}(x)} \geqslant \mathrm{e}^{\gamma t}.$$

Motivated by the presented results, we also consider conditions for which d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Here the randomly stopped sum S_η contains independent but not necessarily identically distributed r.v.s. We suppose that some d.f.s from $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ belongs to the exponential class, and we find conditions for

$F_{\xi_1}, F_{\xi_2}, \ldots$ and η such that the distribution of the randomly stopped sum S_η remains in the same class. In this work, we present three collections of such conditions. The proofs of the main results are based on ideas from the papers [6–8, 10, 13] and [15]. The similar results for class $\mathcal{L} = \mathcal{L}(0)$ were obtained in the papers [12] and [14].

The rest of the paper is organized as follows. In Section 2, we present our main results together with two examples of randomly stopped sums having some exponential distributions. Section 3 is a collection of auxiliary lemmas. The proofs of the three main results are presented in Section 4.

2 Main results and examples

At first, in this section, we present three theorems, which deal with situations when the d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. In Theorem 3, the case of a finitely supported counting r.v. η is considered, while Theorems 4 and 5 deal with the case of unbounded right tail of η. In Theorems 3 and 5, we consider nonnegative r.v.s, while in Theorem 4, r.v.s $\xi_1, \xi_2, \ldots$ can be real valued.

Theorem 3. *Let $n \geqslant 1$ and $\{\xi_1, \xi_2, \ldots, \xi_n\}$ be a collection of nonnegative independent r.v.s with d.f.s $\{F_{\xi_1}, F_{\xi_2}, \ldots, F_{\xi_n}\}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots, \xi_n\}$ and having a finite support* $\operatorname{supp}\eta \subseteq \{0, 1, \ldots, n\}$. *Then d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if $F_{\xi_\nu} \in \mathcal{L}(\gamma)$ for some nonrandom $1 \leqslant \nu \leqslant \min\{\operatorname{supp}\eta \setminus \{0\}\}$, and*

$$F_{\xi_k} \in \mathcal{L}(\gamma) \quad \text{or} \quad \overline{F}_{\xi_k}(x) = o\big(\overline{F}_{\xi_\nu}(x)\big)$$

for each $k \in \{1, 2, \ldots, \max\{\operatorname{supp}\eta\}\}$.

Theorem 4. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of real valued independent r.v.s such that for some $\gamma \geqslant 0$,*

$$\sup_{k\geqslant 1}\left|\frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y}\right| \underset{x\to\infty}{\longrightarrow} 0 \tag{1}$$

for each fixed $y \geqslant 0$. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ such that

$$\frac{\mathbf{P}(\eta = k+1)}{\mathbf{P}(\eta = k)} \underset{k\to\infty}{\longrightarrow} 0. \tag{2}$$

Then $F_{S_\eta} \in \mathcal{L}(\gamma)$.

Here we observe that condition (1) is equivalent to the two-sided estimate

$$\mathrm{e}^{-\gamma y} \leqslant \liminf_{x\to\infty}\inf_{k\geqslant 1}\frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} \leqslant \limsup_{x\to\infty}\sup_{k\geqslant 1}\frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} \leqslant \mathrm{e}^{-\gamma y},$$

which holds for some $\gamma \geqslant 0$ and for each fixed $y \geqslant 0$.

In addition, we observe that condition (2) implies that $\mathbf{P}(\eta = k) > 0$ for all sufficiently large k.

Theorem 5. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of nonnegative independent r.v.s with d.f.s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. D.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if there exist $\varkappa \geqslant 1$ and $1 \leqslant \nu \leqslant \varkappa$ such that*

(i) $\nu \leqslant \min\{\operatorname{supp} \eta \setminus \{0\}\}$,
(ii) $F_{\xi_\nu} \in \mathcal{L}(\gamma)$,
(iii) *for each* $1 \leqslant k \leqslant \varkappa$, $F_{\xi_k} \in \mathcal{L}(\gamma)$ *or* $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_\nu}(x))$,
(iv) *for each* $y \geqslant 0$,
$$\sup_{k \geqslant \varkappa+1} \left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y} \right| \underset{x\to\infty}{\longrightarrow} 0,$$
(v) $\mathbf{P}(\eta = k+1)/\mathbf{P}(\eta = k) \to 0$ *as* $k \to \infty$.

Further in this section, we present two examples, which illustrate several applications of our theorems. In both examples, we construct randomly stopped sums that belong to the class of exponential distributions.

Example 1. Suppose that we have a three-seasonal sequence of independent Erlang r.v.s with d.f.s from class $\mathcal{L}(2)$, i.e.
$$F_{\xi_k}(x) = \begin{cases} (1 - \mathrm{e}^{-2x}(1+2x))\mathbf{1}_{[0,\infty)}(x) & \text{if } k \equiv 1 \bmod 3, \\ (1 - \mathrm{e}^{-2x}(1+2x+2x^2))\mathbf{1}_{[0,\infty)}(x) & \text{if } k \equiv 2 \bmod 3, \\ (1 - \mathrm{e}^{-2x}(1+2x+2x^2+4x^3/3))\mathbf{1}_{[0,\infty)}(x) & \text{if } k \equiv 0 \bmod 3. \end{cases}$$
In addition, suppose that the counting r.v. η is independent of $\{\xi_1, \xi_2, \ldots\}$ and has Poisson distribution with parameter $\lambda > 0$.

It is clear that
$$\begin{aligned} &\sup_{k\geqslant 1} \left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-2y} \right| \\ &\quad = \max\left\{ \left| \frac{\overline{F}_{\xi_1}(x+y)}{\overline{F}_{\xi_1}(x)} - \mathrm{e}^{-2y} \right|, \left| \frac{\overline{F}_{\xi_2}(x+y)}{\overline{F}_{\xi_2}(x)} - \mathrm{e}^{-2y} \right|, \left| \frac{\overline{F}_{\xi_3}(x+y)}{\overline{F}_{\xi_3}(x)} - \mathrm{e}^{-2y} \right| \right\} \underset{x\to\infty}{\longrightarrow} 0 \end{aligned}$$
and
$$\frac{\mathbf{P}(\eta = k+1)}{\mathbf{P}(\eta = k)} = \frac{\lambda}{k+1} \underset{k\to\infty}{\longrightarrow} 0.$$

We see that all conditions of Theorem 3 are satisfied. Consequently, d.f. $F_{S_\eta} \in \mathcal{L}(2)$.

Example 2. Suppose that $\{\xi_1, \xi_2, \ldots\}$ is a sequence of nonnegative r.v.s such that
$$\begin{aligned} \overline{F}_{\xi_1}(x) &= \mathrm{e}^{-x}, \quad x \geqslant 0, \\ \overline{F}_{\xi_k}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y, \quad x \geqslant 0,\ k \in \{2, 3, \ldots, 10\}, \\ \overline{F}_{\xi_k}(x) &= \mathrm{e}^{-x}\left(1 + \frac{x}{k-10}\right), \quad x \geqslant 0,\ k \in \{11, 12, \ldots\}. \end{aligned}$$

In addition, let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ distributed according to law

$$\mathbf{P}(\eta = k) = \frac{1}{\hat{c}} \mathrm{e}^{-k^2}, \quad k \in \{0, 1, 2, \ldots\},$$

where

$$\hat{c} = \sum_{k=0}^{\infty} \mathrm{e}^{-k^2} \approx 1.3863.$$

The described sequence $\{\xi_1, \xi_2, \ldots\}$ and the counting r.v. η satisfy conditions of Theorem 5 with $\gamma = 1$, $\nu = 1$ and $\varkappa = 10$ because:

$$F_{\xi_1} \in \mathcal{L}(1), \qquad \operatorname{supp} \eta \setminus \{0\} = \mathbb{N},$$
$$\overline{F}_{\xi_k}(x) = o\big(\overline{F}_{\xi_1}(x)\big) \quad \text{if } k \in \{2, 3, \ldots, 10\},$$
$$\frac{\mathbf{P}(\eta = k+1)}{\mathbf{P}(\eta = k)} = \mathrm{e}^{-2k-1}, \quad k \in \mathbb{N},$$

and

$$\left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-y} \right| = \frac{y\mathrm{e}^{-y}}{x + k - 10}$$

for all $k \geqslant 11$, $x > 0$ and $y \geqslant 0$.

Consequently, d.f. F_{S_η} belongs to the class $\mathcal{L}(1)$ due to assertion of Theorem 5.

3 Auxiliary lemmas

In this section, we give all auxiliary assertions, which we use in the proofs of our main results. The first lemma was proved by Embrechts and Goldie (see [9, Thm. 3]).

Lemma 1. *Let F and G be two d.f.s, and let F belong to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Then convolution $F * G$ belongs to the class $\mathcal{L}(\gamma)$ if one of the following conditions holds*:

(i) *d.f. G belongs to the class $\mathcal{L}(\gamma)$,*
(ii) $\overline{G}(x) = o(\overline{F}(x))$.

The second lemma is the inhomogeneous case of the upper estimate, which was presented in the proof of Proposition 6.1 from [13].

Lemma 2. *Let $\xi_1, \xi_2, \ldots$ be real valued independent r.v.s such that*

$$\limsup_{x \to \infty} \sup_{k \geqslant 1} \frac{\overline{F}_{\xi_k}(x+a)}{\overline{F}_{\xi_k}(x)} \leqslant \mathrm{e}^{-\gamma a} \tag{3}$$

for some $\gamma \geqslant 0$ and $a > 0$. Then, for any $\varepsilon \in (0, 1)$, there exists $b = b(a, \varepsilon) > 0$ such that

$$\overline{F}_{S_{n+1}}(x+a) \leqslant (1+\varepsilon)\mathrm{e}^{-\gamma a} \overline{F}_{S_{n+1}}(x) + \overline{F}_{S_n}(x-b)$$

for all x and all $n \geqslant 1$.

Proof. For any x and any $b > 0$, we have

$$\begin{aligned}
\overline{F}_{S_{n+1}}(x) &= \mathbf{P}(S_n + \xi_{n+1} > x) \\
&= \int\limits_{(-\infty,\, x-b]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) + \int\limits_{(x-b,\,\infty)} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\
&=: \mathcal{J}_1(x,b) + \mathcal{J}_2(x,b). \qquad (4)
\end{aligned}$$

Condition (3) implies that

$$\sup_{n\geqslant 1} \frac{\overline{F}_{\xi_{n+1}}(x-y+a)}{\overline{F}_{\xi_{n+1}}(x-y)} \leqslant (1+\varepsilon)\,\mathrm{e}^{-\gamma a}$$

for any fixed $\varepsilon \in (0,1)$ if $y \leqslant x+a-b$ (then $x-y \geqslant b-a$) and b is sufficiently large. For such b, we get

$$\begin{aligned}
\mathcal{J}_1(x+a,\, b) &= \int\limits_{(-\infty,\, x+a-b]} \frac{\overline{F}_{\xi_{n+1}}(x+a-y)}{\overline{F}_{\xi_{n+1}}(x-y)} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\
&\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma a} \int\limits_{(-\infty, x-b]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\
&\quad + (1+\varepsilon)\mathrm{e}^{-\gamma a} \int\limits_{(x-b,\, x+a-b]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\
&\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma a}\big(\mathcal{J}_1(x,b) + \mathcal{J}_2(x,b)\big) \\
&= (1+\varepsilon)\mathrm{e}^{-\gamma a}\overline{F}_{S_{n+1}}(x).
\end{aligned}$$

On the other hand, it is obvious that

$$\mathcal{J}_2(x+a,\, b) \leqslant \int\limits_{(x+a-b,\infty)} \mathrm{d}F_{S_n}(y) \leqslant \overline{F}_{S_n}(x-b).$$

Therefore, for any $\varepsilon \in (0,1)$ and sufficiently large $b = b(a,\varepsilon)$, we obtain

$$\begin{aligned}
\overline{F}_{S_{n+1}}(x+a) &= \mathcal{J}_1(x+a,b) + \mathcal{J}_2(x+a,b) \\
&\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma a}\overline{F}_{S_{n+1}}(x) + \overline{F}_{S_n}(x-b).
\end{aligned}$$

Lemma 2 is proved. □

The next lemma deals with the lower estimate of $\overline{F}_{S_{n+1}}(x+a)$ in the case of non-identical d.f.s $F_{\xi_1}, F_{\xi_2}, \dots$.

Lemma 3. *Let $\xi_1, \xi_2, \dots$ be real valued independent r.v.s such that*

$$\liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x+a)}{\overline{F}_{\xi_k}(x)} \geqslant \mathrm{e}^{-\gamma a} \tag{5}$$

for some $\gamma \geqslant 0$ and $a > 0$. Then, for any $\varepsilon \in (0, 1/2)$, there exists $\hat{b} = \hat{b}(a, \varepsilon) > 0$ such that

$$\overline{F}_{S_{n+1}}(x+a) \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a}\overline{F}_{S_{n+1}}(x) - \overline{F}_{S_n}(x-\hat{b})$$

for all x and $n \geqslant 1$.

Proof. Due to representation (4), we have

$$\overline{F}_{S_{n+1}}(x) = \mathcal{J}_1(x, \hat{b}) + \mathcal{J}_2(x, \hat{b})$$

for arbitrary real x and positive $\hat{b}$.

According to (5), for fixed $\varepsilon \in (0, 1/2)$, we have

$$\inf_{n\geqslant 1} \frac{\overline{F}_{\xi_{n+1}}(x-y+a)}{\overline{F}_{\xi_{n+1}}(x-y)} \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a}$$

for all $y \leqslant x + a - \hat{b}$ and sufficiently large $\hat{b} = \hat{b}(a, \varepsilon)$.

Similarly as in the proof of Lemma 2, for such $\hat{b}$, we get

$$\begin{aligned}
\mathcal{J}_1(x+a,\, \hat{b}) &= \int\limits_{(-\infty,\, x+a-\hat{b}]} \frac{\overline{F}_{\xi_{n+1}}(x+a-y)}{\overline{F}_{\xi_{n+1}}(x-y)} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\
&\geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a} \int\limits_{(-\infty, x+a-\hat{b}]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\
&\geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a} \int\limits_{(-\infty, x-\hat{b}]} \overline{F}_{\xi_{n+1}}(x-y)\,\mathrm{d}F_{S_n}(y) \\
&= (1-\varepsilon)\mathrm{e}^{-\gamma a}\mathcal{J}_1(x, \hat{b}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{F}_{S_{n+1}}(x+a) &\geqslant \mathcal{J}_1(x+a, \hat{b}) \\
&\geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a}\mathcal{J}_1(x, \hat{b}) \\
&= (1-\varepsilon)\mathrm{e}^{-\gamma a}\bigl(\mathcal{J}_1(x, \hat{b}) + \mathcal{J}_2(x, \hat{b})\bigr) - (1-\varepsilon)\mathrm{e}^{-\gamma a}\mathcal{J}_2(x, \hat{b}) \\
&\geqslant (1-\varepsilon)\mathrm{e}^{-\gamma a}\overline{F}_{S_{n+1}}(x) - \overline{F}_{S_n}(x-\hat{b}),
\end{aligned}$$

and the assertion of Lemma 3 follows. □

The last auxiliary assertion is a mild generalization of Braverman's lemma (see [3, Lemma 1]).

Lemma 4. *Let $\{\xi_1, \xi_2, \dots\}$ be independent r.v.s, and let $F_{\xi_1} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Then, for each $a > 0$, there exists a constant c_a such that*

$$\mathbf{P}(S_n > x - a) \leqslant c_a \mathbf{P}(S_n > x)$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proof. The definition of the class $\mathcal{L}(\gamma)$ implies that

$$\overline{F}_{S_1}(x-a) = \overline{F}_{\xi_1}(x-a) \leqslant 2\mathrm{e}^{\gamma a}\overline{F}_{\xi_1}(x)$$

if $x > x_a$ and x_a is sufficiently large.

If $x \leqslant x_a$, then, obviously,

$$\frac{\overline{F}_{S_1}(x-a)}{\overline{F}_{S_1}(x)} = \frac{\overline{F}_{\xi_1}(x-a)}{\overline{F}_{\xi_1}(x)} \leqslant \frac{1}{\overline{F}_{\xi_1}(x_a)}.$$

Consequently,

$$\sup_{x\in\mathbb{R}} \frac{\overline{F}_{S_1}(x-a)}{\overline{F}_{S_1}(x)} \leqslant \max\left\{2\mathrm{e}^{\gamma a}, \frac{1}{\overline{F}_{\xi_1}(x_a)}\right\} =: c_a.$$

If $n \geqslant 2$, then, for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
\overline{F}_{S_n}(x-a) &:= \mathbf{P}(\xi_1 + S_{2,n} > x - a) \\
&= \int\limits_{-\infty}^{\infty} \frac{\overline{F}_{\xi_1}(x-y-a)}{\overline{F}_{\xi_1}(x-y)} \overline{F}_{\xi_1}(x-y)\,\mathrm{d}\mathbf{P}(S_{2,n} \leqslant y) \\
&\leqslant \sup_{z\in\mathbb{R}} \frac{\overline{F}_{\xi_1}(z-a)}{\overline{F}_{\xi_1}(z)} \int\limits_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y)\,\mathrm{d}\mathbf{P}(S_{2,n} \leqslant y) \\
&\leqslant c_a \mathbf{P}(\xi_1 + S_{2,n} > x) \\
&= c_a \mathbf{P}(S_n > x).
\end{aligned}$$

So, the assertion of Lemma 4 follows. □

4 Proofs of main results

In this section, we present detailed proofs of all our main results. For these proofs, we use essentially approaches from [6, 10] and [13].

Proof of Theorem 3. For each positive x, we have

$$\overline{F}_{S_\eta}(x) = \sum_{k \in \operatorname{supp} \eta} \mathbf{P}(\eta = k) \overline{F}_{S_k}(x).$$

Since support $\operatorname{supp} \eta$ is finite, for an arbitrary positive y, we have

$$\min_{k \in \operatorname{supp} \eta} \left\{ \frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)} \right\} \leqslant \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant \max_{k \in \operatorname{supp} \eta} \left\{ \frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)} \right\} \tag{6}$$

due to the two-sided inequality

$$\min\left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_m}{b_m} \right\} \leqslant \frac{a_1 + a_2 + \ldots + a_m}{b_1 + b_2 + \cdots + b_m} \leqslant \max\left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_m}{b_m} \right\},$$

provided that $a_i \geqslant 0$, $b_i > 0$, $i \in \{1, 2, \dots, m\}$ and $m \in \mathbb{N}$.

If $k \in \operatorname{supp} \eta$, then

$$S_k = \sum_{i \in \mathcal{K}} \xi_i + \sum_{i \notin \mathcal{K}} \xi_i,$$

where $\mathcal{K} = \{1 \leqslant i \leqslant k\colon F_{\xi_i} \in \mathcal{L}(\gamma)\}$.

Since $F_{\xi_\nu} \in \mathcal{L}(\gamma)$ for some $1 \leqslant \nu \leqslant \min\{\operatorname{supp} \eta \setminus \{0\}\}$, the set of indices $\mathcal{K}$ is not empty. Lemma 1 implies that d.f. $F_{\mathcal{K}}$ of sum $\sum_{i \in \mathcal{K}} \xi_i$ belongs to the class $\mathcal{L}(\gamma)$.

Further, if $i^* \notin \mathcal{K}$, then $\overline{F}_{\xi_{i^*}}(x) = o(\overline{F}_{\xi_\nu}(x))$ because of the theorem's conditions. Therefore,

$$\frac{\overline{F}_{\xi_{i^*}}(x)}{\overline{F}_{\mathcal{K}}(x)} = \frac{\mathbf{P}(\xi_{i^*} > x)}{\mathbf{P}(\sum_{i \in \mathcal{K}} \xi_i > x)} \leqslant \frac{\overline{F}_{\xi_{i^*}}(x)}{\overline{F}_{\xi_\nu}(x)} \underset{x \to \infty}{\longrightarrow} 0,$$

and consequently, $F_{\mathcal{K}} * F_{\xi_{i^*}}$ belongs to the class $\mathcal{L}(\gamma)$ due to the second part of Lemma 1. Continuing our considerations, we get that d.f.

$$F_{S_k} = F_{\mathcal{K}} * \left\{ \mathop{\prod\!\!\!\!\!*}_{i \notin \mathcal{K}} F_{\xi_i} \right\}$$

belongs to the class $\mathcal{L}(\gamma)$ as well for an arbitrary index $k \in \operatorname{supp} \eta$. Here $\prod^{*}_{i \notin \mathcal{K}} F_{\xi_i}$ denotes d.f. of sum $\sum_{i \notin \mathcal{K}} \xi_i$.

Consequently, the double estimate (6) implies the following two inequalities:

$$\limsup_{x \to \infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant \max_{k \in \operatorname{supp} \eta} \left\{ \limsup_{x \to \infty} \frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)} \right\} = \mathrm{e}^{-\gamma y},$$

$$\liminf_{x \to \infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \geqslant \min_{k \in \operatorname{supp} \eta} \left\{ \liminf_{x \to \infty} \frac{\overline{F}_{S_k}(x+y)}{\overline{F}_{S_k}(x)} \right\} = \mathrm{e}^{-\gamma y}$$

for each positive y. The last two estimates finish the proof of Theorem 3. □

Proof of Theorem 4. In order to show that $F_{S_\eta} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$, it is sufficient to derive the following two estimates:

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant \mathrm{e}^{-\gamma y}, \tag{7}$$

$$\liminf_{x\to\infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \geqslant \mathrm{e}^{-\gamma y}, \tag{8}$$

which both should valid for each positive y.

(I) At first, we show inequality (7). For this, we suppose that y is an arbitrary positive number, and we choose $\varepsilon \in (0,1)$. According to condition (2), we have

$$\mathbf{P}(\eta = n+1) \leqslant \varepsilon \mathbf{P}(\eta = n) \tag{9}$$

for all $n \geqslant N = N(\varepsilon) \geqslant 2$. For such N, we get that

$$\overline{F}_{S_\eta}(x+y) = \sum_{n=1}^{N} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x+y) + \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x+y). \tag{10}$$

Using Lemma 2, we obtain

$$\begin{aligned}\sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x+y) \quad &\leqslant \sum_{n=N+1}^{\infty} (1+\varepsilon)\mathrm{e}^{-\gamma y}\,\mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x) \\ &\quad + \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_{n-1}}(x-b)\end{aligned}$$

for some $b = b(y,\varepsilon) > 0$. This relation together with inequality (9) shows that

$$\begin{aligned}\overline{F}_{S_\eta}(x+y) \leqslant &\sum_{n=1}^{N} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x+y) \\ &+ (1+\varepsilon)\mathrm{e}^{-\gamma y} \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x) \\ &+ \varepsilon \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n-1)\,\overline{F}_{S_{n-1}}(x-b).\end{aligned} \tag{11}$$

Condition (1) implies that

$$\mathrm{e}^{-\gamma u} \leqslant \liminf_{x\to\infty} \frac{\overline{F}_{\xi_k}(x+u)}{\overline{F}_{\xi_k}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(x+u)}{\overline{F}_{\xi_k}(x)} \leqslant \mathrm{e}^{-\gamma u}$$

for all fixed k and u. It follows that $\overline{F}_{\xi_k} \in \mathcal{L}(\gamma)$ for each k. Hence, according to Lemma 1, we obtain that $\overline{F}_{S_n} \in \mathcal{L}(\gamma)$ for each fixed $n \in \mathbb{N}$. Therefore,

$$\max_{1\leqslant n\leqslant N} \frac{\overline{F}_{S_n}(x+y)}{\overline{F}_{S_n}(x)} \leqslant (1+\varepsilon)\mathrm{e}^{-\gamma y} \tag{12}$$

if $x \geqslant \hat{x} = \hat{x}(N, y, \varepsilon)$.

So, for the chosen N, b and for all $x \geqslant \hat{x}$, we have

$$\begin{aligned}
\overline{F}_{S_\eta}(x+y) &\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma y} \sum_{n=1}^{N} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x) \\
&\quad + (1+\varepsilon)\mathrm{e}^{-\gamma y} \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x) \\
&\quad + \varepsilon \sum_{n=N}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x-b) \\
&= (1+\varepsilon)\mathrm{e}^{-\gamma y}\overline{F}_{S_\eta}(x) + \varepsilon \sum_{n=N}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x-b).
\end{aligned}$$

According to Lemma 4, $\overline{F}_{S_n}(x-b) \leqslant c_1\overline{F}_{S_n}(x)$ for some positive constant $c_1 = c_1(b(y,\varepsilon))$.

Therefore,

$$\begin{aligned}
\overline{F}_{S_\eta}(x+y) &\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma y}\overline{F}_{S_\eta}(x) + \varepsilon c_1 \sum_{n=N}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x) \\
&\leqslant (1+\varepsilon)\mathrm{e}^{-\gamma y}\overline{F}_{S_\eta}(x) + \varepsilon c_1\overline{F}_{S_\eta}(x)
\end{aligned}$$

for all sufficiently large x.

The last inequality implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \leqslant (1+\varepsilon)\mathrm{e}^{-\gamma y} + \varepsilon c_1.$$

Since $\varepsilon \in (0,1)$ is arbitrarily chosen, the desired inequality (7) holds for each positive y.

(II) In this part, we show inequality (8). We fix positive y and choose $\varepsilon \in (0, 1/2)$. Let N be a natural number such that, for $n \geqslant N$, inequality (9) holds. Due to Lemma 3, we have

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x+y) &\geqslant \sum_{n=N+1}^{\infty} (1-\varepsilon)\mathrm{e}^{-\gamma y}\mathbf{P}(\eta = n)\,\overline{F}_{S_n}(x) \\
&\quad - \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\,\overline{F}_{S_{n-1}}(x-\hat{b})
\end{aligned} \tag{13}$$

for some $\hat{b} = \hat{b}(y, \varepsilon) > 0$. Substituting estimates (9) and (13) into equality (10), we get

$$\begin{aligned} \overline{F}_{S_\eta}(x+y) \geqslant \sum_{n=1}^{N} \mathbf{P}(\eta = n)\, \overline{F}_{S_n}(x+y) \\ + (1-\varepsilon)\,\mathrm{e}^{-\gamma y} \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n)\, \overline{F}_{S_n}(x) \\ - \varepsilon \sum_{n=N+1}^{\infty} \mathbf{P}(\eta = n-1)\, \overline{F}_{S_{n-1}}(x - \hat{b}). \end{aligned} \tag{14}$$

Since $F_{\xi_k} \in \mathcal{L}(\gamma)$ for each fixed k, using Lemma 1, we obtain $F_{S_n} \in \mathcal{L}(\gamma)$ for $n \in \mathbb{N}$. Similarly as in derivation of (12), this implies that

$$\min_{1 \leqslant n \leqslant N} \frac{\overline{F}_{S_n}(x+y)}{\overline{F}_{S_n}(x)} \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma y} \tag{15}$$

if $x \geqslant \tilde{x} = \tilde{x}(N, y, \varepsilon)$.

For such $x \geqslant \tilde{x}$, due to (14) and (15), we have

$$\overline{F}_{S_\eta}(x+y) \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma y} \overline{F}_{S_\eta}(x) - \varepsilon \sum_{n=N}^{\infty} \mathbf{P}(\eta = n)\, \overline{F}_{S_n}(x - \hat{b}).$$

According to Lemma 4, we have that $\overline{F}_{S_n}(x - \hat{b}) \leqslant c_2 \overline{F}_{S_n}(x)$ for some positive constant $c_2 = c_2(\hat{b}(y, \varepsilon))$. Therefore,

$$\overline{F}_{S_\eta}(x+y) \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma y} \overline{F}_{S_\eta}(x) - \varepsilon c_2 \overline{F}_{S_\eta}(x)$$

when $x \geqslant \tilde{x}$. This last estimate implies that

$$\liminf_{x \to \infty} \frac{\overline{F}_{S_\eta}(x+y)}{\overline{F}_{S_\eta}(x)} \geqslant (1-\varepsilon)\mathrm{e}^{-\gamma y} - \varepsilon c_2$$

for an arbitrary $\varepsilon \in (0, 1/2)$.

Letting ε tend to zero, from the last inequality we get the desired estimate (8). The theorem is proved. □

Proof of Theorem 5. If $\varkappa = 1$, then the assertion of Theorem follows from Theorem 4. So, we suppose that $\varkappa \geqslant 2$, and we split our proof into two parts.

(I) If $\mathbf{P}(\eta \leqslant \varkappa) = 0$, then r.v. η has an infinite support

$$\operatorname{supp} \eta \subset \{\varkappa + 1,\ \varkappa + 2,\ \ldots\}.$$

Conditions (i)–(iii) imply that $F_{S_\varkappa} \in \mathcal{L}(\gamma)$ due to Theorem 3. Since $F_{\xi_{\varkappa+1}} \in \mathcal{L}(\gamma)$ according to condition (iv), the convolution $F_{S_{\varkappa+1}} = F_{S_\varkappa} * F_{\xi_{\varkappa+1}}$ belongs to the class $\mathcal{L}(\gamma)$

as well due to Lemma 1. This and condition (iv) imply that

$$\sup_{k\geqslant 1}\left|\frac{\overline{F}_{\hat{\xi}_k}(x+y)}{\overline{F}_{\hat{\xi}_k}(x)} - \mathrm{e}^{-\gamma y}\right| \underset{x\to\infty}{\longrightarrow} 0$$

for each fixed $y \geqslant 0$, where $\hat{\xi}_1 = S_{\varkappa+1}$, $\hat{\xi}_2 = \xi_{\varkappa+2}$, $\hat{\xi}_3 = \xi_{\varkappa+3}, \dots$.

Let $\hat{\eta}$ be the counting r.v. defined by equality $\mathbf{P}(\hat{\eta} = k) = \mathbf{P}(\eta = \varkappa + k)$, where $k = 1, 2, \dots$, and let $\hat{S}_n = \hat{\xi}_1 + \hat{\xi}_2 + \dots + \hat{\xi}_n$ for each $n \geqslant 1$.

R.v.s $\{\hat{\xi}_1, \hat{\xi}_2, \dots\}$ and $\hat{\eta}$ satisfy conditions of Theorem 4. Hence, d.f. $F_{\hat{S}_{\hat{\eta}}}$ belongs to the class $\mathcal{L}(\gamma)$. We observe that

$$\begin{aligned}\overline{F}_{\hat{S}_{\hat{\eta}}}(x) &= \mathbf{P}(\hat{\eta} = 1)\mathbf{P}(\hat{S}_1 > x) + \sum_{k=2}^{\infty}\mathbf{P}(\hat{\eta} = k)\mathbf{P}(\hat{S}_k > x)\\ &= \mathbf{P}(\eta = \varkappa + 1)\mathbf{P}(S_{\varkappa+1} > x) + \sum_{k=2}^{\infty}\mathbf{P}(\eta = \varkappa + k)\mathbf{P}(S_{\varkappa+k} > x)\\ &= \overline{F}_{S_\eta}(x)\end{aligned}$$

for an arbitrary nonnegative x. Consequently, F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ as well in the case under consideration.

(II) Let now $\mathbf{P}(\eta \leqslant \varkappa) > 0$. Since $\mathbf{P}(\eta \geqslant \varkappa + 1) > 0$ due to condition (v), we have that

$$\overline{F}_{S_\eta}(x) = \mathbf{P}(\eta \leqslant \varkappa)\overline{F}_{S_{\tilde{\eta}}}(x) + \mathbf{P}(\eta \geqslant \varkappa + 1)\,\overline{F}_{S_{\hat{\eta}}}(x) \qquad (16)$$

for an arbitrary nonnegative x, where $\tilde{\eta}$ and $\hat{\eta}$ are two counting r.v.s independent of $\{\xi_1, \xi_2, \dots\}$ with distributions

$$\begin{aligned}\mathbf{P}(\tilde{\eta} = k) &= \frac{\mathbf{P}(\eta = k)}{\mathbf{P}(\eta \leqslant \varkappa)}, \quad k \in \{0, 1, \dots, \varkappa\},\\ \mathbf{P}(\hat{\eta} = k) &= \frac{\mathbf{P}(\eta = k)}{\mathbf{P}(\eta \geqslant \varkappa + 1)}, \quad k \in \{\varkappa + 1,\ \varkappa + 2,\ \dots\}.\end{aligned}$$

Theorem 3 implies that $F_{S_{\tilde{\eta}}} \in \mathcal{L}(\gamma)$ because of the finiteness of support $\operatorname{supp}\tilde{\eta}$. The investigation analogous to that in part (II) implies that d.f. $F_{S_{\hat{\eta}}}$ belongs to the class $\mathcal{L}(\gamma)$ as well. Now the statement of theorem follows immediately from relation (16). Theorem 5 is proved. □

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