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MOMENTS OF ADDITIVE FUNCTIONS DEFINED ON RANDOM ASSEMBLIES

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Notation

$\mathbb{N}$	the set of positive integers
$\mathbb{N}_0$	the set of nonnegative integers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
n	some positive integer
$\bar{s}$	generic vector $(s_1, s_2, ..., s_n) \in \mathbb{N}_0^n$
$\ell(\bar{s})$	the sum $1s_1 + \cdots + ns_n$
$\mathbf{S}_n$	the symmetric group of permutations of order n
$\mathcal{G}$	the class of assemblies
$\mathcal{G}_n$	the set of assemblies of order $n, \mathcal{G}_n \subset \mathcal{G}$
$G(n)$	the cardinality of the set $\mathcal{G}_n$
σ	generic element of the $\mathcal{G}_n$
$k_j(\sigma)$	the number of components of size j in $\sigma, 1 \leq j \leq n; k_j(\sigma) \geq 0$
$\bar{k}(\sigma)$	the component vector $\big(k_1(\sigma), \ldots, k_n(\sigma)\big)$
$w(\sigma)$	the number-of-component function $k_1(\sigma) + \cdots + k_n(\sigma)$
$\#\{\cdot\}$	the cardinality of a set
$\mathbf{1}\{\cdot\}$	the indicator function
$\Gamma(\cdot)$	the Euler gamma function
$\ll$	the analog of the symbol $O(\cdot)$
$a(x) \asymp b(x)$	means $a(x) \ll b(x)$ and $b(x) \ll a(x)$
$a(x) \sim b(x)$	means $\lim\limits_{x\to\infty} (a(x)/b(x)) = 1$

Introduction

The dissertation work is devoted to random decomposable combinatorial structures. The highly developed probabilistic number theory dealing with product decomposition of a random natural number into primes served as a great pattern for our research. In particular, we concentrate on additive statistics, therefore it is worth to recall corresponding results from number theory.

We call a function $f : \mathbb{N} \to \mathbb{C}$ *additive* if $f(nm) = f(n) + f(m)$ whenever n, m are coprime integers. Established by P. Turán ([54]) and generalized by J. Kubilius in 1956 (see [30] for an historical account) the famous Turán-Kubilius inequality states that

$$\sum_{n \le x} |f(n) - A(x)|^2 \ll xB(x)^2$$

uniformly for all real $x \ge 2$ and additive functions f. Here, the estimates of the "expectation" $A(x)$ and the "variance" $B(x)^2$ are defined as

$$A(x) := \sum_{p^k \le x} \frac{f(p^k)}{p^k},$$

$$B(x)^2 := \sum_{p^k \le x} \frac{|f(p^k)|^2}{p^k},$$

where p are prime numbers.

The system of events $\{n : \ n \equiv 0 \bmod p\}$, $p \le x$, when n is taken uniformly from the set $\{1, 2, \ldots, [x]\}$ is dependent; however, the inequality demonstrates that the variance of $f(n)$ can be estimated via a sum of variances of the summands. In this regard, the result has a form close to that for the sums of independent random variables. In fact, the absolute constant in the symbol $\ll$ absorbed influence of dependency. This phenomenon repeated itself in the subsequent generalizations of the above inequality. We gained a lot studying the paper by A. Biró and T. Szamuely [7] exploring the case when a natural number n is taken with a weighted probability.

A higher power analogue of the Turán-Kubilius inequality was established by P.D.T.A. Elliott in [12]. It has the following form:

Let α be a real number. Then there is a constant c, depending at most upon α, so that the inequality

$$x^{-1}\sum_{n\le x}|f(n)-A(x)|^{\alpha}\le c\begin{cases}B(x)^{\alpha}+\sum_{p^k\le x}p^{-k}|f(p^k)|^{\alpha} & \text{if } \alpha\ge 2,\\ B(x)^{\alpha} & \text{if } 0\le\alpha\le 2,\end{cases}\tag{1}$$

holds uniformly for all additive functions f and real $x\ge 2$.

Furthermore, Elliott established an inequality, dual to (1), in [13]. The method of dualization is explained in [11], and the whole monograph [13] is about duality and its applications. That also gave us an impulse to obtain some combinatorial analogs of this type. Various generalizations of the power moment estimates (1) followed. We mention papers by I.Z. Ruzsa [51] and K.-H. Indlekofer [20] to list but few.

The Turán-Kubilius inequality was also extended to additive functions defined on arithmetical semigroups (for definitions and motivation, we refer to books [28], [29]). A lot of work has been done by, for example, Z. Juškys ([25]) and J.-L. Mauclaire ([45], [46]). Elements of an additive semigroups can be interpreted as weighted multisets laying within the frames of combinatorics. Taking them at random, one can raise and solve problems analogues to that cultivated in probabilistic number theory. The variance of an additive function defined in such semigroups was examined by W.-B. Zhang ([65]). The result was considerably extended by K.-H. Indlekofer's student S. Wehmeier in the dissertation [56] and in paper [57] a result of which we now present.

An additive arithmetic semigroup $\mathcal{G}$ is a monoid with a countable generating set $\mathcal{P}$ of "primes" which admits a degree mapping $\partial:\mathcal{G}\to\mathbb{N}_0$ such that $\partial(ab)=\partial(a)+\partial(b)$ for all $a,b\in\mathcal{G}$, $\partial(p)\ge 1$ for any $p\in\mathcal{P}$ and that $\mathbf{G}(n):=\#\{a\in\mathcal{G}|\partial(a)=n\}$ is finite for all n. Let a function $f:\mathcal{G}\to\mathbb{R}$ be such that $f(a)=\sum_{p|a}f(p)$ for all $a\in\mathcal{G}$. Assume that $\mathbf{G}(n)=Aq^n(1+R(n))$ with constants $A>0$, $q>1$. Suppose Chebyshev bound condition $\mathbf{P}(n)\ll\mathbf{G}(n)/n$, where $\mathbf{P}(n):=\#\{p\in\mathcal{P}|\partial(p)=n\}$, and the condition $R(n)=O(\log(n)^{-1})$ holds. Let

$$A(n):=\frac{1}{\mathbf{G}(n)}\sum_{\partial(p)\le n}f(p)\mathbf{G}(n-\partial(p))$$

and

$$B(n)^2:=\frac{1}{\mathbf{G}(n)}\sum_{\partial(p)\le n}f(p)^2\mathbf{G}(n-\partial(p)).$$

Then we have

$$\frac{1}{\mathbf{G}(n)}\sum_{\partial(a)=n}\big(f(a)-A(n)\big)^2 \ll B(n)^2.$$

In this research, the main goal was finding the most general conditions assumed for the semigroup under which an analogue of the Turán-Kubilius still holds. Similar task but for another class of combinatorial structures is raised in the present thesis.

The success, importance and great scale of applications of the Turán-Kubilius inequality and it's generalizations in number theory gives us a believe that analogues established for additive functions defined on various combinatorial structures will be of great value. The first such appearance has been made by E. Manstavičius in paper [31] devoted to random permutations. To display the result, we introduce some notation and definitions.

Let $\mathbf{S}_n$ be the symmetric group of permutations σ acting on n letters. If the canonical representation of $\sigma \in \mathbf{S}_n$ into a product of independent cycles has $k_j(\sigma) \in \mathbb{N}_0$ cycles of length $1 \le j \le n$, then the so-called cycle vector

$$\bar{k}(\sigma) := \big(k_1(\sigma), \dots, k_n(\sigma)\big)$$

satisfies a relation $\ell(\bar{k}(\sigma)) = n$ for each $\sigma \in \mathbf{S}_n$. Here $\ell(\bar{s}) := 1s_1 + \cdots + ns_n$ if $\bar{s} = (s_1, \dots, s_n) \in \mathbb{N}_0^n$. Further, given a real two-dimensional array $\{h_j(s)\}$, where $1 \le j \le n$ and $s \ge 0$, such that $h_j(0) := 0$ for all $j \le n$, we define an additive function $h : \mathbf{S}_n \to \mathbb{R}$ by setting

$$h(\sigma) := \sum_{j \le n} h_j\big(k_j(\sigma)\big). \tag{2}$$

One can easily see resemblance to a number-theoretic additive function if cycles of different lengths are understood as analogy of coprime numbers.

Now, Corollary 5.3 in [31] can be perceived as the following result.

For an additive function $h : \mathbf{S}_n \to \mathbb{R}$, $A \in \mathbb{R}$ and $\alpha \ge 0$ we have

$$\frac{1}{n!}\sum_{\sigma \in \mathbf{S}_n} |h(\sigma) - A|^{\alpha} \ll \mathbb{E}\Big|\sum_{j \le n} h_j(\xi_j) - A\Big|^{\alpha}, \tag{3}$$

uniformly for all $n \in \mathbb{N}$, where ξ_j, $j \le n$, are independent Poisson random variables with parameters $1/j$, $\mathbb{E}$ denotes the expectation and constant in $\ll$ depends only on α.

As noted in [31] and showed in [35], the inequality (3), using well known results regarding moments of sums of independent random variables (see, for example, [48]), leads to:

For an additive function $h : \mathbf{S}_n \to \mathbb{R}$*, and* $\alpha \geq 0$ *we have*

$$\frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} |h(\sigma) - A_n|^\alpha \ll \begin{cases} B_n^\alpha + B_n(\alpha) & \text{if } \alpha \geq 2, \\ B_n^\alpha & \text{if } 0 \leq \alpha \leq 2, \end{cases}$$

uniformly for all $n \in \mathbb{N}$*. Here the constant in* $\ll$ *depends only on* α*,*

$$A_n := \sum_{jk \leq n} \frac{h_j(k)}{j^k k!}, \qquad B_n(\alpha) := \sum_{jk \leq n} \frac{|h_j(k)|^\alpha}{j^k k!},$$

and $B_n := \big(B_n(2)\big)^{1/2}$.

The latter result is nothing less but an analogue of Elliott's result (1) for additive functions defined on random permutations. Subsequently, an inequality on random permutations, taken according to the Ewens measure, was established by G.J. Babu and E. Manstavičius [3]. Later an inequality on mappings of a finite set into itself was proved by Manstavičius [41]. We proceed the work by obtaining moment estimates, not yet known, for the class of combinatorial structures, called *assemblies*.

By the definition given in Section 2.2 of the book [2], an assembly is a construction defined on a finite set by its partition into blocks and some combinatorial structure introduced in all these blocks, afterwards called components of the assembly. In permutations components are cycles, in labeled graphs they are connected components and so on. More examples are given in the Chapter 2 of the present thesis. The notion of an additive function on assemblies remains the same as in the case of permutations; it suffice to substitute the cycle vector by a corresponding component vector. Taking an assembly from a given class at random we go ahead in obtaining moment estimates.

The dissertation is organized as follows:

Chapter 1 deals with additive functions defined on the symmetric group, where a permutation is taken according to a generalized Ewens probability. Here we establish an upper bound of its variance via a sum of variances of the summands. The idea of our approach goes back to the above mentioned paper by Biró and Szamuely [7].

Chapter 2 presents an analogue of Turán-Kubilius inequality for an additive function defined on random assemblies. The result generalizes estimates obtained earlier in the cases of permutations and mappings of a finite set into itself, but is also slightly different from the results obtained in Chapter 1.

Chapter 3 manages the additive semigroup of vectors with non-negative integer coordinates endowed with the Ewens Probability Measure, which plays an important role as a probabilistic space for many statistical models. In them, additive and multiplicative statistics defined on the semigroup having decompositions via dependent random variables raise an interest from many points of view. We obtain upper estimates of the power moments of additive statistics defined on the semigroup. Our result is an analogue of the result obtained by Elliott in [12].

Actuality

Random discrete structures appear modelling various objects in biology, computer science, physics, etc. As witnessed by H. Crane in the survey paper [10], the only Ewens Sampling Formula and distributions defined via it contribute to the foundations of evolutionary molecular genetics, the neutral theory of biodiversity, Bayesian nonparametrics, combinatorial stochastic processes. They also emerge from fundamental concepts in probability theory, algebra, and number theory. Value distribution of additive statistics defined on decomposable combinatorial structures is a fairly important and complex problem. Moment estimates of the statistics become very desirable dealing with it. One can observe that the latter line is less developed in probabilistic combinatorics than that in probabilistic number theory (papers by P. Turán, J. Kubilius, P.D.T.A. Elliott, I.Z. Ruzsa, I. Kátai, K.-H. Indlekofer, etc.) and that in the parallel theory of additive arithmetical semigroups (W.-B. Zhang, S. Wehmeier, etc.). A few papers by E. Manstavičius devoted for random mappings do not fill up this gap. It is our main purpose to extend the results of the mentioned authors.

Methods

We use combinatorial, probabilistic and analytical methods. The technical approaches applied in probabilistic number theory are adopted and further enriched.

Novelty

All the results stated in this dissertation are new.

Dissemination of the results

The results, exposed in Chapters 1 and 2, are already published in the papers:

- E. Manstavičius and V. Stepanauskas (Stepas), *On variance of an additive function with respect to a generalized Ewens probability.* Proceedings of the 25th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms, 301–312, Discrete Math. Theor. Comput. Sci. Proc., BA, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2014.

- E. Manstavičius and V. Stepas, *Variance of additive functions defined on random assemblies.* Lith. Math. J. 57 (2017), no. 2, 222–235.

The papers are filed in Bibliography as [39] and [42] respectively. The result, stated in Chapter 3, is to be submitted as:

- E. Manstavičius and V. Stepas, *Moments of additive statistics with respect to the Ewens Sampling Formula.*

All of the results were exposed and discussed in the seminars organized by Department of Probability Theory and Number Theory of Faculty of Mathematics and Informatics of Vilnius University during 2012-2017 year period. Also, they were presented at the following conferences:

- E. Manstavičius and V. Stepanauskas (Stepas), *On influence of probabilistic number theory to probabilistic combinatorics.* The 13th Serbian Mathematical Congress, Vrnjačka Banja, Serbia, 22-25 May 2014.

- E. Manstavičius and V. Stepanauskas (Stepas), *On variance of an additive function with respect to a generalized Ewens probability.* The 25th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms, AofA'14, Paris, France, 16-20 June 2014.

- V. Stepanauskas (Stepas), *On variance of an additive function on permutations.* The 11th international Vilnius Conference on Probability Theory and Mathematical Statistics, Vilnius, Lithuania, 30 June - 4 July 2014.

- E. Manstavičius and V. Stepas, *On variance of an additive function defined on random permutations.* The 57th Conference of the Lithuanian Mathematical Society, Vilnius, Lithuania, 20-21 June 2016.

- ▷ E. Manstavičius and V. Stepas, *Variance of additive functions defined on random assemblies.* The 27th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms, AofA'16, Krakow, Poland, 4-8 July 2016.

In addition, there are few conference abstracts:

- ▷ E. Manstavičius and V. Stepanauskas (Stepas), *On influence of probabilistic number theory to probabilistic combinatorics.* http://tesla.pmf.ni.ac.rs/people/smak/book_of_abstracts.pdf (2014), 15.
- ▷ V. Stepanauskas (Stepas), *On variance of an additive function on permutations.* 11th international Vilnius Conference on Probability Theory and Mathematical Statistics, Abstracts of Communications (2014), 234.
- ▷ E. Manstavičius and V. Stepas, *Variance of additive functions defined on random assemblies.*http://aofa.tcs.uj.edu.pl/proceedings/aofa2016.pdf (2016), 278-280 or arXiv:1605.04239v1 (2016).

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Chapter 1

Variance of additive functions with respect to a generalized Ewens probability

1.1 Basics and motivation

Let $\mathbf{S}_n$ be the symmetric group of permutations σ acting on n letters. If the canonical representation of $\sigma \in \mathbf{S}_n$ into a product of independent cycles has $k_j(\sigma) \in \mathbb{N}_0$ cycles of length $1 \le j \le n$, then the so-called cycle vector

$$\bar{k}(\sigma) := \big(k_1(\sigma), \ldots, k_n(\sigma)\big)$$

satisfies a relation $\ell(\bar{k}(\sigma)) = n$ for each $\sigma \in \mathbf{S}_n$. Here $\ell(\bar{s}) := 1s_1 + \cdots + ns_n$ if $\bar{s} = (s_1, \ldots, s_n) \in \mathbb{N}_0^n$. As in (2), an *additive function* $h : \mathbf{S}_n \to \mathbb{R}$ is defined by a real two-dimensional array $\{h_j(k)\}$, where $j, k \in \mathbb{N}$, $jk \le n$, and $h_j(0) := 0$ for all $j \le n$, by setting

$$h(\sigma) := \sum_{j=1}^{n} h_j\big(k_j(\sigma)\big).$$

Apart from the most popular example of the number-of-cycles function $w(\sigma) := k_1(\sigma) + \cdots + k_n(\sigma)$, they appear in many algebraic and combinatorial problems. In the so-called Erdős-Turán problem they are used to approximate the logarithm of group theoretical order of $\sigma \in \mathbf{S}_n$ (see [63], [18] and the references therein). Particular additive functions appear in physical models as a part of Hamiltonians in the Bose gas theory (see, for example, [5]).

Moreover, one may mention additive functions related to a permutation matrix

$$M(\sigma) := \big(m_{ij}(\sigma)\big), \quad 1 \le i, j \le n,\ \sigma \in \mathbf{S}_n.$$

Here $m_{ij}(\sigma) := 1$ if $i = \sigma(j)$ and $m_{ij}(\sigma) := 0$ otherwise. It is known (see, for example, [64]) that the characteristic polynomial is

$$Z_n(x;\sigma) := \det\big(I - xM(\sigma)\big) = \prod_{j\le n}(1-x^j)^{k_j(\sigma)}.$$

Let $\mathrm{e}^{2\pi i\varphi_j(\sigma)}$, where $\varphi_j(\sigma) \in [0,1)$ and $j \le n$, be its eigenvalues. A lot of work has been done on $\log|Z_n(x;\sigma)|$, imaginary part of $\log Z_n(x;\sigma)$ or the trace-related mappings

$$\sum_{j\le n} f\big(\varphi_j(\sigma)\big) = \sum_{j\le n} k_j(\sigma) \sum_{0\le s\le j-1} f\Big(\frac{s}{j}\Big),$$

where $f : [0,1] \to \mathbb{R}$ is an arbitrary function. We just mention [1], [58], [64], and [14] to name but a few. The papers confirm a need to examine the value distribution of general additive functions (*separable statistics*, as the authors of [18] propose) as $n \to \infty$ if σ is taken at random. One can also observe the recent trend to do this with respect to a generalized Ewens probability measure endowed in $\mathbf{S}_n$ (see, for example, [6], [47], [44], [14], [8]). The measure has been introduced in 2002 [32] where some limit theorems for additive functions have been proved. Later this line of research was continued in a few of E. Manstavičius papers.

Let $\theta_j \ge 0$, $1 \le j \le n$, be an arbitrary, maybe, dependent on n, and not identical to zero sequence, then the generalized Ewens probability measure $\nu_n^{(\bar\theta)}$ is defined by

$$\nu_n^{(\bar\theta)}(\{\sigma\}) := (n!Q(n))^{-1}\prod_{j\le n}\theta_j^{k_j(\sigma)}, \qquad Q(n) := \sum_{\ell(\bar s)=n}\prod_{j\le n}\Big(\frac{\theta_j}{j}\Big)^{s_j}\frac{1}{s_j!}, \qquad \sigma \in \mathbf{S}_n,$$

provided that $Q(n) > 0$.

If $\theta_j \equiv \theta > 0$, some fixed constant, then $\nu_n^{(\bar\theta)} =: \nu_n^{(\theta)}$ is the classical Ewens measure on $\mathbf{S}_n$. In this case

$$Q(n) = \Theta(n) := \binom{\theta + n - 1}{n} \tag{1.1}$$

and the cycle vector has a distribution

$$\nu_n^{(\theta)}(\bar k(\sigma) = \bar s) = \mathbf{1}\{\ell(\bar s) = n\}\Theta(n)^{-1}\prod_{j\le n}\frac{1}{s_j!}\Big(\frac{\theta}{j}\Big)^{s_j} =: P_n(\{\bar s\}), \tag{1.2}$$

where $\bar{s} \in \Omega_n := \{\bar{s} \in \mathbb{N}_0^n : \ \ell(\bar{s}) = n\}$. The expression of probabilities $P_n(\{\bar{s}\})$ ascribed to $\bar{s} \in \Omega_n$ is well known as the Ewens Sampling Formula (see [15]).

In the present chapter, we focus on the estimates of the variance $\mathbf{V}_n^{(\bar{\theta})} h(\sigma)$ with respect to $\nu_n^{(\bar{\theta})}$. This seemingly simple problem concerns a variance of a sum of dependent random variables, thus, an estimate of $\mathbf{V}_n^{(\bar{\theta})} h(\sigma)$ in terms of a sum of variances of the summands is not that easy if general weights θ_j, $j \leq n$, are involved. Even for the Ewens measure, if $\theta_j \equiv \theta < 1$, we had no decent result so far. As it is shown in Lemma 3.2 in [4], we can expose explicit formulas for factorial moments of additive function h, but no estimates follow. The more simple case with $\theta \geq 1$ has been dealt with in [27]. The second moment estimates are very useful for proving the law of large numbers (see, for example, [27]). Together with the total variation approximation of the distribution of the first cycle vector coordinates by independent random variables (see [38]), they comprise an instrument allowing to estimate the error appearing by truncating sums over long cycles (see, for example, [9]).

1.2 Results

Our first theorem is for simplicity stated for a completely additive function defined via $h_j(s) = sa_j$ with arbitrary $a_j \in \mathbb{R}$, where $j \le n$ and $s \ge 0$, and for the Ewens probability. Let $\mathbb{E}_n^{(\theta)} g(\sigma)$ and $\mathbf{V}_n^{(\theta)} g(\sigma)$ be the expectation and the variance with respect to $\nu_n^{(\theta)}$ of a random variable $g : \mathbf{S}_n \to \mathbb{R}$ and

$$B_n^2 := B_n^2(h) := \theta \sum_{j \le n} \frac{a_j^2}{j} \frac{\Theta(n-j)}{\Theta(n)}.$$

We will establish in the next section that

$$R_n := B_n^2 - \sum_{j \le n} \mathbf{V}_n^{(\theta)}\big(a_j k_j(\sigma)\big) = O\big(n^{-\min\{1,\theta\}} B_n^2\big) \tag{1.3}$$

if $n \to \infty$. This motivates the inequalities proved below and a fairly frequent use of B_n as a scaling sequence in limit theorems for $h(\sigma)$ as well. As it has been shown in [1], for a particular class of additive functions $h(\sigma)$, the relation $\mathbf{V}_n^{(\theta)} h(\sigma) \sim B_n^2(h)$ as $n \to \infty$ holds but this is not the case in general. A complete characterization of the additive functions $h(\sigma)$ satisfying the latter relation for variances seems to be an uneasy problem.

Theorem 1.1. *There exists an absolute constant $C > 1$ such that, for any completely additive function $h(\sigma)$, $\theta > 0$, and for any $n \ge 1$,*

$$\mathbf{V}_n^{(\theta)} h(\sigma) \le C B_n^2. \tag{1.4}$$

If $\theta \ge 1$, one can take $C = 2$. For large n, even smaller constants can be obtained. Indeed, if

$$\tau_n(\theta) := \sup\Big\{\mathbf{V}_n^{(\theta)} h(\sigma) B_n(h)^{-2} :\ h \ne 0\Big\},$$

then $\tau_n(1) = 3/2 + O(n^{-1})$ and $\tau_n(2) = 4/3 + O(n^{-1})$ (see [33] and [37]).

To simplify B_n^2, one can apply the asymptotic formula

$$\Theta(n-j)/\Theta(n) = (1 - j/n)^{\theta-1}\Big(1 + O\big((n-j)^{-1}\big)\Big), \quad 1 \le j \le n-1, \tag{1.5}$$

following from the well known (see [17]) estimate

$$\Theta(m) = [z^m](1-z)^{-\theta} = \frac{m^{\theta-1}}{\Gamma(\theta)}\left(1 + O\Big(\frac{1}{m}\Big)\right), \tag{1.6}$$

where $0 < \theta \leq T$, $m \geq 1$ and constant in $O(\cdot)$ depends on T only. This is implemented in the next inequality valid in a more general case. However, now the dependence on θ of the appearing constant is more involved.

Theorem 1.2. *For an arbitrary real additive function given in* (2) *and all* $n \geq 1$, *there exists a constant* $C(\theta) > 0$ *depending on* θ *only and such that*

$$\mathbf{V}_n^{(\theta)} h(\sigma) \leq C(\theta) \sum_{jk \leq n} \left(\frac{\theta}{j}\right)^k \frac{h_j(k)^2}{k!} \left(1 - \frac{jk}{n+1}\right)^{\theta - 1}.$$

The variance $\mathbf{V}_n^{(\bar\theta)} h(\sigma)$ with respect to the generalized Ewens probability measure will be estimated in terms of the quantity

$$D_n^2 := \sum_{jk \leq n} \left(\frac{\theta_j}{j}\right)^k \frac{h_j(k)^2}{k!} \frac{Q(n-jk)}{Q(n)}.$$

The next result generalizes Theorem 1.2.

Theorem 1.3. *Assume that* $0 < \alpha \leq \theta_j \leq \beta < \infty$ *for all* $j \leq n$. *Then there exist a positive constant* C_1 *depending only on* α *and* β *such that*

$$\mathbf{V}_n^{(\bar\theta)} h(\sigma) \leq C_1 D_n^2. \tag{1.7}$$

As it has been shown in Lemma 1 of [32], under conditions of Theorem 1.3, we have

$$Q(n) \asymp \exp\left\{ \sum_{i \leq n} \frac{\theta_i - 1}{i} \right\}, \tag{1.8}$$

where the constants in $\asymp$ depending on α and β. This allows to change the ratio $Q(n-jk)/Q(n)$ in D_n^2 by other quantities.

A proof of Theorem 1.1 is presented in the next section. The similar argument, refined by some ideas going back to a number-theoretical paper by A.Biró and T. Szamuely [7], is exploited in the proof of Theorem 1.3 which is exposed in the last section of the chapter.

1.3 Proof of Theorem 1.1

We will use the following particular cases of Watterson's formula [55]:

$$\mathbb{E}_n^{(\theta)} k_j(\sigma) = \frac{\theta}{j} \frac{\Theta(n-j)}{\Theta(n)}, \quad j \le n;$$

$$\mathbb{E}_n^{(\theta)} k_j(\sigma)\big(k_j(\sigma)-1\big) = \frac{\theta^2}{j^2} \frac{\Theta(n-2j)}{\Theta(n)}, \quad j \le n/2;$$

and

$$\mathbb{E}_n^{(\theta)} k_i(\sigma) k_j(\sigma) = \mathbf{1}\{i+j \le n\} \frac{\theta^2}{ij} \frac{\Theta(n-i-j)}{\Theta(n)}, \quad i \ne j,\ i,j \le n.$$

Now, to verify the already mentioned relation (1.3), we have

$$R_n = \theta^2 \sum_{j \le n/2} \frac{a_j^2}{j^2} \left[\frac{\Theta(n-2j)}{\Theta(n)} - \frac{\Theta(n-j)^2}{\Theta(n)^2} \right] - \theta^2 \sum_{n/2 < j \le n} \frac{a_j^2}{j^2} \frac{\Theta(n-j)^2}{\Theta(n)^2}.$$

Applying a rough form of (1.5), we can evaluate the last sum by

$$\frac{1}{n} \sum_{n/2 < j \le n} \frac{a_j^2}{j} \frac{\Theta(n-j)}{\Theta(n)} \Big(1 - \frac{j}{n+1}\Big)^{\theta-1} = O\big(n^{-\min\{1,\theta\}} B_n^2\big).$$

The same estimate holds for the partial sum in R_n over $n/4 < j \le n/2$. Finally, combining $(1-x)^u = 1 - ux + O(x^2)$ where $0 \le x \le 1/2$ and $u = \theta - 1$, with the asymptotical formula (1.5), we obtain

$$\sum_{j \le n/4} \frac{a_j^2}{j^2} \left[\frac{\Theta(n-2j)}{\Theta(n)} - \frac{\Theta(n-j)^2}{\Theta(n)^2} \right] = O\big(n^{-1} B_n^2\big).$$

Collecting the above estimates we obtain (1.3) as well as

$$\mathbb{E}_n^{(\theta)} h(\sigma) = \theta \sum_{j \le n} \frac{a_j}{j} \frac{\Theta(n-j)}{\Theta(n)},$$

and

$$\mathbb{E}_n^{(\theta)} h(\sigma)^2 = B_n^2 + \theta^2 \sum_{i+j \le n} \frac{a_i a_j}{ij} \frac{\Theta(n-i-j)}{\Theta(n)}.$$

Hence

$$
\begin{aligned}
\mathbf{V}_n^{(\theta)} h(\sigma) &= \mathbb{E}_n^{(\theta)} h(\sigma)^2 - \left(\mathbb{E}_n^{(\theta)} h(\sigma)\right)^2 \\
&= B_n^2 - \theta^2 \sum_{\substack{i,j \le n \\ i+j>n}} \frac{a_i a_j}{ij} \frac{\Theta(n-i)\Theta(n-j)}{\Theta(n)^2} \\
&\quad + \theta^2 \sum_{i+j \le n} \frac{a_i a_j}{ij} \left[\frac{\Theta(n-i-j)}{\Theta(n)} - \frac{\Theta(n-i)\Theta(n-j)}{\Theta(n)^2} \right]. \qquad (1.9)
\end{aligned}
$$

It is worth to point out that an analysis of the maximal eigenvalues of matrices of the last two quadratic forms with respect to a_j, $j \le n$, as $n \to \infty$ yielded the above mentioned asymptotical formulas for $\tau_n(\theta)$ if $\theta = 1$ or 2.

Proving upper estimates we firstly observe that it suffices to deal with $a_j \ge 0$, $j \le n$, only and later apply the result for positive and negative parts of $h(\sigma)$ separately. Secondly, we may omit the non-positive terms on the right-hand side of (1.9). Such a property has the last sum if $\theta \ge 1$. This yields the desired inequality in this case as has been also observed in [27].

Let us examine the more delicate case $\theta < 1$. We now have

$$
\begin{aligned}
\mathbf{V}_n^{(\theta)} h(\sigma) &\le B_n^2 + \theta^2 \sum_{i+j \le n} \frac{a_i a_j}{ij} \left[\frac{\Theta(n-i-j)}{\Theta(n)} - \frac{\Theta(n-i)\Theta(n-j)}{\Theta(n)^2} \right]^+ \\
&\le B_n^2 + \theta^2 \sum_{j<n} \frac{a_j^2}{j} \sum_{i \le n-j} \frac{1}{i} \left[\frac{\Theta(n-i-j)}{\Theta(n)} - \frac{\Theta(n-i)\Theta(n-j)}{\Theta(n)^2} \right]^+
\end{aligned}
$$

by the inequality $xy \le (x^2+y^2)/2$ for $x, y \in \mathbb{R}$. Here the positive part x^+ of $x \in \mathbb{R}$ is defined by

$$
x^+ := \begin{cases} x & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}
$$

We now see that an inequality

$$
\begin{aligned}
\Delta(m) &:= \left(\sum_{i \le m/2} + \sum_{m/2 < i \le m} \right) \frac{1}{i} \Big[\Theta(n)\Theta(m-i) - \Theta(n-i)\Theta(m) \Big]^+ \\
&=: \Delta_1(m) + \Delta_2(m) \le (C_2/\theta)\Theta(n)\Theta(m), \qquad (1.10)
\end{aligned}
$$

where $m := n - j \ge 1$ and $C_2 > 0$ is absolute constant, suffices to complete the proof.

We have

$$\theta\Delta_2(m) \le \frac{2\theta\Theta(n)}{m}\sum_{k=0}^{m}\Theta(k) = \frac{2\theta\Theta(n)}{m}\binom{\theta+m}{m} = \frac{2(\theta+m)}{m}\Theta(n)\Theta(m) \le 4\Theta(n)\Theta(m).$$

The sum $\Delta_1(m)$ over $i \le m/2$ can be estimated by the use of asymptotic formula (1.5) which is valid with an absolute constant in the symbol $O(\cdot)$ if $\theta \le 1$. Indeed, applying it twice, we have

$$\begin{aligned}
\Delta_1(m) &= \sum_{i\le m/2}\frac{1}{i}\Big[\Theta(n)\Theta(m-i)-\Theta(n-i)\Theta(m)\Big]^{+}\\
&\le \Theta(n)\Theta(m)\sum_{i\le m/2}\frac{1}{i}\bigg[\Big(1-\frac{i}{m}\Big)^{\theta-1}\Big(1+O\Big(\frac{1}{m}\Big)\Big)-\Big(1-\frac{i}{n}\Big)^{\theta-1}\Big(1+O\Big(\frac{1}{n}\Big)\Big)\bigg]^{+}\\
&= \Theta(n)\Theta(m)\sum_{i\le m/2}\frac{1}{i}O\Big(\frac{i}{m}\Big) \le C_3\Theta(n)\Theta(m),
\end{aligned}$$

where $C_3 > 0$ is an absolute constant. In the step we have applied the inequality $(1-x)^{-\alpha}-1 \le 2x$ if $0<\alpha<1$ and $0 \le x \le 1/2$. Adding the estimates of $\Delta_1(m)$ and $\Delta_2(m)$ we obtain (1.10) with $C_2 = 4 + C_3$.

Theorem 1.1 is proved.

□

1.4 Proof of Theorem 1.3

For an idea of the proof, we owe much to A. Biró and T. Szamuely [7] who established an inequality for the weighted variance of an additive number-theoretic function.

Let

$$Q^{\{j\}}(n) := \sum_{\substack{\ell(\bar{s})=n \\ s_j=0}} \prod_{i\le n} \left(\frac{\theta_i}{i}\right)^{s_i} \frac{1}{s_i!}, \qquad Q^{\{i,j\}}(n) := \sum_{\substack{\ell(\bar{s})=n \\ s_i=s_j=0}} \prod_{r\le n} \left(\frac{\theta_r}{r}\right)^{s_r} \frac{1}{s_r!},$$

where $i \neq j$.

We begin with the weighted expectation

$$\mathbb{E}_n^{(\bar{\theta})} h(\sigma) = \frac{1}{Q(n)n!} \sum_{\sigma\in \mathbf{S}_n} h(\sigma) \prod_{r\le n} \theta_r^{k_r(\sigma)}.$$

There are $n!\prod_{r\le n}\left(r^{s_r}s_r!\right)^{-1}$ permutations in a class corresponding to the vector $\bar{s}\in\Omega_n$. Therefore grouping over the classes, we obtain

$$\begin{aligned}
\mathbb{E}_n^{(\bar{\theta})} h(\sigma) &= \frac{1}{Q(n)} \sum_{\ell(\bar{s})=n} \sum_{j\le n} h_j(s_j) \prod_{r\le n} \left(\frac{\theta_r}{r}\right)^{s_r} \frac{1}{s_r!} \\
&= \frac{1}{Q(n)} \sum_{jk\le n} \frac{\theta_j^k h_j(k)}{j^k k!} \sum_{\substack{\ell(\bar{s})=n-jk \\ s_j=0}} \prod_{r\le n-jk} \left(\frac{\theta_r}{r}\right)^{s_r} \frac{1}{s_r!} \\
&=: \sum_{jk\le n} \frac{\theta_j^k h_j(k)}{j^k k!} \frac{Q^{\{j\}}(n-jk)}{Q(n)}.
\end{aligned} \tag{1.11}$$

Here we have changed the order of summation taking sums firstly over natural numbers j and $s_j =: k$ and used the property $js_j = jk \le n$.

Similarly,

$$
\begin{aligned}
Q(n)\mathbb{E}_n^{(\bar{\theta})} h^2(\sigma) &= \sum_{\ell(\bar{s})=n} \sum_{j\le n} h_j(s_j) \sum_{i\le n} h_i(s_i) \prod_{r\le n} \left(\frac{\theta_r}{r}\right)^{s_r} \frac{1}{s_r!} \\
= \sum_{jk\le n} \frac{\theta_j^k h_j^2(k)}{j^k k!} & \sum_{\substack{\ell(\bar{s})=n-jk \\ s_j=0}} \prod_{r\le n-jk} \left(\frac{\theta_r}{r}\right)^{s_r} \frac{1}{s_r!} \\
&+ \sum_{\substack{jk+il\le n \\ i\ne j}} \frac{\theta_j^k \theta_i^l h_j(k) h_i(l)}{j^k k! i^l l!} \sum_{\substack{\ell(\bar{s})=n-il-jk \\ s_i=s_j=0}} \prod_{r\le n-il-jk} \left(\frac{\theta_r}{r}\right)^{s_r} \frac{1}{s_r!} \\
= \sum_{jk\le n} \frac{\theta_j^k h_j^2(k)}{j^k k!} & Q^{\{j\}}(n-jk) + \sum_{\substack{jk+il\le n \\ i\ne j}} \frac{\theta_j^k \theta_i^l h_j(k) h_i(l)}{j^k k! i^l l!} Q^{\{i,j\}}(n-jk-il).
\end{aligned}
$$

As in the proof of Theorem 1.1, it suffices to deal with the nonnegative $h(\sigma)$ only. Omitting a part of summands we have

$$
\left(\mathbb{E}_n^{(\bar{\theta})} h(\sigma)\right)^2 \ge \sum_{jk+il\le n} \frac{\theta_j^k h_j(k)}{j^k k!} \frac{Q^{\{j\}}(n-jk)}{Q(n)} \frac{\theta_i^l h_i(l)}{i^l l!} \frac{Q^{\{i\}}(n-il)}{Q(n)}.
$$

Hence

$$
\begin{aligned}
\mathbf{V}_n^{\bar{\theta}} h(\sigma) \quad \le \quad & D_n^2 + \sum_{jk+il\le n} \frac{\theta_j^k h_j(k) \theta_i^l h_i(l)}{j^k k! i^l l!} \\
& \times \left(\frac{Q^{\{i,j\}}(n-il-jk)}{Q(n)} - \frac{Q^{\{j\}}(n-jk) Q^{\{i\}}(n-il)}{Q(n)^2} \right)^+ .
\end{aligned}
$$

By virtue of $ab \le (1/2)(a^2+b^2)$, this implies

$$
\begin{aligned}
& \mathbf{V}_n^{\bar{\theta}} h(\sigma) \le D_n^2 \\
& + \sum_{jk<n} \frac{\theta_j^k h_j^2(k)}{j^k k!} \sum_{il\le n-jk} \frac{\theta_i^l}{i^l l!} \left(\frac{Q^{\{i,j\}}(n-il-jk)}{Q(n)} - \frac{Q^{\{j\}}(n-jk) Q^{\{i\}}(n-il)}{Q(n)^2} \right)^+ .
\end{aligned}
$$

It remains to estimate the inner sum, namely, we have to prove that

$$
\sum_{il\le m} \frac{\theta_i^l}{i^l l!} \left(Q(n) Q^{\{i,j\}}(m-il) - Q^{\{j\}}(m) Q^{\{i\}}(n-il) \right)^+ \le C_4 Q(n) Q(m), \tag{1.12}
$$

where $1 \le m := n-jk < n$ and $C_4 = C_4(\alpha,\beta) > 0$ is a constant.

It is easy to get rid of the sum over $m/2 < il \le m$ on the left-hand side. Indeed,

$$\begin{aligned}
&\sum_{m/2<il\le m} \frac{\theta_i^l}{i^l l!}\Big(Q(n)Q^{\{i,j\}}(m-il) - Q^{\{j\}}(m)Q^{\{i\}}(n-il)\Big)^+ \\
\le\ & Q(n)\sum_{m/2<il\le m} \frac{\theta_i^l}{i^l l!} Q^{\{i\}}(m-il) = Q(n)\sum_{m/2<il\le m} \frac{\theta_i^l}{i^l l!} \sum_{\substack{\ell(\bar t)=m-il \\ t_i=0}} \prod_{r\le m-il} \Big(\frac{\theta_r}{r}\Big)^{t_r}\frac{1}{t_r!} \\
\le\ & Q(n)Q(m).
\end{aligned}$$

In the last step, we observed that the double summation is over the vectors $\bar s$ satisfying $\ell(\bar s) = m$ and having a unique decomposition $\bar s = \bar t + l\bar e_i \in \mathbb{N}_0^m$ with $\bar t \perp \bar e_i$, where $\bar e_i := (0,\ldots,1,\ldots,0) \in \mathbb{N}_0^m$ with the only 1 at the ith place and $m/2 < il \le m$, while $Q(m)$ sums up the summands over all $\bar s \in \mathbb{N}_0^m$ satisfying the condition $\ell(\bar s) = m$.

Observe that $Q^{\{i,j\}}(m-il) \le Q(m-il) \asymp Q(m)$ for $il \le m/2$ by estimate (1.8). Consequently, if $0 < \delta < 1/2$ be an arbitrary fixed number,

$$\begin{aligned}
&\sum_{\delta m<il\le m/2} \frac{\theta_i^l}{i^l l!}\Big(Q(n)Q^{\{i,j\}}(m-il) - Q^{\{j\}}(m)Q^{\{i\}}(n-il)\Big)^+ \\
\le\ & C_5 Q(n)Q(m)\sum_{\delta m<il\le m/2} \frac{\theta_i^l}{i^l l!} \le C_6(\delta)Q(n)Q(m),
\end{aligned}$$

where $C_5 = C_5(\alpha,\beta)$ and $C_6(\delta) = C_6(\delta,\alpha,\beta)$ are positive constants.

To estimate the remaining sum in (1.12) over $il \le \delta m$, it suffices to insert an appropriate asymptotic formula for the quantity

$$Q(n)Q^{\{i,j\}}(m-il) - Q^{\{j\}}(m)Q^{\{i\}}(n-il).$$

Theorem 1.1, equation (1.9) in [40] gives us the estimate

$$\frac{Q^{\{i\}}(m-il)}{Q^{\{i\}}(m)} = \frac{Q^{\{i,j\}}(m-il)}{Q^{\{i,j\}}(m)} = 1 + O\bigg(\Big(\frac{il}{m}\Big)^{\varepsilon}\bigg),$$

where $\varepsilon > 0$ is a constant depending at most on α and β. Let us note that equation (1.9) in [40] requires $il/m = o(1)$ in its statement, but by inspecting the proof we see that $il/m \le \delta$ is enough for sufficiently small δ, depending at most on α and β.

Further, Proposition 2.1 in [32] asserts that

$$\frac{Q^{\{i\}}(m)}{Q(m)} = \exp\Big\{-\frac{\theta_i}{i}\Big\}\bigg(1 + O\Big(\frac{1}{m^{\varepsilon}}\Big)\bigg)$$

and
$$\frac{Q^{\{i,j\}}(m)}{Q(m)} = \exp\left\{-\frac{\theta_i}{i} - \frac{\theta_j}{j}\right\}\left(1 + O\left(\frac{1}{m^{\varepsilon}}\right)\right),$$
where $\varepsilon > 0$ is a constant depending at most on α and β.

Collecting the last three equations, we have the following key relations:
$$Q^{\{j\}}(m) = \exp\left\{-\frac{\theta_j}{j}\right\}Q(m)\left(1 + O\left(\frac{1}{m^{\varepsilon}}\right)\right),$$
$$Q^{\{i\}}(n - il) = \exp\left\{-\frac{\theta_i}{i}\right\}Q(n)\left(1 + O\left(\left(\frac{il}{n}\right)^{\varepsilon}\right)\right),$$
and
$$Q^{\{i,j\}}(m - il) = \exp\left\{-\frac{\theta_i}{i} - \frac{\theta_j}{j}\right\}Q(m)\left(1 + O\left(\left(\frac{il}{m}\right)^{\varepsilon}\right)\right),$$
where $\varepsilon > 0$ is a constant depending at most on α and β, provided that $il \leq \delta m$ and δ is a sufficiently small constant depending at most on α and β. Fixing so δ, we obtain also the previous estimate with $C_6(\delta)$ depending only on α and β.

Theorem 1.3 is proved.

□

Chapter 2

Variance of additive functions defined on random assemblies

2.1 Basics and motivation

In this chapter, we deal with additive functions defined on decomposable combinatorial structures called *assemblies* (see the Meta-example 2.1 in [2]). If a structure is taken at random, the additive functions are sums of dependent random variables; sometimes, they are called separable statistics. Their value distribution is a complex problem in which estimates of the variance is a fairly useful tool.

Throughout this chapter, $i, j, k, l \in \mathbb{N}$ and $m, s, s_j \in \mathbb{N}_0$. Let us recall the definition of an assembly. Suppose an n set σ of labelled points is partitioned into subsets so that, amog them, there are k_j of size $j, 1 \leq j \leq n$, with $1k_1 + \cdots + nk_n = n$. In each such subset of size j, independent of the choice of elements, let a structure be defined. Let the number of different structures that can be defined on a subset of size j be g_j, where $1 \leq g_j < \infty$. A subset with a given structure is called a component of σ. Suppose the number g_j does not depend on the possibility of other subsets forming components. The set σ with a fixed component structure satisfying the aforementioned properties is called an assembly. The sequence g_j, $j \geq 1$, characterizes the class of assemblies which we will denote by $\mathcal{G}$. Let $\mathcal{G}_n \subset \mathcal{G}$ be the set of assemblies spanned over an n set (assemblies of the order n). Now,

$$G(n) := \#\mathcal{G}_n = n! \sum_{\ell(\bar{s})=n} \prod_{j=1}^{n} \left(\frac{g_j}{j!}\right)^{s_j} \frac{1}{s_j!} =: n! Q(n).$$

Setting also $G(0) = Q(0) := 1$, we have the following formal relation of the corresponding exponential generating series

$$Z_{\mathcal{G}}(z) := \sum_{n=0}^{\infty} Q(n) z^n = \exp\left\{ \sum_{j=1}^{\infty} \frac{g_j}{j!} z^j \right\}. \tag{2.1}$$

All quantitative information about the class is encoded in (2.1). We assume that $G(n) \geq 1$ for all $n \in \mathbb{N}_0$. The latter can fail as the example with $g_j = 0$ for all odd j shows.

Examples of assemblies and their properties can be found in books [2] and [17]. Let us name some of them:

Example 2.1. *Permutations whose components are cycles. Then*

$$G(n) = n!, \qquad g_j = (j-1)!.$$

Example 2.2. *Labelled graphs having connected graphs as components. For them*

$$G(n) = 2^{\binom{n}{2}}, \qquad g_j \sim G(j),$$

where the latter expression is true because random graphs are connected with high probability.

Example 2.3. *Labeled 2-regular graphs comprised from cycles of length $j \geq 3$. Now,*

$$G(n) \sim \frac{\sqrt{2}}{\mathrm{e}^{3/4}} \left(\frac{n}{\mathrm{e}} \right)^n, \qquad g_j = \frac{(j-1)!}{2}.$$

Example 2.4. *Mappings of a finite set into itself, interpreted as functional digraphs, where the components are the connected components of the underlying undirected graph. For them,*

$$G(n) = n^n, \qquad g_j = (j-1)! \sum_{k=0}^{j-1} \frac{j^k}{k!} \sim \frac{1}{2} \mathrm{e}^j (j-1)!.$$

Example 2.5. *Set partitions. Here,*

$$G(n) \sim \frac{\mathrm{e}^{n(r-1-1/r)-1}}{\sqrt{\log n}}, \qquad g_j = 1,$$

where $r\mathrm{e}^r = n$.

Example 2.6. *Forests of labelled unrooted trees. For them,*

$$G(n) \sim \sqrt{e} n^{n-2}, \qquad g_j = j^{j-2}.$$

Example 2.7. *Forests of labelled rooted trees. For them,*

$$G(n) = (n+1)^{n-1}, \qquad g_j = j^{j-1}.$$

Example 2.8. *Cyclations. For the definition of cyclations, we follow [49]. Consider n unit intervals, say* $[1,2], [3,4], ..., [2n-1, 2n]$. *Identify their endpoints in pairs at random, with all* $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$ *pairings being equally likely. The result, which we call a random n-cyclation, is a collection of cycles, which may be looked upon as components of an assembly, of various lengths. For example, if* $n = 3$, *so that we start with the three intervals* $[1,2]$, $[3,4]$ *and* $[5,6]$, *and if we identify the pairs* $1,5$, $2,6$ *and* $3,4$, *then we end with two cycles: a cycle of length one formed from* $[3,4]$, *and a cycle of length two formed from* $[1,2]$ *and* $[5,6]$. *One may easily check, that in this case*

$$G(n) = (2n-1)!!, \qquad g_j = 2^{j-1}(j-1)!.$$

Example 2.9. *Let us discuss permutations with restricted cycle lengths. For the definition, we follow [60]. We fix $\mathcal{A} \subset \mathbb{N}$ and take only such permutations, again denoted by σ, whose cycle lengths $k_j(\sigma) \in \mathcal{A}$ for all $j \leq n$. We call them $\mathcal{A}$-permutations. Then*

$$G(n) = n! \sum_{\substack{\ell(\bar{s})=n \\ s_j=0 \text{ if } j \notin \mathcal{A}}} \prod_{j=1}^{n} \frac{1}{j^{s_j} s_j!}, \qquad g_j = \begin{cases} (j-1)! & \text{if } j \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $G(n)$ can be equal to 0 for some $\mathcal{A}$, but we can avoid that if, for example, $1 \in \mathcal{A}$.

Example 2.10. *It is worth to add that permutations, taken from the symmetric group $\mathbf{S}_n$ according to the generalized Ewens probabilities*

$$\nu_n^{(\bar{\theta})}(\{\sigma\}) := G(n)^{-1} \prod_{j \leq n} \theta_j^{k_j(\sigma)}, \quad G(n) > 0,$$

where the nonnegative numbers θ_j, $j \leq n$, are arbitrary also follow the described scheme. Now, $\theta_j (j-1)!$ substitute for g_j but are not necessarily integers. It is natural to consider such permutations as a particular class of assemblies calling them weighted permutations. They have been introduced in the paper by E. Manstavičius [32]. Later they started to play an important role related to phenomena of statistical physics (see, for example, [5]). So in this case, if we ignored the requirements that g_j and $G(n)$ be integers, we may write

$$G(n) = n! \sum_{\ell(\bar{s})=n} \prod_{j=1}^{n} \Big(\frac{\theta_j}{j}\Big)^{s_j} \frac{1}{s_j!}, \qquad g_j = \theta_j (j-1)!.$$

Example 2.11. *The weighted permutations in the case $\theta_j \equiv \theta > 0$ are well known since the seminal paper by J. Ewens [15]. In this case, recalling (1.1), we have*

$$G(n) = \prod_{j=1}^{n} \big(j + \theta - 1\big), \qquad g_j = \theta (j-1)!.$$

If ν_n denotes the uniform probability measure on the subsets of $\mathcal{G}_n$, then the distribution of the component vector is

$$\nu_n\big(\bar{k}(\sigma) = \bar{s}\big) = \frac{n!}{G(n)} \prod_{j=1}^{n} \Big(\frac{g_j}{j!}\Big)^{s_j} \frac{1}{s_j!}, \tag{2.2}$$

where $\bar{s}$ runs through the set of vectors such that $\ell(\bar{s}) = n$. Observe that, if ξ_j, $j \leq n$, is a family of independent Poisson random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ with the parameters $\lambda_j(x) := \mathbb{E}\xi_j = x^j g_j/j!$, where $x > 0$ is arbitrary, and $\bar{\xi} := (\xi_1, \dots, \xi_n)$, then

$$\nu_n\big(\bar{k}(\sigma) = \bar{s}\big) = P\big(\bar{\xi} = \bar{s} \big| \, \ell(\bar{\xi}) = n\big). \tag{2.3}$$

Let us discuss the working conditions assumed for a class of assemblies under which our probabilistic problem will be explored. For brevity, introduce the notation $\lambda_j := \lambda_j(1) := g_j/j!$, $1 \leq j \leq n$, and

$$\alpha(z, n) := \exp\Big\{ \sum_{j \leq n} \lambda_j z^j \Big\}.$$

Check that $[z^m]\alpha(z; n) = [z^m] Z_{\mathcal{G}}(z) = Q(m)$ if $m \leq n$.

In the past decades much attention was paid to the *logarithmic* class defined by the asymptotic condition $\rho^j j \lambda_j \sim \theta$ for some fixed constants $\theta > 0$ and $\rho > 0$ as $j \to \infty$ (see [2]). Logarithmic class includes the Examples 2.1, 2.3, 2.4, 2.8, 2.10 and 2.11. Extensions were initiated by E. Manstavičius in the paper [32], where a condition

$$0 < \theta \leq \rho^j j \lambda_j \leq \Theta, \quad j \geq 1, \tag{2.4}$$

was used. The lower bound excluded, for example, the class of 2-regular graphs, however. Further generalizations were proposed in papers [36] and [38], where the total variation approximation of the distribution of a vector $\big(k_1(\sigma), \dots, k_r(\sigma)\big)$ by that of $(\xi_1, \dots, \xi_r)$ if $r = r(n) = o(n)$ as $n \to \infty$ was examined.

General classes of assemblies appear in papers by K.-H. Indlekofer [21], [22] and [23] where some specialized Tauberian theorems are proved.

We confine ourselves to a class of assemblies introduced in [38]. It is characterized by a few positive constants ρ, Θ, θ, θ', and $n_0 \geq 1$.

Definition 2.1. *We say that a class of assemblies is weakly logarithmic if the following conditions are satisfied:*

$$\rho^j j\lambda_j \leq \Theta, \quad j \geq 1; \tag{2.5}$$

$$\sum_{j\leq n} \rho^j j\lambda_j \geq \theta n, \quad n \geq n_0; \tag{2.6}$$

$$nQ(n)\rho^n \geq \theta' \alpha(\rho; n), \quad n \geq 1. \tag{2.7}$$

It is worth to stress that the listed conditions assure a lower bound of the probability of the condition present in (2.3). Indeed, in our notation, taking $x = \rho$, we have

$$P\big(\ell(\bar{\xi}) = n\big) = \alpha(\rho; n)^{-1}[z^n]\alpha(\rho z; n) = \alpha(\rho; n)^{-1}\rho^n Q(n) \geq \theta'/n.$$

Moreover, Lemma 2.3 below shows that $n^{\theta-1} \ll \rho^n Q(n) \ll n^{\Theta-1}$ for $n \geq 1$. No approximation, like $\rho^n Q(n) \sim n^{\theta-1}$ holding for logarithmic assemblies, may be expected when only conditions $(2.5-2.7)$ are assumed. To give an example, we follow [60]. Let us fix $\mathcal{A} \subset \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{\#\{k : k \leq n, k \in \mathcal{A}\}}{n} = \kappa > 0 \tag{2.8}$$

and

$$\lim_{n\to\infty} \frac{\#\{k : k \leq n, k \in \mathcal{A}, m-k \in \mathcal{A}\}}{n} = \kappa^2 \tag{2.9}$$

holds uniformly in $m \in [n, O(n)]$. Now, weakly logarithmic class includes random $\mathcal{A}$-*permutations*, satisfying 2.8-2.9 and the condition $Q(n) > 0$ for all $n \in \mathbb{N}$, and many other of such type of structures, while logarithmic class does not. Indeed, our condition (2.6) requires lower bound $\#\{k : k \leq n, k \in \mathcal{A}\}/n \geq \theta > 0$ rather than limit (2.8) and are in this sense weaker. Futhermore, Theorem 3.3.1 in [60], page 126, yields

$$Q(n) \sim Cn^{-1} \exp\Bigg\{ \sum_{j\leq n, j\in\mathcal{A}} \frac{1}{j} \Bigg\}, \quad C > 0,$$

under the conditions (2.8) and (2.9). Our condition (2.7) requires only an inequality for $n \geq 1$. Random $\mathcal{A}$-*permutations* have been studied by a number of authors in recent decades, including V. N. Sachkov (see, for example, monographs [52], [53]) and A. L. Yakymiv (see, for example, [60], [59], [61], [62]).

Now, let us turn to an *additive function* $h : \mathcal{G}_n \to \mathbb{R}$. Similarly to (2), it is defined by a real two-dimensional array $\{h_j(k)\}$, where $j, k \in \mathbb{N}$, $jk \leq n$, and $h_j(0) := 0$ for all $j \leq n$, by setting

$$h(\sigma) := \sum_{j=1}^{n} h_j\big(k_j(\sigma)\big). \tag{2.10}$$

Let $\mathbb{E}_n h$ and $\mathbf{V}_n h$ denote the expectation and the variance of additive function $h := h(\sigma)$ with respect to the uniform measure ν_n. The problem is to estimate

$$\mathbf{V}_n h = \frac{1}{G(n)} \sum_{\sigma \in \mathcal{G}_n} \big(h(\sigma) - \mathbb{E}_n h\big)^2 = \mathbb{E}_n h^2 - (\mathbb{E}_n h)^2$$

in terms of the values $h_j(k)$ where $jk \le n$. By (2.3), the problem is equivalent to estimation of the conditional variance

$$\mathrm{Var}\left(\sum_{j=1}^{n} h_j(\xi_j) \,\middle|\, \ell(\bar{\xi}) = n \right) = \mathbf{V}_n h. \tag{2.11}$$

In the sequel, let $Q^J(m)$, $0 \le m \le n$, be defined by

$$\sum_{m=0}^{\infty} Q^J(m) z^m := \exp\left\{ \sum_{\substack{i \le n \\ i \notin J}} \lambda_i z^i \right\} =: \alpha^J(z, n), \tag{2.12}$$

where $J \subset \{1, 2, \ldots, n\}$. In particular, we have

$$Q^{\{j\}}(n) = \sum_{\substack{\ell(\bar{s}) = n \\ s_j = 0}} \prod_{i \le n} \frac{\lambda_i^{s_i}}{s_i!}, \quad j \le n.$$

In the estimates below, dependence on the parameters ρ, Θ, θ, θ' and $n_0 \ge 1$, indicated in the Definition 2.1, is allowed. However, we will add an extra index, say, ε if dependence on the latter will occur. We now list the results.

2.2 Results

Theorem 2.1. *Assume that $\mathcal{G}$ is weakly logarithmic and $h : \mathcal{G}_n \to \mathbb{R}$ is an arbitrary additive function. Then*

$$\begin{aligned}\mathbf{V}_n h &= \frac{1}{G(n)} \sum_{\sigma \in \mathcal{G}_n} \left[\sum_{jk \le n} h_j(k) \left(\mathbf{1}\{k_j(\sigma) = k\} - \frac{\lambda_j^k}{k!} \frac{Q^{\{j\}}(n-jk)}{Q(n)} \right) \right]^2 \\ &\ll \sum_{jk \le n} \frac{\lambda_j^k h_j(k)^2}{k!} \frac{Q^{\{j\}}(n-jk)}{Q(n)} =: B_n^2 \end{aligned} \tag{2.13}$$

for $n \ge 1$.

Inequality (2.13) sharpens a bit Theorem 1.3 proved for an arbitrary additive function defined on weighted permutations under condition (2.4).

A *completely additive function h* is defined by the array $h_j(k) = a_j k$, where $a_j \in \mathbb{R}$ and $jk \le n$. For such functions, inequality (2.13) takes a simpler form.

Theorem 2.2. *Assume that $\mathcal{G}$ is weakly logarithmic and $h : \mathcal{G}_n \to \mathbb{R}$ is a completely additive function. Then*

$$\begin{aligned}\mathbf{V}_n h &= \frac{1}{G(n)} \sum_{\sigma \in \mathcal{G}_n} \left[\sum_{j \le n} a_j \left(k_j(\sigma) - \lambda_j \frac{Q(n-j)}{Q(n)} \right) \right]^2 \\ &\ll \sum_{j \le n} \lambda_j a_j^2 \frac{Q(n-j)}{Q(n)}. \end{aligned} \tag{2.14}$$

Asymptotically optimal constants in $\ll$ have been found for permutations taken with respect to the Ewens probability if $\theta = 1$ or 2 (see [37] and the references therein). For all mappings of a finite set into itself, inequality (2.14) was established in the paper by E. Manstavičius [41].

As P.D.T.A. Elliott has convinced us by a book [13], both of the inequalities (2.13) and (2.14) has a useful dual form. To present it, we have firstly to exclude the summands with $\lambda_j = 0$ as a factor in all the sums occurring in inequalities (2.13) and (2.14). This makes no harm to their validity because of the relation

$$\nu_n\big(k_j(\sigma) = k\big) = (\lambda_j^k / k!)(n!/G(n)) = 0$$

following from (2.2) if $\lambda_j = 0$ and $k \geq 1$. Afterwards, we put an asterisk to denote that only j for which $\lambda_j \neq 0$ are taken into account.

Theorem 2.3. *We have*

$$\sum_{jk\leq n}^{*} \frac{k!}{\lambda_j^k} \frac{Q(n)}{Q^{\{j\}}(n-jk)} \left[\sum_{\sigma\in\mathcal{G}_n} y(\sigma) \left(\mathbf{1}\{k_j(\sigma)=k\} - \frac{\lambda_j^k}{k!} \frac{Q^{\{j\}}(n-jk)}{Q(n)} \right) \right]^2 \ll G(n) \sum_{\sigma\in\mathcal{G}_n} y(\sigma)^2 \tag{2.15}$$

and

$$\sum_{j\leq n}^{*} \frac{1}{\lambda_j} \frac{Q(n)}{Q(n-j)} \left[\sum_{\sigma\in\mathcal{G}_n} y(\sigma) \left(k_j(\sigma) - \lambda_j \frac{Q(n-j)}{Q(n)} \right) \right]^2 \ll G(n) \sum_{\sigma\in\mathcal{G}_n} y(\sigma)^2 \tag{2.16}$$

for all $y(\sigma) \in \mathbb{R}$ *where* $\sigma \in \mathcal{G}_n$.

It takes just one step to derive a weak law of large numbers for a sequence of real-valued additive functions $h_n(\sigma)$ defined via $h_{nj}(k)$, where $k \geq 0$ and $j \leq n$, using Chebyshev's inequality and (2.13). Combined with the earlier mentioned total variation approximation result from [38], inequality (2.13), assures a short path in proving general limit theorems for the distribution functions $\nu_n\big(h_n(\sigma) - \alpha(n) < x\big)$ as $n \to \infty$. Here $\alpha(n)$ is a centralizing sequence. The idea originated in [30], and already exploited in Section 8.5 of [2], lays in an appropriate splitting $\alpha(n) = \alpha'(n) + \alpha''(n)$ and

$$h_n(\sigma) = \sum_{j\leq r} h_{nj}(k_j(\sigma)) + \sum_{r<j\leq n} h_{nj}(k_j(\sigma)) =: h_n'(\sigma) + h_n''(\sigma).$$

Now firstly, the total variation approximation reduces the problem concerning $\nu_n\big(h_n'(\sigma) - \alpha'(n) < x\big)$ to a problem for sums of independent random variables $h_{nj}(\xi_j)$, $j \leq r = o(n)$. Secondly, under general conditions one can assure the weak law of large numbers for $h''(\sigma) - \alpha''(n)$ and so make the contribution of this part negligible. In this way, we succeed in generalizing many results obtained so far.

Not so straightforward applications of our results include investigations of the asymptotic expectations as $n \to \infty$ of multiplicative functions $f : \mathcal{G}_n \to \mathbb{C}$ defined by

$$f(\sigma) := \prod_{j=1}^{n} f_j\big(k_j(\sigma)\big),$$

where $f_j(0) := 1$ for every $j \geq 1$. Number-theoretical ideas proposed by A. Rényi [50], P.D.T.A. Elliott and others (see [11]) can be adopted to prove an analog of the Delange theorem. The inequalities (2.13) and (2.15) are indispensable in proving sufficiency and necessity of the conditions.

2.3 Expressions for the Moments

We will use the sums $Q^{\{j\}}(n)$ and $Q^{\{i,j\}}(n)$ defined above in (2.12). Let us begin with the expectations.

Lemma 2.1. *For an arbitrary additive function defined on $\mathcal{G}_n$ we have*

$$\mathbb{E}_n h = \sum_{jk \le n} \frac{\lambda_j^k h_j(k)}{k!} \frac{Q^{\{j\}}(n-jk)}{Q(n)}. \tag{2.17}$$

Moreover, if $h_j(s) = a_j s$ for $js \le n$, then

$$\mathbb{E}_n h = \sum_{j \le n} \lambda_j a_j \frac{Q(n-j)}{Q(n)}. \tag{2.18}$$

Proof. By the definition and (2.2), we have

$$\begin{aligned}\mathbb{E}_n h &= \frac{1}{G(n)} \sum_{\sigma \in \mathcal{G}_n} h(\sigma) = \sum_{\ell(\bar{s})=n} \sum_{j \le n} h_j(s_j) \nu_n(\bar{k}(\sigma) = \bar{s}) \\ &= \frac{1}{Q(n)} \sum_{\ell(\bar{s})=n} \sum_{j \le n} h_j(s_j) \prod_{j \le n} \frac{\lambda_j^{s_j}}{s_j!}.\end{aligned}$$

Let us change the order of summation by taking at first the sums over natural numbers j and $s_j =: k$ and using the property $js_j = jk \le n$. So we obtain

$$\mathbb{E}_n h = \frac{1}{Q(n)} \sum_{jk \le n} \frac{\lambda_j^k h_j(k)}{k!} \sum_{\substack{\ell(\bar{s})=n-jk \\ s_j=0}} \prod_{i \le n-jk} \frac{\lambda_i^{s_i}}{s_i!}.$$

This is the desired formula (2.17).

If $h_j(k) = a_j k$ for $jk \le n$, then

$$\mathbb{E}_n h = Q(n)^{-1} \sum_{j \le n} \lambda_j a_j \sum_{1 \le k \le n/j} \frac{\lambda_j^{k-1}}{(k-1)!} Q^{\{j\}}\big(n-jk\big).$$

Now, to establish (2.18), the following identity

$$\sum_{1\le k\le n/j} \frac{\lambda_j^{k-1}}{(k-1)!} Q^{\{j\}}(n-jk) = [z^{n-j}] \exp\left\{\lambda_j z^j\right\} \exp\left\{\sum_{i\ge 1, i\ne j} \lambda_i z^j\right\}$$
$$= Q(n-j) \tag{2.19}$$

suffices.

The lemma is proved.

□

Similarly, we obtain the formulas for the second moment.

Lemma 2.2. *For an arbitrary additive function defined on $\mathcal{G}_n$ we have*

$$\mathbb{E}_n h^2 = B_n^2 + \sum_{\substack{jk+il\le n\\ i\ne j}} \frac{\lambda_j^k \lambda_i^l h_j(k) h_i(l)}{k!l!} \frac{Q^{\{i,j\}}(n-jk-il)}{Q(n)}. \tag{2.20}$$

Moreover, if $h_j(k) = a_j k$ for $jk \le n$, then

$$\mathbb{E}_n h^2 = \sum_{j\le n} \lambda_j a_j^2 \frac{Q(n-j)}{Q(n)} + \sum_{i+j\le n} \lambda_i \lambda_j a_i a_j \frac{Q(n-i-j)}{Q(n)}. \tag{2.21}$$

Proof. Interchanging the summation, we obtain

$$\begin{aligned}
Q(n)\mathbb{E}_n h^2 &= \sum_{\ell(\bar{s})=n} \sum_{j\le n} h_j(s_j) \sum_{i\le n} h_i(s_i) \prod_{j\le n} \frac{\lambda_j^{s_j}}{s_j!} \\
&= \sum_{jk\le n} \frac{\lambda_j^k h_j^2(k)}{k!} \sum_{\substack{\ell(\bar{s})=n-jk\\ s_j=0}} \prod_{i\le n-jk} \frac{\lambda_i^{s_i}}{s_i!} \\
&\quad + \sum_{\substack{jk+il\le n\\ i\ne j}} \frac{\lambda_j^k \lambda_i^l h_j(k) h_i(l)}{k!l!} \sum_{\substack{\ell(\bar{s})=n-il-jk\\ s_i=s_j=0}} \prod_{r\le n-il-jk} \frac{\lambda_r^{s_r}}{s_r!}
\end{aligned}$$

which coincides with (2.20).

Afterwards, let h be completely additive. To simplify the first sum on the right-hand side of (2.20), we use the identities

$$\begin{aligned}
&\sum_{1\le k\le n/j} \frac{\lambda_j^k k}{(k-1)!} Q^{\{j\}}(n-jk) \\
&= \sum_{1\le k\le n/j} \frac{\lambda_j^k}{(k-1)!} Q^{\{j\}}(n-jk) + \sum_{2\le k\le n/j} \frac{\lambda_j^k}{(k-2)!} Q^{\{j\}}(n-jk) \\
&= \lambda_j Q(n-j) + \lambda_j^2 Q(n-2j)
\end{aligned}$$

with an agreement that $Q(-m) = 0$ if $m \in \mathbb{N}$. In the last step, we also applied the argument used in deriving (2.19). If $h_j(k) = a_j k$ for $jk \le n$, this gives

$$B_n^2 = \sum_{j\le n} \lambda_j a_j^2 \frac{Q(n-j)}{Q(n)} + \sum_{j\le n} \lambda_j^2 a_j^2 \frac{Q(n-2j)}{Q(n)}. \tag{2.22}$$

Dealing with the second sum on the right-hand side of (2.20), we firstly observe that

$$\begin{aligned}
&\sum_{1\le k\le (n-i)/j} \frac{\lambda_j^{k-1}}{(k-1)!} \sum_{1\le l\le (n-jk)/i} \frac{\lambda_i^{l-1}}{(l-1)!} Q^{\{i,j\}}(n-jk-il) \\
&= \sum_{1\le k\le (n-i)/j} \frac{\lambda_j^{k-1}}{(k-1)!} Q^{\{j\}}(n-jk-i) \\
&= Q(n-i-j)
\end{aligned}$$

if $i + j \le n$ and $i \ne j$. Now, the examined sum in (2.20) equals

$$\sum_{\substack{i+j\le n\\ i\ne j}} \lambda_i \lambda_j a_i a_j \frac{Q(n-i-j)}{Q(n)}.$$

Adding the latter to (2.22), from (2.20), we obtain (2.21).

The lemma is proved.

□

The formulas of moments show that neither the quantity B_n^2 nor $\mathbf{V}_n h$ changes if we substitute $\lambda_j \rho^{-j}$ for λ_j where $j \le n$.

2.4 Comparative Analysis

By the last remark, without loss of generality, we may focus on a class of weakly logarithmic structures satisfying conditions $(2.5-2.7)$ with $\rho = 1$. Proof of Theorem 2.1 is based upon the following proposition.

Key Lemma. *For $j \le n$ and $1 \le m \le n-1$, we have*

$$\sum_{\substack{il\le m\\ i\neq j}} \frac{\lambda_i^l}{l!}\Big(Q(n)Q^{\{i,j\}}(m-il) - Q^{\{j\}}(m)Q^{\{i\}}(n-il)\Big)^+ \ll Q(n)Q^{\{j\}}(m). \tag{2.23}$$

In a few lemmas, we firstly examine the coefficients $Q^J(m)$ for various m and $J \subset \{1,2,\ldots,n\}$. Set, for brevity, $\alpha(n) := \alpha(1,n)$ and $\alpha^J(n) := \alpha^J(1,n)$.

Lemma 2.3. *For $n \ge 1$, we have*

$$n^\theta \ll nQ(n) \asymp \alpha(n) \ll n^\Theta.$$

Proof. Actually, the estimates have been established in [40]. For reader's convenience we present the details. Differentiating (2.1) and comparing the coefficients, we obtain

$$nQ(n) = \sum_{j\le n} j\lambda_j Q(n-j) \le \Theta \sum_{0\le k\le n-1} Q(k) \le \Theta\alpha(n) \tag{2.24}$$

by condition (2.5). On the other hand, summation by parts yields

$$\begin{aligned}\Sigma(x,y) &:= \sum_{x<j\le y}\lambda_j\\ &= \int_x^y \Big(\sum_{j\le u} j\lambda_j\Big)\frac{du}{u^2} + \frac{1}{y}\sum_{j\le y} j\lambda_j - \frac{1}{x}\sum_{j\le x} j\lambda_j\\ &\ge \theta\log\frac{y}{x} - \Theta \end{aligned}\tag{2.25}$$

by virtue of conditions (2.5) and (2.6) if $n_0 \le x < y \le n$. Hence, by (2.7) and the definitions,

$$nQ(n) \gg \alpha(n) \gg \exp\{\Sigma(n_0,n)\} \gg n^\theta.$$

If $n \le n_0$, the estimates are trivial.

□

Paper [40] provides tools needed comparing $Q^J(m)$ with $Q(n)$. We slightly reformulate a result from it. Let d_j, $j \leq N$, be arbitrary nonnegative numbers, maybe, dependent on N or other parameters,

$$D^J(z,N) := \exp\Big\{ \sum_{\substack{j\leq N\\ j\notin J}} \frac{d_j}{j} z^j \Big\} =: \sum_{n=0}^{\infty} D_n^J z^n,$$

where $J \subset \{1,2,\ldots,N\}$; $r := \max\{j: \ j \in J\}$ and $r = 0$ if $J = \emptyset$; $D(z,N) := D^{\emptyset}(z,N)$, $D_n := D_n^{\emptyset}$, and $n \in \mathbb{N}$.

Lemma 2.4. *Assume that there exist positive constants C, c, c', and $n' \in \mathbb{N}$ such that, for $n' \leq n \leq N$,*

$$d_j \leq C, \quad 1 \leq j \leq N;$$

$$\sum_{j\leq n} d_j \geq cn;$$

$$nD_n \geq c' D(1,n).$$

There exist sufficiently small positive constants $\eta > 0$ and γ such that

$$\frac{D_n^J}{D_N} = \exp\Big\{ -\sum_{j\in J} \frac{d_j}{j} \Big\} \Big(1 + O\Big(\Big(\eta + \frac{r+1}{N}\Big)^{\gamma}\Big)\Big)$$

provided that $r \leq \eta n$ and $(1-\eta)n' \leq (1-\eta)N \leq n \leq N$.

Proof. See Theorem 1.1 in [40].

□

Applied to the weakly logarithmic class of assemblies the last lemma yields the desired asymptotical formulas.

Lemma 2.5. *There exist sufficiently small positive constants ε_1 and δ_1 such that*

$$Q(n-s) = Q(n)\Big(1 + O\Big(\Big(\frac{s}{n}\Big)^{\varepsilon_1}\Big)\Big) \tag{2.26}$$

and

$$Q^{\{j\}}(n-s) = \mathrm{e}^{-\lambda_j} Q(n)\Big(1 + O\Big(\Big(\frac{s+j}{n}\Big)^{\varepsilon_1}\Big)\Big) \tag{2.27}$$

for all $0 \leq s \leq \delta_1 n$ and $1 \leq j \leq \delta_1 n$.

Moreover, there exist further constants $n_1 \geq n_0$ *and* $\theta'' > 0$ *such that*

$$nQ^{\{j\}}(n) \geq \theta'' \alpha^{\{j\}}(n) \tag{2.28}$$

for all $j \leq n$ *if* $n \geq n_1$.

Proof. Relations (2.26) and (2.27) for $n \geq n'$ are just the corollaries of Lemma 2.4. If $1 \leq n \leq n'$, the estimates are trivial because of $Q(n) > 0$.

To prove (2.28), it suffices to show that $Q^{\{j\}}(n) \gg Q(n)$ for sufficiently large n and to use the fact that $\alpha(n) \asymp \alpha^{\{j\}}(n)$. Thus, if $j \leq \delta_1 n$, estimate (2.28) follows from (2.27) and condition (2.7).

If $\delta_1 n < j \leq n$, equality (2.12) and a convolution argument gives

$$\begin{aligned} Q^{\{j\}}(n) &= Q(n) + \sum_{1\leq k\leq n/j} \frac{(-\lambda_j)^k}{k!} Q(n-jk) \\ &\geq Q(n) - \sum_{1\leq k\leq n/j} \frac{\Theta^k}{j^k k!} Q(n-jk) \end{aligned} \tag{2.29}$$

by condition (2.5).

If $K > 1$ is arbitrary, inequality (2.24) allows us to estimate the part of sum in (2.29) over k such that $jk \leq n-K$. Indeed, it can be majorized

$$\begin{aligned} \Theta\alpha(n) \sum_{1\leq k\leq (n-K)/j} \frac{\Theta^k}{j^k k!(n-jk)} &\leq \Theta \frac{\alpha(n)}{K} \left(\mathrm{e}^{\Theta/j} - 1\right) \\ &\leq \Theta^2 \mathrm{e}^{\Theta} \frac{\alpha(n)}{K\delta_1 n} \leq \frac{CQ(n)}{K} \end{aligned}$$

by condition (2.5) again.

If

$$C_K := \max_{0\leq m\leq K-1} Q(m),$$

then (2.29) yields

$$Q^{\{j\}}(n) \geq Q(n) - \frac{CQ(n)}{K} - C_K \frac{\Theta \mathrm{e}^{\Theta}}{\delta_1 n}.$$

Fixing K sufficiently large and applying the lower bound $Q(n) \gg n^{\theta-1}$ from Lemma 2.3, we obtain the desired estimate $Q^{\{j\}}(n) \gg Q(n)$ provided that $n \geq n_1$, where n_1 is sufficiently large.

$\square$

Lemma 2.6. *Let $i \neq j$. There exist sufficiently small positive constants ε_2 and δ_2 such that*

$$Q^{\{j\}}(m-s) = Q^{\{j\}}(m)\left(1 + O\left(\left(\frac{s}{m}\right)^{\varepsilon_2}\right)\right) \tag{2.30}$$

and

$$Q^{\{i,j\}}(m-s) = \mathrm{e}^{-\lambda_i} Q^{\{j\}}(m)\left(1 + O\left(\left(\frac{s+i}{m}\right)^{\varepsilon_2}\right)\right) \tag{2.31}$$

for all $0 \leq s \leq \delta_2 m$, $1 \leq i \leq \delta_2 m$, and $m \geq \max\{n', n_1\}$.

Moreover,

$$Q^{\{i,j\}}(m-s) \asymp_\varepsilon Q^{\{j\}}(m) \asymp Q(m) \tag{2.32}$$

for $0 \leq s \leq (1-\varepsilon)m$, where $0 < \varepsilon < 1$ is arbitrary, $m \geq n_2$, and n_2 is sufficiently large.

Proof. By virtue of (2.27), (2.5), and (2.6), we see that $Q^{\{j\}}(n)$ also satisfies conditions of Lemma 2.4. The latter implies the presented asymptotical formulas.

The inequalities

$$Q^{\{i,j\}}(m) \leq Q^{\{j\}}(m) \leq Q(m)$$

are evident. Repeating the same argument as proving (2.28), we obtain

$$Q^{\{i,j\}}(m) \gg Q^{\{j\}}(m) \gg Q(m)$$

for sufficiently large m. The estimate in (2.32) with a shifted argument follows from the relation

$$\alpha^{\{i,j\}}(m-s) \asymp_\varepsilon \alpha^{\{j\}}(m) \asymp \alpha(m)$$

valid in the indicated range of s.

□

Proof of Key Lemma. We start with an observation that $Q(n) > 0$ does not assure that $Q^{\{j\}}(m) > 0$ for all $m \leq n_1$, where n_1 has been introduced in Lemma 2.5. Nevertheless, (2.23) holds even if $Q^{\{j\}}(m) = 0$. Indeed, in such a case, an identity

$$\begin{aligned}\sum_{0\leq l\leq m/i} \frac{\lambda_i^l}{l!} Q^{\{i,j\}}\big(m-il\big) &= [z^m] \exp\left\{\lambda_i z^i\right\} \exp\left\{\sum_{r\geq 1, r\neq i,j} \lambda_r z^r\right\} \\ &= Q^{\{j\}}(m),\end{aligned}$$

valid for each $i \leq m$, $i \neq j$, shows that also $Q^{\{i,j\}}\big(m-il\big) = 0$ for each $0 \leq l \leq m/i$.

If $Q^{\{j\}}(m) > 0$, then it suffices to prove (2.23) for $m \geq n_3$, where n_3 is an arbitrary large natural number. Firstly, we observe that

$$\begin{aligned}
&\sum_{\substack{m/2<il\leq m\\ i\neq j}} \frac{\lambda_i^l}{l!}\Big(Q(n)Q^{\{i,j\}}(m-il) - Q^{\{j\}}(m)Q^{\{i\}}(n-il)\Big)^+ \\
&\leq Q(n) \sum_{\substack{m/2<il\leq m\\ i\neq j}} \frac{\lambda_i^l}{l!} Q^{\{i,j\}}(m-il) \\
&= Q(n) \sum_{\substack{m/2<il\leq m\\ i\neq j}} \frac{\lambda_i^l}{l!} \sum_{\substack{\ell(\bar{s})=m-il\\ s_i=s_j=0}} \prod_{r=1}^{m-il} \frac{\lambda_r^{s_r}}{s_r!} \\
&\leq Q(n)Q^{\{j\}}(m). \qquad (2.33)
\end{aligned}$$

In the last step, we used the fact that the double summation is over the vectors $\bar{s}$ satisfying $\ell(\bar{s}) = m$ and having a unique decomposition $\bar{s} = \bar{t} + l\bar{e}_i$ with a vector $\bar{t} = (t_1, \ldots, t_n) \in \mathbb{N}_0^m$ such that $t_i = t_j = t_u = 0$ for all $m/2 < u \leq m$ and $\bar{e}_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}_0^m$ with the only 1 at the ith place. On the other hand, $Q^{\{j\}}(m)$ sums up the summands over all $\bar{s} \in \mathbb{N}_0^m$ satisfying $\ell(\bar{s}) = m$ and $s_j = 0$.

Secondly, applying (2.32) with $\varepsilon = 1/2$, for every $0 < \delta < 1$, we obtain

$$\begin{aligned}
&\sum_{\substack{\delta m<il\leq m/2\\ i\neq j}} \frac{\lambda_i^l}{l!}\Big(Q(n)Q^{\{i,j\}}(m-il) - Q^{\{j\}}(m)Q^{\{i\}}(n-il)\Big)^+ \\
&\leq Q(n)Q^{\{j\}}(m) \sum_{\delta m<il\leq m/2} \frac{\lambda_i^l}{l!} \ll_\delta Q(n)Q^{\{j\}}(m). \qquad (2.34)
\end{aligned}$$

Afterwards we choose $\delta = \min\{\delta, \delta_1\}$. Then the asymptotical formulas obtained in Lemmas 2.5 and 2.6 yield

$$\begin{aligned}
R(i,j) :=& Q(n)Q^{\{i,j\}}(m-il) - Q^{\{j\}}(m)Q^{\{i\}}(n-il) \\
&= Q(n)Q^{\{j\}}(m)\mathrm{e}^{-\lambda_i}\left(\left(1+O\left(\left(\frac{il}{m}\right)^{\varepsilon_2}\right)\right) - \left(1+O\left(\left(\frac{il}{n}\right)^{\varepsilon_1}\right)\right)\right) \\
&\ll Q(n)Q^{\{j\}}(m)\left(\frac{il}{m}\right)^{\varepsilon}
\end{aligned}$$

with $\varepsilon = \min\{\varepsilon, \varepsilon_1\}$. Hence

$$\begin{aligned}
\sum_{\substack{il \leq \delta m \\ i \neq j}} \frac{\lambda_i^l}{l!} R(i,j) &\ll Q(n) Q^{\{j\}}(m) \sum_{il \leq \delta m} \frac{\lambda_i^l}{l!} \Big(\frac{il}{m}\Big)^{\varepsilon} \\
&\ll Q(n) Q^{\{j\}}(m) \sum_{l \geq 1} \frac{l^{\varepsilon}}{l!} \Theta^l \left(\frac{1}{m^{\varepsilon}} \sum_{i \leq \delta m} i^{-l+\varepsilon} \right) \\
&\ll Q(n) Q^{\{j\}}(m) \sum_{l \geq 1} \frac{l^{\varepsilon}}{l!} \Theta^l \\
&\ll Q(n) Q^{\{j\}}(m). \qquad (2.35)
\end{aligned}$$

Collecting estimates (2.33-2.35), we complete the proof of Key Lemma.

□

2.5 Proof of Theorem 2.1

Observe that it suffices to establish the desired inequality for a nonnegative $h(\sigma)$ only. The general result follows from an inequality $x^2 = (x^+ - x^-)^2 \leq 2(x^+)^2 + 2(x^-)^2$, where the positive and negative parts of $x \in \mathbb{R}$ are defined by

$$x^+ := \begin{cases} x & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases}, \quad x^- := \begin{cases} -x & \text{if } x < 0, \\ 0 & \text{otherwise} \end{cases}$$

respectively, and then to apply the result to a decomposed additive function $h(\sigma) = h(\sigma)^+ - h(\sigma)^-$.

Omitting a part of summands we have from (2.17) that

$$\Big(\mathbb{E}_n h(\sigma)\Big)^2 \geq \sum_{\substack{jk+il\leq n \\ i\neq j}} \frac{\lambda_j^k h_j(k)}{k!} \frac{Q^{(j)}(n-jk)}{Q(n)} \frac{\lambda_i^l h_i(l)}{l!} \frac{Q^{(i)}(n-il)}{Q(n)}.$$

Hence and from (2.20)

$$\mathbf{V}_n h \leq B_n^2 + \sum_{\substack{jk+il\leq n \\ i\neq j}} \frac{\lambda_j^k h_j(k) \lambda_i^l h_i(l)}{k! l!} \times \left(\frac{Q^{(i,j)}(n-il-jk)}{Q(n)} - \frac{Q^{(j)}(n-jk) Q^{(i)}(n-il)}{Q(n)^2} \right)^+.$$

By virtue of $ab \leq (1/2)(a^2+b^2)$, this implies

$$\mathbf{V}_n h \leq B_n^2 + \sum_{jk<n} \frac{\lambda_j^k h_j^2(k)}{k!} \sum_{\substack{il\leq n-jk \\ i\neq j}} \frac{\lambda_i^l}{l!} \left(\frac{Q^{(i,j)}(n-il-jk)}{Q(n)} - \frac{Q^{(j)}(n-jk) Q^{(i)}(n-il)}{Q(n)^2} \right)^+.$$

By Key Lemma with $m = n - jk$, the second sum is $\ll B_n^2$.

Theorem 2.1 is proved.

□

2.6 Proof of Theorem 2.2

If $h_j(s) = a_j s$ for every $j \leq n$ and $s \in \mathbb{N}_0$, then applying Theorem 2.1 and (2.22) we arrive at

$$\mathbf{V}_n h \ll \sum_{j \leq n} \lambda_j a_j^2 \frac{Q(n-j)}{Q(n)} + \sum_{j \leq n/2} \lambda_j^2 a_j^2 \frac{Q(n-2j)}{Q(n)}.$$

The second sum can be majorized by the first one. Indeed, if $j \leq \delta_1(n-j)$, one can apply estimate $Q(n-2j) \ll Q(n-j)$ following from (2.26). Otherwise, if $\delta_1 n/(1+\delta_1) < j \leq n/2$, by (2.25), we have

$$\begin{aligned} Q(n-2j) \ll \alpha(n-2j) &\ll \alpha(n-j)\Big(\frac{n-2j+1}{n-j}\Big)^{\theta} \\ &\ll \alpha(n-j) \ll nQ(n-j). \end{aligned}$$

Using also condition (2.5), we complete estimation of the second sum.

Theorem 2.2 is proved.

□

2.7 Proof of Theorem 2.3

The proof of Theorem 2.3 is based upon P. Elliott's idea to apply the following inversion.

Lemma 2.7. *Let $C = (c_{ij})$, $1 \leq i \leq m$ and $1 \leq j \leq n$ be a real matrix and $\lambda > 0$. If the inequality*

$$\sum_{j \leq n} \left(\sum_{i \leq m} c_{ij} x_j \right)^2 \leq \lambda \sum_{i \leq m} x_i^2$$

holds for all vectors $X = (x_1, \ldots, x_m) \in \mathbb{R}^m$, then so does

$$\sum_{i \leq m} \left(\sum_{j \leq n} c_{ij} y_j \right)^2 \leq \lambda \sum_{j \leq n} y_j^2$$

for all $Y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

Proof. See Lemma 4.3 on page 150 in [13].

□

Proof of Theorem 2.3. Denote

$$\gamma(j,k,\sigma) = \mathbf{1}\{k_j(\sigma) = k\} \sqrt{\frac{k!}{\lambda_j^k} \frac{Q(n)}{Q^{\{j\}}(n-jk)}} - \sqrt{\frac{\lambda_j^k}{k!} \frac{Q^{\{j\}}(n-jk)}{Q(n)}}$$

and

$$x_j(k) = h_j(k) \sqrt{\frac{\lambda_j^k}{k!} \frac{Q^{\{j\}}(n-jk)}{Q(n)}}$$

if $j, k \in \mathbb{N}$, $jk \leq n$, and $\lambda_j \neq 0$.

Applying Theorem 2.1 for arbitrary $h_j(k) \in \mathbb{R}$, $jk \le n$, we have

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{G}_n}\Bigg(\sum_{jk\le n} x_j(k)\gamma(j,k,\sigma)\Bigg)^2\\
&=\sum_{\sigma\in\mathcal{G}_n}\Bigg(\sum_{jk\le n} h_j(k)\mathbf{1}\{k_j(\sigma)=k\}-\sum_{jk\le n} h_j(k)\frac{\lambda_j^k}{k!}\frac{Q^{\{j\}}(n-jk)}{Q(n)}\Bigg)^2\\
&=\sum_{\sigma\in\mathcal{G}_n}\Bigg(\sum_{j\le n} h_j(k_j(\sigma))-\sum_{jk\le n} h_j(k)\frac{\lambda_j^k}{k!}\frac{Q^{\{j\}}(n-jk)}{Q(n)}\Bigg)^2\\
&=\sum_{\sigma\in\mathcal{G}_n}\big(h(\sigma)-\mathbb{E}_n h\big)^2 = G(n)\mathbf{V}_n h \ll G(n)\sum_{jk\le n} x_j(k)^2
\end{aligned}$$

for all $x_j(k) \in \mathbb{R}$, $jk \le n$. Now, by Lemma 2.7, we obtain the desired dual inequality (2.15).

The proof of (2.16) goes by inversion of (2.14) by repeating the argument.

Denote

$$\gamma^{\{c\}}(j,\sigma) = k_j(\sigma)\sqrt{\frac{1}{\lambda_j}\frac{Q(n)}{Q(n-j)}} - \sqrt{\lambda_j\frac{Q(n-j)}{Q(n)}}$$

and

$$x_j^{\{c\}} = a_j\sqrt{\lambda_j\frac{Q(n-j)}{Q(n)}}$$

if $j \in \mathbb{N}$, $j \le n$, and $\lambda_j \ne 0$.

Applying Theorem 2.2, we have

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{G}_n}\Bigg(\sum_{j\le n} x_j^{\{c\}}(k_j(\sigma))\gamma^{\{c\}}(j,\sigma)\Bigg)^2\\
&=\sum_{\sigma\in\mathcal{G}_n}\Bigg(\sum_{j\le n} a_j k_j(\sigma)-\sum_{j\le n} a_j\lambda_j\frac{Q(n-j)}{Q(n)}\Bigg)^2\\
&=\sum_{\sigma\in\mathcal{G}_n}\big(h(\sigma)-\mathbb{E}_n h\big)^2 = G(n)\mathbf{V}_n h \ll G(n)\sum_{j\le n}\big(x_j^{\{c\}}\big)^2
\end{aligned}$$

for all $x_j^{\{c\}} \in \mathbb{R}$, $j \le n$. Now, by Lemma 2.7, we obtain the desired dual inequality (2.16).

Theorem 2.3 is proved.

□

Chapter 3

Moments of additive functions with respect to the Ewens Sampling Formula

3.1 Basics and motivation

Let us denote by $\Omega := \mathbb{N}_0^n$ the additive semigroup of vectors $\bar{s}$, where $\bar{0} = (0, \ldots, 0)$ is the zero vector. The partial order defined by $\bar{s} = (s_1, \ldots, s_n) \leq \bar{t} = (t_1, \ldots, t_n)$ meaning that $s_j \leq t_j$ for each $1 \leq j \leq n$ will be essential. Moreover, we introduce the orthogonality of $\bar{s}, \bar{t} \in \Omega$, denoted by $\bar{s} \perp \bar{t}$, meaning that $s_1 t_1 + \cdots + s_n t_n = 0$. Afterwards we shall use the notation $\bar{t} \parallel \bar{s}$ to express that $\bar{t}$ *exactly enters* $\bar{s}$. Formally, then $\bar{t} \leq \bar{s}$ and $\bar{t} \perp \bar{s} - \bar{t}$. Using the notation when dealing with functions defined on Ω, we come closer to probabilistic number theory which has been carried out on the multiplicative semigroup $\mathbb{N}$ (see [30] and [11]), in which the partial order is defined by division and the orthogonality of $m, n \in \mathbb{N}$ means that their greatest common divisor equals 1. The semigroup structures and the partial orders in Ω and $\mathbb{N}$ could have played a greater role in developing parallel theories. Advantage of applying this approach has been discussed in [26] and in some papers referred to in it. We further demonstrate the number-theoretic ideas adopted estimating moments of functions defined on Ω.

For a probability measure, we take that proposed by Ewens [15] in the mathematical genetic theory. It continuous to serve in various statistical models and probabilistic combinatorics (see, for example, [24], [2] or [16]). To present it, denote $\Omega_n := \ell^{-1}(n) = \{\bar{s} \in \Omega : \ell(\bar{s}) = n\}$. Set also

$$\Theta(n) := \binom{\theta + n - 1}{n} = [z^n] \frac{1}{(1-z)^\theta},$$

where $\theta > 0$ is a parameter and, as usual, $[z^n]g(z)$ stands for the nth coefficient of the power series $g(z)$ if $n \in \mathbb{N}_0$. Then the celebrated Ewens Sampling Formula defines the probability

$$P_n(\{\bar{s}\}) := \Theta(n)^{-1} \prod_{j=1}^{n} \left(\frac{\theta}{j}\right)^{s_j} \frac{1}{s_j!} =: \Theta(n)^{-1} P(\bar{s}) \tag{3.1}$$

ascribed for each $\bar{s} \in \Omega_n$. For convenience we extend the probability measure to the whole Ω by setting $P_n(\{\bar{s}\}) = 0$ if $\bar{s} \in \Omega \setminus \Omega_n$. Now every mapping $G : \Omega \to \mathbb{C}$ becomes a complex-valued random variable, and

$$\mathbb{E}_n(G) := \Theta(n)^{-1} \sum_{\bar{s} \in \Omega_n} G(\bar{s}) P(\bar{s}) \tag{3.2}$$

is its expectation. Let $\mathbb{E}_0(G) := 1$ for every $G : \Omega \to \mathbb{C}$.

It is worthy to recall the following property of (3.1). If ξ_j, $1 \le j \le n$, are mutually independent Poisson random variables with parameters θ/j given on some probability space and $\bar{\xi} := (\xi_1, \dots, \xi_n)$, then

$$P_n(\{\bar{s}\}) = Pr\big(\bar{\xi} = \bar{s} \mid \ell(\bar{\xi}) = n\big), \quad \bar{s} \in \Omega.$$

This clearly shows the dependence of coordinates s_j, $1 \le j \le n$, under the probability measure P_n. Despite to it, some recent results on the asymptotic behavior as $n \to \infty$ of distributions of the linear statistics $a_{n1}s_1 + \cdots + a_{nn}s_n$, where $a_{nj} \in \mathbb{R}$ and $1 \le j \le n$, give general conditions for weak convergence or sharp estimates of the convergence rates. They are mainly formulated in the terminology of the theory of random permutations; therefore, we now present the connections to the latter.

Once again, let us notice that if we define the Ewens Probability Measure $\nu_n^{(\theta)}$ on $\mathbf{S}_n$ by

$$\nu_n^{(\theta)}(\{\sigma\}) := \theta^{w(\sigma)} / \big(\theta(\theta+1)\cdots(\theta+n-1)\big), \quad \sigma \in \mathbf{S}_n,$$

where $\theta > 0$ is a parameter and $w(\sigma)$ is the number-of-cycles function, an easy combinatorial argument (see [2]) gives the distribution of the cycle vector and the coincidence:

$$\nu_n^{(\theta)}\big(\bar{k}(\sigma) = \bar{s}\big) = P_n(\{\bar{s}\})$$

if $\bar{s} \in \Omega_n$. Thus, dealing with statistics of random permutations expressed via $\bar{k}(\sigma)$, we may examine corresponding statistics of random vectors $\bar{s} \in \Omega_n$ taken with probabilities (3.1).

The linear statistics $a_{1n}k_1(\sigma) + \cdots + a_{nn}k_n(\sigma)$ and, in particular, $w(\sigma)$ have attracted much attention in the recent investigations. However, so far, the advance in probabilistic number theory has not been adequately followed by the corresponding results in probabilistic combinatorics. For instance, the results exposed in Section 8.5 of book [2] did not reach the level of their analogs in $\mathbb{N}$ (compare with [11]). In the recent papers [36] and [4] (see also the references therein), E. Manstavičius did some attempt to fill up this gap.

Let us continue by introducing more definitions. A mapping $F : \Omega \to \mathbb{C}$, $F(\bar{0}) := 1$, is called a *multiplicative function* if $F(\bar{s} + \bar{t}) = F(\bar{s})F(\bar{t})$ holds for every pair $\bar{s}, \bar{t} \in \mathbb{N}_0^n$ such that $\bar{s} \perp \bar{t}$. Denote a generic vector $\bar{e}_j := (0, \ldots, 1, \ldots, 0)$, where the only 1 stands at the jth place. Then the multiplicative function F has the decomposition

$$F(\bar{k}) = \prod_{j \le n} F(k_j \bar{e}_j) =: \prod_{j \le n} f_j(k_j). \tag{3.3}$$

Conversely, given a complex two-dimensional array $\{f_j(k)\}$, $1 \le j, k \le n$, satisfying the condition $f_j(0) \equiv 1$, by the last equality, we can define a multiplicative function. If $f_j(k) = f_j(1) =: f_j$ for all $k \ge 1$ and $j \le n$, the function F is called *strongly* multiplicative and, similarly, if $f_j(k) = f_j^k$ and $0^0 := 1$, then F is called *completely* multiplicative. Denote, respectively, by $\mathfrak{M}$, $\mathfrak{M}_s$, and $\mathfrak{M}_c$ the sets of just introduced multiplicative functions. Stress that $P(\bar{s}) \in \mathfrak{M}$ and $P(k\bar{e}_j) = (\theta/j)^k/k! =: p_j(k)$ if $k \in \mathbb{N}_0$ and $j \in \mathbb{N}$.

Similarly, the condition $H(\bar{s} + \bar{t}) = H(\bar{s}) + H(\bar{t})$ holding for every pair $\bar{s}, \bar{t} \in \mathbb{N}_0^n$ such that $\bar{s} \perp \bar{t}$ defines an *additive function* $H : \Omega \to \mathbb{C}$. Let us set also $h_j(k) := H(k\bar{e}_j)$ where $h_j(0) := 0$. Now, condition $h_j(k) = kh_j(1)$, $j \in \mathbb{N}$, reckons *completely* additive functions.

The purpose of the present chapter is to establish power moment inequalities for a complex-valued additive function $H(\bar{s})$. The number-theoretic analogue of desired result is Elliott's high-power analogue (1) of Turán-Kubilius inequality. The tail probability estimates for additive functions proposed in [31] and refined in [3], together with a subsequent use of relevant results for sums of independent random variables, provide an indirect approach to deal with the problem if $\theta \ge 1$. One may expect (see [35] and [34]) that the direct proof, as exposed below, gives sharper results. This has been evidenced in Chapter 1 and Chapter 2, dealing with the second moment of additive functions defined on general decomposable structures including permutations sampled according to the Ewens probability.

3.2 Result

Denote

$$A := A_n := \sum_{jk\le n} h_j(k) p_j(k) \frac{\Theta(n-jk)}{\Theta(n)},$$

$$B_n(\alpha) := \sum_{jk\le n} |h_j(k)|^{\alpha} p_j(k) \frac{\Theta(n-jk)}{\Theta(n)},$$

and $B := \big(B_n(2)\big)^{1/2}$, where $\alpha > 0$.

Let the constant in $\ll$ depend at most on θ and α.

Theorem 3.1. *If H is an additive function and $\theta \ge 1$, then*

$$\mathbb{E}_n\big(|H(\bar{s}) - A|^{\alpha}\big) \ll \begin{cases} B^{\alpha} + B_n(\alpha) & \text{if } \alpha \ge 2, \\ B^{\alpha} & \text{if } 0 \le \alpha \le 2 \end{cases}$$

uniformly for all $n \ge 1$.

In the proof, we adopt Elliott's [12] argument. The main task is analysis of the expectation of $\mathrm{e}^{zH(\bar{s})/B}$, which is a multiplicative function depending on complex parameter z. The needed technique is developed in the next section. The theorem will be proved at the end of the chapter.

3.3 Expectations of multiplicative functions

This section is devoted to estimates of the expectations $\mathbb{E}_n(F)$ of mappings $F : \Omega \to \mathbb{R}^+$ belonging to more specialized classes. If $F \in \mathfrak{M}$, then (3.2) and (3.3) give the following expressions

$$\begin{aligned}\Theta(n)\mathbb{E}_n(F) &= \sum_{\ell(\bar{s})=n} \prod_{j\le n} p_j(s_j) f_j(s_j) \\ &= [x^n] \prod_{j\ge 1} \left(1 + \sum_{k=1}^{\infty} p_j(k) f_j(k) x^{jk}\right) \\ &=: [x^n] \prod_{j\ge 1} \chi_j(x;F) =: [x^n] Z(x;F).\end{aligned} \tag{3.4}$$

Actually, the values $f_j(k)$ if $jk > n$, which do not appear in the quantity on the left-hand side, can be chosen in a convenient way, say, equal to zeros or ones. Here and in the sequel, $j, k \in \mathbb{N}$. On the other hand, if $F \in \mathfrak{M}_c$, that is, for $f_j(k) = f_j(1)^k$, $j, k \ge 1$, there is no need to do so.

Set $\Pi_0(F) := 1$, and

$$\Pi_m(F) := \prod_{j\le m} \left(1 + \sum_{k=1}^{\infty} p_j(k) f_j(k)\right), \quad 1 \le m \le n.$$

From the definitions, one directly obtains the inequality

$$\Sigma_n(F) := \sum_{0\le m<n} \mathbb{E}_m(F)\Theta(m) \le \Pi_{n-1}(F) \tag{3.5}$$

if F is a non-negative multiplicative function and $n \in \mathbb{N}_0$.

We begin with a convenient identity. Introduce a function $P^{(j)} \in \mathfrak{M}$ such that $p_i^{(j)}(k) = p_i(k)$ if $i \ne j$ and $p_i^{(j)}(k) = 0$ if $i = j$ and $k \in \mathbb{N}$. Let

$$\Theta^{(j)}(n) := \sum_{\ell(\bar{s})=n} P^{(j)}(\bar{s}) = [z^n] \mathrm{e}^{-\theta_j x^j/j} (1-z)^{-\theta}.$$

Define the following conditional expectation

$$\mathbb{E}_n^{(j)}(F) = \frac{1}{\Theta^{(j)}(n)} \sum_{\ell(\bar{s})=n} \prod_{i\le n} p_i^{(j)}(s_i) f_i(s_i) = \frac{1}{\Theta^{(j)}(n)} \sum_{\ell(\bar{s})=n} P^{(j)}(\bar{s}) F(\bar{s}).$$

Lemma 3.1. *Let $F : \Omega \to \mathbb{C}$ be a multiplicative function, then*

$$n\Theta(n)\mathbb{E}_n(F) = \sum_{jk\le n} jkf_j(k)p_j(k)\Theta^{(j)}(n-jk)\mathbb{E}^{(j)}_{n-jk}(F).$$

In particular, if $F \in \mathfrak{M}_c$, then

$$n\Theta(n)\mathbb{E}_n(F) = \theta\sum_{j\le n} f_j(1)\Theta(n-j)\mathbb{E}_{n-j}(F).$$

Proof. This is just the identity

$$[x^n]\big(xZ'(x;F)\big) = [x^n]\bigg(x\sum_{j\ge1}\chi_j'(x;F)\prod_{i\ge1,i\ne j}\chi_i(x;F)\bigg);$$

nevertheless, we provide an elementary proof exposing the idea used in the sequel.

Let $\bar{e} = k\bar{e}_j$ if $1\le k\le n/j$ and $1\le j\le n$. Then $\ell(\bar{e}_j) = jk$, $F(\bar{e}) = f_j(k)$, $P(\bar{e}) = p_j(k)$, and

$$\sum_{\bar{e}\|\bar{t}}\ell(\bar{e}) = \ell(\bar{t})$$

for an arbitrary $\bar{t}\in\Omega$. Hence

$$\begin{aligned}
n\Theta(n)\mathbb{E}_n(F) &= \sum_{\ell(\bar{k})=n} F(\bar{k})P(\bar{k})\sum_{\bar{e}\|\bar{k}}\ell(\bar{e})\\
&= \sum_{\substack{\bar{t}\perp\bar{e}\\ \ell(\bar{t})+\ell(\bar{e})=n}} F(\bar{t})F(\bar{e})P(\bar{t})P(\bar{e})\ell(\bar{e})\\
&= \sum_{\ell(\bar{e})\le n} F(\bar{e})\ell(\bar{e})P(\bar{e})\sum_{\substack{\ell(\bar{t})=n-\ell(\bar{e})\\ \bar{t}\perp\bar{e}}} F(\bar{t})P(\bar{t})\\
&= \sum_{jk\le n} jkf_j(k)p_j(k)\sum_{\substack{\ell(\bar{t})=n-jk\\ t_j=0}} F(\bar{t})P(\bar{t})\\
&= \sum_{jk\le n} jkf_j(k)p_j(k)\Theta^{(j)}(n-jk)\mathbb{E}^{(j)}_{n-jk}(F).
\end{aligned}$$

The lemma is proved.

□

Corollary 3.1. *If $\theta > 0$ and $F\ge 0$ is a multiplicative function, then*

$$n\Theta(n)\mathbb{E}_n(F) \le \sum_{jk\le n} jkf_j(k)p_j(k)\Theta(n-jk)\mathbb{E}_{n-jk}(F).$$

Proof. By the definitions above, $\Theta^{(j)}(m)\mathbb{E}_m^{(j)}(F) \le \Theta(m)\mathbb{E}_m(F)$ for each $1 \le j \le m \le n$. □

Lemma 3.2. *If $\theta > 0$ and $F \in \mathfrak{M}$ is such that $0 \le f_j(k) \le K$ for $k, j \le n$ and $n \ge 1$, then*

$$n\Theta(n)\mathbb{E}_n(F) \le \theta K \Sigma_n(F)\Big(1 + O\Big(\frac{K\log(n+1)+1}{n}\Big)\Big).$$

Proof. By Corollary 3.1,

$$\begin{aligned} n\Theta(n)\mathbb{E}_n(F) &\le \theta K \Sigma_n(F) \\ &\quad + K \sum_{0\le m<n} \Theta(m)\mathbb{E}_m(F)\Big|(n-m)\sum_{jk=n-m} p_j(k) - \theta\Big| \\ &=: \theta K\Sigma_n(F) + KR. \end{aligned} \tag{3.6}$$

Observe that, for $m \ge 2$,

$$\begin{aligned} m\sum_{jk=m} p_j(k) - \theta &= \frac{\theta^m}{(m-1)!} + \frac{\theta^2}{m}\sum_{\substack{jk=m,\\ j,k\ge 2}} \Big(\frac{\theta}{j}\Big)^{k-2}\frac{k}{(k-1)!} \\ &\le \frac{\theta^m}{(m-1)!} + \frac{2\theta^2}{m}\sum_{2\le k\le m/2}\Big(\frac{\theta k}{m}\Big)^{k-2}\frac{1}{(k-2)!} \\ &\le \frac{\theta^m}{(m-1)!} + \frac{2\theta^2}{m}\mathrm{e}^{\theta/2} \le \frac{\theta C(\theta)}{m}, \end{aligned}$$

where $C(\theta) > 0$ is a constant depending only upon θ. Plugging this into the previous inequality, we obtain

$$n\Theta(n)\mathbb{E}_n(F) \le \theta K \sum_{0\le m<n} \Theta(m)\mathbb{E}_m(F)\Big(1 + \frac{C(\theta)}{n-m}\Big). \tag{3.7}$$

This also yields a rough estimate $m\Theta(m)\mathbb{E}_m(F) \ll K\Sigma_m(F)$, $1 \le m \le n$, needed below. Now

$$\begin{aligned} R &\ll \Big(\sum_{1\le m\le n/2} + \sum_{n/2\le m<n}\Big)\frac{\Theta(m)\mathbb{E}_m(F)}{n-m} \\ &\ll \frac{\Sigma_n(F)}{n} + K\sum_{n/2\le m<n}\frac{1}{m(n-m)}\Sigma_m(F) \\ &\ll \frac{1+K\log(n+1)}{n}\Sigma_n(F). \end{aligned}$$

Inserting the estimate into (3.6), we complete the proof.

□

In the sequel, we will apply the just proved result in a more convenient form.

Corollary 3.2. *If $\theta > 0$ and $F \in \mathfrak{M}$ is such that $0 \le f_j(k) \le 1$ for $k, j \le n$ and $n \ge 1$, then*

$$\mathbb{E}_n(F) \ll \exp\left\{\theta \sum_{j \le n} \frac{f_j(1) - 1}{j}\right\}.$$

Proof. Applying well-known (see [17]) asymptotic formula

$$\Theta(n) = \frac{n^{\theta-1}}{\Gamma(\theta)}\left(1 + O\left(\frac{1}{n}\right)\right)$$

and (3.5), we obtain from Lemma 3.2 that

$$\begin{aligned}\mathbb{E}_n(F) \le \Gamma(\theta+1) \exp&\left\{\theta\gamma + \theta \sum_{j \le n} \frac{f_j(1) - 1}{j}\right\}\left(1 + O\Big(\frac{\log(n+1)}{n}\Big)\right)\\ &\times \prod_{j \le n} \mathrm{e}^{-\theta f_j(1)/j}\left(1 + \sum_{k=1}^{\infty}\left(\frac{\theta}{j}\right)^k \frac{f_j(k)}{k!}\right).\end{aligned} \tag{3.8}$$

Here γ and Γ denotes the Euler-Masheroni constant and the Gamma function respectively.

Applying the inequality

$$|\mathrm{e}^x - 1 - x| \le |x|^2 \mathrm{e}^{|x|} \quad \text{if} \quad x \in \mathbb{R}, \tag{3.9}$$

we further obtain

$$\begin{aligned}&\prod_{j \le n} \mathrm{e}^{-\theta f_j(1)/j}\left(1 + \frac{\theta f_j(1)}{j} + \sum_{k=2}^{\infty}\left(\frac{\theta}{j}\right)^k \frac{f_j(k)}{k!}\right)\\ &\le \prod_{j \le n} \mathrm{e}^{-\theta f_j(1)/j}\left(1 + \frac{\theta f_j(1)}{j} + \mathrm{e}^{\frac{\theta}{j}} - 1 - \frac{\theta}{j}\right)\\ &\le \prod_{j \le n} \mathrm{e}^{-\theta f_j(1)/j}\left(\mathrm{e}^{\frac{\theta f_j(1)}{j}} + \left(\frac{\theta}{j}\right)^2 \mathrm{e}^{\frac{\theta}{j}}\right)\\ &\le \prod_{j \le n}\left(1 + \left(\frac{\theta}{j}\right)^2 \mathrm{e}^{\frac{\theta}{j}}\right) \le C.\end{aligned}$$

□

Observing that $\frac{\Theta(n-j)}{\Theta(n)} \leq 1$ if $\theta \geq 1$ for all $j \leq n$, we also have

$$\mathbb{E}_n(F) \ll \exp\left\{\theta \sum_{j\leq n} \frac{f_j(1)-1}{j} \frac{\Theta(n-j)}{\Theta(n)}\right\}. \tag{3.10}$$

Remarks. We firstly stress the parallelism with the Hall's [19] paper exploring number-theoretic submultiplicative functions. Inequality (3.8) also holds for a *submultiplicative* function $G : \Omega \to \mathbb{R}$ which by definition satisfies the inequality $G(\bar{s}+\bar{t}) \leq G(\bar{s})G(\bar{t})$ for all $\bar{s}, \bar{t} \in \Omega$ if $\bar{s} \perp \bar{t}$. For example, such is the statistics $G(\bar{s}) = l.c.m.\{j : \ s_j \geq 1\}$, related to the group theoretical order of a permutation in the group $\mathbf{S}_n$. Here the letters $l.c.m.$ stand for the least common multiplier of the indicated natural numbers. This, more general case can be dealt with by a repetition of the used argument or by a direct application of Lemma 3.2. Indeed, given a submultiplicative function $G : \Omega \to [0,1]$, we have

$$G(\bar{k}) \leq \prod_{j\leq n} G(k_j \bar{e}_j) =: \prod_{j\leq n} g_j(k_j).$$

Further, one can define $F \in \mathfrak{M}$ so that $f_j(k) = g_j(k)$ to obtain $G(\bar{k}) \leq F(\bar{k})$ and a subsequent ability to apply the Corollary for the function F.

Secondly, if $\theta = 1$ and F is a multiplicative function satisfying the conditions in Corollary 3.2 and the values $f_j(1)$ for $\varepsilon n < j \leq n$, where $0 < \varepsilon < 1$, are close to 1, one can substitute e^{γ} in (3.8) by a smaller quantity (see [43]). Constructing appropriate indicator functions and using Lemma 3.2 or (3.8), one can obtain sharp estimates of the probabilities of vectors with a forbidden pattern. Note, that the lower estimates have been discussed in [26]. This ends our remarks.

In the next step, we will need some knowledge about the algebraic structure $(\mathfrak{G}, *)$, where $\mathfrak{G} := \{G : \Omega \to \mathbb{C}\}$ and $*$ is the convolution defined as follows

$$F * G(\bar{t}) := \sum_{\bar{s}\leq\bar{t}} F(\bar{s})G(\bar{t}-\bar{s}), \quad \bar{s}, \bar{t} \in \Omega.$$

Let $I(\bar{t}) \equiv 1$ and $E(\bar{t}) = \mathbf{1}\{\bar{t} = \bar{0}\}$ be the indicator function of the subset $\{\bar{0}\}$. It is straightforward to check that $(\mathfrak{G}, *)$ is an Abelian group in which E serves for the neutral element. The inverse of I in the group is an analogue of the Möbius function. Let us leave the notation μ for it. The latter is a multiplicative function such that $\mu(\bar{e}_j) = -1$, and $\mu(r\bar{e}_j) = 0$ if $r \geq 2$, where $1 \leq j \leq n$. It is easy to check that the relations $F = I * G$ and $G = \mu * F$ are equivalent. Finally, we stress that $\mathfrak{M}$ is a subgroup in $\mathfrak{G}$.

Lemma 3.3. *Let $\theta \geq 1$ and $F \in \mathfrak{M}$ be such that $f_j(1) \geq 1$ for each $j \leq n$, then*

$$\mathbb{E}_n(F) \leq \exp\left\{ \sum_{jk \leq n} f_j(k) p_j(k) \frac{\Theta(n-jk)}{\Theta(n)} - \sum_{j \leq n} p_j(1) \frac{\Theta(n-j)}{\Theta(n)} \right\} \tag{3.11}$$

for all $n \geq 1$.

Proof. Let F be as in the lemma. Define a function $G = F * \mu$. Then $g_j(1) = f_j(1) - 1 \geq 0$ and hence $G(\bar{t})\mu^2(\bar{t}) \geq 0$. Moreover, if $\mu^2(\bar{t} + \bar{s}) \neq 0$, then $\bar{t} \perp \bar{s}$. If $\bar{s} \in \Omega$, we have

$$\begin{aligned}
\mathbb{E}_m(F\mu^2) &= \Theta(m)^{-1} \sum_{\ell(\bar{k})=m} F(\bar{k})\mu^2(\bar{k})P(\bar{k}) \\
&= \Theta(m)^{-1} \sum_{\substack{\ell(\bar{s}+\bar{t})=m \\ \bar{t}\perp\bar{s}}} G(\bar{t})\mu^2(\bar{s}+\bar{t})P(\bar{s}+\bar{t}) \\
&\leq \Theta(m)^{-1} \sum_{\ell(\bar{t})\leq m} G(\bar{t})\mu^2(\bar{t})P(\bar{t}) \sum_{\ell(\bar{s})=m-\ell(\bar{t})} P(\bar{s}) \\
&= \sum_{\ell(\bar{t})\leq m} G(\bar{t})\mu^2(\bar{t})P(\bar{t}) \frac{\Theta\big(m-\ell(\bar{t})\big)}{\Theta(m)} \\
&\leq \prod_{j\leq m} \left(1 + (f_j(1)-1)P(\bar{e}_j)\frac{\Theta(m-j)}{\Theta(m)} \right).
\end{aligned}$$

In the last step, we used the inequality

$$\frac{\Theta(n-\ell(\bar{s}))}{\Theta(n)} \leq \prod_{j\leq n} \frac{\Theta(n-js_j)}{\Theta(n)}$$

valid for all $n \geq 1$, $\ell(\bar{s}) \leq n$ and $\theta \geq 1$. The latter implies

$$\mathbb{E}_m(F\mu^2) \leq \exp\left\{ \sum_{j\leq m} (f_j(1)-1)p_j(1)\frac{\Theta(m-j)}{\Theta(m)} \right\}$$

if $0 \leq m \leq n$.

In the general case, we reorganize the expression of $\mathbb{E}_n(F)$. We firstly split uniquely $\bar{k} = \bar{t} + \bar{s}$ with $\bar{t} \perp \bar{s}$, where $\bar{s} \in \big(\mathbb{N}_0 \setminus \{1\}\big)^n$, $\bar{t} \in \{0,1\}^n$, and $t_j = 1$ if and only if $k_j = 1$.

Then, keeping this agreement in summation carried out in the next few lines, we proceed

$$\begin{aligned}\mathbb{E}_n(F) &= \Theta(n)^{-1} \sum_{\substack{\ell(\bar{t}+\bar{s})=n \\ \bar{t}\perp\bar{s}}} F(\bar{t})P(\bar{t})\mu^2(\bar{t})F(\bar{s})P(\bar{s}) \\ &\leq \sum_{\ell(\bar{s})\leq n} F(\bar{s})P(\bar{s})\frac{\Theta(n-\ell(\bar{s}))}{\Theta(n)}\Theta(n-\ell(\bar{s}))^{-1} \sum_{\ell(\bar{t})=n-\ell(\bar{s})} F(\bar{t})\mu^2(\bar{t})P(\bar{t}).\end{aligned}$$

The inner sum was just estimated. Since

$$\begin{aligned}\sum_{\ell(\bar{s})\leq r} F(\bar{s})P(\bar{s})\frac{\Theta(r-\ell(\bar{s}))}{\Theta(r)} &\leq \prod_{j\leq r}\left(1+\sum_{2\leq k\leq r/j} f_j(k)p_j(k)\frac{\Theta(r-jk)}{\Theta(r)}\right) \\ &\leq \exp\left\{\sum_{\substack{jk\leq r \\ k\geq 2}} f_j(k)p_j(k)\frac{\Theta(r-jk)}{\Theta(r)}\right\}\end{aligned}$$

for $0 \leq r \leq n$, we further obtain

$$\begin{aligned}\mathbb{E}_n(F) &\leq \exp\left\{\sum_{j\leq n}(f_j(1)-1)p_j(1)\frac{\Theta(n-j)}{\Theta(n)} + \sum_{\substack{jk\leq n \\ k\geq 2}} f_j(k)p_j(k)\frac{\Theta(n-jk)}{\Theta(n)}\right\} \\ &= \exp\left\{\sum_{jk\leq n} f_j(k)p_j(k)\frac{\Theta(n-jk)}{\Theta(n)} - \sum_{j\leq n} p_j(1)\frac{\Theta(n-j)}{\Theta(n)}\right\}.\end{aligned}$$

The lemma is proved.

□

3.4 Moments of an additive function

We now embark on the power moments of a complex-valued additive function $H(\bar{s})$. Let A and B be the quantities defined in Section 3.2. Define the multiplicative function $F(\bar{s}) = \mathrm{e}^{zH(\bar{s})/B}$, where $z \in \mathbb{C}$, and set $\varphi(z) = \mathrm{e}^{-zA/B}\mathbb{E}_n(F)$. Afterwards, we adopt Elliott's [12] argument.

Lemma 3.4. *Assume that $0 \le h_j(k) \le \delta B$ holds for some $\delta > 0$ and all products $jk \le n$. Then there is a constant c_1 depending on δ such that $|\varphi(z)| \le c_1$ uniformly in z if $|z| \le 1$.*

Proof. The inequality (3.9) gives us

$$\begin{aligned} \sum_{\substack{jk\le n\\ k\ge 2}} P(k\bar{e}_j)\frac{\Theta(n-jk)}{\Theta(n)} &\le \sum_{\substack{jk\le n\\ k\ge 2}} P(k\bar{e}_j) \le \sum_{j\le n}\sum_{k\ge 2}\left(\frac{\theta}{j}\right)^k\frac{1}{k!} \\ &= \sum_{j\le n}\left(\mathrm{e}^{\theta/j} - 1 - \frac{\theta}{j}\right) \le C. \end{aligned} \tag{3.12}$$

Applying this and the Cauchy-Schwarz inequality, we have

$$\left| A - \sum_{j\le n} h_j(1)P(\bar{e}_j)\frac{\Theta(n-j)}{\Theta(n)}\right| \le B\Bigg(\sum_{\substack{jk\le n\\ k\ge 2}} P(k\bar{e}_j)\frac{\Theta(n-jk)}{\Theta(n)}\Bigg)^{1/2} \le C_1 B.$$

In the case $z = r \le 0$, we have $0 \le F(\bar{s}) \le 1$. Thus, by Corollary 3.2, the inequality (3.9) and the inequality above,

$$\begin{aligned} \varphi(r) &\le C_2 \exp\left\{\sum_{j\le n}\left(\mathrm{e}^{rh_j(1)/B} - 1\right)P(\bar{e}_j)\frac{\Theta(n-j)}{\Theta(n)} - \frac{rA}{B}\right\} \\ &\le C_3 \exp\left\{\sum_{j\le n}\left(\mathrm{e}^{rh_j(1)/B} - 1 - \frac{rh_j(1)}{B}\right)P(\bar{e}_j)\frac{\Theta(n-j)}{\Theta(n)}\right\} \\ &\le C_3 \exp\left\{\sum_{j\le n}\left|\frac{rh_j(1)}{B}\right|^2 \mathrm{e}^{|r\delta|}P(\bar{e}_j)\frac{\Theta(n-j)}{\Theta(n)}\right\} \\ &\le C_3 \exp\left\{r^2\mathrm{e}^{|r\delta|}\right\} \le C_4. \end{aligned}$$

In the case $z = r \geq 0$, we have $F(\bar{s}) \geq 1$. Argueing with Lemma 3.3, we obtain

$$\begin{aligned}
\varphi(r) &\leq \exp\left\{ \sum_{jk\leq n} \mathrm{e}^{rh_j(k)/B} P(k\bar{e}_j) \frac{\Theta(n-jk)}{\Theta(n)} - \sum_{j\leq n} P(\bar{e}_j) \frac{\Theta(n-j)}{\Theta(n)} - \frac{rA}{B} \right\} \\
&= \exp\left\{ \sum_{jk\leq n} P(k\bar{e}_j) \frac{\Theta(n-jk)}{\Theta(n)} \left(\mathrm{e}^{rh_j(k)/B} - 1 - \frac{rh_j(k)}{B} \right) \right. \\
&\left. + \sum_{jk\leq n} P(k\bar{e}_j) \frac{\Theta(n-jk)}{\Theta(n)} - \sum_{j\leq n} P(\bar{e}_j) \frac{\Theta(n-j)}{\Theta(n)} \right\} \\
&\leq C_5 \exp\left\{ \frac{r^2 \mathrm{e}^{r\delta}}{B^2} \sum_{jk\leq n} h_j(k)^2 P(k\bar{e}_j) \frac{\Theta(n-jk)}{\Theta(n)} + \sum_{\substack{jk\leq n \\ k\geq 2}} P(k\bar{e}_j) \frac{\Theta(n-jk)}{\Theta(n)} \right\} \\
&\leq C_6 \exp\left\{ r^2 \mathrm{e}^{r\delta} \right\} \leq C_7.
\end{aligned}$$

In general case when $r = Re(z)$, $z \in \mathbb{C}$, we have

$$|\varphi(z)| \leq \Theta(n)^{-1} \sum_{\ell(\bar{s})=n} P(\bar{s}) \left| \mathrm{e}^{z(H(\bar{s})-A)/B} \right| \leq \varphi(r) \leq \max(C_4, C_7).$$

□

Lemma 3.5. *Assume $H(\bar{s})$ is complex-valued additive function such that $|h_j(k)| \leq \delta B$ holds for some $\delta > 0$ and all products $jk \leq n$. Then for each $\alpha > 0$ there is a constant c_2, depending on α and δ, so that the inequality*

$$\mathbb{E}_n\left(|H(\bar{s}) - A|^\alpha\right) \leq c_2 B^\alpha \tag{3.13}$$

holds for all $n \geq 1$.

Proof. Since the weighted power means

$$\left(\mathbb{E}_n(|H(\bar{s}) - A|^\alpha)\right)^{1/\alpha}$$

do not decrease as α increases, it will suffice to prove the inequality (3.13) for integer values of α.

By considering real and imaginary parts separately we see that there is no loss in generality in assuming that $H(\bar{s})$ takes only real values, and, indeed, only non-negative real values. For example, we can define additive functions $H_i(\bar{s})$, $i = 1, 2$, by

$$h_{1,j}(k) := \begin{cases} h_j(k) & \text{if } h_j(k) > 0 \\ 0 & \text{otherwise} \end{cases}, \quad h_{2,j}(k) := \begin{cases} -h_j(k) & \text{if } h_j(k) < 0 \\ 0 & \text{otherwise} \end{cases},$$

$$H_i(\bar{s}) := \sum_{j \le n} h_{i,j}(s_j)$$

and

$$A_i := \sum_{jk \le n} h_{i,j}(k) P(k\bar{e}_j) \frac{\Theta(n-jk)}{\Theta(n)}, \quad i = 1, 2.$$

Then

$$|H(\bar{s}) - A|^\alpha \le 2^\alpha \sum_{i=1}^{2} |H_i(\bar{s}) - A_i|^\alpha.$$

Summing over vectors $\bar{s}$ such that $\ell(\bar{s}) = n$ justifies our last assertion.

For every positive integer k we calculate the kth derivative of $\varphi(z)$ evaluated at $z = 0$. Namely,

$$\varphi^{(k)}(0) = \frac{\Theta(n)^{-1}}{B^k} \sum_{\ell(\bar{s})=n} P(\bar{s}) \big(H(\bar{s}) - A\big)^k = \frac{\mathbb{E}_n((H(\bar{s}) - A)^k)}{B^k}.$$

By Cauchy's integral representation theorem

$$\varphi^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=1} z^{-k-1} \varphi(z) dz$$

and by Lemma 3.4

$$|\varphi^{(k)}(0)| \le \frac{k!}{2\pi} 2\pi \max_{|z|=1} |z^{-k-1} \varphi(z)| \le k! c_1.$$

□

Let us remark that those j, k, $jk \le n$, for which $|h_j(k)| > \delta B$ holds satisfy

$$\sum_{\substack{jk \le n \\ |h_j(k)| > \delta B}} P(k\bar{e}_j) \frac{\Theta(n-jk)}{\Theta(n)} \le \sum_{\substack{jk \le n \\ |h_j(k)| > \delta B}} P(k\bar{e}_j) \le \sum_{jk \le n} P(k\bar{e}_j) \left| \frac{h_j(k)}{\delta B} \right|^2 = \delta^{-2}, \tag{3.14}$$

and are in this sense few in number.

Lemma 3.6. *Suppose $J \subset \{1, 2, ..., n\}$ and define*

$$L := L(n) = \sum_{j \le n, j \in J} P(\bar{e}_j) \frac{\Theta(n-j)}{\Theta(n)}.$$

Let $\omega(\bar{s})$ denote the number of non-zero coordinates s_j such that $j \in J$ or $s_j \ge 2$. Then there is a constant c_3 depending on α such that the inequality

$$\mathbb{E}_m\big(\omega(\bar{s})^\alpha\big) \le c_3(\alpha)(L+1)^\alpha \tag{3.15}$$

holds uniformly for all $n, m, 1 \le m \le n$, *and* $\alpha \ge 0$.

Proof. Since the weighted power means

$$\left(\mathbb{E}_n(\omega(\bar{s})^\alpha)\right)^{1/\alpha}$$

do not decrease as α increases, it will suffice to prove the inequality (3.15) for all integers $k \ge 0$.

We argue inductively on k.

For $k = 0$ the inequality (3.15) is trivially valid. Assume that it holds for $k = 0, 1, ..., v-1, v \ge 1$. Then

$$\begin{aligned}
\mathbb{E}_m(\omega(\bar{s})^v) &= \Theta(m)^{-1} \sum_{\ell(\bar{s})=m} P(\bar{s})\omega(\bar{s})^v \\
&= \Theta(m)^{-1} \sum_{\ell(\bar{s})=m} P(\bar{s})\omega(\bar{s})^{v-1} \sum_{k\bar{e}_j||\bar{s}}^{*} 1 \\
&= \Theta(m)^{-1} \sum_{jk\le m}^{*} \sum_{\substack{\ell(\bar{s})=m \\ s_j=k}} P(\bar{s})\omega(\bar{s})^{v-1}.
\end{aligned}$$

Here and further on, the asterisk means that in the case $k = 1$ the summation is taken only over $j \in J$. If $s_j = k$, say $\bar{s} = k\bar{e}_j + \bar{t}$ where $k\bar{e}_j \perp \bar{t}$, then $\ell(\bar{t}) = \ell(\bar{s}) - jk$ and $\omega(\bar{s}) \le 1 + \omega(\bar{t})$. According to our induction hypothesis the inner sum

$$\begin{aligned}
&\sum_{\substack{\ell(\bar{t})=m-jk \\ k\bar{e}_j\perp\bar{t}}} P(\bar{t}+k\bar{e}_j)\omega(\bar{t}+k\bar{e}_j)^{v-1} \le P(k\bar{e}_j) \sum_{\ell(\bar{t})=m-jk} P(\bar{t})(1+\omega(\bar{t}))^{v-1} \\
&= P(k\bar{e}_j) \sum_{\ell(\bar{t})=m-jk} P(\bar{t}) \sum_{i=0}^{v-1} \binom{v-1}{i} \omega(\bar{t})^i \\
&\le \Theta(m-jk)P(k\bar{e}_j) \sum_{i=0}^{v-1} \binom{v-1}{i} c_3(i)(L+1)^i \\
&\le \Theta(m-jk)P(k\bar{e}_j) \max_{0\le i\le v-1} c_3(i)(L+1+1)^{v-1} \\
&\le C_8(v)\Theta(m-jk)P(k\bar{e}_j)(L+1)^{v-1}.
\end{aligned}$$

Hence, by (3.12),

$$\mathbb{E}_m(\omega(\bar{s})^v) \le C_8 \sum_{jk\le m}^{*} \frac{\Theta(m-jk)}{\Theta(m)} P(k\bar{e}_j)(L+1)^{v-1} \le C_9(L+1)^v.$$

Setting $c_3(\alpha) = C_9$ we complete the proof of Lemma 3.6.

$\square$

Lemma 3.7. *Let complex-valued additive function $H(\bar{s})$ and $\delta > 0$ be such that either $|h_j(k)| > \delta B$ or $h_j(k) = 0$ is true for each of the products $jk \le n$. Then for each $\alpha \ge 1$ there is a constant c_4, depending on α and δ, so that the inequality*

$$\mathbb{E}_n\big(|H(\bar{s}) - A|^\alpha\big) \le c_4 \sum_{jk \le n} |h_j(k)|^\alpha P(k\bar{e}_j) \frac{\Theta(n-jk)}{\Theta(n)} \tag{3.16}$$

holds for all $n \ge 1$.

Proof. Let J be the set of indexes $j \le n$ such that $h_j(1) \ne 0$ and $L, \omega(\bar{s})$ be as in the statement of Lemma 3.6.

By Hölder's inequality we see that

$$H(\bar{s}) \le \omega(\bar{s})^{\alpha-1} \sum_{k\bar{e}_j || \bar{s}} |h_j(k)|^\alpha.$$

Hence

$$\sum_{\ell(\bar{s})=n} P(\bar{s})|H(\bar{s})|^\alpha \le \sum_{jk \le n} |h_j(k)|^\alpha \sum_{\substack{\ell(\bar{s})=n \\ s_j=k}} P(\bar{s})\omega(\bar{s})^{\alpha-1}. \tag{3.17}$$

Since

$$L \le \sum_{\substack{jk \le n \\ |h_j(k)| > \delta B}} P(k\bar{e}_j) \frac{\Theta(n-jk)}{\Theta(n)} \le \delta^{-2}$$

from our remark (3.14), the inner sum on the right hand side of (3.17) is by Lemma 3.6 no more than

$$C_{10}\Theta(n-jk)P(k\bar{e}_j)(L+1)^\alpha \le C_{11}\Theta(n-jk)P(k\bar{e}_j) \tag{3.18}$$

where C_{11} is a constant depending on α, δ.

The inequalities (3.17) and (3.18) show that

$$\Theta(n)^{-1} \sum_{\ell(\bar{s})=n} P(\bar{s})|H(\bar{s})|^\alpha \le C_{11} \sum_{jk \le n} |h_j(k)|^\alpha P(k\bar{e}_j) \frac{\Theta(n-jk)}{\Theta(n)}.$$

Moreover, Hölder's inequality and our remark (3.14) show that

$$|A|^\alpha \le \Bigg(\sum_{\substack{jk\le n \\ |h_j(k)|>\delta B}} P(k\bar{e}_j)\frac{\Theta(n-jk)}{\Theta(n)} \Bigg)^{\alpha-1} \sum_{jk\le n} |h_j(k)|^\alpha P(k\bar{e}_j)\frac{\Theta(n-jk)}{\Theta(n)}$$

$$\le C_{12} \sum_{jk\le n} |h_j(k)|^\alpha P(k\bar{e}_j)\frac{\Theta(n-jk)}{\Theta(n)}.$$

Collecting the last two inequalities and using Hölder's inequality once again we finish the proof of Lemma 3.7.

□

3.5 Proof of Theorem 3.1

We define additive functions $H_i(\bar{s})$, $i = 1, 2$, by

$$h_{1,j}(k) := \begin{cases} h_j(k) & \text{if } |h_j(k)| \le B, \\ 0 & \text{otherwise} \end{cases}, \quad h_{2,j}(k) := \begin{cases} h_j(k) & \text{if } |h_j(k)| > B, \\ 0 & \text{otherwise} \end{cases},$$

$$H_i(\bar{s}) := \sum_{j \le n} h_{i,j}(s_j)$$

and

$$A_i := \sum_{jk \le n} h_{i,j}(k) P(k\bar{e}_j) \frac{\Theta(n - jk)}{\Theta(n)}, \quad i = 1, 2.$$

Since

$$|H(\bar{s}) - A|^\alpha \le 2^\alpha \sum_{i=1}^{2} |H_i(\bar{s}) - A_i|^\alpha,$$

the upper of the desired inequalities of Theorem 3.1 follows from Lemma 3.5 applied to the function $H_1(\bar{s})$ with $\delta = 1$, together with Lemma 3.7 applied to the function $H_2(\bar{s})$ with $\delta = 1$.

The lower of the desired inequalities of Theorem 3.1 follows from the fact that the value of the expression

$$\mathbb{E}_n \left(|H(\bar{s}) - A|^\alpha \right)^{1/\alpha}$$

is no larger than that of the similar expression with α replaced by 2.

□

Conclusions

- The results, obtained in probabilistic number theory, transfers well into probabilistic combinatorics promising further results to be done.
- The thesis shows once more that probabilistic theories developed on additive arithmetic semigroups and assemblies have many parallel lines.
- Moment estimates obtained for additive functions have shapes close to that appearing for independent random variables; this supports a thought that, apart of the constants, the inequalities are sharp.
- It is natural to expect that lower estimates for the moments of additive functions can be obtained.

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