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RUIN PROBABILITY FOR INHOMOGENEOUS RENEWAL RISK MODEL

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VILNIAUS UNIVERSITETAS

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BANKROTO TIKIMYBĖ NEHOMOGENINIAM RIZIKOS ATSTATYMO MODELIUI

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Emilija Bernackaitė
Vilnius
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Contents

Notations

$\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N} = \{1, 2, \ldots\}$.

$\mathbb{R}$ denotes the set of real numbers.

$\mathbb{R}^+$ denotes the positive real half-line $[0, \infty)$.

$[x]$ and $\lfloor x \rfloor$ denote the largest integer less than or equal to x.

$R(t)$ denotes the surplus process of an insurance company.

$\Theta(t)$ denotes the renewal process.

Z denotes the size of a claim.

θ denotes the inter-arrival time, i.e. the time between two claims.

$\psi(x)$ denotes the ultimate ruin probability.

$\psi(x, t)$ denotes the finite-time ruin probability.

$\mathbb{P}$ denotes the probability.

$\mathbb{E}X$ denotes the expectation of a random variable X.

$\mathbb{D}X$ denotes the variation of a random variable X.

F_Z denotes the distribution function of the random variable Z.

$\overline{F_Z}$ denotes the survival function of the random variable Z or the tail of distribution function F_Z.

F_Z^{*2} denotes the convolution of the function F_Z with itself.

F_e denotes the equilibrium distribution function of the random variable generated by distribution function F_Z.

$\mathcal{S}_*$ denotes the class of strongly subexponential functions.

$\mathcal{C}$ denotes the class of functions, which have a consistent variation.

$\mathcal{S}$ denotes the class of subexponential functions.

$\mathcal{L}$ denotes the class of long-tailed functions.

$\mathbb{J}_F^+$ denotes the upper Matuszevska index.

$\prod$ denotes the product.

$\bigcap$ denotes the intersection.

$||$ denotes the modulus.

sup denotes the supremum value.

inf denotes the infimum value.

$\limsup$ denotes the limit superior.

$\liminf$ denotes the limit inferior.

ξ^+ denotes the positive part of a random variable ξ.

$\xrightarrow{\mathbb{P}}$ denotes convergence in probability.

$\mathbb{1}_{x\in A}$ denotes the indicator function. The function is equal to 1, when $x \in A$ and is equal to 0, when $x \notin A$.
$d.f.$ denotes the abbreviation for distribution function.
$r.v.s$ denotes the abbreviation for random variables.
$r.v.$ denotes the abbreviation for random variable.
$i.i.d.$ denotes the abbreviation for independent identically distributed.
$UEND$ denotes the abbreviation for upper extended negatively dependent.
$LEND$ denotes the abbreviation for lower extended negatively dependent.
$f(x) \lesssim g(x)$ denotes that $\limsup\limits_{x\to\infty} \frac{f(x)}{g(x)} \leqslant 1$.
$f(x) \sim g(x)$ denotes that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$.
$f(x) = o((g(x))$ denotes that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

Introduction

Research problem and actuality

Actuarial science and applied probability ruin theory use mathematical models to describe an insurer's vulnerability to insolvency/ruin. In such models key quantities of interest are the probability of ruin, distribution of surplus immediately prior to ruin and deficit at time of ruin. In this thesis we concentrate on the characteristics and asymptotic behaviour of ruin probability.

The theoretical foundation of ruin theory, known as the Cramér–Lundberg model was introduced in 1903 by the Swedish actuary Filip Lundberg (see [Lundberg, 1903]). Lundberg's work was republished in the 1930s by Harald Cramér (see [Cramér, 1930]).

The model describes an insurance company who experiences two opposing cash flows: incoming cash premiums and outgoing claims. Premiums arrive at a constant rate $c > 0$ from customers and claims $Z_1, Z_2, \ldots$ arrive according to a Poisson process with intensity ν and are independent and identically distributed (i.i.d.) non-negative random variables (r.v.s) with distribution F and mean β (they form a compound Poisson process). So an insurer's surplus process at time t is described in the following way:

$$R(t) = x + ct - \sum_{i=1}^{\Theta(t)} Z_i,\ t \geqslant 0,$$

where:

- $x \geqslant 0$ is the initial reserve;
- claim sizes $\{Z_1, Z_2, ...\}$ form a sequence of i.i.d. non-negative r.v.s;
- $c > 0$ represents the constant premium rate;
- $\Theta(t)$ is the number of claims in the interval $[0, t]$, indeed it is a renewal counting process generated by r.v.s (inter-arrival times) $\{\theta_1, \theta_2, \ldots\}$, which are distributed according to the Exponential law with mean $1/\nu$;
- sequences $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are mutually independent.

The central object of the model is to investigate the probability that the insurer's surplus level eventually or at some particular time falls below zero (making the firm bankrupt). This quantity may be defined as a probability of ultimate ruin or finite-time ruin probability.

E. Sparre Andersen (see [Sparre, 1957]) extended the classical model in 1957 by allowing claim inter-arrival times (θ) to have arbitrary distribution functions. Further, by allowing inter-arrival times to have non-identical distributions or dependent in some way, this model became inhomogeneous. Insurance companies usually encounter different types of claims, that is why, nowadays, risk model with inhomogeneous claims becomes more actual. Some authors like [Albrecher and Teugels, 2006], [Li et al., 2010] investigated ruin probability in the renewal risk model with dependent, but identically distributed claims and inter-arrival times.

In this thesis we concentrate on not necessarily identically distributed claims and inter-arrival times. We derive estimates and asymptotic expressions of ultimate ruin probability and finite-time ruin probability for an inhomogeneous renewal risk model.

Aims and tasks

The main purpose of the thesis is to find realistic conditions so that we could apply similar estimations of ruin probability for an inhomogeneous renewal risk model like for the homogeneous one. To be more precise we aim to:

- Establish the requirements under which Lunberg-type inequality would be valid for an inhomogeneous renewal risk model.
- Investigate the asymptotic behaviour of the exponential moment of the renewal counting process in an inhomogeneous renewal risk model.
- Find an asymptotic formula for the finite-time ruin probability in an inhomogeneous renewal risk model.

Novelty

We prove that well-known estimates and asymptotic expressions for the homogeneous renewal risk model can be extended to a much more general case of inhomogeneous claims and inter-arrival times. The assumptions of the theorems are new and they help to apply the results in more realistic cases of insurance. They extend, generalize and supplement the results on finding ruin probability obtained by other authors (e.g. [Andrulytė et al., 2015], [Kočetova et al., 2009], [Tang, 2004]).

Defended propositions

- Established conditions for the Lundberg-type inequality in an inhomogeneous renewal risk model.
- Established assumptions for the evaluation of the exponential moment tail of renewal counting process in an inhomogeneous renewal risk model.
- Derived asymptotic formula of finite-time ruin probability for an inhomogeneous renewal risk model.

Structure of the thesis

Chapter 1 contains the outlines of classical risk theory. In this chapter we overview the homogeneous renewal risk model, present all the necessary definitions and the main critical characteristics.

In Chapter 2 we describe an inhomogeneous renewal risk model and present the differences from the homogeneous renewal risk model. In this chapter there are also provided the formulations of the main theorems for inhomogeneous renewal risk model. In Theorem 2.1 we present the conditions for Lundberg-type inequality. Theorems 2.2, 2.3 and 2.4 consider an inhomogeneous renewal counting process generated by inter-arrival times, which may dependent in some way. Finally, in Theorem 2.5 we provide a formula to estimate the finite-time ruin probability.

In Section 3.1 of Chapter 3 we formulate and prove an auxiliary lemma about large values of a sum of random variables asymptotically drifted in the negative direction. The proof of Theorem 2.1 we present in Section 3.2.

Chapter 4 consists of four parts. In Sections 4.1, 4.2, 4.3 we provide the proofs of Theorems 2.2, 2.3 and 2.4. In the last Section 4.4 we derive and proove the corollaries, which reassure the existence of our selected inhomogeneous renewal processes.

Finally, in Chapter 5, Theorem 2.5 is prooved. In Section 5.1 we give all the auxiliary results which we need. In Section 5.2 we obtain lower estimate of the finite-time ruin probability, while in the next Section 5.3 we prove the upper estimate for the same probability. Lastly, in Section 5.4 we derive additional Corollary 5.1.

Chapter 1

Outlines of Clasical Risk Theory

1.1 Homogeneous Renewal Risk Model

The theoretical foundation of ruin theory, known as the Cramér–Lundberg model (or classical compound-Poisson risk model, classical risk process or Poisson risk process) was introduced in 1903 by the Swedish actuary Filip Lundberg (see [Lundberg, 1903]). Lundberg's work was republished in the 1930s by Harald Cramér (see [Cramér, 1930]).

The model describes an insurance company which experiences two opposing cash flows: incoming cash premiums and outgoing claims. Premiums from customers arrive at a constant rate c > 0 and claims arrive according to a Poisson process $\Theta(t)$ with intensity ν and are i.i.d. non-negative r.v.s with distribution function (d.f.) F and mean β (they form a compound Poisson process). So an insurer's surplus process at time t is described in the following way:

$$R(t) = x + ct - \sum_{i=1}^{\Theta(t)} Z_i, t \geqslant 0, \tag{1.1}$$

where:

- $x = R(0)$ is the initial surplus;
- $c > 0$ represents the constant premium rate;
- the sequence $\{Z_1, Z_2, ...\}$ represents claim sizes, wich are i.i.d. non-negative r.v.s;
- $\Theta(t)$ is a renewal counting process generated by random variable (r.v.) θ, which is distributed according to the Exponential law with mean $1/\nu$.

Definition 1.1. *Let $\theta_1, \theta_2, \ldots$ be a sequence of i.i.d. nonnegative r.v.s. Then the process*

$$\Theta(t) = \sup\{n \geqslant 1 : \theta_1 + \theta_2 + \ldots + \theta_n \leqslant t\} \tag{1.2}$$

is called a renewal process (renewal counting process).

In Figure 1.1 we can see the behaviour of the surplus process $R(t)$.

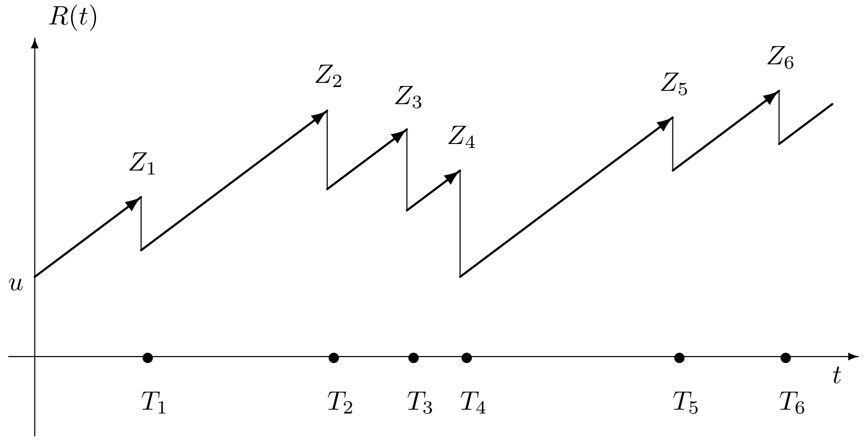

Figure 1.1. Behaviour of the surplus process $R(t)$

E. Sparre Andersen extended the classical model in 1957 (see [Sparre, 1957]) by allowing claim inter-arrival times to have arbitrary distribution functions. Nowadays the Sparre Andersen model is one of the most popular and used models in non-life insurance mathematics.

The models described above are examples of a homogeneous renewal risk model.

Definition 1.2. *We say that the insurer's surplus $R(t)$ varies according to the homogeneous renewal risk model if* (1.1) *holds together with the following conditions:*

- $x \geqslant 0$ *is the initial reserve;*
- *claim sizes $\{Z_1, Z_2, ...\}$ form a sequence of i.i.d. non-negative r.v.s;*
- $c > 0$ *represents the constant premium rate;*
- $\Theta(t) = \sum\limits_{n=1}^{\infty} \mathbb{1}_{\{T_n \leqslant t\}} = \sup\{n \geqslant 0 : T_n \leqslant t\}$ *is the number of claims in the interval* $[0, t]$*, where $T_0 = 0$, $T_n = \theta_1 + \theta_2 + ... + \theta_n$, $n \geqslant 1$, and the inter-arrival times $\{\theta_1, \theta_2, \ldots\}$ are i.i.d. non-negative and non-degenerated at zero r.v.s;*
- *sequences $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots ...\}$ are mutually independent.*

The time of ruin and the ruin probability are the main critical characteristics of any risk model. Let $\mathcal{B}$ denote the event of ruin. We suppose that

$$\mathcal{B} = \bigcup_{t>0} \{\omega : R(\omega, t) < 0\} = \bigcup_{t>0} \Big\{\omega : x + ct - \sum_{i=1}^{\Theta(t)} Z_i < 0\Big\}.$$

That is, we suppose that ruin occurs if at some time $t > 0$ the surplus of the insurance company becomes negative or, in other words, the insurer becomes unable to pay all the claims. The first time τ when the surplus drops to a level less than zero is called the time of ruin, i.e. τ is the extended r.v. for which

$$\tau = \tau(\omega) = \begin{cases} \inf\{t > 0 : R(\omega, t) < 0\}, & \text{if } \omega \in \mathcal{B}, \\ \infty, & \text{if } \omega \notin \mathcal{B}. \end{cases}$$

The ultimate ruin probability ψ is defined by the equality

$$\psi(x) = \mathbb{P}(\mathcal{B}) = \mathbb{P}(\tau < \infty).$$

The probability of ruin within time s is a bivariate function

$$\psi(x, s) = \mathbb{P}(\tau \leqslant s). \tag{1.3}$$

Usually we suppose that the main argument of the ruin probability is the initial reserve x, though actually the ruin probability together with time of ruin depends on all components of the renewal risk model.

All trajectories of the process $R(t)$ are non-decreasing functions between times T_n and T_{n+1} for all $n = 0, 1, 2, \ldots$. Therefore, random variables $R(\theta_1 + \theta_2 + ... + \theta_n)$, $n \geqslant 1$, are the local minimums of the trajectories. Consequently, we can express the ultimate ruin probability in the following manner (for details see [Embrechts et al., 1997a] or [Mikosch, 2009])

$$\begin{aligned}
\psi(x) &= \mathbb{P}\left(\inf_{n\in\mathbb{N}} R(\theta_1 + \theta_2 + ... + \theta_n) < 0\right) \\
&= \mathbb{P}\left(\inf_{n\in\mathbb{N}} \left\{ x + c(\theta_1 + \theta_2 + ... + \theta_n) - \sum_{i=1}^{\Theta(\theta_1+...+\theta_n)} Z_i \right\} < 0\right) \\
&= \mathbb{P}\left(\inf_{n\in\mathbb{N}} \left\{ x - \sum_{i=1}^{n} (Z_i - c\theta_i) \right\} < 0\right) \\
&= \mathbb{P}\left(\sup_{n\in\mathbb{N}} \left\{ \sum_{i=1}^{n} (Z_i - c\theta_i) \right\} > x\right)
\end{aligned}$$

and the finite-time ruin probability by equality

$$\psi(x, s) := \mathbb{P}\left(\inf_{0<t\leqslant s} R(t) < 0\right) = \mathbb{P}\left(\max_{1\leqslant k\leqslant\Theta(s)} \sum_{i=1}^{k} (Z_i - c\,\theta_i) > x\right). \tag{1.4}$$

1.2 Lundberg-type Inequality for Homogeneous Renewal Risk Model

Below we give a well known exponential bound for $\psi(x)$ in a homogeneous renewal risk model. (see, for instance, Chapters "Lundberg Inequality for Ruin Probability", "Collective Risk Theory", "Adjustment Coefficient" or "Cramer-Lundberg Asymptotics" in [Teugels and Sundt, 2004]).

Theorem 1.1. *Let the net profit condition $\mathbb{E}Z_1 - c\mathbb{E}\theta_1 < 0$ hold and $\mathbb{E}\mathrm{e}^{hZ_1} < \infty$ for some $h > 0$ in the homogeneous renewal risk model. Then, there is a positive H such that*

$$\psi(x) \leqslant \mathrm{e}^{-Hx}. \tag{1.5}$$

for all $x \geqslant 0$. If the equality $\mathbb{E}\mathrm{e}^{R(Z_1 - c\theta_1)} = 1$ holds for a positive R, then we can choose $H = R$ in (1.5).

There exist a lot of different proofs of this theorem. The main ways to prove the above inequality are described in Chapter "Lundberg Inequality for Ruin Probability" of encyclopedia by [Teugels and Sundt, 2004]. Details of some existing proofs were given, for instance, by [Asmussen and Albrecher, 2010], [Embrechts et al., 1997a], [Embrechts and Veraverbeke, 1982a], [Gerber, 1973], [Mikosch, 2009]. We note only that bound (1.5) can be proved using exponential tail bound of [Sgibnev, 1997] and inequality $\psi(0) < 1$.

1.3 Properties of Renewal Process

In the studies of finite-time ruin probability many authors considered renewal processes, which satisfy the following properties:

$$
\begin{aligned}
&(\mathcal{A}1): \quad \frac{\Theta(t)}{\mathbb{E}\Theta(t)} \underset{t\to\infty}{\overset{\mathbb{P}}{\to}} 1,\\
&(\mathcal{A}2): \quad \sum_{k>(1+\delta)\,\mathbb{E}\Theta(t)} \mathbb{P}(\Theta(t) \geqslant k)(1+\varepsilon)^k \underset{t\to\infty}{\to} 0\\
&\text{for any } \ \delta > 0 \text{ and some small } \ \varepsilon > 0.
\end{aligned}
$$

It is not difficult to find examples of counting processes satisfying condition ($\mathcal{A}1$). For instance, this condition holds for every Poisson process with unboundedly increasing accumulated intensity function and for every renewal process generated by a r.v. θ with finite expectation $\mathbb{E}\theta$. Meanwhile, assumption ($\mathcal{A}2$) is quite complex to verify. [Klüpellberg and Mikosch, 1997] (see Lemma 2.1) and [Yang et al., 2013] (see Lemma 1) proved that this assumption is satisfied for a Poisson process with unboundedly increasing function $\mathbb{E}\Theta(t)$.

[Tang et al., 2001] instead of assumptions ($\mathcal{A}1$) and ($\mathcal{A}2$), supposed that the counting process $\Theta(t)$ satisfies the following assumption:

$$
\begin{aligned}
&(\mathcal{A}3): \quad \sum_{k>(1+\delta)\,\mathbb{E}\Theta(t)} k^{\beta}\, \mathbb{P}(\Theta(t) = k) = O(\mathbb{E}\Theta(t))\\
&\text{for any } \ \delta > 0 \text{ and some small } \ \varepsilon > 0,
\end{aligned}
$$

where $\beta > 1$ is a certain number related to the regularity of d.f. $\mathbb{P}(X \leqslant x)$.

If $\mathbb{E}\Theta(t) \to \infty$ as $t \to \infty$, then assumption ($\mathcal{A}3$) follows from ($\mathcal{A}2$). The results of [Tang et al., 2001] generalize the ones of [Klüpellberg and Mikosch, 1997] since [Tang et al., 2001] showed that assumption ($\mathcal{A}3$) implies assumption ($\mathcal{A}1$) (see Lemma 3.3) and showed that each renewal process satisfies condition ($\mathcal{A}3$) in the case where it is generated by a r.v. having a finite expectation (see Lemma 3.5).

[Leipus and Šiaulys, 2009] considered the asymptotic behavior of finite-time ruin probability in the renewal risk model

$$
x + ct - \sum_{i=1}^{\Theta(t)} Z_i\,, \ t \geqslant 0.
$$

Here $x \geqslant 0$, $c > 0$, $Z_1, Z_2, \ldots$ are i.i.d random variables with strongly subexponential d.f., and $\Theta(t)$ is a renewal process, defined in (1.2), where $\theta_1, \theta_2, \ldots$ are independent copies of a nonnegative r.v. θ nondegenerate at zero. The authors of this paper supposed that the renewal process $\Theta(t)$ also satisfies condition ($\mathcal{A}2$) because assumption ($\mathcal{A}3$) is not sufficient to obtain the desired asymptotic formulas in the case of strongly subexponential claims $Z_1, Z_2, \ldots$. Continuing their studies on the asymptotic behavior of ruin probability, [Kočetova et al., 2009] obtained that each renewal process fulfils condition ($\mathcal{A}2$) in the case where the process generator θ has a finite positive expectation. Namely, the following assertion was proved.

Theorem 1.2. *Let the renewal process $\Theta(t)$ be defined in (1.2) with a sequence $\theta, \theta_1, \theta_2, \ldots$ of independent identically distributed r.v.s. If $\mathbb{E}\theta = 1/\lambda \in (0, \infty)$, then for every real number $a > \lambda$, there exists $b > 1$ such that*

$$\lim_{t\to\infty} \sum_{k>at} \mathbb{P}(\Theta(t) \geqslant k)\, b^k = 0. \tag{1.6}$$

[Chen and Yuen, 2012] and [Lu, 2011] used this assertion considering the large deviation problem, whereas [Chen et al., 2010], [Bi and Zhang, 2013], and [Wang et al., 2012] obtained analogous assertions when the generating random variables $\theta_1, \theta_2, \ldots$ are identically distributed but dependent in some sense.

1.4 Asymptotic Properties of Finite-time Ruin Probability in a Homogeneous Renewal Risk Model

The renewal risk model has been extensively investigated in the literature since it was introduced by Sparre Andersen half a century ago. In this risk model, the claim sizes $Z_1, Z_2, ...$ form a sequence of i.i.d. nonnegative r.v.s with a common d.f. $F_Z(u) = P(Z_1 \leqslant u)$ and a finite mean $\beta = \mathbb{E}Z_1$, while the inter arrival times $\theta_1, \theta_2, \ldots$ are i.i.d. nonnegative r.v.s with common finite positive mean $\mathbb{E}\theta_1 = 1/\lambda$. In addition, it is assumed that $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are mutually independent. In this model, the number of accidents in the interval [0, t] is given by a renewal counting process

$$\Theta(t) = \sup\{n \geqslant 1 : \theta_1 + \theta_2 + \ldots + \theta_n \leqslant t\}$$

which has a mean function $\lambda(t) = \mathbb{E}\Theta(t)$ with $\lambda(t) \sim \lambda t$ as $t \to \infty$. The surplus process of the insurance company is then expressed as

$$R(t) = x + ct - \sum_{i=1}^{\Theta(t)} Z_i\,,\ t \geqslant 0,$$

where $x \geqslant 0$ is the initial risk reserve and $c > 0$ represents the constant premium rate.

As mentioned before finite-time ruin probability is a bivariate function, defined by equation (1.4).

Under the assumptions that $\mu = c\mathbb{E}\theta_1 - \mathbb{E}Z_1 = c/\lambda - \beta > 0$ and the equilibrium d.f. of F_Z

$$F_e(x) = \frac{1}{\beta} \int_0^x \overline{F_Z}(u)\,\mathrm{d}u$$

is subexponential, [Veraverbeke, 1977] and [Embrechts and Veraverbeke, 1982b] established a celebrated asymptotic relation for the ultimate ruin probability:

$$\psi(x,\infty) \underset{x\to\infty}{\sim} \frac{1}{\mu}\int\limits_x^\infty \overline{F_Z}(u)\,\mathrm{d}u, . \tag{1.7}$$

Definition 1.3. *We recall that a d.f. F supported on $[0,\infty)$ is subexponential (F belongs to the class $\mathcal{S}$) if*

$$\overline{F^{*2}}(x) \underset{x\to\infty}{\sim} 2\overline{F}(x),$$

where F^{*2} denotes the convolution of F with itself.

[Tang, 2004] showed that a formula similar to (1.7) holds for the finite-time ruin probability as well. More exactly, the following statement was proved in that paper.

Theorem 1.3. *If d.f. F_Z has a consistent variation and $\mathbb{E}\,\theta_1^p < \infty$ for some $p > 1 + \mathbb{J}^+_{F_Z}$, where*

$$\mathbb{J}^+_{F_Z} = -\lim_{y\to\infty}\frac{1}{\log y}\liminf_{x\to\infty}\frac{\overline{F_Z}(xy)}{\overline{F_Z}(x)},$$

then

$$\psi(x,t) \underset{x\to\infty}{\sim} \frac{1}{\mu}\int\limits_x^{x+\mu\lambda(t)} \overline{F_Z}(u)\,\mathrm{d}u, \tag{1.8}$$

uniformly for all t such that $t \in \Lambda = \{t : \lambda(t) > 0\}$.

Definition 1.4. *We say that a d.f. F concentrated on $[0,\infty)$ (or on $\mathbb{R}$) has a consistent variation (F belongs to the class $\mathcal{C}$) if*

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

If d.f. $F \in \mathcal{C}$ has a finite mean m, then the equilibrium d.f. F_e is subexponential (see, for instance, Proposition 1.4.4 in [Embrechts et al., 1997b]). In addition, the upper Matuszevska index $\mathbb{J}^+_F$ is finite for each $F \in \mathcal{C}$ (see, for instance, Section 2.1 in [Bingham et al., 1987]).

In [Leipus and Šiaulys, 2009] and [Kočetova et al., 2009], it was proved that the asymptotic formula (1.8) holds uniformly for $t \in [a(x),\infty)$ with an arbitrary unboundedly increasing function $a(x)$ if d.f. $F_Z \in \mathcal{S}_*$.

Definition 1.5. *A d.f. F belongs to class $\mathcal{S}_*$ (F is strongly subexponential according to the definition in [Korshunov, 2002]) if*

$$\int\limits_0^\infty \overline{F}(u)\,\mathrm{d}u < \infty \quad \text{and} \quad \lim_{x\to\infty}\frac{\overline{F_v^{*2}}(x)}{\overline{F_v}(x)} = 2$$

uniformly in $v \in [1,\infty)$, where

$$\overline{F_v}(x) = \begin{cases} \min\left\{1, \int\limits_x^{x+v} \overline{F}(u)\,\mathrm{d}u\right\}, & \text{if } x \geqslant 0,\\ 1, & \text{if } x < 0. \end{cases}$$

It follows from Lemma 4 of [Korshunov, 2002] that each d.f. $F \in \mathcal{C}$ with finite mean value is strongly subexponential.

[Wang et al., 2012] (see also [Yang et al., 2011] and [Wang et al., 2013]) generalized the above results. It was showed that the asymptotic formula (1.8) preserves its form in the case when the inter occurrence times $\theta_1, \theta_2, \ldots$ obey to certain dependence structures. In the latter publications already an inhomogeneous renewal risk model was considered. It will be described in the next chapter.

Chapter 2

Inhomogeneous Renewal Risk Model

2.1 Differences From Homogeneous Renewal Risk Model

In this thesis, we assume that inter-arrival times and claim sizes are non-negative r.v.s which are not necessarily identically distributed. We call such model the inhomogeneous model and we present below the exact definition of such renewal risk model.

Definition 2.1. *We say that the insurer's surplus $R(t)$ varies according to an inhomogeneous risk renewal model if*

$$R(t) = x + ct - \sum_{i=1}^{\Theta(t)} Z_i \tag{2.1}$$

for all $t \geqslant 0$. Here:

- *$x \geqslant 0$ is the initial reserve;*
- *claim sizes $\{Z_1, Z_2, ...\}$ form a sequence of independent (not necessarily identically distributed) non-negative r.v.s;*
- *$c > 0$ represents the constant premium rate;*
- *$\Theta(t) = \sum\limits_{n=1}^{\infty} \mathbb{1}_{\{T_n \leqslant t\}} = \sup\{n \geqslant 0 : T_n \leqslant t\}$ is the number of claims in the interval $[0, t]$, where $T_0 = 0$, $T_n = \theta_1 + \theta_2 + ... + \theta_n$, $n \geqslant 1$, and the inter-arrival times $\{\theta_1, \theta_2, \ldots\}$ are independent (not necessarily identically distributed), non-negative and non-degenerated at zero r.v.s. $\Theta(t)$ is called an inhomogeneous renewal process;*
- *sequences $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots ...\}$ are mutually independent.*

It is evident that the inhomogeneous renewal risk model reflects better the real insurance activities in comparison with the classical risk model or with the homogeneous renewal risk model.

The inhomogeneous risk renewal model differs from the homogeneous one because independence and/or homogeneous distribution of sequences of random variables $\{Z_1, Z_2, ...\}$ and/or

$\{\theta_1, \theta_2, ...\}$ are no longer required. The changes depend on how the inhomogeneity in a particular model is understood. In Definition 2.1 we have chosen one of two possible directions used in numerous articles that deal with inhomogeneous renewal risk models. This is due to the fact that an inhomogeneity can be considered as the possibility to have either differently distributed or dependent r.v.s in sequences.

The possibility to have differently distributed random variables was considered, e.g. in the articles [Bieliauskienė and Šiaulys, 2010], [Blaževičius et al., 2010], [Lefèvre and Picard, 2006], and [Raducan et al., 2015]. In the first three works the discrete time inhomogeneous risk model was considered. In such model, the inter-arrival times are fixed and claims $\{Z_1, Z_2, ...\}$ are independent, not necessarily identically distributed, integer valued r.v.s. In [Raducan et al., 2015], the authors considered the model where inter-arrival times are identically distributed and have the special distribution, while claims are differently distributed with distributions belonging to the special class. In [Bernackaitė and Šiaulys, 2015], [Bernackaitė and Šiaulys, 2017] we deal with an inhomogeneous renewal risk model, where r.v.s $\{\theta_1, \theta_2, ...\}$ are not necessarily identically distributed, but the claim sizes $\{Z_1, Z_2, ...\}$ have a common distribution function.

There is another approach to the inhomogeneous renewal risk models, which implies the possibility to have dependence in sequences and mainly found in works by Chinese researchers. In this kind of models, sequences $\{Z_1, Z_2, ...\}$ and $\{\theta_1, \theta_2, ...\}$ consist of identically distributed r.v.s, but there may be some kind of dependence between them. Results for such models can be found, for instance, in [Chen and Ng, 2007] and [Wang et al., 2013]. Another interpretation of dependence is also possible, where r.v.s in both sequences $\{Z_1, Z_2, ...\}$ and $\{\theta_1, \theta_2, ...\}$ still remain independent. Instead of that, mutual independence between these two sequences is no longer required. The idea of this kind of dependence belongs to [Albrecher and Teugels, 2006], and this encouraged Li, [Li et al., 2010] to study renewal risk models having this dependence structure.

2.2 Main Theorems of the Thesis

In this section we collected all the main assertions of the thesis:

First theorem is formulated to represent Lundberg-type inequality for inhomogeneous renewal risk model.

Theorem 2.1. *Let the claim sizes $\{Z_1, Z_2, ...\}$ and the inter-arrival times $\{\theta_1, \theta_2, ...\}$ form an inhomogeneous renewal risk model described in Definition 2.1. Further, let the following three conditions be satisfied:*

$$(\mathcal{B}1)\ \sup_{i\in\mathbb{N}} \mathbb{E}\mathrm{e}^{\gamma Z_i} < \infty \quad \textit{with some} \quad \gamma > 0,$$

$$(\mathcal{B}2)\ \lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}(\theta_i \mathbb{1}_{\{\theta_i > u\}}) = 0,$$

$$(\mathcal{B}3)\ \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\mathbb{E}Z_i - c\mathbb{E}\theta_i) < 0.$$

Then, there are constants $c_1 > 0$ and $c_2 \geqslant 0$ such that $\psi(x) \leqslant \mathrm{e}^{-c_1 x}$ for all $x \geqslant c_2$.

In the next three theorems we present generalizations of Theorem 1.2. In Theorems 2.2 and 2.4 we consider an inhomogeneous renewal process generated by LEND r.v.s. In Theorem 2.3, r.v.s can be dependent in any way.

Definition 2.2. *R.v.s $\xi_1, \xi_2, \dots$ are said to be upper extended negatively dependent (UEND) if there exists a dominating constant α_ξ such that*

$$\mathbb{P}\left(\bigcap_{k=1}^{n}\{\xi_k > x_k\}\right) \leqslant \alpha_\xi \prod_{k=1}^{n} \mathbb{P}(\xi_k > x_k)$$

for all $n \in \mathbb{N}$ and all $x_1, x_2, \dots, x_n$.

Definition 2.3. *R.v.s $\xi_1, \xi_2, \dots$ are said to be lower extended negatively dependent (LEND) if there exists a dominating constant β_ξ such that*

$$\mathbb{P}\left(\bigcap_{k=1}^{n}\{\xi_k \leqslant x_k\}\right) \leqslant \beta_\xi \prod_{k=1}^{n} \mathbb{P}(\xi_k \leqslant x_k)$$

for all $n \in \mathbb{N}$ and all $x_1, x_2, \dots, x_n$.

One can find related concepts of negative dependence and useful properties of negatively dependent r.v.s , for instance, in [Tang, 2006], [Liu, 2009], and [Chen et al., 2010].

So the first assertion describes the asymptotic behavior of the exponential moment tail in the case of uniformly integrable inter-arrival times.

Theorem 2.2. *Let $\theta_1, \theta_2, \dots$ be LEND nonnegative r.v.s. Suppose that these r.v.s are uniformly integrable, that is,*

$$\lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}\left(\theta_i \mathbb{1}_{\{\theta_i \geqslant u\}}\right) = 0, \tag{2.2}$$

and for some $\lambda \in (0, \infty)$,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\,\theta_i \geqslant \frac{1}{\lambda}. \tag{2.3}$$

If $\Theta(t)$ is an inhomogeneous renewal process (see Definition 2.1) generated by r.v.s $\theta_1, \theta_2, \dots$, then for every $a > \lambda$, there exists $b > 1$ such that

$$\lim_{t\to\infty} \sum_{k>at} \mathbb{P}(\Theta(t) \geqslant k)\, b^k = 0. \tag{2.4}$$

Next theorem shows that the uniform integrability of inter-arrival times is not necessary if all these times are bounded from below.

Theorem 2.3. *Let $\theta_1, \theta_2, \dots$ be arbitrarily dependent random variables. Suppose that there exists a positive constant c such that $\theta_n \geqslant c$ for all $n \in \mathbb{N}$. If $\Theta(t)$ is an inhomogeneous renewal process (see Definition 2.1) generated by r.v.s $\theta_1, \theta_2, \dots$, then for every $a > 1/c$, there exists $b > 1$ such that relation (2.4) holds.*

Further theorem shows that there are cases where relation (2.4) holds for an arbitrary positive a.

Theorem 2.4. *Let $\theta_1, \theta_2, \ldots$ be LEND nonnegative r.v.s for which*

$$\lim_{u\to\infty,\, n\to\infty} u\left(\mathbb{E}e^{-\theta_n/u} - 1\right) = -\infty. \tag{2.5}$$

If $\Theta(t)$ is an inhomogeneous renewal process (see Definition 2.1) generated by r.v.s $\theta_1, \theta_2, \ldots$, then for every $a > 0$, there exists $b > 1$ such that relation (2.4) holds.

Finally, we show that the asymptotic formula of finite-time ruin probability (1.8) preserves its form in the case when the inter-arrival times $\theta_1, \theta_2, \ldots$ satisfy some additional requirements. We suppose that inter occurrence times $\theta_1, \theta_2, \ldots$ are independent but not necessarily identically distributed. In fact, we consider an inhomogeneous renewal risk model defined by equation (2.1) under the following three main assumptions:

Assumption $\mathcal{C}_1$**.** The claim sizes $\{Z_1, Z_2, \ldots\}$ are i.i.d. nonnegative r.v.s with common distribution function F_Z and finite positive mean β.

Assumptions $\mathcal{C}_2$. The inter occurrence times $\{\theta_1, \theta_2, \ldots\}$ are independent nonnegative r.v.s such that:

$$(\mathcal{C}_{21}) \qquad \lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}\left(\theta_i \mathbb{1}_{\{\theta_i \geqslant u\}}\right) = 0\ ,$$

$$(\mathcal{C}_{22}) \qquad \sum_{i=1}^{\infty} \frac{\mathbb{D}\theta_i}{i^2} < \infty,$$

$$(\mathcal{C}_{23}) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\theta_i = \frac{1}{\lambda},$$

for some finite positive λ.

Assumption $\mathcal{C}_3$**.** The sequences $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are mutually independent.

In the presented model analogously as in the classical Sparre Andersen model, the finite-time ruin probability $\psi(x,t)$ has expression (1.4), and we denote the mean function of the inhomogeneous renewal counting process $\Theta(t)$ by $\lambda(t) = \mathbb{E}\Theta(t)$, where $t \geqslant 0$. The model assumptions $\mathcal{C}_1$ and $\mathcal{C}_3$ are natural, while assumption $\mathcal{C}_2$ needs some additional comments. Hypothesis $\mathcal{C}_{21}$ requires that r.v.s $\{\theta_1, \theta_2, \ldots\}$ should be uniformly integrable. Such requirement is used sufficiently frequently in the study of non identically distributed r.v.s (see, for instance, [Smith, 1964a] or Chapter II in [Shiryaev, 1996]). We use assumption $\mathcal{C}_{21}$ together with $\mathcal{C}_{23}$ to obtain an asymptotic formula for the exponential moment tail of renewal process (see Theorem 2.2) and to obtain an exponential estimate for maxima of sums of uniformly integrable r.v.s (see Lemma 5.4). These both auxiliary results are crucial to get the upper bound of Proposition 2.7. Requirements $\mathcal{C}_{22}$ and $\mathcal{C}_{23}$ are sufficient in order that the sequence $\{\theta_1, \theta_2, \ldots\}$ satisfies the strong law of large numbers (see Lemma 5.3), which we use to obtain the lower bound for the finite-time ruin probability (see Proposition 2.6). Below we present two sequences of r.v.s $\{\theta_1, \theta_2, \ldots\}$ satisfying assumption $\mathcal{C}_2$.

Example 1. Let $\{\theta_1, \theta_2, \ldots\}$ be independent r.v.s, such that $\theta_1, \theta_4, \theta_7, \ldots$ be distributed according to the Poisson law with parameter $1/\lambda_1$, r.v.s $\theta_2, \theta_5, \theta_8, \ldots$ be distributed according to the Poisson law with parameter $1/\lambda_2$ and $\theta_3, \theta_6, \theta_9, \ldots$ be distributed according to the Poisson law with parameter $1/\lambda_3$. If $\lambda_1 \neq \lambda_2 \neq \lambda_3$ then the renewal counting process $\Theta(t)$ is inhomogeneous but assumption $\mathcal{C}_2$ holds with $\lambda = 3\lambda_1\lambda_2\lambda_3/(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)$.

Example 2. Let $\{\theta_1, \theta_2, \ldots\}$ be independent r.v.s distributed in the following way:

$$\mathbb{P}(\theta_i = 0) = \frac{1}{2},\ \mathbb{P}(\theta_i = 1) = \frac{1}{2} - \frac{1}{i+3},\ \mathbb{P}(\theta_i = \sqrt{i+3}) = \frac{1}{i+3}.$$

The renewal process with such inter occurrence times is also inhomogeneous and assumption $\mathcal{C}_2$ holds again with $\lambda = 2$ because:

$$\sup_{i\in\mathbb{N}} \mathbb{E}\big(\theta_i \mathbb{1}_{\{\theta_i \geqslant u\}}\big) \leqslant \frac{1}{u},$$
$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\theta_i = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \left(\frac{1}{2} - \frac{1}{i+3} + \frac{1}{\sqrt{i+3}}\right) = \frac{1}{2},$$
$$\mathrm{Var}(\theta_i) = \frac{5}{4} - \frac{1}{i+3} - \frac{1}{(i+3)^2} - \frac{i+1}{(i+3)\sqrt{(i+3)}} < \frac{5}{4},\ i \in \mathbb{N}.$$

Theorem 2.5. *If Assumptions $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ hold, $\mu := c/\lambda - \beta > 0$ and d.f. $F_Z \in \mathcal{S}_*$, then*

$$\psi(x,t) \underset{x\to\infty}{\sim} \frac{1}{\mu} \int\limits_{x}^{x+\mu\lambda(t)} \overline{F_Z}(u)\,\mathrm{d}u$$

uniformly for $t \in [T, \infty)$, where $T \in \Lambda := \{t > 0 : \lambda(t) > 0\}$.

It is evident that Theorem 2.5 follows immediately from two propositions below. Before the formulation of these propositions we recall definition of long tailed distribution.

Definition 2.4. *A d.f. F supported on $[0, \infty)$ (or on $\mathbb{R}$) belongs to class $\mathcal{L}$ (is long tailed) if for each positive y*

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1.$$

Note that $\mathcal{S}_* \subset \mathcal{S} \subset \mathcal{L}$ due to Lemma 1 of [Kaas and Tang, 2003] (see Lemma A.5 in Appendix) and Lemma 1.3.5(a) of [Embrechts et al., 1997b] (see Lemma A.3 in Appendix).

Proposition 2.6. *Let Assumptions $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ hold, $\mu > 0$ and $F_Z \in \mathcal{L}$. Then for each $T \in \Lambda$*

$$\inf_{t\in[T,\infty)} \psi(x,t) \underset{x\to\infty}{\gtrsim} \frac{1}{\mu} \int\limits_{x}^{x+\mu\lambda(t)} \overline{F_Z}(u)\,\mathrm{d}u.$$

Proposition 2.7. *Let conditions $\mathcal{C}_1$, $\mathcal{C}_{21}$, $\mathcal{C}_{23}$, $\mathcal{C}_3$ are satisfied, $\mu > 0$ and $F_Z \in \mathcal{S}_*$. Then*

$$\sup_{t\in[T,\infty)} \psi(x,t) \underset{x\to\infty}{\lesssim} \frac{1}{\mu} \int\limits_{x}^{x+\mu\lambda(t)} \overline{F_Z}(u)\,\mathrm{d}u$$

with an arbitrary $T \in \Lambda$.

Chapter 3

Lundberg-type Inequality for Inhomogeneous Renewal Risk Model

3.1 Auxiliary Lemma

In this chapter we proove Theorem 2.1. For this we use an auxiliary lemma formulated below. In Lemma 3.1, the form of conditions for r.v.s $\eta_1, \eta_2, \eta_3, \ldots$ is taken from articles by [Smith, 1964b] and Theorem 2.2. Details of the proof can be also found in Lema 5.4, where a similar assertion was proved but for bounded r.v.s.

Lemma 3.1. *Let $\eta_1, \eta_2, \eta_3, \ldots$ be independent r.v.s, such that*

$$(\mathcal{D}1^*) \ \sup_{i\in\mathbb{N}} \mathbb{E}\mathrm{e}^{\delta\eta_i} < \infty \ \textit{with some } \delta > 0,$$
$$(\mathcal{D}2^*) \ \lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}(|\eta_i| \mathbb{1}_{\{\eta_i \leqslant -u\}}) = 0,$$
$$(\mathcal{D}3^*) \ \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\eta_i < 0.$$

Then, there are some constants $c_3 > 0$ and $c_4 > 0$ such that

$$\mathbb{P}\left(\sup_{k\geqslant 1}\sum_{i=1}^{k}\eta_i > x\right) \leqslant c_3\mathrm{e}^{-c_4 x}$$

for all $x \geqslant 0$.

Proof. First of all, we observe that for all $x \geqslant 0$

$$\mathbb{P}\left(\sup_{k\geqslant 1}\sum_{i=1}^{k}\eta_i > x\right) = \mathbb{P}\left(\bigcup_{k=1}^{\infty}\left\{\sum_{i=1}^{k}\eta_i > x\right\}\right)$$

$$\leqslant \sum_{k=1}^{\infty} \mathbb{P}\Big(\sum_{i=1}^{k} \eta_i > x\Big). \tag{3.1}$$

According to Markov's inequality, for all $x \geqslant 0$, $0 < y \leqslant \delta$ and an arbitrary $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^{k} \eta_i > x\Big) &= \mathbb{P}\Big(\mathrm{e}^{y\sum\limits_{i=1}^{k}\eta_i} > \mathrm{e}^{yx}\Big)\\ &\leqslant \mathrm{e}^{-yx}\prod_{i=1}^{k}\mathbb{E}\mathrm{e}^{y\eta_i}. \end{aligned}\tag{3.2}$$

Moreover, for an arbitrary $i \in \mathbb{N}$ and all $0 < y \leqslant \delta$, $u > 0$, we have

$$\mathbb{E}\mathrm{e}^{y\eta_i} = 1 + y\mathbb{E}\eta_i + \mathbb{E}(\mathrm{e}^{y\eta_i} - 1 - y\eta_i) \tag{3.3}$$

and

$$\begin{aligned}&\mathbb{E}(\mathrm{e}^{y\eta_i} - 1 - y\eta_i)\\ =\ &\mathbb{E}((\mathrm{e}^{y\eta_i} - 1)\mathbb{1}_{\{\eta_i \leqslant -u\}}) - y\mathbb{E}(\eta_i\mathbb{1}_{\{\eta_i \leqslant -u\}})\\ +\ &\mathbb{E}((\mathrm{e}^{y\eta_i} - 1 - y\eta_i)\mathbb{1}_{\{-u<\eta_i \leqslant 0\}}) + \mathbb{E}((\mathrm{e}^{y\eta_i} - 1 - y\eta_i)\mathbb{1}_{\{\eta_i>0\}}).\end{aligned}$$

In order to evaluate the absolute value of the remainder term in (3.3), we need the following inequalities

$$\begin{aligned}&|\mathrm{e}^v - 1| \leqslant |v|,\ \ v \leqslant 0,\\ &|\mathrm{e}^v - v - 1| \leqslant \frac{v^2}{2},\ \ v \leqslant 0,\\ &|\mathrm{e}^v - v - 1| \leqslant \frac{v^2}{2}\mathrm{e}^v,\ \ v \geqslant 0.\end{aligned}$$

Using these inequalities we get

$$\begin{aligned}&|\mathbb{E}(\mathrm{e}^{y\eta_i} - 1 - y\eta_i)|\\ \leqslant\ &2y\mathbb{E}(|\eta_i|\mathbb{1}_{\{\eta_i \leqslant -u\}}) + \frac{y^2}{2}\mathbb{E}(\eta_i^2\mathbb{1}_{\{-u<\eta_i \leqslant 0\}}) + \frac{y^2}{2}\mathbb{E}(\eta_i^2\mathrm{e}^{y\eta_i}\mathbb{1}_{\{\eta_i>0\}})\\ \leqslant\ &2y\sup_{i\in\mathbb{N}}\mathbb{E}(|\eta_i|\mathbb{1}_{\{\eta_i \leqslant -u\}}) + \frac{y^2u^2}{2} + \frac{y^2}{2}\sup_{i\in\mathbb{N}}\mathbb{E}(\eta_i^2\mathrm{e}^{y\eta_i}\mathbb{1}_{\{\eta_i>0\}}),\end{aligned}\tag{3.4}$$

where $i \in \mathbb{N}$, $0 < y \leqslant \delta$ and $u > 0$.

Since

$$\lim_{v\to\infty}\frac{\mathrm{e}^{\delta v/2}}{v^2} = \infty,$$

we have

$$\mathrm{e}^{\delta v/2} \geqslant v^2$$

for all $v \geqslant c_5$, where $c_5 = c_5(\delta) > 0$.

Therefore,

$$\begin{aligned} & \sup_{i\in\mathbb{N}} \mathbb{E}(\eta_i^2 \mathrm{e}^{\delta\eta_i/2} \mathbb{1}_{\{\eta_i>0\}}) \\ \leqslant\; & \sup_{i\in\mathbb{N}} \mathbb{E}(\eta_i^2 \mathrm{e}^{\delta\eta_i/2} \mathbb{1}_{\{0<\eta_i\leqslant c_5\}}) + \sup_{i\in\mathbb{N}} \mathbb{E}(\eta_i^2 \mathrm{e}^{\delta\eta_i/2} \mathbb{1}_{\{\eta_i>c_5\}}) \\ \leqslant\; & (c_5^2+1) \sup_{i\in\mathbb{N}} \mathbb{E}\mathrm{e}^{\delta\eta_i} < \infty. \end{aligned} \tag{3.5}$$

Choosing $u = \frac{1}{\sqrt[4]{y}}$ in (3.4) and using (3.5) we get

$$\begin{aligned} & |\mathbb{E}(\mathrm{e}^{y\eta_i} - 1 - y\eta_i)| \\ \leqslant\; & 2y \sup_{i\in\mathbb{N}} \mathbb{E}(|\eta_i| \mathbb{1}_{\{\eta_i \leqslant -\frac{1}{\sqrt[4]{y}}\}}) + \frac{y^{\frac{3}{2}}}{2} + \frac{y^2}{2} \sup_{i\in\mathbb{N}} \mathbb{E}(\eta_i^2 \mathrm{e}^{y\eta_i} \mathbb{1}_{\{\eta_i>0\}}) \\ \leqslant\; & y\left(2\sup_{i\in\mathbb{N}} \mathbb{E}(|\eta_i| \mathbb{1}_{\{\eta_i \leqslant -\frac{1}{\sqrt[4]{y}}\}}) + \frac{y^{\frac{1}{2}}}{2} + \frac{y}{2}(c_5^2+1) \sup_{i\in\mathbb{N}} \mathbb{E}\mathrm{e}^{\delta\eta_i}\right) \\ =:\; & y\alpha(y), \end{aligned} \tag{3.6}$$

where $i \in \mathbb{N}$, $y \in (0, \delta/2]$, $c_5 = c_5(\delta)$ and

$$\alpha(y) = 2\sup_{i\in\mathbb{N}} \mathbb{E}(|\eta_i| \mathbb{1}_{\{\eta_i \leqslant -\frac{1}{\sqrt[4]{y}}\}}) + \frac{y^{\frac{1}{2}}}{2} + \frac{y}{2}(c_5^2+1) \sup_{i\in\mathbb{N}} \mathbb{E}\mathrm{e}^{\delta\eta_i}.$$

Conditions ($\mathcal{D}1^*$) and ($\mathcal{D}2^*$) imply that $\alpha(y) \downarrow 0$ as $y \to 0$.

For an arbitrary positive v we have

$$\begin{aligned} \sup_{i\in\mathbb{N}} \mathbb{E}\big(|\eta_i| \mathbb{1}_{\{\eta_i<0\}}\big) &= \sup_{i\in\mathbb{N}} \mathbb{E}\big(|\eta_i| \mathbb{1}_{\{-v<\eta_i<0\}} + |\eta_i| \mathbb{1}_{\{\eta_i\leqslant -v\}}\big) \\ &\leqslant v + \sup_{i\in\mathbb{N}} \mathbb{E}\big(|\eta_i| \mathbb{1}_{\{\eta_i\leqslant -v\}}\big). \end{aligned}$$

So, condition ($\mathcal{D}2^*$) implies that

$$\sup_{i\in\mathbb{N}} \mathbb{E}\big(|\eta_i| \mathbb{1}_{\{\eta_i<0\}}\big) < \infty. \tag{3.7}$$

Denote

$$\widehat{y} = \min\left\{\delta/2, 1/\Big(2\sup_{i\in\mathbb{N}} \mathbb{E}\big(|\eta_i| \mathbb{1}_{\{\eta_i<0\}}\big)\Big)\right\}.$$

If $y \in (0, \widehat{y}]$, then

$$\begin{aligned} y(\mathbb{E}\eta_i + \alpha(y)) &> y\mathbb{E}\eta_i \\ &= y\mathbb{E}\big(\eta_i \mathbb{1}_{\{\eta_i\geqslant 0\}} + \eta_i \mathbb{1}_{\{\eta_i<0\}}\big) \\ &\geqslant y\mathbb{E}\big(\eta_i \mathbb{1}_{\{\eta_i<0\}}\big) \\ &\geqslant \widehat{y} \inf_{i\in\mathbb{N}} \mathbb{E}\big(\eta_i \mathbb{1}_{\{\eta_i<0\}}\big) \\ &= -\widehat{y} \sup_{i\in\mathbb{N}} \mathbb{E}\big(|\eta_i| \mathbb{1}_{\{\eta_i<0\}}\big) \\ &\geqslant -1/2 \end{aligned}$$

for all $i \in \mathbb{N}$.

Therefore, (3.2), (3.3), (3.6) and the well known inequality

$$\ln(1+u) \leqslant u,\ u > -1,$$

imply that

$$\begin{aligned}
\mathbb{P}\Big(\sum_{i=1}^{k} \eta_i > x\Big) &\leqslant \mathrm{e}^{-yx} \prod_{i=1}^{k} (1 + y\mathbb{E}\eta_i + \mathbb{E}(\mathrm{e}^{y\eta_i} - 1 - y\eta_i)) \\
&\leqslant \mathrm{e}^{-yx} \prod_{i=1}^{k} (1 + y(\mathbb{E}\eta_i + \alpha(y))) \\
&= \exp\Big\{ -yx + \sum_{i=1}^{k} \ln(1 + y(\mathbb{E}\eta_i + \alpha(y))\Big\} \\
&\leqslant \exp\Big\{ -yx + y\sum_{i=1}^{k} \mathbb{E}\eta_i + yk\alpha(y)\Big\}, \qquad (3.8)
\end{aligned}$$

where $k \in \mathbb{N}$, $x \geqslant 0$ and $y \in (0, \widehat{y}\,]$.

By estimate (3.7) and condition ($\mathcal{D}3^*$) we can suppose that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\eta_i = -c_6,$$

for some positive constant c_6. Then we have

$$\frac{1}{k} \sum_{i=1}^{k} \mathbb{E}\eta_i \leqslant -\frac{c_6}{2}.$$

for $k \geqslant M+1$ with some $M \geqslant 1$. Moreover, there exists $y^* \in (0, \widehat{y}\,]$ such that $\alpha(y^*) \leqslant c_6/4$, because of $\alpha(y) \downarrow 0$ as $y \to 0$.

Using results from (3.1), (3.2) and (3.8) we derive

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{k\geqslant 1} \sum_{i=1}^{k} \eta_i > x\Big) \\
\leqslant\ & \sum_{k=1}^{M} \mathbb{P}\Big(\sum_{i=1}^{k} \eta_i > x\Big) + \sum_{k=M+1}^{\infty} \mathbb{P}\Big(\sum_{i=1}^{k} \eta_i > x\Big) \\
\leqslant\ & \sum_{k=1}^{M} \mathrm{e}^{-y^*x} \prod_{i=1}^{k} \mathbb{E}\mathrm{e}^{y^*\eta_i} + \sum_{k=M+1}^{\infty} \mathbb{P}\Big(\sum_{i=1}^{k} \eta_i > x\Big) \\
\leqslant\ & \sum_{k=1}^{M} \mathrm{e}^{-y^*x} \prod_{i=1}^{k} \mathbb{E}\mathrm{e}^{y^*\eta_i} + \sum_{k=M+1}^{\infty} \mathrm{e}^{-y^*x + y^* \sum_{i=1}^{k} \mathbb{E}\eta_i + y^* k\alpha(y^*)}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \quad \mathrm{e}^{-y^* x}\Bigg(\sum_{k=1}^{M}\prod_{i=1}^{k}\mathbb{E}\mathrm{e}^{y^*\eta_i}+\sum_{k=0}^{\infty}\mathrm{e}^{-ky^*c_6/4}\Bigg)\\
&\leqslant \quad \mathrm{e}^{-y^* x}\Bigg(\sum_{k=1}^{M}\prod_{i=1}^{k}\Delta+\frac{1}{1-\mathrm{e}^{-y^*c_6/4}}\Bigg)\\
&= \quad \mathrm{e}^{-y^* x}\Bigg(\frac{\Delta(\Delta^M-1)}{\Delta-1}+\frac{\mathrm{e}^{y^*c_6/4}}{\mathrm{e}^{y^*c_6/4}-1}\Bigg)=:c_3\mathrm{e}^{-c_4x},
\end{aligned}
$$

where:

$$
\begin{aligned}
&x \geqslant 0,\\
&\Delta = 1+\sup_{i\in\mathbb{N}}\mathbb{E}\mathrm{e}^{\delta\eta_i},\\
&c_3 = \frac{\Delta(\Delta^M-1)}{\Delta-1}+\frac{e^{y^*c_6/4}}{e^{y^*c_6/4}-1},\\
&c_4 = y^* \in (0,\widehat{y}\,]
\end{aligned}
$$

with quantities $M \geqslant 1$, $c_6 > 0$ and $\widehat{y} > 0$ which are defined above. The assertion of lemma is now proved. □

3.2 Proof of Theorem 2.1

In this section we derive the assertion of Theorem 2.1.

Proof. Since

$$
\psi(x)=\mathbb{P}\Bigg(\sup_{n\geqslant 1}\Bigg\{\sum_{i=1}^{n}(Z_i-c\theta_i)\Bigg\}>x\Bigg)
$$

the desired bound of Theorem 2.1 can be derived from auxiliary Lemma 3.1.

Namely, supposing that r.v.s $Z_i - c\theta_i$, $i \in \{1, 2, \ldots\}$, satisfy all conditions of Lemma 3.1, we get

$$
\psi(x) \leqslant c_7\mathrm{e}^{-c_8x}
$$

for all $x \geqslant 0$ with some positive c_7, c_8 irrespective of x.

Therefore,

$$
\psi(x) \leqslant c_7\mathrm{e}^{-c_8x/2}\mathrm{e}^{-c_8x/2} \leqslant \mathrm{e}^{-c_8x/2},
$$

with $x \geqslant \max\{0, (2\ln c_7)/c_8\}$,

Thus, it is enough to check weather all three assumptions in our lemma are true with random variables $Z_i - c\theta_i, i \in \mathbb{N}$. The requirement $(\mathcal{D}3^*)$ of Lemma 3.1 is evidently satisfied by condition $(\mathcal{B}3)$.

Next, it follows from $(\mathcal{D}1^*)$ that

$$
\sup_{i\in\mathbb{N}}\mathbb{E}\mathrm{e}^{\gamma(Z_i-c\theta_i)} \leqslant \sup_{i\in\mathbb{N}}\mathbb{E}\mathrm{e}^{\gamma Z_i} < \infty.
$$

So, the requirement $(\mathcal{D}1^*)$ holds too.

It remains to prove that

$$\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(|Z_i-c\theta_i|\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\big)=0. \tag{3.9}$$

To establish this, we use the inequality

$$\begin{aligned}\sup_{i\in\mathbb{N}}\mathbb{E}\big(|Z_i-c\theta_i|\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\big) \quad &\leqslant \quad \sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\big)\\ &+ \quad c\sup_{i\in\mathbb{N}}\mathbb{E}\big(\theta_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\big).\end{aligned} \tag{3.10}$$

Taking the limit as $u\to\infty$ in the first summand of the right side of inequality (3.10) we get

$$\begin{aligned}&\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\big)\\ \leqslant\ &\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\mathbb{1}_{\{\theta_i\leqslant \frac{u}{2c}\}}\big)\\ +\ &\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\mathbb{1}_{\{\theta_i> \frac{u}{2c}\}}\big)\\ \leqslant\ &\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i\leqslant -u/2\}}\big)\\ +\ &\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\mathbb{1}_{\{\theta_i> \frac{u}{2c}\}}\big)\\ =\ &\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\mathbb{1}_{\{\theta_i> \frac{u}{2c}\}}\big)\\ \leqslant\ &\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(Z_i\mathbb{1}_{\{\theta_i> \frac{u}{2c}\}}\big)\\ =\ &\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}Z_i\mathbb{P}\left(\theta_i>\frac{u}{2c}\right)\\ \leqslant\ &\sup_{i\in\mathbb{N}}\mathbb{E}Z_i\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{P}\left(\theta_i>\frac{u}{2c}\right).\end{aligned} \tag{3.11}$$

Since $x\leqslant \mathrm{e}^{\gamma x}/\gamma$, $x\geqslant 0$, condition $(\mathcal{D}1^*)$ implies that

$$\sup_{i\in\mathbb{N}}\mathbb{E}Z_i<\infty. \tag{3.12}$$

In addition,

$$\begin{aligned}\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{P}\left(\theta_i>\frac{u}{2c}\right) &= \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\left(\frac{\theta_i\mathbb{1}_{\{\theta_i>\frac{u}{2c}\}}}{\theta_i}\right)\\ &\leqslant \lim_{u\to\infty}\frac{2c}{u}\sup_{i\in\mathbb{N}}\mathbb{E}\big(\theta_i\mathbb{1}_{\{\theta_i>\frac{u}{2c}\}}\big)=0\end{aligned} \tag{3.13}$$

by condition $(\mathcal{B}2)$.

Therefore, relations (3.11), (3.12) and (3.13) imply that

$$\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}(Z_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}})=0. \tag{3.14}$$

Now take the limit as $u\to\infty$ in the second summand of the right side of inequality (3.10).

By condition ($\mathcal{B}2$) we have

$$\begin{aligned}\lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(\theta_i\mathbb{1}_{\{Z_i-c\theta_i\leqslant -u\}}\big) &= \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(\theta_i\mathbb{1}_{\{\theta_i\geqslant\frac{1}{c}(Z_i+u)\}}\big)\\ &\leqslant \lim_{u\to\infty}\sup_{i\in\mathbb{N}}\mathbb{E}\big(\theta_i\mathbb{1}_{\{\theta_i\geqslant\frac{u}{c}\}}\big)=0. \qquad (3.15)\end{aligned}$$

We now see that the desired equality (3.9) follows from (3.10), (3.14) and (3.15). This means that all requirements of Lemma 3.1 hold for r.v.s $Z_i - c\theta_i$, $i \in \mathbb{N}$. □

Chapter 4

Exponential Moment Tail for Inhomogeneous Renewal Risk Model

4.1 Proof of Theorem 2.2

In this section, we present detailed proofs of the theorems 2.2, 2.3 and 2.4. For this, we need an auxiliary lemma about negatively dependent r.v.s.

Lemma 4.1. *(see Lemma 2.2 in [Chen et al., 2010]) If r.v.s $\xi_1, \xi_2, \dots$ are UEND with dominating constant α_ξ, then*

$$\mathbb{E}\left(\prod_{k=1}^{n} \xi_k^+\right) \leqslant \alpha_\xi \prod_{k=1}^{n} \mathbb{E}\xi_k^+.$$

If r.v.s $\xi_1, \xi_2, \dots$ are UEND with dominating constant α_ξ and $g_1, g_2, \dots$ are all nondecreasing real functions, then the r.v.s $g_1(\xi_1), g_2(\xi_2), \dots$ are UEND with the same dominating constant. If r.v.s $\xi_1, \xi_2, \dots$ are LEND with dominating constant α_ξ and $g_1, g_2, \dots$ are all nonincreasing real functions, then the r.v.s $g_1(\xi_1), g_2(\xi_2), \dots$ are UEND with the same dominating constant.

Now we are in the position to prove Theorem 2.2.

Proof. Let us define

$$\varphi_{a,b}(t) := \sum_{k>at} \mathbb{P}(\Theta(t) \geqslant k) b^k = \sum_{k>at} \mathbb{P}(\theta_1 + \theta_2 + \cdots + \theta_k \leqslant t) b^k$$

for all $a > \lambda$, $b > 0$, and $t > 0$. The random variables $\theta_1, \theta_2, \dots$ are *LEND* with some dominating constant, say κ. According to the Markov's inequality and Lemma 4.1, we have that for all $t > 0$, $y > 0$, and $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}(\theta_1 + \theta_2 + \cdots + \theta_k \leqslant t) &= \mathbb{P}\left(\mathrm{e}^{-y(\theta_1+\theta_2+\cdots+\theta_k)} \geqslant \mathrm{e}^{-yt}\right)\\ &\leqslant \kappa \mathrm{e}^{yt} \prod_{i=1}^{k} \mathbb{E}\mathrm{e}^{-y\theta_i}\\ &:= \kappa \mathrm{e}^{yt} \mathrm{g}_k(y).\end{aligned}$$

Therefore, for all $a > \lambda$, $b > 0$, $t > 0$, and $y > 0$, we get

$$\varphi_{a,b}(t) \leqslant \kappa\ \mathrm{e}^{yt} \sum_{k>at} \mathrm{g}_k(y) b^k. \tag{4.1}$$

Since $\log(1+x) \leqslant x$ for $x > -1$, we have that for all $k \in \mathbb{N}$ and $y \geqslant 0$,

$$\begin{aligned}\log \mathrm{g}_k(y) \leqslant \sum_{i=1}^{k} \log(\mathbb{E}\mathrm{e}^{-y\theta_i}) \leqslant &\quad \textstyle\sum_{i=1}^{k} \left(\mathbb{E}\mathrm{e}^{-y\theta_i} - 1\right)\\ = &\quad \textstyle\sum_{i=1}^{k} \left(-y\,\mathbb{E}\theta_i + \varepsilon_i(y)\right),\end{aligned} \tag{4.2}$$

where

$$\varepsilon_i(y) = \mathbb{E}\mathrm{e}^{-y\theta_i} - 1 + y\,\mathbb{E}\theta_i = \int\limits_{[0,\infty)} \left(\mathrm{e}^{-yu} - 1 + yu\right) \mathrm{d}\,\mathbb{P}(\theta_i \leqslant u).$$

It is evident that for every $M > 0$,

$$\begin{aligned}|\varepsilon_i(y)| \quad &\leqslant \quad \int\limits_{[0,M]} \left|\mathrm{e}^{-yu} - 1 + yu\right| \mathrm{d}\,\mathbb{P}(\theta_i \leqslant u)\\ &\quad + \int\limits_{(M,\infty)} \left|\mathrm{e}^{-yu} - 1\right| \mathrm{d}\,\mathbb{P}\big(\theta_i \le u\big) + y \int\limits_{(M,\infty)} u\,\mathrm{d}\,\mathbb{P}(\theta_i \leqslant u)\\ &\leqslant \quad y^2 M^2 + 2y\mathbb{E}\left(\theta_i \mathbb{1}_{\{\theta_i > M\}}\right)\end{aligned} \tag{4.3}$$

because of the estimates

$$\left|\mathrm{e}^{-x} - 1 + x\right| \leqslant x^2 \quad \text{and} \quad \left|\mathrm{e}^{-x} - 1\right| \leqslant x$$

for nonnegative x. Choosing $M = y^{-1/4}$, from the uniform integrability (2.2) we obtain that for every $i \in \mathbb{N}$,

$$|\varepsilon_i(y)| \leqslant y\left(y^{\frac{1}{2}} + 2\sup_{i\in\mathbb{N}} \mathbb{E}\left(\theta_i \mathbb{1}_{\{\theta_i \geqslant y^{-1/4}\}}\right)\right) := y\varepsilon(y)$$

with a positive function $\varepsilon(y)$ satisfying the following condition

$$\lim_{y\downarrow 0} \varepsilon(y) = 0. \tag{4.4}$$

From the obtained relation and inequality (4.2) we get the following estimate, which holds for every $k \in \mathbb{N}$ and $y > 0$:

$$\frac{1}{k} \log \ \mathrm{g}_k(y) \leqslant -\frac{y}{k}\sum_{i=1}^{k} \mathbb{E}\theta_i + y\varepsilon(y).$$

Assumption (2.3) implies that for sufficiently large k ($k \geqslant K_{a,\lambda}$),

$$\frac{1}{k}\sum_{i=1}^{k} \mathbb{E}\theta_i \geqslant \frac{1}{\lambda} - \frac{a-\lambda}{6a\lambda} = \frac{5a+\lambda}{6a\lambda}.$$

Thus, for all $y > 0$ and $k \geqslant K_{a,\lambda}$, we get that

$$\frac{1}{k}\log \mathrm{g}_k(y) \leqslant -y\left(\frac{5a+\lambda}{6a\lambda} - \varepsilon(y)\right).$$

According to relation (4.4), we can find $\hat{y} > 0$ such that for every $y \in (0, \hat{y})$, the following estimate holds:

$$\varepsilon(y) \leqslant \frac{a-\lambda}{6a\lambda}.$$

Therefore, for every $y \in (0, \hat{y})$ and every $k \geqslant K_{a,\lambda}$, we have

$$\frac{1}{k}\log \mathrm{g}_k(y) \leqslant -y\,\frac{2a+\lambda}{3a\lambda}.$$

Consequently, by (4.1) we obtain the estimate

$$\varphi_{a,b}(t) \leqslant \kappa\ \mathrm{e}^{yt}\sum_{k>at} \mathrm{e}^{-yk\frac{2a+\lambda}{3a\lambda}}\, b^k$$

for all $t > K_{a,\lambda}/a$, $y \in (0, \hat{y})$, and $b > 1$. By choosing

$$y^* = \frac{\hat{y}}{2} \in (0, \hat{y}) \quad \text{and} \quad b^* = \mathrm{e}^{\ y^*\frac{a-\lambda}{6a\lambda}} > 1,$$

for $t > K_{a,\lambda}/a$, we get

$$\begin{aligned}
\varphi_{a,b^*}(t) &\leqslant \kappa\ \mathrm{e}^{\,y^*t}\sum_{k>at}\left(\mathrm{e}^{-y^*\frac{2a+\lambda}{3a\lambda}}\, b^*\right)^k \\
&= \kappa\ \mathrm{e}^{\,y^*t}\sum_{k>at}\left(e^{-y^*\frac{a+\lambda}{2a\lambda}}\right)^k = \kappa \mathrm{e}^{\,y^*t}\,\frac{\mathrm{e}^{-y^*\frac{a+\lambda}{2\lambda a}([at]+1)}}{1-\mathrm{e}^{-y^*\frac{a+\lambda}{2\lambda a}}} \\
&\leqslant \kappa\frac{\mathrm{e}^{-y^*t\left(\frac{a-\lambda}{2\lambda a}a-1\right)}}{1-\mathrm{e}^{-y^*\frac{a+\lambda}{2\lambda a}}} = \kappa\frac{\mathrm{e}^{-y^*t\frac{a-\lambda}{2\lambda}}}{1-\mathrm{e}^{-y^*\frac{a+\lambda}{2\lambda a}}}.
\end{aligned}$$

The desired relation (2.4) immediately follows from the last estimate. Theorem 2.2 is proved. □

4.2 Proof of Theorem 2.3

Proof. The statement of Theorem 2.3 is evident because the conditions of theorem imply that

$$\sum_{k>at} \mathbb{P}(\theta_1+\theta_2+\cdots+\theta_k \leqslant t) b^k \leqslant \sum_{at<k\leqslant t/c} b^k = 0$$

for an arbitrary $t>0$. □

4.3 Proof of Theorem 2.4

The proof of Theorem 2.4 is similar to the proof of Theorem 2.2. We further present the details.

Proof. According to (4.1) and (4.2), we have that

$$\varphi_{a,b}(t) := \sum_{k>at} \mathbb{P}(\Theta(t) \geqslant k) b^k \leqslant \kappa \exp\{y\,t\} \sum_{k>at} b^k \exp\left\{\sum_{i=1}^{k} \left(\mathbb{E}\mathrm{e}^{-y\theta_i} - 1\right)\right\} \tag{4.5}$$

for all $a>0$, $b>0$, $t>0$, and $y>0$.

Condition (2.5) implies that

$$\mathbb{E}\left(\mathrm{e}^{-\theta_i/u} - 1\right) \leqslant -\frac{3}{a\,u}$$

for all $u \geqslant U$ and $i \geqslant K$.

Therefore, for all $k>K$ and $y \leqslant 1/U$, we have

$$\sum_{i=1}^{k} \left(\mathbb{E}\mathrm{e}^{-y\theta_i} - 1\right) \leqslant -\frac{3y}{a}\,(k-K).$$

Substituting this estimate into (4.5), we get that

$$\varphi_{a,b}(t) \leqslant \kappa\, \mathrm{e}^{3yK/a+yt} \sum_{k>at} \left(\frac{b}{\mathrm{e}^{3y/a}}\right)^k$$

if $a>0$, $b>0$, $0<y\leqslant 1/U$ and t is sufficiently large ($t \geqslant (K+1)/a$).

We can choose $y=y^*=1/(2U)$ and $b=b^*=\mathrm{e}^{y^*/a}>1$. Then we have the estimate

$$\begin{aligned}\varphi_{a,b^*}(t) &\leqslant \kappa\, \mathrm{e}^{3K/(2Ua)+y^*t} \sum_{k>at} \left(\frac{1}{\mathrm{e}^{2y^*/a}}\right)^k \\ &\leqslant \kappa\, \frac{\mathrm{e}^{3K/(2Ua)+1/(Ua)}}{\mathrm{e}^{1/(Ua)}-1} \exp\left\{-\frac{t}{2U}\right\}\end{aligned}$$

for sufficiently large t, from which the statement of Theorem 2.4 follows. □

4.4 Corollaries

In this section we formulate and derive the assertions of the corollaries, which proove the existence of inhomogeneous renewal processes satisfying assumptions ($\mathcal{A}1$) and ($\mathcal{A}2$).

Corollary 4.1. *Let r.v.s $\theta_1, \theta_2, \dots$ satisfy all conditions of Theorem 2.2. Then,*

$$\lim_{t\to\infty} \mathbb{E}\left(\Theta^r(t)\mathbb{1}_{\{\Theta(t)>(1+\delta)\lambda t\}}\right) = 0 \tag{4.6}$$

for all fixed $r > 0$ and $\delta > 0$.

Proof. Let r and δ be fixed positive numbers. We have

$$\mathbb{E}\left(\Theta^r(t)\mathbb{1}_{\{\Theta(t)>(1+\delta)\lambda t\}}\right) = \sum_{k>(1+\delta)\lambda t} k^r \mathbb{P}(\Theta(t) = k). \tag{4.7}$$

According to Theorem 2.2, there exists $\varepsilon = \varepsilon(\delta)$ such that

$$\lim_{t\to\infty} \sum_{k>(1+\delta)\lambda t} (1+\varepsilon)^k \, \mathbb{P}(\Theta(t) \geqslant k) = 0. \tag{4.8}$$

Equations (4.7) and (4.8) imply the statement of the corollary because $k^r/(1+\varepsilon)^k \leqslant c_{r,\varepsilon}$ for some positive $c_{r,\varepsilon}$ irrespective of $k \in \{1, 2, \dots\}$. ☐

Corollary 4.2. *Let $\theta_1, \theta_2, \dots$ be independent nonnegative and uniformly integrable r.v.s. If*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\theta_i = \frac{1}{\lambda}$$

for some $\lambda \in (0, \infty)$, then $\mathbb{E}\Theta^r(t) \sim \lambda^r t^r$ (as $t \to \infty$) for each fixed $r > 0$.

Proof. If $\delta \in (0, 1)$, then

$$\mathbb{E}\Theta^r(t) = \sum_{k\leqslant(1+\delta)\lambda t} k^r \mathbb{P}(\Theta(t) = k) + \sum_{k>(1+\delta)\lambda t} k^r \mathbb{P}(\Theta(t) = k).$$

Therefore, due to Corollary 4.1, we get that

$$\limsup_{t\to\infty} \frac{\mathbb{E}\Theta^r(t)}{\lambda^r t^r} \leqslant (1+\delta)^r \tag{4.9}$$

for arbitrary $\delta \in (0, 1)$.

On the other hand, if $0 < \delta < \min\{1/2, 1/2\lambda\}$ and t is sufficiently large, then

$$\begin{aligned} \mathbb{E}\Theta^r(t) &\geqslant \sum_{k\geqslant(1-\delta)\lambda t} k^r \mathbb{P}(\Theta(t) = k) \\ &\geqslant (1-\delta)^r \lambda^r t^r (1 - \mathbb{P}(\theta_1 + \theta_2 + \dots + \theta_\tau > t)), \end{aligned} \tag{4.10}$$

where $\tau = \lfloor (1 - \delta/2)\lambda t \rfloor$.

If t is sufficiently large, then

$$\mathbb{P}(\theta_1+\theta_2+\ldots+\theta_\tau > t) = \mathbb{P}\left(\frac{1}{\tau}\sum_{i=1}^{\tau}(\theta_i - \mathbb{E}\theta_i) > \frac{1}{\tau}\sum_{i=1}^{\tau}(t-\mathbb{E}\theta_i)\right) \leqslant \mathbb{P}\left(\frac{1}{\tau}\sum_{i=1}^{\tau}(\theta_i - \mathbb{E}\theta_i) > \frac{\delta}{2(2-\delta)\lambda}\right) \tag{4.11}$$

because for such t,

$$\begin{aligned} \frac{1}{\tau}\left(t-\sum_{i=1}^{\tau}\mathbb{E}\theta_i\right) &= \frac{1}{\tau}\left(t-\frac{\tau}{\lambda}-\tau\left(\frac{1}{\tau}\sum_{i=1}^{\tau}\mathbb{E}\theta_i-\frac{1}{\lambda}\right)\right) \\ &\geqslant \frac{1}{\tau}\left(t-\frac{\tau}{\lambda}-\tau\left|\frac{1}{\tau}\sum_{i=1}^{\tau}\mathbb{E}\theta_i-\frac{1}{\lambda}\right|\right) \\ &\geqslant \frac{\delta}{2(2-\delta)\lambda} \end{aligned}$$

according to the conditions of the corollary and the choice of δ.

We observe that the weak law of large numbers holds for r.v.s $\theta_1, \theta_2, \ldots$ satisfying the conditions of the corollary. Namely, for all $\varepsilon > 0$, $L \geqslant 1$, and $N \geqslant L$, we have that

$$\begin{aligned} \mathbb{P}\left(\frac{1}{N}\left|\sum_{i=1}^{N}(\theta_i-\mathbb{E}\theta_i)\right| > \varepsilon\right) &\leqslant \mathbb{P}\left(\left|\sum_{i=1}^{N}\left(\theta_i\mathbb{1}_{\{\theta_i\leqslant N/L\}}-\mathbb{E}\theta_i\mathbb{1}_{\{\theta_i\leqslant N/L\}}\right)\right| > \frac{\varepsilon N}{2}\right) \\ &+ \mathbb{P}\left(\left|\sum_{i=1}^{N}\left(\theta_i\mathbb{1}_{\{\theta_i> N/L\}}-\mathbb{E}\theta_i\mathbb{1}_{\{\theta_i> N/L\}}\right)\right| > \frac{\varepsilon N}{2}\right) \\ &\leqslant \frac{4}{\varepsilon^2N^2}\sum_{i=1}^{N}\mathbb{E}\left(\theta_i^2\mathbb{1}_{\{\theta_i\leqslant N/L\}}\right)+\frac{4}{\varepsilon N}\sum_{i=1}^{N}\mathbb{E}\left(\theta_i\mathbb{1}_{\{\theta_i>N/L\}}\right) \\ &\leqslant \frac{4}{L\varepsilon^2}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\theta_i+\frac{4}{\varepsilon}\max_{1\leqslant i\leqslant N}\mathbb{E}\left(\theta_i\mathbb{1}_{\{\theta_i>N/L\}}\right), \end{aligned}$$

which tends to $4/(L\varepsilon^2\lambda)$ as N tends to infinity. By the arbitrariness of $L \geqslant 1$ we get that

$$\mathbb{P}\left(\frac{1}{N}\left|\sum_{i=1}^{N}(\theta_i-\mathbb{E}\theta_i)\right| > \varepsilon\right) \underset{N\to\infty}{\to} 0$$

for each fixed positive ε.

Now estimate (4.11) implies that

$$\lim_{t\to\infty}\mathbb{P}(\theta_1+\theta_2+\cdots+\theta_\tau > t) = 0,$$

whereas inequality (4.10) implies that

$$\liminf_{t\to\infty}\frac{\mathbb{E}\Theta^r(t)}{\lambda^r t^r} \geqslant (1-\delta)^r \tag{4.12}$$

for an arbitrary $\delta \in (0, \min\{1/2, 1/2\lambda\})$. The assertion of the corollary immediately follows from (4.9) and (4.12). □

Corollary 4.3. *If r.v.s $\theta_1, \theta_2, \ldots$ satisfy the conditions of Corollary 4.2, then*

$$\frac{\Theta(t)}{\mathbb{E}\Theta(t)} \xrightarrow{\mathbb{P}} 1 \quad as \quad t \to \infty.$$

Proof. We can use Lemma 3.3 from [Tang et al., 2001]. According to this lemma, it suffices to prove that

$$\mathbb{E}\left(\frac{\Theta(t)}{\mathbb{E}\Theta(t)} \mathbb{1}_{\{\Theta(t)>(1+\delta)\mathbb{E}\Theta(t)\}}\right) \underset{t\to\infty}{\to} 0$$

for arbitrary $\delta > 0$. But this is obvious due to Corollaries 4.1 and 4.2 and the estimate

$$\mathbb{E}\left(\Theta(t)\mathbb{1}_{\{\Theta(t)>(1+\delta)\mathbb{E}\Theta(t)\}}\right) \leqslant \frac{1}{(1+\delta)\mathbb{E}\Theta(t)}\, \mathbb{E}\left(\Theta^2(t)\mathbb{1}_{\{\Theta(t)>(1+\delta/2)\lambda t\}}\right),$$

which holds for all sufficiently large t. □

By showing assertions of our corollaries we prove a so-called elementary renewal theorem for an inhomogeneous renewal process. Of course, this elementary renewal theorem can be derived from well-known classical results (see, for instance, [Kawata, 1956], [Hatori, 1959], [Hatori, 1960], [Smith, 1964a]). However, we have showed that this theorem can be also obtained using an analog of Theorem 1.2.

Chapter 5

Finite-time Ruin Probability for Inhomogeneous Renewal Risk Model

5.1 Auxiliary Lemmas

In this section, we present lemmas which we use in the proof of Theorem 2.5.

Lemma 5.1. *(see Lemma 1 in [Korshunov, 2002]) Let $\xi_1, \xi_2, \ldots$ be independent copies of r.v ξ with d.f. F_ξ and negative mean $\mathbb{E}\xi < 0$. If $F_\xi \in \mathcal{L}$, then*

$$\liminf_{x\to\infty} \inf_{n\geqslant 1} \left\{ \mathbb{P}\left(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k} \xi_i > x \right) \bigg/ \frac{1}{|\mathbb{E}\xi|} \int\limits_{x}^{x+|\mathbb{E}\xi|n} \overline{F}_\xi(v)\,\mathrm{d}v \right\} \geqslant 1\,.$$

Lemma 5.2. *(see Lemma 9 in [Korshunov, 2002]) Let $\xi_1, \xi_2, \ldots$ be independent copies of r.v ξ with d.f. F_ξ and negative mean $\mathbb{E}\xi < 0$. If $F_\xi \in \mathcal{S}_*$, then*

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \left\{ \mathbb{P}\left(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k} \xi_i > x \right) \bigg/ \frac{1}{|\mathbb{E}\xi|} \int\limits_{x}^{x+|\mathbb{E}\xi|n} \overline{F}_\xi(v)\,\mathrm{d}v \right\} \leqslant 1\,.$$

Lemma 5.3. *(see Theorem 6.7 and Lemma 6.8 in [Petrov, 1995]) If $\eta_1, \eta_2, \ldots$ are independent r.v.s such that $\sum\limits_{i=1}^{\infty} \mathbb{D}\,\eta_i/i^2 < \infty$, then*

$$\frac{1}{n}\sum_{k=1}^{n} \eta_i - \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\eta_i \underset{n\to\infty}{\to} 0$$

almost surely, or equivalently

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{m\geqslant n} \left| \frac{1}{m}\sum_{k=1}^{m} \eta_i - \frac{1}{m}\sum_{k=1}^{m} \mathbb{E}\eta_i \right| > \epsilon \right) = 0$$

for an arbitrary positive ϵ.

Lemma 5.4. *Let $\eta_1, \eta_2, \ldots$ be independent r.v.s such that:*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\eta_i = -\,d_1, \quad \lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}\big(|\eta_i| \mathbb{1}_{\{\eta_i \leqslant -u\}}\big) = 0, \quad \eta_i \leqslant d_2,\ i \in \mathbb{N},$$

for some positive constants d_1 and d_2. Then there exist positive constants d_3 and d_4, may be depending on d_1, d_2, for which

$$\mathbb{P}\Big(\sup_{k\geqslant 1}\sum_{i=1}^{k}\eta_i > x\Big) \leqslant d_3 \mathrm{e}^{-d_4\, x},\ x > 0.$$

Proof. It is obvious that

$$\mathbb{P}\Big(\sup_{k\geqslant 1}\sum_{i=1}^{k}\eta_i > x\Big) = \mathbb{P}\Big(\bigcup_{k=1}^{\infty}\Big\{\sum_{i=1}^{k}\eta_i > x\Big\}\Big) \leqslant \sum_{k=1}^{\infty}\mathbb{P}\Big(\sum_{i=1}^{k}\eta_i > x\Big) \tag{5.1}$$

for an arbitrary positive x.

According to the Markov's inequality we obtain

$$\mathbb{P}\Big(\sum_{i=1}^{k}\eta_i > x\Big) \quad \leqslant \quad \mathrm{e}^{-yx}\prod_{i=1}^{k}\mathbb{E}\,\mathrm{e}^{y\eta_i} \tag{5.2}$$

for each $x, y > 0$.

We have

$$\mathbb{E}\mathrm{e}^{y\eta_i} \quad = \quad 1 + y\mathbb{E}\eta_i + \mathbb{E}(\mathrm{e}^{y\eta_i} - 1 - y\eta_i), \tag{5.3}$$

and

$$\begin{aligned} \mathbb{E}(\mathrm{e}^{y\eta_i} - 1 - y\eta_i) \quad &= \quad \mathbb{E}\big((\mathrm{e}^{y\eta_i} - 1)\mathbb{1}_{\{\eta_i \leqslant -\,z\}}\big) \\ &- \quad y\mathbb{E}\big(\eta_i \mathbb{1}_{\{\eta_i \leqslant -\,z\}}\big) \\ &+ \quad \mathbb{E}\Big((\mathrm{e}^{y\eta_i} - 1 - y\eta_i)\mathbb{1}_{\{-z<\eta_i \leqslant 0\}}\Big) \\ &+ \quad \mathbb{E}\Big((\mathrm{e}^{y\eta_i} - 1 - y\eta_i)\mathbb{1}_{\{0<\eta_i \leqslant d_2\}}\Big) \end{aligned} \tag{5.4}$$

if $i \in \mathbb{N}$, $y > 0$ and $z > 0$.

Due to estimates

$$|\mathrm{e}^x - 1| \leqslant |x|,\ x \leqslant 0; \quad |\mathrm{e}^x - x - 1| \leqslant x^2,\ x \leqslant 0; \quad |\mathrm{e}^x - x - 1| \leqslant x^2\mathrm{e}^x,\ x \geqslant 0,$$

expression (5.4) implies that

$$\begin{aligned} \big|\mathbb{E}(\mathrm{e}^{y\eta_i} - 1 - y\eta_i)\big| \quad &\leqslant \quad 2y\,\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i \leqslant -\,z\}}\big) + y^2\mathbb{E}\big(\eta_i^2\mathbb{1}_{\{-z<\eta_i\leqslant 0\}}\big) + y^2\mathbb{E}\big(\eta_i^2\mathrm{e}^{y\eta_i}\mathbb{1}_{\{0<\eta_i\leqslant d_2\}}\big) \\ &\leqslant \quad 2y\,\sup_{i\in\mathbb{N}}\,\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i \leqslant -\,z\}}\big) + y^2z^2 + y^2d_2^{\,2}\,\mathrm{e}^{\,yd_2} \end{aligned}$$

for all $i \in \mathbb{N}$, $y > 0$ and $z > 0$.

If we choose $z = 1/\sqrt[4]{y}$, then we obtain

$$\big|\mathbb{E}(\mathrm{e}^{y\eta_i} - 1 - y\eta_i)\big| \leqslant y\epsilon(y), \tag{5.5}$$

where $\epsilon(y) = \Big(y^{1/2} + yd_2^2\mathrm{e}^{yd_2} + 2\sup\limits_{i\in\mathbb{N}}\mathbb{E}\big(|\eta_i|\mathbb{1}_{\{\eta_i\leqslant -y^{-1/4}\}}\big)\Big)$ is vanishing function as $y \downarrow 0$ according to conditions of Lemma 5.4.

Relations (5.2), (5.3) and (5.5) imply that

$$\mathbb{P}\Big(\sum_{i=1}^{k}\eta_i > x\Big) \leqslant \exp\Big\{ - yx + y\sum_{i=1}^{k}\mathbb{E}\eta_i + yk\epsilon(y)\Big\}, \tag{5.6}$$

where $k \in \mathbb{N}$, $x > 0$ and $y > 0$.

If k is sufficiently large, say $k \geqslant K + 1$, then

$$\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}\eta_i \leqslant -\frac{d_1}{2}$$

because of the first condition of Lemma 5.4 . On the other hand, there exists $y^* > 0$ such that

$$\epsilon(y^*) \leqslant \frac{d_1}{4}$$

because of vanishing function $\epsilon(y)$.

Using the last two estimations and inequalities (5.1), (5.2), (5.6) we get that

$$\begin{aligned}
\mathbb{P}\Big(\sup_{k\geqslant 1}\sum_{i=1}^{k}\eta_i > x\Big) &\leqslant \sum_{k=1}^{K}\mathbb{P}\Big(\sum_{i=1}^{k}\eta_i > x\Big) + \sum_{k=K+1}^{\infty}\mathbb{P}\Big(\sum_{i=1}^{k}\eta_i > x\Big)\\
&\leqslant \mathrm{e}^{-y^*x}\sum_{k=1}^{K}\prod_{i=1}^{k}\mathbb{E}\,\mathrm{e}^{y^*\eta_i} + \mathrm{e}^{-y^*x}\sum_{k=K+1}^{\infty}\mathrm{e}^{-y^*d_1k/4}\\
&\leqslant \mathrm{e}^{-y^*x}\Bigg(\sum_{k=1}^{K}\mathrm{e}^{\,y^*d_2k} + \frac{\mathrm{e}^{\,y^*d_1/4}}{\mathrm{e}^{\,y^*d_1/4} - 1}\Bigg),
\end{aligned}$$

and the assertion of Lemma follows. □

Remark 5.1. *It is not difficult to observe that the assertion of Lemma 5.4 follows directly from Lemma 3.1 if we change the condition $\eta_i \leqslant d_2$, $i \in \mathbb{N}$, by condition $\sup\limits_{i\in\mathbb{N}}\mathbb{E}\,e^{\gamma\eta_i} < \infty$ provided for some positive γ. Indeed, Lema 3.1 is a generalization of Lemma 5.4.*

In the next two sections the proof of Theorem 2.5 is presented. Esentially, we keep in our proof the way of [Wang et al., 2012].

5.2 Proof of Proposition 2.6 (Lower Bound)

Proof. Let, as usual, $\varepsilon, \delta \in (0,1)$, $L \in \mathbb{N}$ and $\widehat{Z}_i = Z_i - c\,(1+\delta)/\lambda$, $\widehat{\theta}_i = (1+\delta)/\lambda - \theta_i$ for $i \in \mathbb{N}$. For such i we have $\widehat{Z}_i + c\,\widehat{\theta}_i = Z_i - c\,\theta_i$. So, according to (1.4) we get that

$$\begin{aligned}
&\psi(x,t)\\
\geqslant\ &\mathbb{P}\Big(\max_{1\leqslant k\leqslant\Theta(t)} \sum_{i=1}^{k}\Big(\widehat{Z}_i + c\,\widehat{\theta}_i\Big) > x,\ \min_{1\leqslant k\leqslant\Theta(t)} \sum_{i=1}^{k} \widehat{\theta}_i > -L\Big)\\
=\ &\sum_{n=1}^{\infty}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k}\Big(\widehat{Z}_i + c\,\widehat{\theta}_i\Big) > x,\ \min_{1\leqslant k\leqslant n} \sum_{i=1}^{k} \widehat{\theta}_i > -L, \Theta(t)=n\Big)\\
\geqslant\ &\sum_{n=1}^{\infty}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k}\Big(\widehat{Z}_i - cL\Big) > x,\ \max_{1\leqslant k\leqslant n} \sum_{i=1}^{k} (-\widehat{\theta}_i) < L, \Theta(t)=n\Big)\\
\geqslant\ &\sum_{n=1}^{\infty}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k}\widehat{Z}_i > x+cL,\ \sup_{k\geqslant 1} \sum_{i=1}^{k} (-\widehat{\theta}_i) < L, \Theta(t)=n\Big)\\
\geqslant\ &\sum_{n\geqslant(1-\varepsilon)\lambda t}\mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k}\widehat{Z}_i > x+c\,L\Big)\mathbb{P}\Big(\sup_{k\geqslant 1} \sum_{i=1}^{k} (-\widehat{\theta}_i) < L, \Theta(t)=n\Big)
\end{aligned}\tag{5.7}$$

for all positive x and t.

Since d.f. F_Z is long-tailed we obtain using Lemma 5.1 that

$$\begin{aligned}
&\mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k}\widehat{Z}_i > x+c\,L\Big)\\
&\geqslant \frac{1-\varepsilon}{|\widehat{\beta}|}\int_{x+c\,L}^{x+|\widehat{\beta}|n+c\,L} \mathbb{P}\Big(\widehat{Z}_1 > v\Big)\,\mathrm{d}v\\
&\geqslant \frac{1-\varepsilon}{|\widehat{\beta}|}\int_{x}^{x+|\widehat{\beta}|n} \mathbb{P}\Big(\widehat{Z}_1 > u+c\,L\Big)\,\mathrm{d}u\\
&\geqslant (1-\varepsilon)\frac{1}{|\widehat{\beta}|}\int_{x}^{x+|\widehat{\beta}|n} \overline{F_Z}(u+cL+c(1+\delta)/\lambda)\,\mathrm{d}u\\
&\geqslant \frac{1-\varepsilon}{|\widehat{\beta}|}\inf_{u\geqslant x}\frac{\overline{F_Z}(u+cL+c(1+\delta)/\lambda)}{\overline{F_Z}(u)}\int_{x}^{x+\mu\,n} \overline{F_Z}(u)\,\mathrm{d}u
\end{aligned}$$

for $n \geqslant 1$ if x is sufficiently large ($x \geqslant x_1 = x_1(\delta)$), where $\widehat{\beta} = \mathbb{E}\widehat{Z_1} = -\mu(1+\delta+\delta\beta/\mu) < 0$.

Substituting the last estimate into (5.7) we get

$$\begin{aligned}
&\liminf_{x\to\infty}\inf_{t\geqslant T_1}\left(\psi(x,t)\Big/\int\limits_x^{x+\mu\,(1-\varepsilon)\lambda t}\overline{F_Z}(u)\,\mathrm{d}u\right)\\
&\qquad\geqslant\frac{(1-\varepsilon)}{\mu(1+\delta)+\delta\beta}\inf_{t\geqslant T_1}\mathbb{P}\left(\sup_{k\geqslant1}\sum_{i=1}^{k}(-\widehat{\theta}_i)<L,\Theta(t)\geqslant(1-\varepsilon)\lambda t\right) \qquad (5.8)
\end{aligned}$$

for all for $\varepsilon,\delta\in(0,1)$, $L\in\mathbb{N}$ and $T_1>0$.

It is obvious that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{k\geqslant1}\sum_{i=1}^{k}(-\widehat{\theta}_i)<L,\Theta(t)\geqslant(1-\varepsilon)\lambda t\right)\\
\geqslant\ &\mathbb{P}\left(\sup_{k\geqslant1}\sum_{i=1}^{k}(-\widehat{\theta}_i)<L\right)+\mathbb{P}\Big(\Theta(t)\geqslant(1-\varepsilon)\lambda t\Big)-1. \qquad (5.9)
\end{aligned}$$

Conditions of Theorem 2.5 imply that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{k\geqslant1}\sum_{i=1}^{k}(-\widehat{\theta}_i)<L\right)\\
\geqslant\ &\mathbb{P}\left(\bigcap_{k=1}^{\infty}\Big\{\sum_{i=1}^{k}(-\widehat{\theta}_i)<L-1\Big\}\right)\\
\geqslant\ &\mathbb{P}\left(\bigcap_{k=1}^{K}\Big\{\sum_{i=1}^{k}(-\widehat{\theta}_i)<L-1\Big\}\right)\\
+\ &\mathbb{P}\left(\bigcap_{k=K+1}^{\infty}\Big\{\sum_{i=1}^{k}(-\widehat{\theta}_i)<L-1\Big\}\right)-1\\
\geqslant\ &\mathbb{P}\left(\Big\{\max_{1\leqslant k\leqslant K}\sum_{i=1}^{k}(-\widehat{\theta}_i)<L-1\Big\}\cap\Big\{\sum_{i=1}^{k}(-\widehat{\theta}_i)<L-1\ \text{ for }\ k\geqslant K+1\Big\}\right)\\
\geqslant\ &\mathbb{P}\left(\max_{1\leqslant k\leqslant K}\sum_{i=1}^{k}(-\widehat{\theta}_i)<L-1\right)\\
+\ &\mathbb{P}\left(\frac{1}{k}\sum_{i=1}^{k}(-\widehat{\theta}_i)+\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}\,\widehat{\theta}_i<\frac{1+\delta}{\lambda}-\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}\,\theta_i\ \text{ for }\ k\geqslant K+1\right)-1\\
\geqslant\ &\mathbb{P}\left(\bigcap_{k=1}^{K}\Big\{\sum_{i=1}^{k}(-\widehat{\theta}_i)<L-1\Big\}\right)\\
+\ &\mathbb{P}\left(\frac{1}{k}\sum_{i=1}^{k}(-\widehat{\theta}_i)+\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}\,\widehat{\theta}_i<\frac{1+\delta}{\lambda}-\frac{1}{\lambda}-\frac{\delta}{2\lambda}\ \text{for}\ \ k\geqslant K+1\right)-1\\
\geqslant\ &\mathbb{P}\left(\bigcap_{k=1}^{K}\Big\{\sum_{i=1}^{k}(-\widehat{\theta}_i)<L-1\Big\}\right)\\
+\ &\mathbb{P}\left(\sup_{k\geqslant K+1}\Big|\,\frac{1}{k}\sum_{i=1}^{k}(-\widehat{\theta}_i)+\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}\,\widehat{\theta}_i\,\Big|<\frac{\delta}{2\lambda}\right)-1.
\end{aligned}$$

for each sufficiently large $K = K(\delta)$ and $L \geqslant 2$

So, due to Lemma 5.3,

$$\lim_{L\to\infty} \mathbb{P}\Big(\sup_{k\geqslant 1} \sum_{i=1}^{k} (-\widehat{\theta}_i) < L \Big) = 1. \tag{5.10}$$

In addition, according to Corollaries 4.2 and 4.3

$$\inf_{t\geqslant T_2} \mathbb{P}\Big(\Theta(t) \geqslant (1-\varepsilon)\,\lambda t \Big) \geqslant 1-\varepsilon \tag{5.11}$$

for some sufficiently large $T_2 = T_2(\varepsilon, \delta)$.

The derived estimates (5.8) – (5.11) and the assumption $\mathcal{C}_2$ imply that

$$\begin{aligned}
&\liminf_{x\to\infty} \inf_{t\geqslant T_2} \left(\psi(x,t) \Big/ \int\limits_{x}^{x+\mu\,(1-\varepsilon)\lambda t} \overline{F_Z}(u)\,\mathrm{d}u \right) \\
&\geqslant \frac{1-\varepsilon}{\mu(1+\delta)+\delta\beta} \Bigg(\mathbb{P}\Big(\bigcap_{k=1}^{K} \Big\{ \sum_{i=1}^{k} (-\widehat{\theta}_i) < L-1 \Big\} \Big) \\
&+\mathbb{P}\Big(\sup_{k\geqslant K+1} \Big| \frac{1}{k} \sum_{i=1}^{k} (-\widehat{\theta}_i) + \frac{1}{k} \sum_{i=1}^{k} \mathbb{E}\,\widehat{\theta}_i \Big| < \frac{\delta}{2\lambda} \Big) - 1 \\
&+\mathbb{P}\Big(\Theta(t) \geqslant (1-\varepsilon)\lambda t \Big) - 1 \Bigg) \\
&\geqslant \frac{1-\varepsilon}{\mu(1+\delta)+\delta\beta}(1-\varepsilon) \geqslant \frac{(1-\varepsilon)^2}{\mu(1+\delta)+\delta\beta}
\end{aligned} \tag{5.12}$$

for all for $\varepsilon, \delta \in (0,1)$ and sufficiently large T_2.

Due to Lemma 4.2 $\mathbb{E}\Theta(t) \sim \lambda t$. Therefore

$$\int_x^{x+\mu\lambda(1-\varepsilon)t} \overline{F_Z}(u)\,\mathrm{d}u \Big/ \int_x^{x+\mu\lambda(t)} \overline{F_Z}(u)\,\mathrm{d}u$$

$$\geqslant \int_x^{x+\mu\lambda(1-\varepsilon)t} \overline{F_Z}(u)\,\mathrm{d}u \Big/ \int_x^{x+\mu\lambda(1+\varepsilon)t} \overline{F_Z}(u)\,\mathrm{d}u$$

$$= 1 - \int_{x+\mu\lambda(1-\varepsilon)t}^{x+\mu\lambda(1+\varepsilon)t} \overline{F_Z}(u)\,\mathrm{d}u \Big/ \int_x^{x+\mu\lambda(1+\varepsilon)t} \overline{F_Z}(u)\,\mathrm{d}u$$

$$\geqslant 1 - (\overline{F_Z}(x+\mu\lambda(1-\varepsilon)t)\mu\lambda t 2\varepsilon) \Big/ \int_x^{x+\mu\lambda(1+\varepsilon)t} \overline{F_Z}(u)\,\mathrm{d}u$$

$$\geqslant 1 - (\overline{F_Z}(x+\mu\lambda(1-\varepsilon)t)\mu\lambda t 2\varepsilon) \Big/ \int_x^{x+\mu\lambda(1-\varepsilon)t} \overline{F_Z}(u)\,\mathrm{d}u$$

$$\geqslant 1 - \frac{\overline{F_Z}(x+\mu\lambda(1-\varepsilon)t)\mu\lambda t 2\varepsilon}{\overline{F_Z}(x+\mu\lambda(1-\varepsilon)t)\mu\lambda t(1-\varepsilon)} \geqslant \frac{1-3\varepsilon}{1-\varepsilon}$$

if $x > 0$, $\varepsilon \in (0, 1/3)$ and $t \geqslant T_3$ ($T_3 \geqslant T_2$).

The last estimate substituting into (5.12) we obtain

$$\liminf_{x\to\infty} \inf_{t\geqslant T_3} \left(\psi(x,t) \Big/ \frac{1}{\mu} \int_x^{x+\mu\,\lambda(t)} \overline{F_Z}(u)\,\mathrm{d}u \right) \tag{5.13}$$

$$\geqslant \frac{1-3\varepsilon}{1-\varepsilon} \liminf_{x\to\infty} \inf_{t\geqslant T_3} \left(\psi(x,t) \Big/ \frac{1}{\mu} \int_x^{x+\mu(1-\varepsilon)\lambda(t)} \overline{F_Z}(u)\,\mathrm{d}u \right) \tag{5.14}$$

$$\geqslant \frac{1-3\varepsilon}{1-\varepsilon} \frac{1}{1+\delta+\delta\beta/\mu} \tag{5.15}$$

for all for $\varepsilon \in (0, 1/3)$, $\delta \in (0, 1)$ and sufficiently large T_3.

Now let T be such that $\lambda(T) > 0$. If $x > 0$ and $t \in [T, T_3]$, then due to expression (1.4) we have

$$\begin{aligned}
\psi(x,t) &\geqslant \mathbb{P}\Big(\sum_{i=1}^{\Theta(t)}(Z_i - c\,\theta_i) > x\Big)\\
&= \sum_{n=1}^{\infty}\mathbb{P}\Big(\sum_{i=1}^{n} Z_i - c\sum_{i=1}^{n}\theta_i > x, \sum_{i=1}^{n}\theta_i \leqslant t, \sum_{i=1}^{n+1}\theta_i > t\Big)\\
&\geqslant \sum_{n=1}^{\infty}\mathbb{P}\Big(\sum_{i=1}^{n} Z_i > x + c\,t, \Theta(t) = n\Big)\\
&= \sum_{n=1}^{\infty}\mathbb{P}\Big(\max_{1\leqslant m\leqslant n}\sum_{i=1}^{m} Z_i > x + ct\Big)\,\mathbb{P}(\Theta(t)=n)\\
&\geqslant \sum_{n=1}^{\infty}\mathbb{P}\Big(\max_{1\leqslant m\leqslant n}\sum_{i=1}^{m} Z_i > x + c\,T_3\Big)\,\mathbb{P}(\Theta(t)=n)\\
&\geqslant \sum_{n=1}^{\infty}\mathbb{P}\Big(\max_{1\leqslant m\leqslant n}\sum_{i=1}^{m}(Z_i - c/\lambda) > x + c\,T_3\Big)\,\mathbb{P}(\Theta(t)=n).
\end{aligned}$$

Suppose that $\varphi(x) \geqslant 1$ is some unboundedly increasing function under condition

$$\overline{F_Z}(x+\mu\varphi(x))/\overline{F_Z}(x) \underset{x\to\infty}{\sim} 1. \tag{5.16}$$

The existence of such function follows from condition $F_Z \in \mathcal{L}$. According to Lemma 5.1 we have

$$\begin{aligned}
\psi(x,t) &\geqslant \sum_{n=1}^{\infty}\mathbb{P}(\Theta(t)=n)\overline{F_Z}(x + c/\lambda + +c\,T_3)\\
&\geqslant \frac{1-\varepsilon}{\mu}\sum_{n=1}^{\infty}\mathbb{P}(\Theta(t)=n)\int\limits_{x+c\,T_3}^{x+c\,T_3+\mu n}\overline{F_Z}(u+c/\lambda)\,\mathrm{d}u\\
&\geqslant (1-\varepsilon)\sum_{n=1}^{\lfloor\varphi(x)\rfloor} n\,\mathbb{P}(\Theta(t)=n)\overline{F_Z}(x + c\,T_3 + c/\lambda + \mu\varphi(x))\\
&\geqslant (1-\varepsilon)^2\,\overline{F_Z}(x)\,\mathbb{E}\Theta(t)\mathbb{1}_{\{\Theta(t)\leqslant\varphi(x)\}}
\end{aligned} \tag{5.17}$$

if $t \in [T, T_3]$ and $x \geqslant x_2 = x_2(\delta,\varepsilon,T_3)$.

The Hölder inequality implies that for sufficiently large x ($x \geqslant x_3 = x_3(\varepsilon, T, T_3) \geqslant x_2$)

$$\begin{aligned}
\mathbb{E}\Theta(t)\mathbb{1}_{\{\Theta(t)\leqslant\varphi(x)\}} &\leqslant \Big(\mathbb{E}\Theta^2(t)\Big)^{1/2}\sqrt{\mathbb{P}\big(\Theta(t)\leqslant\varphi(x)\big)}\\
&\leqslant \Big(\mathbb{E}\Theta^2(T_3)\Big)^{1/2}\sqrt{\mathbb{P}\big(\Theta(T_3)\leqslant\varphi(x)\big)}\,\frac{\lambda(t)}{\lambda(T)}\\
&\leqslant \varepsilon\lambda(t).
\end{aligned}$$

The last estimate and (5.17) imply that

$$\psi(x,t) \geqslant (1-\varepsilon)^3\,\overline{F_Z}(x)\lambda(t) \geqslant \frac{(1-\varepsilon)^3}{\mu}\int_x^{x+\mu\lambda(t)}\overline{F_Z}(u)\,\mathrm{d}u \tag{5.18}$$

for all $\varepsilon \in (0,1)$, $x \geqslant x_3$ and $t \in [T, T_3]$. Consequently,

$$\liminf_{x\to\infty} \inf_{t\in[T,T_3]} \left(\psi(x,t) \Big/ \frac{1}{\mu} \int\limits_x^{x+\mu\,\lambda(t)} \overline{F_Z}(u)\,\mathrm{d}u \right) \geqslant (1-\varepsilon)^3 \tag{5.19}$$

The desired lower estimate of Proposition 2.6 follows now from (5.13) and (5.19) immediately because of arbitrariness of $\varepsilon \in (0, 1/3)$ and $\delta \in (0,1)$. □

5.3 Proof of Proposition 2.7 (Upper bound)

Proof. In this section, we obtain the assertion of Proposition 2.7. The proof of the assertion consists of two parts. In the first part of proof we use the way from [Leipus and Šiaulys, 2009]. In the second part of proof we use mainly the consideration from [Wang et al., 2012].

Let $\varepsilon, \delta \in (0,1)$, $T \in \Lambda$ and $\widetilde{Z}_i = Z_i - c(1-\delta)/\lambda$, $\widetilde{\theta}_i = (1-\delta)/\lambda - \theta_i$ for each $i \in \mathbb{N}$. According to (1.4) we have that

$$\begin{aligned} \psi(x,t) &\leqslant \mathbb{P}\Big(\max_{1\leqslant k\leqslant(1+\varepsilon)\lambda(t)} \sum_{i=1}^{k} \widetilde{Z}_i + c\sup_{k\geqslant 1}\sum_{i=1}^{k}\widetilde{\theta}_i > x \Big) \\ &+ \mathbb{P}\Big(\max_{1\leqslant k\leqslant\Theta(t)} \sum_{i=1}^{k}(Z_i - c\theta_i) > x,\ \Theta(t) > (1+\varepsilon)\lambda(t) \Big) \\ &:= \psi_1(x,t) + \psi_2(x,t) \end{aligned} \tag{5.20}$$

if $x > 0$ and $t \geqslant T$. Denoting

$$\zeta_t = \max_{1\leqslant k\leqslant(1+\varepsilon)\lambda(t)} \sum_{i=1}^{k} \widetilde{Z}_i, \quad \chi = c\sup_{k\geqslant 1}\sum_{i=1}^{k}\widetilde{\theta}_i, \quad \chi^+ = \chi \mathbb{1}_{\{\chi\geqslant 0\}},$$

we obtain

$$\begin{aligned} \psi_1(x,t) &= \mathbb{P}\big(\zeta_t + \chi > x\big) \\ &\leqslant \int\limits_{[0,x-y]} \mathbb{P}\big(\zeta_t > x-u\big)\mathrm{d}\mathbb{P}(\chi^+ \leqslant u) + \mathbb{P}(\chi^+ > x-y) \\ &:= \psi_{11}(x,y,t) + \psi_{12}(x,y,t), \end{aligned} \tag{5.21}$$

where $0 < y \leqslant x/2$.

If $0 < \delta < 1 - \lambda\beta/c = \mu/(\mu+\beta)$, then $\widetilde{\beta} := \mathbb{E}\widetilde{Z}_1 = -\mu + \delta(\mu+\beta) < 0$. In addition, we have that d.f. $\mathbb{P}(\widetilde{Z}_1 \leqslant u) = F_Z(u + c(1-\delta)/\lambda)$ belongs to the class $\mathcal{S}_*$ due to Lemma 3 of [Korshunov, 2002] (see Lemma A.1 in Appendix). So, applying Lemma 5.2, we get that

$$\psi_{11}(x,y,t) \leqslant \frac{1+\varepsilon}{|\widetilde{\beta}|} \int\limits_{[0,\,x-y]} \left(\int\limits_{x-u}^{x-u+|\widetilde{\beta}|(1+\varepsilon)\lambda(t)} \overline{F_Z}(w + c(1-\delta)/\lambda)\,\mathrm{d}w \right) \mathrm{d}F_{\chi^+}(u),$$

where $x \geqslant 2y$, y is sufficiently large ($y \geqslant x_1 = x_1(\delta, \varepsilon)$) and F_{χ^+} denote d.f. of r.v. χ^+.

By the Fubini-Tonelli theorem

$$\begin{aligned}\psi_{11}(x,y,t) &\leqslant \frac{1+\varepsilon}{|\widetilde{\beta}|} \int\limits_{[0,\infty)} \left(\int\limits_{x}^{x+\mu(1+\varepsilon)\lambda(t)} \overline{F_Z}(w-u)\,\mathrm{d}w \right) \mathrm{d}F_{\chi^+}(u) \\ &= \frac{1+\varepsilon}{|\widetilde{\beta}|} \int\limits_{x}^{x+\mu(1+\varepsilon)\,\lambda(t)} \overline{F_Z * F_{\chi^+}}(w)\,\mathrm{d}w. \end{aligned} \tag{5.22}$$

Conditions of Proposition 2.7 imply that:

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\widetilde{\theta}_i \underset{n\to\infty}{\to} -\frac{\delta}{\lambda}\,; \\ &\widetilde{\theta}_i \leqslant \frac{1-\delta}{\lambda} \quad \text{for each} \quad i \in \mathbb{N}\,; \\ &\lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}\Big(|\widetilde{\theta}_i| \mathbb{1}_{\{\widetilde{\theta}_i \leqslant -u\}}\Big) \leqslant 2 \lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}\Big(|\theta_i| \mathbb{1}_{\{\theta_i \geqslant u\}}\Big) = 0. \end{aligned}$$

So, due to Lemma 5.4,

$$\overline{F}_{\chi^+}(w) = \mathbb{P}(\chi > w) \leqslant c_1 \mathrm{e}^{-\,c_2 w}, \tag{5.23}$$

for some positive constants $c_1 = c_1(\delta)$ and $c_2 = c_2(\delta)$. Applying, for instance, Corollary 2 from [Pitman, 1980](see Lemma A.2 in Appendix) we obtain

$$\overline{F_Z * F_{\chi^+}}(w) \underset{w\to\infty}{\sim} \overline{F_Z}(w).$$

because of $F_Z \in \mathcal{S}_* \subset \mathcal{L}$.

Therefore, estimate (5.22) implies that

$$\psi_{11}(x,y,t) \leqslant \frac{(1+\varepsilon)^2}{|\widetilde{\beta}|} \int\limits_{x}^{x+\mu(1+\varepsilon)\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w, \tag{5.24}$$

where $\varepsilon \in (0,1)$, $\delta \in \big(0, \mu/(\mu+\beta)\big)$, $t \geqslant T$ and $x \geqslant 2y$ and y is sufficiently large, i.e. $y \geqslant x_2(\delta,\varepsilon) \geqslant x_1$

If $t \geqslant T$, then

$$\begin{aligned} \int\limits_{x}^{x+\mu(1+\varepsilon)\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w &= \int\limits_{x}^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w \left(1 + \frac{\int_{x+\mu\lambda(t)}^{x+\mu(1+\varepsilon)\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w}{\int_{x}^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \right) \\ &\leqslant (1+\varepsilon) \int\limits_{x}^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w. \end{aligned}$$

The last inequality and estimate (5.24) imply that

$$\limsup_{x\to\infty} \sup_{t\in[T,\infty)} \frac{\psi_{11}(x,y,t)}{\int\limits_x^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \leqslant \frac{(1+\varepsilon)^3}{\mu-\delta(\mu+\beta)} \tag{5.25}$$

if $\varepsilon \in (0,1)$, $\delta \in (0, \mu/(\mu+\beta))$ and $y \geqslant x_2$.

To estimate the term $\psi_{12}(x,y,t)$ from (5.21) we use (5.23) again. If $y \geqslant x_2$, then we have

$$\begin{aligned} \limsup_{x\to\infty} \sup_{t\in[T,\infty)} \frac{\psi_{12}(x,y,t)}{\int\limits_x^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} &\leqslant \limsup_{x\to\infty} \frac{\mathbb{P}(\chi^+ > x/2)}{\mu\lambda(T)\,\overline{F_Z}\,(x+\mu\lambda(T))} \\ &\leqslant \frac{c_1}{\mu\lambda(T)} \limsup_{x\to\infty} \frac{\mathrm{e}^{-c_2 x/2}}{\overline{F_Z}(x+\mu\lambda(T))} \\ &= 0 \end{aligned} \tag{5.26}$$

because of $F_Z \in \mathcal{S}_* \subset \mathcal{S} \subset \mathcal{L}$ and Lemma 1.3.5 (b) from [Embrechts et al., 1997b] (see Lemma A.3 in Appendix).

Relations (5.21), (5.24) and (5.26) hold for all $y \geqslant x_2$. So, these relations imply that

$$\limsup_{x\to\infty} \sup_{t\in[T,\infty)} \frac{\psi_{1}(x,t)}{\int\limits_x^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \leqslant \frac{(1+\varepsilon)^3}{\mu-\delta(\mu+\beta)} \tag{5.27}$$

for all $\varepsilon \in (0,1)$, $\delta \in (0, \mu/(\mu+\beta))$ and $t \in \Lambda$.

It remains to get a similar inequality for $\psi_2(x,t)$. Corollary 4.2 implies that

$$\begin{aligned} \psi_2(x,t) &\leqslant \sum_{n>(1+\varepsilon)\lambda(t)} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k} Z_i > x, \Theta(t)=n\Big) \\ &\leqslant \sum_{n>(1+\varepsilon/2)\lambda t} \overline{F_Z^{*n}}(x)\,\mathbb{P}(\Theta(t)=n), \end{aligned} \tag{5.28}$$

where $x > 0$ and t is sufficiently large ($t \geqslant T_4 = T_4(\varepsilon) \geqslant T$). According to the Kesten estimate for d.f. $F_Z \in \mathcal{S}_* \subset \mathcal{S}$ (see, for instance, Lemma 1.3.5 (c) in [Embrechts et al., 1997b] (see Lemma A.3 in Appendix)) we have that

$$\sup_{x\geqslant 0} \frac{\overline{F_Z^{*n}}(x)}{\overline{F_Z}(x)} \leqslant c_3(1+\Delta)^n, \tag{5.29}$$

where $\Delta > 0$ and $c_3 = c_3(\Delta)$ is a suitable positive constant.

For each $x > 0$ and $T_5 \geqslant T_4$ relations (5.28), (5.29) imply that

$$\begin{aligned}
&\sup_{t\in[T_5,\infty)} \frac{\psi_2(x,t)}{\int\limits_x^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \\
&\leqslant \frac{1}{\mu\lambda(T_5)} \sup_{t\in[T_5,\infty)} \sum_{n>(1+\varepsilon/2)\lambda t} \sup_{x\geqslant 0} \frac{\overline{F_Z^{*n}}(x)}{\overline{F_Z}(x+\mu\lambda(T_5))}\, \mathbb{P}(\Theta(t)=n) \\
&\leqslant \frac{c_3}{\mu\lambda(T_5)} \sup_{x\geqslant 0} \frac{\overline{F_Z}(x)}{\overline{F_Z}(x+\mu\lambda(T_5))} \sup_{t\in[T_5,\infty)} \sum_{n>(1+\varepsilon/2)\lambda t} (1+\Delta)^n\, \mathbb{P}(\Theta(t)=n).
\end{aligned}$$

If $b = 1+\Delta$ is chosen for $a = (1+\varepsilon/2)\lambda$ according to Theorem 2.2, then the last inequality implies that

$$\limsup_{x\to\infty} \sup_{t\in[T_5,\infty)} \frac{\psi_2(x,t)}{\int\limits_x^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \leqslant \varepsilon$$

where $T_5 = T_5(\varepsilon) \in \Lambda$ is sufficiently large.

The last inequality together with equality (5.20) and estimate (5.27) implies that

$$\limsup_{x\to\infty} \sup_{t\in[T_5,\infty)} \frac{\psi(x,t)}{\int\limits_x^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \leqslant \frac{(1+\varepsilon)^3}{\mu-\delta(\mu+\beta)} + \varepsilon \tag{5.30}$$

It remains to estimate $\psi(x,t)$ in the case when $t \in [T, T_5]$. Suppose that function $1 \leqslant \varphi(x) \leqslant \sqrt{x}$, $x \geqslant 1$, satisfies property (5.16). If $x \geqslant 1$ and $t \geqslant T$, then due to (1.4) we have

$$\begin{aligned}
\psi(x,t) \quad &\leqslant \quad \mathbb{P}\Big(\max_{1\leqslant k\leqslant\Theta(t)} \sum_{i=1}^{k} Z_i > x,\ \Theta(t) \leqslant \varphi(x) \Big) \\
&+ \quad \mathbb{P}\Big(\max_{1\leqslant k\leqslant\Theta(t)} \sum_{i=1}^{k} \widetilde{Z}_i + c \sup_{k\geqslant 1} \sum_{i=1}^{k} \widetilde{\theta}_i > x, \Theta(t) > \varphi(x) \Big) \\
&:= \quad \widehat{\psi}_1(x,t) + \widehat{\psi}_2(x,t)
\end{aligned} \tag{5.31}$$

Applying Lemma 5.2 we obtain

$$\begin{aligned}
\widehat{\psi}_1(x,t) &= \mathbb{P}\Big(\sum_{i=1}^{\Theta(t)} Z_i > x,\ \Theta(t) \leqslant \varphi(x)\Big) \\
&\leqslant \sum_{n \leqslant \varphi(x)} \mathbb{P}\Big(\sum_{i=1}^{n} Z_i > x,\ \Theta(t) = n\Big) \\
&\leqslant \sum_{n \leqslant \varphi(x)} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k} \Big(Z_i - \frac{c}{\lambda}\Big) > x - \frac{c\,\varphi(x)}{\lambda}\Big)\,\mathbb{P}\big(\Theta(t)=n\big) \\
&\leqslant \frac{(1+\varepsilon)}{\mu} \sum_{n=1}^{\infty} \mathbb{P}\big(\Theta(t)=n\big) \int\limits_{x-c\,\varphi(x)/\lambda}^{x-c\,\varphi(x)/\lambda+\mu n} \overline{F_Z}(u)\,\mathrm{d}u \\
&\leqslant (1+\varepsilon)\,\lambda(t)\,\overline{F_Z}\Big(x - \frac{c\,\varphi(x)}{\lambda}\Big) \\
&\leqslant \frac{1+\varepsilon}{\mu}\,\frac{\overline{F_Z}\Big(x-\frac{c\,\varphi(x)}{\lambda}\Big)}{\overline{F_Z}\Big(x+\mu\lambda(T_5)\Big)} \int\limits_{x}^{x+\mu\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w
\end{aligned}$$

if $t \leqslant T_5$ and x is sufficiently large. Consequently,

$$\limsup_{x\to\infty} \sup_{t\in[T,T_5]} \frac{\widehat{\psi}_1(x,t)}{\int\limits_{x}^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \leqslant \frac{1+\varepsilon}{\mu} \tag{5.32}$$

because of condition (5.16).

On the other hand,

$$\begin{aligned}
\widehat{\psi}_2(x,t) &\leqslant \sum_{n>\varphi(x)} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k} \widetilde{Z}_i + \chi^+ > x,\ \chi^+ \leqslant x-\varphi(x), \Theta(t)=n\Big) \\
&+\ \mathbb{P}\big(\chi^+ > x-\varphi(x)\big) \\
&:= \widehat{\psi}_{21}(x,t) + \widehat{\psi}_{22}(x,t).
\end{aligned} \tag{5.33}$$

Using (5.23), the fact that $F_Z \in \mathcal{L}$ and Lemma 1.3.5 (b) from [Embrechts et al., 1997b] (see Lemma A.3 in Appendix) we have

$$\limsup_{x\to\infty} \sup_{t\in[T,T_5]} \frac{\widehat{\psi}_{22}(x,t)}{\int\limits_{x}^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \leqslant \limsup_{x\to\infty} \frac{c_1 \mathrm{e}^{-c_2(x-\varphi(x))}}{\mu\lambda(T)\overline{F_Z}(x+\mu\lambda(T_5))} = 0. \tag{5.34}$$

If x is sufficiently large, then Lemma 5.2 implies

$$
\begin{aligned}
\widehat{\psi}_{21}(x,t) &= \sum_{n>\varphi(x)} \int_{[0,\,x-\varphi(x)]} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k} \widetilde{Z}_i > x-y\Big)\,\mathrm{d}\mathbb{P}\big(\chi^+ \leqslant y, \Theta(t)=n\big)\\
&\leqslant \frac{1+\varepsilon}{|\widetilde{\beta}|} \sum_{n>\varphi(x)} \int_{[0,\,x-\varphi(x)]} \Bigg(\int_{x-y}^{x-y+|\widetilde{\beta}|n} \overline{F_Z}(w)\,\mathrm{d}w\Bigg)\,\mathrm{d}\mathbb{P}\big(\chi^+ \leqslant y, \Theta(t)=n\big)\\
&\leqslant (1+\varepsilon) \sum_{n>\varphi(x)} n \int_{[0,\,x-\varphi(x)]} \overline{F_Z}(x-y)\,\mathrm{d}\mathbb{P}\big(\chi^+ \leqslant y, \Theta(t)=n\big)\\
&= (1+\varepsilon) \sum_{n>\varphi(x)} n\, \mathbb{P}\big(Z+\chi^+ > x,\, \chi^+ \leqslant x-\varphi(x), \Theta(t)=n\big)\\
&\leqslant (1+\varepsilon) \sum_{n>\varphi(x)} n\, \mathbb{P}\big(Z+\chi^+ > x,\, \chi^+ \leqslant x-\varphi(x),\, Z \leqslant x-\varphi(x),\, \Theta(t)=n\big)\\
&+ (1+\varepsilon) \sum_{n>\varphi(x)} n\, \mathbb{P}\big(Z > x-\varphi(x),\, \Theta(t)=n\big)\\
&:= (1+\varepsilon)(\widehat{\psi}_{211}(x,\,t) + \widehat{\psi}_{212}(x,\,t)).
\end{aligned}
\tag{5.35}
$$

Using the Hölder inequality we get

$$
\begin{aligned}
\widehat{\psi}_{212}(x,t) &= \overline{F_Z}(x-\varphi(x))\,\mathbb{E}\Theta(t)\mathbb{1}_{\{\Theta(t)>\varphi(x)\}}\\
&\leqslant \overline{F_Z}\big(x-\varphi(x)\big)\Big(\mathbb{E}\Theta^2(t)\Big)^{1/2}\Big(\mathbb{P}\big(\Theta(t)>\varphi(x)\big)\Big)^{1/2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\limsup_{x\to\infty} \sup_{t\in[T,T_5]} \frac{\widehat{\psi}_{212}(x,t)}{\int_x^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w}\\
&\qquad\leqslant \frac{1}{\mu\lambda(T)} \limsup_{x\to\infty} \frac{\overline{F_Z}(x-\varphi(x))}{\overline{F_Z}(x+\mu\lambda(T_5))}\, \Big(\mathbb{E}\Theta^{\,2}(T_5)\Big)^{1/2}\Big(\mathbb{P}\big(\Theta(T_5)>\varphi(x)\big)\Big)^{1/2}\\
&\qquad= 0
\end{aligned}
\tag{5.36}
$$

according to property (5.16).

Finally, if $t \in [T, T_5]$ and x is sufficiently large, then

$$\begin{aligned}
\widehat{\psi}_{211}(x,t) &\leqslant \sum_{n>\varphi(x)} n\, \mathbb{P}\big(Z+\chi^+ > x,\, \varphi(x) < Z \leqslant x-\varphi(x), \Theta(t)=n\big) \\
&= \int\limits_{\varphi(x)}^{x-\varphi(x)} \sum_{n>\varphi(x)} n\, \mathbb{P}\big(\chi^+ > x-y,\, \Theta(t)=n\big)\, \mathrm{d}F_Z(y) \\
&= \int\limits_{\varphi(x)}^{x-\varphi(x)} \mathbb{E}\Big(\Theta(t)\mathbb{1}_{\{\chi^+>x-y\}}\mathbb{1}_{\{\Theta(t)>\varphi(x)\}}\Big)\, \mathrm{d}F_Z(y) \\
&\leqslant \big(\mathbb{E}\Theta^2(t)\big)^{1/2} \int\limits_{\varphi(x)}^{x-\varphi(x)} \Big(\mathbb{P}(\chi^+ > x-y)\Big)^{1/2}\, \mathrm{d}F_Z(y) \\
&\leqslant \big(c_1\, \mathbb{E}\Theta^2(T_5)\big)^{1/2} \int\limits_{\varphi(x)}^{x-\varphi(x)} \mathrm{e}^{-c_2(x-y)/2}\, \mathrm{d}F_Z(y) \\
&\leqslant \varepsilon \int\limits_{\varphi(x)}^{x-\varphi(x)} \overline{F_Z}(x-y)\, \mathrm{d}F_Z(y)
\end{aligned}$$

because of the Hölder inequality, estimate (5.23) and Lemma 1.3.5 (b) from [Embrechts et al., 1997b] (see Lemma A.3 in Appendix). Therefore, property (5.16) and Theorem 3.7 from [Foss et al., 2011] (see Lemma A.4 in Appendix) imply that

$$\begin{aligned}
&\limsup_{x\to\infty} \sup_{t\in[T,T_5]} \frac{\widehat{\psi}_{211}(x,t)}{\int_x^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \\
&\leqslant \frac{\varepsilon}{\mu\lambda(T)} \limsup_{x\to\infty} \frac{\overline{F_Z}(x)}{\overline{F_Z}(x+\mu\lambda(T_5))} \frac{1}{\overline{F_Z}(x)} \int\limits_{\varphi(x)}^{x-\varphi(x)} \overline{F_Z}(x-y)\,\mathrm{d}F_Z(y) \\
&\quad = 0.
\end{aligned}$$

The last inequality together with relations (5.31) – (5.36) implies that

$$\limsup_{x\to\infty} \sup_{t\in[T,T_5]} \frac{\psi(x,t)}{\int_x^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \leqslant \frac{1+\varepsilon}{\mu}.$$

Consequently, due to estimate (5.30), we have that

$$\limsup_{x\to\infty} \sup_{t\in[T,\infty)} \frac{\psi(x,t)}{\int_x^{x+\mu\,\lambda(t)} \overline{F_Z}(w)\,\mathrm{d}w} \leqslant \max\left\{\frac{(1+\varepsilon)^3}{\mu-\delta(\mu+\beta)}+\varepsilon,\, \frac{1+\varepsilon}{\mu}\right\},$$

where $\varepsilon \in (0,1)$, $\delta \in (0, \mu/(\mu+\beta)$ and $T \in \Lambda$. We obtain the assertion of Proposition 2.7 by letting ε and δ to zero in the last estimate. □

5.4 Corollary

According to Corollary 4.2 $\lambda(t) \sim \lambda t$ if $t \to \infty$. Therefore, Theorem 2.5 implies the following, more simple asymptotic formula for the finite-time ruin probability in the case when the horizon of time t is restricted to a smaller region.

Corollary 5.1. *Under conditions of Theorem 2.5*

$$\psi(x,t) \underset{x\to\infty}{\sim} \frac{1}{\mu} \int\limits_{x}^{x+\mu\lambda t} \overline{F_Z}(u)\,\mathrm{d}u$$

uniformly with respect to $t \in [a(x), \infty)$*, where* $a(x)$ *is an unboundedly increasing function.*

.

Obviously, Corollary 5.1 follows immediately from Theorems 2.5 and 2.2 because $S_* \subset S \subset L$ due to Lemma 1 of [Kaas and Tang, 2003] (see Lemma A.5 in Appendix), Lema 2 of [Chistyakov, 1964] (see Lemma A.6 in Appendix) and Lemma 1.3.5(a) of [Embrechts et al., 1997b] (see Lemma A.3 in Appendix). According to Corollary 4.2 $\lambda(t) \sim \lambda t$ if $t \to \infty$. Therefore, Corollary 5.1 implies more simple asymptotic formula for the finite-time ruin probability in the case when the horizon of time t is restricted to a smaller region.

Conclusions

In this last Chapter, a brief summary of the results obtained is given.

- We proved a theorem about the possibility to apply Lundberg-type inequality in an inhomogeneous renewal risk model. We consider the model with independent, but not necessarily identically distributed claim sizes and the inter-occurrence times.
- We obtained that the exponential moment tail of an inhomogeneous renewal process vanishes exponentially at infinity. This property holds for inter-arrival times having different distributions and satisfying certain dependence structures. The obtained property can be used to prove weak law of large numbers for an inhomogeneous renewal process.
- By showing assertions of our corollaries we proved a so-called elementary renewal theorem for an inhomogeneous renewal process. This elementary renewal theorem can be derived from well-known classical results (see, for instance, [Kawata, 1956], [Hatori, 1959] [Hatori, 1960], [Smith, 1964a]). We showed that this theorem can be also obtained using the derived property of the exponential moment tail of inhomogeneous renewal process.
- We gave an asymptotic formula for the finite-time ruin probability for an inhomogeneous renewal risk model and we found out that it was insensitive to the homogeneity of inter-occurrence times.
- Possibly, the asymptotic formula for the finite-time ruin probability for an inhomogeneous renewal risk model holds uniformly for all $t \in \Lambda$, not only for $t \in [T, \infty)$ with $T \in \Lambda$. At the moment, we do not know how we can extend the region of uniformity without additional requirements.

Appendix

Lemma A.1.*(see Lemma 3 in [Korshunov, 2002]) Let G and H be two long-tailed distributions on $\mathbb{R}^+$. If $G \in \mathcal{S}_*$ and $c_1\overline{G}(x) \leqslant \overline{H}(x) \leqslant c_2\overline{G}(x)$ for some c_1 and c_2, $0 < c_1 < c_2 < \infty$, then $H \in \mathcal{S}_*$.*

Lemma A.2. *(see Corollary 2 in [Pitman, 1980]) If $F_X \in \mathcal{S}$, $\overline{F_Y}(x) = o\left(\overline{F_X}(x)\right)$, $x \to \infty$, then $\overline{F_{X+Y}}(x) \sim \overline{F_X}(x)$, $x \to \infty$ and $F_{X+Y} \in \mathcal{S}$.*

Lemma A.3. *(see Lemma 1.3.5 in [Embrechts et al., 1997b])*
textit(a) If $F \in \mathcal{S}$ then, uniformly on compact y-sets of $(0, \infty)$,

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1. \tag{A.1}$$

(b) If (A.1) holds, then for all $\varepsilon > 0$,

$$e^{\varepsilon x}\overline{F}(x) \to \infty, \quad \text{x} \to \infty$$

(c) If $F \in \mathcal{S}$ then, given $\varepsilon > 0$, there exists a finite constant K so that for all $n \geqslant 2$

$$\frac{\overline{F^{n*}}(x)}{\overline{F}(x)} \leqslant K(1+\varepsilon)^n, \quad \text{x} \geqslant 0.$$

Lemma A.4. *(see Theorem 3.7 in [Foss et al., 2011]) Let the distribution F on $\mathbb{R}$ be long-tailed. Then the following are equivalent:*
(a) F is whole-line subexponential, i.e. $F \in \mathcal{S}_{\mathbb{R}}$.
(b) For every function h with $h(x) < x/2$ for all x and such that $h(x) \to \infty$ as $x \to \infty$,

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)\, dF(y) = o(\overline{F}(x)) \quad \text{as } x \to \infty. \tag{A.2}$$

(c) There exists a function h with $h(x) < x/2$ for all x, such that $h(x) \to \infty$ as $x \to \infty$ and F

is h-insensitive and the relation (A.2) holds.

Definition A.1. *(see Definition 3.5 in [Foss et al., 2011]) Let F be a distribution on $\mathbb{R}$ with right-unbounded support. We say that F is whole-line subexponential, and write $F \in \mathcal{S}_{\mathbb{R}}$, if F is long-tailed and*

$$\overline{F * F}(x) \sim 2\overline{F}(x) \quad \text{as } x \to \infty.$$

Lemma A.5. *(see Lemma 1 in [Kaas and Tang, 2003]) Let F be a d.f. supported on $(-\infty, +\infty)$ with finite $\int\limits_0^\infty \overline{F}(u)\,du$. If condition*

$$\lim_{x\to\infty} \frac{\overline{F_v^{*2}}(x)}{\overline{F_v}(x)} = 2$$

holds for some fixed $0 < v < \infty$, then $F \in \mathcal{S}$. Here

$$\overline{F_v}(x) = \begin{cases} \min\left\{1, \int\limits_x^{x+v} \overline{F}(u)\,\mathrm{d}u\right\} & \text{if } x \geqslant 0, \\ 1 \quad \text{if } x < 0. \end{cases}$$

for some $v > 0$.

Lemma A.6. *(see Lema 2 in [Chistyakov, 1964]) If a d.f. F of a non-negative r.v. belongs to the class $\mathcal{S}$, then F is long-tailed ($F \in \mathcal{S}$).*

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