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ON a-VALUES OF ZETA-FUNCTIONS

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Contents

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Notation

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{H}$	the upper half-plane of the complex plane
$\|x\|$	the absolute value of x
$[x]$	the integer part of x
$\{x\}$	the fractional part of x
$a_n \to a,\ n \to \infty$	the sequence (a_n) converges to a
$\lim_{n\to\infty} a_n = a$	same as above
$f_n(x) \to g(x),\ n \to \infty$	f converges to g pointwise
$\lim_{n\to\infty} f_n(x) = g(x)$	same as above
$f(x) = O(g(x))$	there exists a fixed $C > 0$ such that $\|f(x)\| \leq Cg(x)$ as $x \to \infty$
$f(x) \ll g(x)$	same as above
$f(x) \ll_\epsilon g(x)$	same as above except that the implicit constant depends on ϵ
$f(x) = o(g(x))$	the ratio $\|f(x)\|/g(x) \to 0$ as $x \to \infty$
(x_n)	a sequence of numbers x_n, $n = 1, 2, \ldots$

Chapter 1

Introduction

In this work, we use the notation $s = \sigma + it$ to denote a complex number. Here σ and t are real numbers, σ denotes the real part and t—the imaginary part of the complex number s. Suppose f is a function, mapping set X into set Y. We call x belonging to X an a-value of f (a belongs to Y) if and only if $f(x) = a$. In case $a = 0$, we call x a zero of f. Since we often use it later, we give the definition of the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This Dirichlet series converges absolutely in the complex half-plane $\{s \in \mathbb{C} : \sigma > 1\}$. The Riemann zeta-function has a simple pole with residue 1 at $s = 1$ and is defined as the meromorphic continuation into the rest of the complex plane.

As the title suggests, later on in this dissertation we investigate the distribution of a-values of zeta-functions, in particular the Selberg zeta-functions and the zeta-functions belonging to the extended Selberg class. Chapter 1 gives an account of the major novelties attained in this study together with the bibliographical data of our articles and the list of our conferences and visits. In Chapter 2, we discuss some well-known results which are later employed in our original proofs. In Chapter 3, we show that almost all a-values of Selberg zeta-functions associated to a compact Riemann surface are clustered around the critical line $\sigma = 1/2$. This result was proved in the joint work with our adviser Ramūnas Garunkštis [25]. Later on, we prove an analogous result for the Selberg zeta-function associated to a finite volume Riemann surface. Then we turn our attention to the vertical distribution of the a-values of Selberg zeta-functions associated to a compact Riemann surface. We show that the imaginary parts of these values are uniformly distributed modulo one. This result appeared in the joint work with Ramūnas Garunkštis and Jörn Steuding in [28]. In the end, we consider the relationship between the zeros of the functions belonging to the extended Selberg class and of their derivatives. We show that both of them have approximately the same number of zeros left of the critical line $\sigma = 1/2$. This result was obtained together with Ramūnas Garunkštis in [26].

1.1 Methods

The methods used in this dissertation mostly come from Complex Analysis. Almost all results employ the residue theorem which connects the value of the integral of some function f along a contour with the appropriately weighted sum of the residues of f inside the contour. Often we need to find the number of a-values of f inside some region. In order to do so, we employ the Littlewood's lemma. We also use the Hadamard's factorization theorem and the Jensen's theorem.

1.2 Novelties

The results obtained in this dissertation are all original. Most of them are based on some classical results. The following main theorems have been attained:

- This is Theorem 3.3. Suppose $N(a,T)$ is the number of non-trivial a-values of the Selberg zeta-function associated to a compact Riemann surface of genus $g \geq 2$ in the region $0 \leq \tau < t < T$. Here τ is such that the Selberg zeta-function does not have any trivial a-values in the region $\tau < t$. For $a \neq 1$,

  $$N(a,T) = (g-1)T^2 + o(T)$$

  and, for $a = 1$,

  $$N(1,T) = (g-1)T^2 - \frac{T}{2\pi}\log N(P_{00}) + o(T).$$

Here $N(P_{00})$ is the norm of a prime element in a certain conjugacy class of a Fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$, Γ giving rise to the Riemann surface.

- This is Theorem 3.5. Let $N^{-}(a,\delta,T)$ denote the number of non-trivial a-values of the Selberg zeta-function Z associated to a compact Riemann surface of genus $g \geq 2$ in the region $\tau < t < T$ and $\sigma < 1/2 - \delta$. The number $N^{+}(a,\delta,T)$ corresponds to the non-trivial a-values of Z in the region $\tau < t < T$ and $\sigma > 1/2 + \delta$, while the number $N_0(a,\delta,T)$ denotes the non-trivial a-values of Z in the region $\tau < t < T$ and $1/2 - \delta < \sigma < 1/2 + \delta$, where

$$\delta := \frac{(\log\log T)^2}{\log T}.$$

  We have

$$N^{-}(a,\delta,T) + N^{+}(a,\delta,T) \ll \frac{T^2}{\log\log T}$$

  and

$$N_0(a,\delta,T) = (g-1)T^2 + O\left(\frac{T^2}{\log\log T}\right).$$

- This is Theorem 4.2, which is a generalization of Theorem 3.3 for the Selberg zeta-function Z associated to a finite volume Riemann surface. We have the following estimate for the number of non-trivial a-values of Z up to T, that is, where the imaginary parts of the a-values are between 0 and $T > 0$, provided $a \neq 1$

$$N(a,T) = \frac{\mathrm{vol}(M)}{4\pi}T^2 - \frac{n_1}{\pi}T\log T + \frac{n_1 - \log\mathfrak{g}_1 - n_1\log 2}{\pi}T + o(T).$$

  If $a = 1$, then

$$\begin{aligned} N(1,T) = &\frac{\mathrm{vol}(M)}{4\pi}T^2 - \frac{n_1}{\pi}T\log T + \frac{n_1 - \log\mathfrak{g}_1 - n_1\log 2}{\pi}T \\ &- \frac{T}{2\pi}\log N(P_{00}) + o(T). \end{aligned}$$

  Here n_1 denotes the number of cusps of the Riemann surface. The real constants $d(1)$ and $\mathfrak{g}_1$ come from the scattering matrix determinant. They are defined in Section 4.1. The constant $N(P_{00})$ is defined as above.

- This is Theorem 4.3, which again is a generalization of Theorem 3.5 for the Selberg zeta-function associated to a finite volume Riemann surface. The following estimates hold

$$N^{-}(a,\delta,T)+N^{+}(a,\delta,T)\ll\frac{T^2}{\log\log T}$$

and

$$N^{0}(a,\delta,T)=\frac{\mathrm{vol}(M)}{4\pi}T^2+O\left(\frac{T^2}{\log\log T}\right).$$

Here the values $N^{-}(a,\delta,T)$, $N^{+}(a,\delta,T)$, and $N^{0}(a,\delta,T)$ are defined as above.

- This is Theorem 4.4. Suppose $\rho_a=\beta_a+i\gamma_a$ is an a-value of the Selberg zeta-function associated to finite volume Riemann surface. For non-trivial $0<\gamma_a\leq T$ and $a\neq 1$, we have

$$\sum_{0<\gamma_a\leq T}\left(\frac{1}{2}-\beta_a\right)=\frac{n_1}{4\pi}T\log T-\frac{T}{2\pi}\left(\frac{n_1}{2}+\frac{n_1}{2}\log\pi+\log|d(1)|-\log\mathfrak{g}_1\right)$$
$$-\frac{T}{2\pi}\log|1-a|+o(T),$$

and, for $a=1$ respectively,

$$\sum_{0<\gamma_a\leq T}\left(\frac{1}{2}-\beta_a\right)=\frac{n_1}{4\pi}T\log T-\frac{T}{2\pi}\left(\frac{n_1}{2}+\frac{n_1}{2}\log\pi+\log|d(1)|-\log\mathfrak{g}_1\right)$$
$$-\frac{T}{2\pi}\log m_0+\frac{T}{4\pi}\log N(P_{00})+o(T).$$

Again, for the precise meaning of the real constants $d(1)$ and $\mathfrak{g}_1$, see Section 4.1. The constants m_0 and $N(P_{00})$ are defined as above.

- This is Theorem 5.1. Let $a\in\mathbb{C}$. The imaginary parts of non-trivial a-points of the Selberg zeta-function $Z(s)$ associated to a compact Riemann surface are uniformly distributed modulo one. The precise meaning of the uniform distribution modulo one will be explained in Section 2.5.

- This is Theorem 6.2. Let F belong to the extended Selberg class, the degree of F strictly greater than 0, and $\sigma_0>\sigma_F$, σ_F being the least real number greater or equal to $1/2$ such that F does not have any non-trivial zeros in the region $\sigma > \sigma_F$. Let τ be such that F' does not have any zeros in the region $t > \tau$ and $\sigma < 1 - \sigma_F$. Let $N(T)$ and $N_1(T)$ respectively denote the number of zeros of $F(s)$ and $F'(s)$ in the region $\tau < t < T$, $\sigma < 1/2$. Then

$$N(T) = N_1(T) + O(\log T).$$

Moreover, if $N(T) < T/(2\sigma_0 - 1) + O(1)$, then there is a monotonic sequence $\{T_j\}$, $T_j \to \infty$, $j \to \infty$ such that

$$N(T_j) - N(T_1) = N_1(T_j) - N_1(T_1).$$

- This is Theorem 6.3. Let F be a non-constant function belonging to the extended Selberg class with degree 0. Let $N(T)$ and $N_1(T)$ respectively denote the number of zeros of F and F' in the region $0 < t < T$, $\sigma < 1/2$. Then

$$N(T) = N_1(T) + O(1).$$

1.3 Articles, conferences, and visits

For the reader's convenience, we reproduce the list of our publications:

- On the Speiser equivalent for the Riemann hypothesis. *Eur. J. Math.*, 1(2):337–350, 2015 (with Ramūnas Garunkštis).
- The a-points of the Selberg zeta-function are distributed uniformly modulo one. *Illinois J. Math.*, 58(1):207–218, 2014 (with Ramūnas Garunkštis and Jörn Steuding).
- The a-values of the Selberg Zeta-function. *Lith. Math. J.*, 52(2):145–154, April 2012 (with Ramūnas Garunkštis).

The following article is under submission:

- On the distribution of the a-values of the Selberg zeta-function associated to finite volume Riemann surfaces (with Ramūnas Garunkštis).

Here is the list of our conferences and visits:

- On the distribution of the a-values of the Selberg zeta-function, *5th International Conference on Uniform Distribution Theory*, University of West Hungary, Sopron, Hungary, July 5–8, 2016.
- On the distribution of the a-values of the Selberg zeta-function, *57th Lithuanian Mathematical Society Conference*, Vilnius Gediminas University of Technology, Vilnius, Lithuania, June 20–21, 2016.
- Research school *L-Functions and Automorphic Forms*, Heidelberg University, Heidelberg, Germany, February 17–26, 2016.
- Research school *Analytic Number Theory and Diophantine Geometry*, Hannover University, Hannover, Germany, September 7–11, 2015.
- Research school *Galois Theory and Number Theory*, Konstanz University, Konstanz, Germany, July 18–24, 2015.
- On the zeros of the extended Selberg class functions and of their derivatives, *56th Lithuanian Mathematical Society Conference*, Kaunas University of Technology, Kaunas, Lithuania, June 16–17, 2015.
- Visit at Würzburg University, Würzburg, Germany, November 24–30, 2014.
- On the zeros of the extended Selberg class functions and of their derivatives, *27th Journées Arithmétiques*, Vilnius University, Vilnius, Lithuania, June 27–July 1, 2011.

1.4 Acknowledgements

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Chapter 2

Classical theory

In this chapter, we discuss the classical results pertaining to our dissertation. As for the proofs in this section, we do not claim any originality on our part.

First, we discuss the historical context of the problems in this dissertation. Then we present Speiser's [86] result regarding the Riemann zeta-function. Speiser proves that the Riemann hypothesis is equivalent to the fact that the derivative of the Riemann zeta-function does not have any zeros left of the critical line $\sigma = 1/2$. Speiser's proof employs geometrical means. Later on, Levinson and Montgomery [62] came up with an analytic proof of the Speiser's result. Essentially, Levinson and Montgomery prove that the Riemann zeta-function and its derivative have approximately the same number of zeros left of the critical line. Later on in this chapter, we consider the Lindelöf hypothesis which is a consequence of the Riemann hypothesis. One of the most beautiful results in Analytic Number Theory is the Selberg trace formula, which we discuss next. In order to prove that a number sequence is uniformly distributed modulo one, we need the Weyl criterion. We give an account of that. We formulate and prove the Littlewood's lemma and the Jensen's theorem. Finally, we discuss a variety of results related to the a-value theory of zeta-functions, paying a particular attention to the case of the Riemann zeta-function.

2.1 Historical background

In this dissertation we discuss the distribution of the a-values of zeta-functions. The zeta-functions analyzed here are the Selberg zeta-functions for a compact Riemann surface and the Selberg zeta-function for a finite volume Riemann surface. In addition, we present a result concerning the trajectories of the zeros of the functions belonging to the extended Selberg class (not to be confused with Selberg zeta-functions) and of their derivatives.

Analytic number theory is the study of integers, first of all the prime numbers, by using the tools of mathematical analysis. Its beginning could be traced back to 19th century. It was the famous German mathematician Peter Gustav Lejeune Dirichlet who in 1837 used the so-called Dirichlet L-functions to prove the theorem that every arithmetic progression, whose first member and the common difference are co-prime, contains an infinite number of primes (see Dirichlet [14]).

The topics studied in this dissertation are originally related to the work of Bernhard Riemann, again a German mathematician. He published little. However, his publications were of enormous influence to the subsequent development of mathematics. In one of his papers, which appeared in 1859, he defined a function, which later became known as the Riemann zeta-function, and proved some of its properties, most importantly, its analytic continuation, the functional equation which it satisfies, and the Euler product (see Riemann [74]). Each of these concepts are discussed later. In addition, Bernhard Riemann claimed that all the non-trivial zeros of the Riemann zeta-function are located on a certain vertical line in the complex plane, called the critical line. This claim became known as the Riemann hypothesis. Here it is appropriate to note that x is a zero of a function f if and only if $f(x) = 0$.

Quite unexpectedly, the distribution of the zeros of the Riemann zeta-function turned out to be closely related to the distribution of primes. Suppose $\pi(x)$ is a function measuring the quantity of prime numbers up to x. The Prime Number Theorem in one of its forms claims that

$$\pi(x) \sim \frac{x}{\log x}.$$

The truth (or falsity) of the Riemann hypothesis plays a role in the error term in the Prime Number Theorem. What this exactly amounts to is explained in our discussion of this theorem.

The Prime Number Theorem in its modern form was first conjectured in 1797 or 1798 by the French mathematician Adrien-Marie Legendre, claiming that $\pi(x)$ is approximated by $x/(A \log x + B)$, where A and B are some constants. Later on in 1838, Dirichlet in his communication to the German mathematician Carl Friedrich Gauss stated that $\pi(x)$ is approximated by the logarithmic integral $\mathrm{li}(x) = \int_2^x (ds/\log s)$.

The Russian mathematician Pafnuty Lvovich Chebyshev in two papers dating from 1848 to 1850 proved a version of the asymptotic law of the distribution of primes. Quite importantly, in his proof Chebyshev used the Riemann zeta-function. This was about ten years before Riemann's article on this zeta-function. Differently from the Riemann's approach, Chebyshev considered the zeta-function only as a function of the real argument, while Riemann extended the study of this function to the complex plane. It should be mentioned that the zeta-function as a function of the real argument had already been introduced by the Swiss mathematician Leonhard Euler in 18th century.

Chebyshev's version of the asymptotic law of the distribution of primes states that, if the limit of the expression $\pi(x)/(x/\log x)$ as x approaches infinity exists at all, then it is necessarily equal to one. In addition, Chebyshev proved unconditionally that the ratio of $\pi(x)/(x/\log x)$ as x approaches infinity is bounded from above and below by some constants close to 1. Chebyshev's result is not quite the Prime Number Theorem, but it provides a sufficiently deep understanding about the distribution of primes to prove the Bertrand's postulate.

The strict version of the Bertrand's postulate states that for any integer $n > 3$, there exists a prime number p with the property that $n < p < 2n - 2$. This postulate was later proved by using more simple means by the Indian mathematician Srinivasa Ramanujan (see [70]) and, yet later, by the Hungarian mathematician Paul Erdős (see [19]).

The Prime Number Theorem as such was first proven independently by two French mathematicians Jacques Hadamard (see [32]) and Charles Jean de la Valée-Poussin (see [94]). In their proofs, both of these mathematicians used the Riemann zeta-function as defined by Riemann. They employed the fact that the Riemann zeta-function does not vanish on the vertical line $\sigma = 1$ of the complex plane.

Later on, different approaches to prove the Prime Number Theorem have been used. In 1949, the Norwegian mathematician Atle Selberg and Erdős proved the theorem by using elementary methods (see [79, 20]). The shortest proof of this theorem up to date was discovered by the American mathematician Donald J. Newman in 1980.

The Riemann hypothesis turned out to be closely linked to the distribution of primes. At the turn of the previous century, in 1901, the Swedish mathematician Helge von Koch (see [52]) noticed this relationship. He proved that if the Riemann hypothesis was correct, then the error in the Prime Number Theorem is in a sense minimal. In 1976, this result was refined by the American mathematician Lowell Schoenfeld (see [77]), claiming that

$$|\pi(x) - \mathrm{li}(x)| < \frac{1}{8\pi}\sqrt{x}\log x,$$

here $x \geq 2657$.

2.2 Lindelöf hypothesis

In this section we discuss the Lindelöf hypothesis and its equivalents. Our discussion is based on Titchmarsh [92, Chapter 13]. As for the motivational part of this section, Theorem 2.4 due to Backlund [3] relates the number of the zeros of the Riemann zeta-function right of the critical line and the Lindelöf hypothesis. Later on, Garunkštis [23] extended this result to the case of any a-values, not only zeros. For a more detailed discussion on this topic, see the beginning of Chapter 3.

In a nutshell, the Lindelöf hypothesis specifies the growth of the Riemann zeta-function on the critical line as

$$\zeta(1/2 + it) = O(t^{\epsilon})$$

for any $\epsilon > 0$. It turns out that the Lindelöf hypothesis follows from the Riemann hypothesis. However, it is thought that the converse is not necessarily true, so the latter is a stronger statement than the former.

In Hardy and Littlewood [33], the following equivalents for the Lindelöf hypothesis were obtained. This is also Theorem 13.2 in Titchmarsh [92].

Theorem 2.1. *The following statements are equivalent to the Lindelöf hypothesis*

$$\frac{1}{T}\int_1^T |\zeta(1/2 + it)|^{2k}\ dt = O(T^{\epsilon}),\ \ (k = 1, 2, \ldots);$$

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T^\epsilon),\ (\sigma > 1/2,\ k = 1, 2, \ldots);$$

and

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt \sim \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}},\ (\sigma > 1/2,\ k = 1, 2, \ldots).$$

Here $d_k(n)$ denotes the number of ways of expressing n as product of k factors, $k = 2, 3, \ldots$.

Here is another theorem about the necessary and sufficient conditions of the Lindelöf hypothesis. This is Theorem 13.3 in Titchmarsh [92].

Theorem 2.2. *The Lindelöf hypothesis is equivalent to*

$$\zeta^k(s) = \sum_{n \le t^\delta} \frac{d_k(n)}{n^s} + O(t^{-\lambda}).$$

Here k is any integer, $\sigma > 1/2$, $t > 0$, δ is any fixed positive number less that 1, and $\lambda = \lambda(k, \delta, \sigma) > 0$.

Now we are ready to move on to the next set of conditions for the Lindelöf hypothesis. Let us introduce $D_k(x)$ as

$$D_k(x) := \sum_{n \le x} d_k(n).$$

Suppose $d(n)$ denotes the number of divisors of n, including 1 and n. Then define

$$D(x) := \sum_{n \le x} d(n).$$

Dirichlet proved that

$$D(x) = x \log x + (2\gamma - 1)x + O(x^{1/2}).$$

Here γ is the Euler constant. Define Δ by

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x).$$

So we have $\Delta(x) = O(x^{1/2})$. Assume $P_k(x)$ is a polynomial of degree $k-1$. We have

$$D_k(x) = xP_k(\log x) + \Delta_k(x).$$

Obviously, $\Delta_2(x) = \Delta(x)$. In [54], Landau proved that

$$\Delta_k(x) = O(x^{1-1/k} \log^{k-2} x).$$

Let us define the *order* α_k of $\Delta_k(x)$ as the least number satisfying

$$\Delta_k(x) = O(x^{\alpha_k+\epsilon})$$

for any $\epsilon > 0$.

Let us introduce the notion of the *average order of* $\Delta_k(x)$, which we denote by β_k, as the least number satisfying

$$\frac{1}{x}\int_0^x \Delta_k^2(y)\,dy = O(x^{2\beta_k+\epsilon}),$$

again, ϵ is any positive number.

We have the theorem (see Titchmarsh [92, Theorem 13.4])

Theorem 2.3. *Lindelöf hypothesis is equivalent to the following conditions on the numbers α_k and β_k*

$$\alpha_k \le \frac{1}{2}, \ \ (k = 2, 3, \ldots);$$

$$\beta_k \le \frac{1}{2}, \ \ (k = 2, 3, \ldots);$$

and

$$\beta_k = \frac{k-1}{2k}, \ \ (k = 2, 3, \ldots).$$

The following theorem is due to Backlund [3] (see also Titchmarsh [92, Theorem 13.5]).

Theorem 2.4. *Let $N(\sigma_0, T)$ denote the number of zeros of the Riemann zeta-function ζ in the region $\sigma > \sigma_0$ and $0 < t < T$. Then the Lindelöf hypothesis is equivalent to*

$$N(\sigma_0, T+1) - N(\sigma_0, T) = o(\log T)$$

for any $\sigma_0 > 1/2$.

2.3 Selberg zeta-function

First, we discuss Selberg zeta-function asscociated to a compact Riemann surface. Let F denote a compact Riemann surface of genus $g \geq 2$. The surface F can be represented as a quotient space $\Gamma \backslash \mathbb{H}$, where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a strictly hyperbolic Fuchsian group and $\mathbb{H}$ is the upper half-plane. The Γ conjugacy class determined by $P \in \Gamma$ will be denoted by $\{P\}$ and its norm is defined by $N(\{P\}) = N(P) = \alpha^2$, if the eigenvalues of P are α and α^{-1} ($|\alpha| > 1$). By P_0 we denote the primitive element of Γ. The Selberg zeta-function for $\sigma > 1$ is given by (Hejhal [35, Chapter 2, Definition 4.1])

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}). \tag{2.1}$$

By the Selberg trace formula the Selberg zeta-function is extended to an entire function (Hejhal [35, Chapter 2, Theorem 4.11]) of order 2 with the functional equation (Hejhal [35, Chapter 2, Theorem 4.12])

$$Z(s) = X(s)Z(1-s), \tag{2.2}$$

where

$$X(s) = \exp\left(4\pi(g-1)\int_0^{s-\frac{1}{2}} v\tan(\pi v)\, dv\right). \tag{2.3}$$

The so-called trivial zeros of Z are located at $1, 0, -1, -2, \ldots$ and non-trivial zeros on the critical line $\sigma = 1/2$ with at most finitely many exceptions of zeros on the real segment $0 < s < 1$ (Hejhal [35, §2.4, Theorem 4.11] and Randol [71]). All non-trivial zeros $s_j = 1/2 \pm it_j$ correspond to eigenvalues

$$0 < \lambda_j = s_j(1 - s_j) = 1/4 + t_j^2 \tag{2.4}$$

of the hyperbolic Laplacian Δ on $M = \Gamma\backslash\mathbb{H}$ (Hejhal [35, §2.4, Theorem 4.11]).

Now let us proceed to the definition of the Selberg zeta-function associated to a finite volume Riemann surface. This discussion is based on Hejhal [36] and Jorgenson and Smajlović [44]. The definition is very similar to that of the Selberg zeta-function associated to a compact Riemann surface. However, for the sake of completeness, we provide the full definition of the Selberg zeta-function associated to a finite volume Riemann surface. Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian group of the first kind acting on the upper half-plane $\mathbb{H}$. Then $M := \Gamma\backslash\mathbb{H}$ denotes the Riemann surface generated by Γ. There exists a canonic metric with curvature -1 acting on the upper half-plane. This metric induces a metric acting on M with $\mathrm{vol}(M)$ finite. By n_1 we will always denote the number of cusps of M.

Suppose $\mathcal{H}(\Gamma)$ is the set of all representatives of inconjugate hyperbolic elements of Γ. For each $P \in \mathcal{H}(\Gamma)$, there exists $P_0 \in \mathcal{H}(\Gamma)$ such that $P = P_0^n$. Then P_0 is called a primitive element of $\mathcal{H}(\Gamma)$. The norm of P is defined by $N(P) = \alpha^2$ if the eigenvalues of P are α and α^{-1} ($|\alpha| > 1$). The Selberg zeta-function for M, which is associated to Γ, is defined by the following Euler product (see Hejhal [36, Section 10.5] with $m = 0$, $r = 1$, and $W = \mathrm{Id}$ in the notations of [36])

$$Z(s) = \prod_{P_0 \in \mathcal{H}(\Gamma)} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}). \tag{2.5}$$

Here Z also depends on the underlying Riemann surface M. However, here and later on we implicitly assume that the relevant objects depend on M and omit the subscript M. The Euler product (2.5) converges absolutely for $\sigma > 1$. Z has a meromorphic continuation into the rest of the complex plane (see Hejhal [36, Section 10.5, Theorem 5.3]).

Here a discussion on the zeros of the Selberg zeta-function associated to a finite volume Riemann surface is in order. The Selberg zeta-function Z associated to a finite volume Riemann surface satisfies the functional equation $Z(s)\phi(s) = \eta(s)Z(1-s)$. For the precise definition of the factors $\phi(s)$ and $\eta(s)$, see Section 4.1. Some well-understood zeros and poles of Z lying on the real line come from the factor η. Other zeros are related to the discrete eigenvalues of the Laplacian. They lie on the critical line $\sigma = 1/2$. The remaining zeros of Z are the poles of the scattering matrix determinant ϕ. They are located in the half-plane $\sigma < 1/2$ aside from a finite number in $(1/2, 1]$ (see Hejhal [36], formula (2.10), p.437 and Hejhal [36], Theorem 5.3, formula (5.3), p.498). The total number of zeros, counting with multiplicities, of Z in the region $0 < t < T$ is asymptotically $\mathrm{vol}(M)T^2/4\pi$.

The strong version of the Selberg conjecture claims that there exists some constant $\delta > 0$ such that the number of zeros of Z (or the number of poles of ϕ) in the region $t > 0$, $\sigma < 1/2$ is $O(T^{2-\delta})$. In the case $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, Selberg proved that (see Hejhal [36, p.508, formulas (2.2), (2.3), (2.4)])

$$\phi(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}.$$

Therefore, in the region $0 < t \leq T$, $\sigma < 1/2$, the function Z has asymptotically $\frac{T}{\pi}\log T$ zeros. Analogous bounds are known for congruence subgroups (see Huxley [40], Hejhal [36], pp.532-538).

Suppose $\rho = \beta + i\gamma$ is a zero of Z such that $\beta, \gamma \in \mathbb{R}$. Selberg showed that (see Hejhal [36, p.456, Theorem 2.22], Selberg [82], Jorgenson and Smajlović [44, formula (91)])

$$\begin{aligned}\sum_{\substack{0<\gamma\leq T\\ \beta<1/2}}\left(\frac{1}{2}-\beta\right) =& \frac{n_1}{4\pi}T\log T - \frac{T}{2\pi}\left(\frac{n_1}{2}+\frac{n_1}{2}\log\pi + \log|d(1)| - \log\mathfrak{g}_1\right) \\ &+ O(\log T).\end{aligned} \tag{2.6}$$

We recall that n_1 is the number of cusps of M, constants $d(1)$ and $\mathfrak{g}_1$ are related to the scattering matrix determinant ϕ, see formula (4.6) in Section 4.1. This result can be regarded as supporting the Selberg conjecture. A version of formula (2.6) for higher dimensional hyperbolic spaces is obtained in Kelmer [51, Theorem 1.3].

However, in [67] and [68], Phillips and Sarnak proved that the Selberg conjecture is false for generic noncompact M, provided that certain “standard conjectures” are true. They also guess that the Selberg zeta-function has finitely many zeros on the line $\sigma = 1/2$ for generic noncompact M.

2.4 Selberg trace formula

In this section, we provide a somewhat in depth discussion of the Selberg trace formula related to a compact Riemann surface. A detailed account of this formula is Hejhal [35]. Our overview is based on Elstrodt [18]. Selberg himself left few publications on his trace formula, out of which perhaps the most cited is [80].

Suppose M is a compact Riemann surface of genus $g \geq 2$. Informally, a Riemann surface is a complex manifold of dimension one, and its genus is the number of holes the surface contains. According to the uniformization theory, M is conformally equivalent to the quotient $\Gamma\backslash\mathbb{H}$, where Γ is a certain subgroup of the projective special linear group $\mathrm{PSL}(2,\mathbb{R})$, the latter consisting of 2 by 2 matrices with real coefficients and determinant 1. $\mathbb{H}$ is the upper complex half-plane. The notation $\Gamma\backslash\mathbb{H}$ means that the points in the upper complex half-plane are partitioned into equivalence classes such that $x \sim y$ if and only if there exists $A \in \Gamma$ such that $A(x) = y$. The map $A : \mathbb{H} \to \mathbb{H}$ is defined as

$$A(x) = \frac{a_{11}x + a_{12}}{a_{21}x + a_{22}},$$

where a_{11}, a_{12}, a_{21}, a_{22} are the coefficients of the matrix A. The requirements on Γ are that Γ be discontinuous and free of fixed points. Discontinuity means that the set of matrices belonging to Γ viewed as a subset of $\mathrm{PSL}(2,\mathbb{R})$ is discrete. The condition that Γ is free of fixed points means that Γ does not contain a matrix A other than identity such that there exists $x \in \mathbb{H}$ with $A(x) = x$.

Let us introduce the notion of the fundamental domain $\mathcal{F} \subset \mathbb{H}$ associated to the group Γ. The fundamental domain $\mathcal{F}$ contains exactly one representative from each of the equivalence classes of $\Gamma\backslash\mathbb{H}$. The fundamental domain $\mathcal{F}$ could be chosen so that it is a relatively compact subset of $\mathbb{H}$.

The upper complex half-plane is endowed with a $\mathrm{PSL}(2,\mathbb{R})$-invariant hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Here $s = x + iy$, $s \in \mathbb{H}$, $x \in \mathbb{R}$, and $y > 0$. Attached to this metric, we get a $\mathrm{PSL}(2,\mathbb{R})$-invariant area measure ω

$$d\omega = \frac{dx\,dy}{y^2}.$$

The $\mathrm{PSL}(2,\mathbb{R})$-invariant Laplace-Beltrami operator is defined as

$$\Delta := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The fact that the Laplace-Beltrami operator Δ is $\mathrm{PSL}(2,\mathbb{R})$-invariant means

$$\Delta(f \circ S) = (\Delta f) \circ S,$$

where $f \in C^2(\mathbb{H})$ and $S \in \mathrm{PSL}(2,\mathbb{R})$.

The symbol Δ is a linear operator in the Hilbert space

$$\mathcal{H} = \left\{ f : \mathbb{H} \to \mathbb{C} : f \text{ is measurable, } \Gamma\text{-invariant }, \int_{\mathcal{F}} |f|^2 \, d\omega < \infty \right\} \cong$$
$$\cong L^2(M).$$

The domain of Δ is

$$\mathcal{D} = \{ f \in \mathcal{H} : f \text{ is twice continuously differentiable} \}.$$

So Δ is a linear operator mapping $\mathcal{D}$ into $\mathcal{H}$.

Suppose $-\Delta$ is a self-adjoint operator to Δ. Then we have the following theorem regarding the spectrum of $-\Delta$

Theorem 2.5. *The operator $-\Delta : \mathcal{D} \to \mathcal{H}$ has an orthnormal system $\{\phi_n\}_{n\geq 0}$ of real eigenfunctions with eigenvalues*

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

satisfying

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty.$$

It is worth noting that Γ does not contain any parabolic elements, that is, elements with the property $|a_{11} + a_{22}| = 2$, or elliptic elements with $|a_{11} + a_{22}| < 2$. This result can be proved using the facts that X is compact and that Γ is free of fixed points while acting on $\mathbb{H}$. Thus all the non-identity elements of Γ are hyperbolic with $|a_{11} + a_{22}| > 2$. Such hyperbolic elements contain exactly two fixed points in the set $\mathbb{R} \cup \infty$. If one transforms these fixed points by a fractional linear transformation associated with $P \in \mathrm{PSL}(2, \mathbb{R})$ so that they map to the fixed points to 0 or ∞, one gets that P is associated to the map

$$s \mapsto N(P)s$$

with $N(P) > 1$, which we call the *norm* of P. It can be shown that $N(P)$ equals to the square of the greater of the two eigenvalues of the matrix P.

The elements belonging to the same conjugacy class $\{P\}_\Gamma$ have the same norm, which we call the norm of $\{P\}_\Gamma$. Now consider the elements of Γ which leave both of the fixed points of P intact. These elements form an infinite cyclic subgroup of Γ. Out of the two generators of this subgroup, we choose P_0 satisfying

$$P = P_0^m$$

for some natural $m \geq 1$. We call P_0 the *primitive transformation* belonging to P. Observe that as Q runs through all the primitive elements of $\Gamma \backslash I$, I denoting the identity matrix, the powers Q^m with $m = 1, 2, \ldots$ run through all the elements of $\Gamma \backslash I$, and assume the value of every such element exactly once. Hence the primitive transformations of Γ play a similar role to that of the prime numbers in $\mathbb{N}$.

Suppose λ_n, $n \geq 0$ are the eigenvalues of the operator $-\Delta$. Then write

$$\lambda_n = \frac{1}{4} + r_n^2.$$

We get the following theorem about the Selberg trace formula (see Elstrodt [18, Proposition 3.1])

Theorem 2.6. *Suppose*

$$S_\epsilon := \left\{ r \in \mathbb{C} : \Im r < \left| \frac{1}{2} + \epsilon \right| \right\},$$

where $\epsilon > 0$. *Let* $h : S_\epsilon \to \mathbb{C}$ *is a holomorphic function which satisfies the growth condition as* $|r| \to \infty$ *in the strip* S_ϵ

$$h(r) = O((1 + |r|^2)^{-1-\epsilon})$$

uniformly in r. *Define the Fourier transformation*

$$g(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iru} h(r)\, dr.$$

Then we have the Selberg trace formula

$$\sum_{n=0}^{\infty} h(r_n) = \frac{\omega(\mathcal{F})}{4\pi} \int_{-\infty}^{\infty} h(r) r \tanh(\pi r)\, dr + \\ + \sum_{\{P\}_\Gamma} \frac{\log N(P_0)}{N(P)^{1/2} - N(P)^{-1/2}} g(\log N(P)).$$

Here $\{P\}_\Gamma$ *and* P_0 *are as defined above. In the formula, both sums and the* *integral converge absolutely.*

Ideas for proof. First, we introduce the notion of *point-pair invariants*, that is, functions $k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ obeying the rule

$$k(Sx, Sy) = k(x, y)$$

for all $S \in \mathrm{PSL}(2, \mathbb{R})$. It could be shown that any such invariant could be written in the form

$$k(z, w) = \psi\left(\frac{|z - w|^2}{\Im z \Im w}\right)$$

with an appropriate function $\psi : [0, \infty) \to \mathbb{C}$. Conversely, any k of the above form is a point-pair invariant.

Let us take a continuous function $\psi : [0, \infty) \to \mathbb{C}$ with compact support. Define the kernel function K as

$$K(z, w) := \sum_{M \in \Gamma} k(z, Mw).$$

Now we can get to the linear operator $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ by

$$(\mathcal{K}(f))(z) := \int_{\mathcal{F}} K(z, w) f(w)\, d\omega(w)$$

with $f \in \mathcal{H}$, $z \in \mathbb{H}$ and $d\omega$ as the hyperbolic area differential.

It can be proved that the eigenfunctions of the Laplace-Beltrami operator Δ are also eigenvalues of the operator $\mathcal{K}$. Let us denote such eigenfunctions by ϕ_n. Further, we could obtain

$$\mathcal{K}\phi_n = h(r_n)\phi_n. \tag{2.7}$$

Here $h : \mathbb{C} \to \mathbb{C}$ is a certain function which we do not discuss here. We have $K(z, \cdot) \in \mathcal{H}$ and

$$K(z, \cdot) = \sum_{n=0}^{\infty} c_n(z)\phi_n.$$

We can calculate the coefficients c_n explicitly. Suppose $\langle \cdot, \cdot \rangle$ is a scalar product of functions belonging to $\mathcal{H}$. We get

$$c_n(z) = \langle K(z,\cdot), \phi_n \rangle = \int_{\mathcal{F}} K(z,w)\phi_n(w)\, d\omega(w) =$$
$$= (\mathcal{K}\phi_n)(z) = h(r_n)\phi_n(z).$$

The last equality holds by (2.7). So we have

$$K(z,w) = \sum_{n=0}^{\infty} h(r_n)\phi_n(z)\phi_n(w).$$

Here we do not discuss the convergence of the above series. By integrating both sides of the above, we get

$$\sum_{n=0}^{\infty} h(r_n) = \int_{\mathcal{F}} K(z,z)\, d\omega(z).$$

This is our trace formula. □

2.5 Weyl criterion

In this section we develop the theory leading to the Weyl criterion concerning uniform distribution modulo one of sequences. Our discussion is based on the book by Kuipers and Niederreiter [53]. The concept of the uniform distribution modulo one was first introduced by Weyl [96, 97].

First, let us proceed to the basic definitions. Assume $x \in \mathbb{R}$. Then by $[x]$ we denote the *integer part* of x, which is the greatest integer less or equal to x. By $\{x\}$ we denote the *fractional part* of x, given by $\{x\} := x - [x]$. Obviously, $\{x\} \in [0,1) := I$.

Suppose $\omega = (x_n)$ is a sequence of real numbers. Suppose $A(E;N;\omega)$ is a counting function, which counts how many members of ω with $1 \le n \le N$ lie inside the set $E \subset I$. A real sequence ω is said to be *uniformly distributed modulo one* if and only if for any pair of real numbers a and b, satisfying $0 \le a < b \le 1$, we have

$$\frac{A([a,b);N;\omega_1)}{n} \to b - a, \ \ N \to \infty,$$

where ω_1 is the sequence of the fractional parts of the members of ω.

Let us now reformulate the definition above. Suppose $c_{[a,b)}$ is the characteristic function, where $[a,b) \subseteq I$, that is, $c_{[a,b)}(x) = 1$ if $x \in [a,b)$ and $c_{[a,b)}(x) = 0$ otherwise. Then the definition of the uniform distribution modulo one becomes

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{[a,b)}(\{x_n\}) = \int_0^1 c_{[a,b)}(x)\, dx.$$

The following theorem obtains (see Kuipers and Niederreiter [53, Theorem 1.1])

Theorem 2.7. *The sequence ω is uniformly distributed modulo one if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\, dx$$

for any real-valued continuous function defined on $[0,1]$.

We get the corollary (see Kuipers and Niederreiter [53, Corollary 1.2])

Corollary 2.8. *The sequence ω is uniformly distributed modulo one if and only if for any $f : \mathbb{R} \to \mathbb{C}$ with period* 1, *we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\, dx. \tag{2.8}$$

Observe that functions $e^{2\pi i h x}$ with h a non-zero integer and real x satisfy the conditions of Corollary 2.8. This leads to the theorem (see Kuipers and Niederreiter [53, Theorem 2.1])

Theorem 2.9. (Weyl criterion) *The sequence ω is uniformly distributed modulo one if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0, \tag{2.9}$$

for any non-zero integer h.

Proof. The fact that the condition is necessary follows immediately from Corollary 2.8. It remains to demonstrate the sufficiency. Suppose (2.9) holds. We need to show that if (2.9) holds, then for any f satisfying the conditions in Corollary 2.8, the condition (2.8) obtains.

Assume f is any function satisfying the conditions in Corollary 2.8. Let $\epsilon > 0$ be arbitrary. From the Weierstrass approximation theorem it follows that there exists a trigonometric polynomial $\Psi(x)$ with complex coefficients satisfying

$$\sup_{0 \le x \le 1} |f(x) - \Psi(x)| < \epsilon.$$

We get

$$\begin{aligned} \left| \int_0^1 f(x)\,dx - \frac{1}{N}\sum_{n=1}^{N} f(x_n) \right| \le & \left| \int_0^1 (f(x) - \Psi(x))\,dx \right| + \\ & + \left| \int_0^1 \Psi(x)\,dx - \frac{1}{N}\sum_{n=1}^{N} \Psi(x_n) \right| + \\ & + \left| \frac{1}{N}\sum_{n=1}^{N} (f(x_n) - \Psi(x_n)) \right|. \end{aligned}$$

See that the first and the third term on the right of the above inequality are less than ϵ by the choice of Ψ. The second term on the right is small because of (2.9) and Ψ, being a trigonometric polynomial, can be expressed as a linear combination of the powers of e. □

There is a generalization of the concept of the uniform distribution modulo one. It is called *uniform distribution modulo a subdivision*. This concept was introduced by LeVeque [59]. The topic has been studied by Cigler [10], Davenport and LeVeque [12], Davenport et al. [13], and Schmidt [76]. Define $\Delta : 0 = z_0 < z_1 < \ldots$ as a subdivision of the interval $[0, \infty)$ and $z_k \to \infty$ as $k \to \infty$. Let us generalize the notions of the integral and the fractional parts of a real number x, satisfying $z_{k-1} \le x < z_k$, in the following way

$$[x]_\Delta := z_{k-1}$$

and

$$\{x\}_\Delta := \frac{x - z_{k-1}}{z_k - z_{k-1}}.$$

Observe that $0 \le \{x\}_\Delta < 1$.

We call a sequence of nonnegative real numbers (x_n), $n = 1, 2, \ldots$ *uni-*

formly distributed modulo Δ if and only if the sequence $(\{x_n\}_\Delta)$ is uniformly distributed modulo one. Obviously, the concept of the uniform distribution modulo subdivision Γ reduces to the familiar concept of the uniform distribution modulo one if for all k we have $z_k = k$.

Now suppose we have an increasing sequence of nonnegative real numbers (x_n) with $x_n \to \infty$ as $n \to \infty$. Define $A(x, \alpha)$ to be the number of $x_n < x$ such that $\{x_n\}_\Delta < \alpha$ and $A(x) := A(x, 1)$. Then the sequence (x_n) is uniformly distributed modulo one if and only if

$$\lim_{x\to\infty} \frac{A(x,\alpha)}{A(x)} = \alpha,$$

for all $\alpha \in (0, 1)$.

We have the theorem (Kuipers and Niederreiter [53, Theorem 1.3])

Theorem 2.10. *Suppose* (x_n) *is an unboundedly increasing sequence of nonnegative numbers. Then* (x_n) *is uniformly distributed modulo subdivision* Δ *only if*

$$\lim_{k\to\infty} \frac{A(z_{k+1})}{A(z_k)} = 1.$$

2.6 Results from Complex Analysis

In this section we introduce several results from Complex Analysis which we later use in our proofs. First, we discuss the Littlewood's lemma. Then we turn our attention to the Jensen's theorem.

Theorem 2.11. (Littlewood's lemma, see Titchmarsh [92, Section 9.9]) *Suppose* $\mathcal{C}$ *is a rectangle in the complex plane with vertices* σ_0, σ_1, $\sigma_1 + iT$, *and* $\sigma_0 + iT$ *with* $\sigma_0 < \sigma_1$. *Let* f *is an analytic function which does not vanish on the border of* $\mathcal{C}$. *Let* $\rho = \beta + i\gamma$ *is a zero of* f *belonging to* $\mathcal{C}$ *with* $\rho \in \mathbb{C}$ *and* β, $\gamma \in \mathbb{R}$. *Then*

$$2\pi \sum_{\rho\in\mathcal{C}} (\beta - \sigma_0) = \int_0^T \log|f(\sigma_0 + it)|\,dt - \int_0^T \log|f(\sigma_1 + it)|\,dt + \\ + \int_{\sigma_0}^{\sigma_1} \arg f(\sigma + iT)\,d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma)\,d\sigma.$$

Proof. Assume $\mathcal{C}'$ is the contour involving $\mathcal{C}$ together with the loops around each zero ρ, denoted by $\mathcal{L}_\rho$. We have

$$\int_{\mathcal{C}} \log f(s)\, ds = \int_{\mathcal{C}'} \log f(s)\, ds + \sum_{\rho\in\mathcal{C}} \int_{\mathcal{L}_\rho} \log f(s)\, ds.$$

Observe that $\log f(s)$ is analytic in $\mathcal{C}'$, so we get

$$\int_{\mathcal{C}'} \log f(s)\, ds = 0.$$

This leads to

$$\int_{\mathcal{C}} \log f(s)\, ds = \sum_{\rho\in\mathcal{C}} \int_{\mathcal{L}_\rho} \log f(s)\, ds.$$

As usual, for the zeros of f we use the notation $\rho = \beta + i\gamma$. Suppose the radius of $\mathcal{L}_\rho$ is r. Then

$$\int_{\mathcal{L}_\rho} \log f(s)\, ds = \int_{\sigma_0}^{\beta-r} \log f(\sigma + i\gamma^-)\, d\sigma + \int_0^{2\pi} \log f(re^{i\theta}) ire^{i\theta}\, d\theta - \\ - \int_{\sigma_0}^{\beta-r} \log f(\sigma + i\gamma^+)\, d\sigma.$$

It is easy to see that the integral $\int_0^{2\pi} \log f(re^{i\theta}) ire^{i\theta}\, d\theta \to 0$ as $r \to 0^+$. We also have

$$\int_{\sigma_0}^{\beta-r} \log f(\sigma + i\gamma^+)\, d\sigma = \int_{\sigma_0}^{\beta-r} (f(\sigma + i\gamma^-) + 2\pi i)\, d\sigma.$$

Now

$$\int_{\mathcal{L}_\rho} \log f(s)\, ds \to -2\pi i \int_{\sigma_0}^{\beta} d\sigma = -2\pi i(\beta - \sigma_0),\ \ r \to 0^+.$$

Hence

$$\int_{\mathcal{C}} \log f(s)\, ds = -2\pi i \sum_{\rho\in\mathcal{C}} (\beta - \sigma_0).$$

We obtain

$$-2\pi i \sum_{\rho\in\mathcal{C}} (\beta - \sigma_0) = \int_0^T \log f(\sigma_1 + it) i\, dt - \int_0^T \log f(\sigma_0 + it) i\, dt + \\ + \int_{\sigma_0}^{\sigma_1} \log f(\sigma)\, d\sigma - \int_{\sigma_0}^{\sigma_1} f(\sigma + iT)\, d\sigma.$$

The theorem follows by equating the imaginary parts. □

Jensen's theorem, which is stated below, relates the distribution of zeros of some holomorphic function f inside a disc with the behavior of f along the border of the disc. Our discussion of this theorem is based on Rudin [75]. Originally, the theorem was proved in Jensen [43].

In order to prove the Jensen's theorem, first we need the concept of a harmonic function. Suppose f is a complex function on an open set Ω belonging to the complex plane. Suppose second order derivatives $f_{\sigma\sigma}$ and f_{tt} exist at every point in Ω. Suppose Δ is the Laplacian operator, that is, $\Delta f = f_{\sigma\sigma} + f_{tt}$. The function f is called *harmonic* if and only if f is continuous in Ω and $\Delta f = 0$ at every point of Ω.

In addition, we need the following theorem and lemma, which we give without proof

Theorem 2.12. (see Rudin [75, Theorem 13.12]) *Suppose Ω is an open set in the complex plane. Suppose f is holomorphic in Ω and f does not vanish in Ω. Then $\log|f|$ is harmonic in Ω.*

Lemma 2.13. (see Rudin [75, Lemma 15.17])

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1-e^{i\theta}|\, d\theta = 0.$$

Theorem 2.14. (Jensen's theorem, see Rudin [75, Theorem 15.18]) *Suppose $\Omega = D(0;R)$ is a disc with the center on the origin of the complex plane and radius R. Suppose f is a holomorphic function on Ω, $f(0) \neq 0$, $0 < r < R$, and $\alpha_1, \ldots, \alpha_N$ are the zeros of f in $\overline{D}(0;r)$. Then the following holds*

$$|f(0)| \prod_{n=1}^{N} \frac{r}{|\alpha_n|} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left|f(re^{i\theta})\right|\, d\theta \right\}.$$

Proof. Suppose $D(0;r)$ is an open disc with its center at the origin and radius r. First, order the points α_j so that α_1, ..., α_m belong to $D(0;r)$ and α_{m+1}, ..., α_N lie on the border of $\overline{D}(0;r)$. Define

$$g(z) := f(z) \prod_{n=1}^{m} \frac{r^2 - \overline{\alpha}_n z}{r(\alpha_n - z)} \prod_{n=m+1}^{N} \frac{\alpha_n}{\alpha_n - z}. \tag{2.10}$$

It follows that g is a holomorphic function on $D := D(0; r+\epsilon)$ for some $\epsilon > 0$.

In addition, g has no zero in D. By Theorem 2.12, $\log|g|$ is harmonic in D. By a standard result from Complex Analysis, we get

$$\log|g(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g(re^{i\theta})|\,d\theta. \tag{2.11}$$

By (2.10),

$$|g(0)| = |f(0)|\prod_{n=1}^{m}\frac{r}{|\alpha_n|}. \tag{2.12}$$

The factors in (2.10) have absolute value 1 for $1 \leq n \leq m$, provided $|z| = r$. As for $\alpha_n = re^{i\theta_n}$ for $m < n \leq N$, we get

$$\log|g(re^{i\theta})| = \log|f(re^{i\theta})| - \sum_{n=m+1}^{N}\log|1 - e^{i(\theta-\theta_n)}|.$$

By Lemma 2.13, f and g are interchangeable in (2.11). The Theorem follows from (2.12). □

One more theorem which we use throughout our dissertation is the *Rouché's theorem*. We state it without proof

Theorem 2.15. *Suppose f and g are complex holomorphic functions inside and on some closed contour K. In addition, suppose $|g(s)| < |f(s)|$ on K. Then f and $f+g$ have the same number of zeros inside K, counted with multiplicities.*

In addition, we make use of the Phragmén-Lindelöf principle. The general case of the principle deals with sectors of the complex plane. For our purposes, the special case dealing with strips of the complex plane suffices.

Theorem 2.16. *Suppose Ω is a half-strip of the complex plane such that*

$$\Omega = \{s \in \mathbb{C} : \sigma_1 \leq \Re s \leq \sigma_2, \Im s \geq t_0\}.$$

Suppose f is holomorphic on Ω and that there exist constants N, A, and B such that

$$|f(s)| \leq N \text{ for all } s \in \partial\Omega$$

and

$$\frac{|f(\sigma+it)|}{t^A} \leq B \text{ for all } \sigma+it \in \Omega.$$

Then f is bounded by N on all of Ω.

2.7 Notes on *a*-value theory and universality

A detailed account of a-value theory of zeta-functions can be found in a book by Steuding [89]. In the first part of this section, we cite results primarily concerned with the a-value distribution of the Riemann zeta-function. On our way, we also mention some results on the a-value distribution of more general L-functions. Later on, we consider the topic of universality. A detailed account of universality could be found in Laurinčikas [58].

In the beginning, the a-value theory concentrated on the a-values of the Riemann zeta-function. In particular, this theory was a study of the zeros of the Riemann zeta-function. As it was mentioned above, the zeros of the Riemann zeta-function are divided into non-trivial and trivial. The trivial zeros are located at points $-2n$, $n \in \mathbb{N}$. The rest of the zeros are non-trivial. They are located in the critical strip $0 < \Re s < 1$.

One of the earliest studies on the zeros of the Riemann zeta-function is Gram [31]. In this study, Gram located the first 15 non-trivial zeros. He observed that all of them are located on the critical line. In order to find these zeros, Gram investigated the location of certain points on the critical line where the Riemann zeta-function assumes real values. Later on, these points became known as the Gram points. Let us denote their imaginary part by g_n. It was conjectured (see Hutchinson [39]) that there is exactly one zero $\rho = \beta + i\gamma$ of the Riemann zeta-function satisfying $\gamma \in [g_n, g_{n+1})$. This is called the Gram's law. Recently, Trudgian [93] demonstrated that the Gram's law fails a positive proportion of time.

Let $N(T)$ denote the number of non-trivial zeros in the region $0 < t \leq T$. Then we have

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T), \ \ (T \to \infty).$$

An important question is whether all the non-trivial zeros are located on the critical line $\Re s = 1/2$. In a famous article, Conrey [11] showed that at least two fifths of the non-trivial zeros are simple and lie on the critical line.

In [7], Bohr and Jessen demonstrated that $\log \zeta(s)$ assumes any complex value infinitely often in the strip $\sigma_1 < \Re s < \sigma_2$, satisfying $1/2 < \sigma_1 < \sigma_2 < 1$.

Garunkštis and Steuding [27, Lemma 6] showed that the Riemann zeta-function has an a-value near each trivial zero $-2n$ for high enough n. Apart from these a-values, there exist only finitely many a-values in the half-plane $\sigma < 0$. Let us call these a-values trivial. In addition, let us call the rest of the a-values non-trivial. For any a, there exist left and right half-planes free of non-trivial a-values. Similarly to the formula 2.7, we have a formula for the number of a-values for generic a (see Landau [8])

$$N_a(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e c_a} + O(\log T), \ (T \to \infty),$$

where $c_a = 1$ for $a \neq 1$ and $c_1 = 2$.

A significant result regarding the a-values of the Riemann zeta-function is due to Levinson [61]. He proved that almost all the a-values of the Riemann zeta-function are clustered around the critical line. This is to be understood in the sense that all but $O(N_a(T)/\log\log T)$ of the a-values with imaginary parts $T < \gamma_a < 2T$ lie in the strip

$$\left|\Re s - \frac{1}{2}\right| < \frac{(\log\log T)^2}{\log T}, \ (T \to \infty).$$

In [27], Garunkštis and Steuding showed that the equation $\zeta(s) = a$ has infinitely many simple roots with arbitrarily large imaginary parts. In a way, this result was generalized by Gonek et al. [29]. They proved that a positive proportion of a-values for the Riemann zeta-function are simple. They also extended this result for the Dirichlet L-functions with primitive characters and some other functions belonging to the Selberg class. For a further discussion on this result, see Selberg [81].

A similar result in this direction is Kalpokas and Steuding [49]. In their article, the authors investigate the behavior of the curve $\mathbb{R} \ni t \mapsto \zeta(\sigma + it)$. They prove that the mean value of this curve exists and is equal to 1 provided the Riemann hypothesis is true. They also prove, this time unconditionally, that the Riemann zeta-function takes arbitrarily large real values on the critical line. As for the negative real values on the critical line, there is a result of Kalpokas et al. [50] showing that the Riemann zeta-function assumes arbitrarily large negative values on the critical line.

Christ and Kalpokas [9] investigated the behavior of the discrete moments of the Riemann zeta-function on the critical line. In particular, they sum up the values of the Riemann zeta-function on what they call generalized Gram points. Assuming the Riemann hypothesis, they obtain upper bounds for these moments.

As for the uniform distribution modulo one, Rademacher [69] proved, assuming the truth of the Riemann hypothesis, that the imaginary parts of the non-trivial zeros of the Riemann zeta-function are uniformly distributed modulo one. Later on, Hlawka [37, 38] and Elliott [17] showed the same result without the assumption of the Riemann hypothesis. Akbary and Murty [1] studied the uniform distribution modulo one for a large family of L-functions, which includes the Selberg class. Jakhlouti et al. [42] demonstrated that the imaginary parts of the a-values of the zeta-functions, which belong to the Selberg class, which have a polynomial Euler product, and which satisfy the Lindelöf hypothesis, are uniformly distributed modulo one.

Although not directly relevant to the results described in this dissertation, the concept of universality is prevalent in the theory of zeta-functions, so we briefly discuss it here. Originally, the universality property was formulated by Voronin [95]. It concerned the Riemann zeta-function. In its modern form (see Laurinčikas [58, 57], Matsumoto [64], and Steuding [89]), the universality theorem is as follows

Theorem 2.17. *Suppose U is a compact subset of the strip $\{s \in \mathbb{C} : 1/2 < \Re s < 1\}$ such that the complement of U is connected. Suppose $f : U \to \mathbb{C}$ is a continuous function on U which is holomorphic in the interior of U and does not vanish on U. Then for any $\epsilon > 0$, there exists $t \geq 0$ such that*

$$|\zeta(s + it) - f(s)| < \epsilon \text{ for all } s \in U.$$

In the version of the universality theorem by Voronin, the set U is a disk of radius $1/4$, centered on the vertical line $\sigma = 3/4$ of the complex plane.

There is a stronger version of this theorem, which essentially states that there are a lot of "shifts" of the Riemann zeta-function which approximate the given holomorphic function f satisfying the conditions of Theorem 2.17. Suppose U, f, and ϵ are as in Theorem 2.17. Then the following inequality holds

$$0 < \liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ t \in [0.T] : \max_{s\in U} |\zeta(s + it) - f(s)| < \epsilon \right\}.$$

Here meas is the Lebesgue measure.

In Laurinčikas [57], the universality property is extended to a wide class of zeta-functions. More specifically, in [56], Laurinčikas extended the universality property to the Lerch zeta-function, which is a generalization of the Riemann zeta-function. Bagchi [4] coined the notion of joint universality. The idea is to consider several zeta-functions (in the case of Bagchi, these are Dirichlet L-functions) and to look how they approximate some given functions f_k, satisfying the conditions of Theorem 2.17. We pair each function f_k with a different Dirichlet L-function. Then there exists some t such that each f_k is approximated by their respective Dirichlet L-function "shifted" by the same value of t. This joint universality is also exhibited by the Lerch zeta-function.

2.8 Speiser's result

This discussion of the Speiser's result is based on Arias de Reyna [2]. We should note here that we only reproduce the main ideas of the proof, not the proof itself. In his article, Speiser [86] studies trajectories of points where the Riemann zeta-function ζ assumes constant argument.

Theorem 2.18. (Speiser's theorem) *The Riemann hypothesis is equivalent to the fact that the derivative of the Riemann zeta-function does not have any zeros left of the critical line.*

Ideas for proof. Suppose a is a zero of ζ left of the critical line $\sigma = 1/2$. Consider the trajectories going through this point. As we move along these trajectories, the absolute value of the Riemann zeta-function $|\zeta|$ is increasing. Then no trajectory could cross the critical line because in doing so, the absolute value upon crossing the critical line should decrease. In addition, such trajectories cannot be tangent to the critical line, since at the point of tangency $\zeta(1/2 + it)$ would assume the value of zero, which leads to the contradiction of the fact that $|\zeta|$ is increasing. Therefore, the trajectories would eventually turn left. Some of them would leave the point a above, some below. The trajectory separating the two would reach a zero of the derivative of the Riemann zeta-function which would allow it to turn left. Thus if the Riemann hypothesis is false, then the derivative of the Riemann zeta-function has a zero left of the critical line.

Conversely, assume the derivative of the Riemann zeta-function vanishes left of the critical line at point a. If $\zeta(a) = 0$, we are done. Suppose $|\zeta(a)| \neq 0$. Hence there exist two opposite trajectories of constant argument along which $|\zeta|$ decreases and thus there exists a point at which the absolute value of the Riemann zeta-function vanishes. Again, if $|\zeta|$ vanishes left of the critical line, then we are finished. So assume both trajectories cross the critical line. Both trajectories and the segment of the critical line they determine enclose a region Ω. Now, there are two paths coming out of a of constant argument along which $|\zeta|$ increases. One of these paths must enter the region Ω. The value of $|\zeta(s)|$ approaches $+\infty$ as we go along this path. Therefore this path must leave the region Ω. But in doing so, it must either cross one of the two paths coming out of a and enclosing the region, which leads to the contradiction $|\zeta(a)| > |\zeta(s)|$. If this path crosses the critical line, then in doing so it should also decrease, since it crosses the line from left to right. Again, we get a contradiction. Thus if ζ' vanishes left of the critical line, the Riemann hypothesis fails. In the end, the Riemann hypothesis is true if and only if the derivative of the Riemann zeta-function does not vanish left of the critical line. ☐

Let us consider Levinson and Montgomery's article [62]. To use their original notation, suppose $N^-(T)$ represents the number of zeros of the Riemann zeta-function in the area $0 < t < T$ and $0 < \sigma < 1/2$. Similarly, let $N_1^-(T)$ denote the number of zeros of the derivative of the Riemann zeta-function in the same area. Theorem 1 in Levinson and Montgomery [62] states that

Theorem 2.19.

$$N_1^-(T) = N^-(T) + O(\log T).$$

In addition, if $N^-(T) < T/2$, then there exists a sequence $\{T_j\}$ with $T_j \to \infty$ as $j \to \infty$ such that

$$N_1^-(T_j) = N^-(T_j).$$

As a corollary, Levinson and Montgomery get the result that the Riemann hypothesis is equivalent to the fact that the derivative of the Riemann zeta-function does not have any zeros left of the critical line $\sigma = 1/2$.

In addition to the derivative of the Riemann zeta-function, Levinson and Montgomery also study the zeros of higher derivatives. Theorem 2 in Levinson and Montgomery [62] states that

Theorem 2.20. *Suppose the number of non-real zeros of* $\zeta^{(k)}(s)$ *in the region* $0 < t < T$ *and* $\sigma \leq c$ *is* $N_k^-(c, T)$*, the number of non-real zeros of* $\zeta^{(k)}(s)$ *in the region* $0 < t < T$ *and* $\sigma \geq c$ *is* $N_k^+(c, T)$ *and the total number of zeros of* $\zeta^{(k)}(s)$ *in the region* $0 < t < T$ *is* $N_k(T)$*. Then*

$$N_k^+(1/2 + \delta, T) + N_k^-(1/2 - \delta, T) \ll \delta^{-1} T \log\log T$$

for given k *and uniformly in* $\delta > 0$*.*

In [6], Berndt proved that

$$N_k(T) = \frac{T}{2\pi}\left(\log\frac{T}{4\pi} - 1\right) + O(\log T).$$

So Theorem 2 in Levinson and Montgomery [62] becomes

$$N_k^+(1/2 + \delta, T) + N_k^-(1/2 - \delta, T) \ll \frac{N_k(T)\log\log T}{\delta \log T}.$$

We should note here that Levinson and Montgomery [62] served as the basis for the subsequent proof by Levinson [60] that at least 1/3 zeros of the Riemann zeta-function lie on the critical line $\sigma = 1/2$ (see also Selberg [78]).

Chapter 3

Horizontal distribution of the *a*-values of the Selberg zeta-function associated to a compact Riemann surface

This chapter is based on Garunkštis and Šimėnas [25]. Let $s = \sigma + it$ be a complex variable. We start with the Riemann zeta-function, which for $\sigma > 1$ is given by the following Dirichlet series or Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The Lindelöf hypothesis (LH) states, that for any $\epsilon > 0$,

$$\zeta(1/2 + it) \ll_\epsilon |t|^\epsilon \qquad (|t| \to \infty).$$

In [23], Garunkštis extended the Backlund equivalent for the LH to the following statement: *Let a be a complex number. The Lindelöf hypothesis for $\zeta(s)$ is true if and only if for any $\sigma' > 1/2$ the number of roots of $\zeta(s) - a$ in the region $\sigma > \sigma'$, $T \le t \le T + 1$ is* $o(\log T)$ as $T \to \infty$. The original Backlund equivalent (see Backlund [3] or Titchmarsh [92, Section 13.5]) corresponds to the case $a = 0$. Here and further the number of roots are always counted according to multiplicities. Solutions of $f(s) = a$ are called a-values of $f(s)$.

We see that the Lindelöf hypothesis follows from the Riemann hypothesis, which states that the Riemann zeta-function has no zeros to the right of the critical line $\sigma = 1/2$.

Moreover, in [23] Garunkštis proved that the above extended Backlund equivalent can be generalized to some other functions, for example, to the Selberg zeta-function $Z(s)$ associated to a compact Riemann surface (for the full definition see Section 2.3). In this case we have the following equivalent: *Let a be a complex number. For any $\epsilon > 0$,*

$$Z(1/2 + it) \ll_\epsilon \exp(\epsilon|t|) \qquad (|t| \to \infty) \tag{3.1}$$

if and only if for any $\sigma' > 1/2$ the number of roots of $Z(s) - a$ in the region $\sigma > \sigma'$, $T \le t \le T + 1$ is $o(T)$ as $T \to \infty$. The bound (3.1) corresponds to the LH for the Riemann zeta-function. For the Selberg zeta-function the analog of RH is true (Hejhal [35, Chapter 2, Theorem 4.11]), thus in the last equivalence both statements are true.

The Selberg and the Riemann zeta-functions have many similar properties: the functional equation, the Euler product, the Selberg trace formula – the Weil explicit formula. There are many results about a-values of $\zeta(s)$ (see Titchmarsh [92], Levinson [61], Steuding [89]). In view of the above it would be of interest to compare the a-value distribution for both zeta-functions. In the remaining part of this chapter we consider the a-values of the Selberg zeta-function.

3.1 Statement of results

The following lemma shows the behavior of the factor $X(s)$ from the functional equation (see Section 2.3) for large t (cf. Hejhal [35, Chapter 2, formula (4.4)]).

Lemma 3.1. *For $t > 1$,*

$$X(s) = \exp\left(2\pi i(g-1)\left(s - \frac{1}{2}\right)^2 + \frac{\pi i(g-1)}{6} + O\left(\frac{t}{e^{2\pi t}}\right) + O\left(\frac{(\sigma - 1/2)^2}{e^{2\pi t}}\right) + O\left(\frac{(\sigma - 1/2)t}{e^{2\pi t}}\right)\right) \qquad (t \to \infty)$$

uniformly in σ.

The lemma will be proved in the next section.

Here a is always a fixed complex number and T always tends to plus infinity. By β_a and γ_a we denote the real and the imaginary parts of the a-value. Let

$$N(P_{00}) = \min_{P_0}\{N(P_0)\} \quad \text{and} \quad m_0 = \#\{\{P_0\} : N(P_0) = N(P_{00})\}.$$

By definition of the Selberg zeta-function (2.1) we find that

$$Z(s) = 1 + m_0 N(P_{00})^{-\sigma} + o(N(P_{00})^{-\sigma}) \quad (\sigma \to \infty). \tag{3.2}$$

Note that $N(P_{00}) > 1$ (Hejhal [35, Chapter 2, Section 2]). Thus there exists a number $A > 1$ depending on a such that $Z(s) \neq a$ for $\sigma \geq A$. Further, by $\overline{Z(s)} = Z(\overline{s})$, the functional equation (2.2), and by Lemma 3.1 we see that there is $\tau \geq 0$ such that

$$Z(s) \neq a \qquad \text{for} \quad \sigma \leq 1 - A \quad \text{and} \quad |t| \geq \tau. \tag{3.3}$$

In view of this we say that an a-value $\rho_a = \beta_a + i\gamma_a$ is *non-trivial* if $1 - A < \beta_a < A$. If $\beta_a \leq 1 - A$ (and $|\gamma_a| < \tau$) then $\rho_a = \beta_a + i\gamma_a$ is called a *trivial a-value.*

The following proposition will be the main tool in the investigation of a-values.

Proposition 3.2. *Let τ be defined by formula* (3.3)*. Let $b = b(T) = o(\log T)$ be a positive and unboundedly increasing function. Then*

$$\sum_{\tau < \gamma_a \leq T} (\beta_a + b) = (g-1)(\frac{1}{2} + b)T^2 - \frac{T}{2\pi}\log|1-a| + o(T),$$

if $a \neq 1$, and

$$\sum_{\tau < \gamma_1 \leq T} (\beta_1 + b) = (g-1)(\frac{1}{2} + b)T^2 - \frac{bT}{2\pi}\log N(P_{00}) - \frac{T}{2\pi}\log m_0 + o(T),$$

otherwise.

The proposition will be proved in the next section. The accuracy $o(T)$ is due to the bound (3.1).

Define by $N(a, T)$ the number of non-trivial a-values with $\tau < \gamma_a \leq T$. In

Proposition 3.2 subtracting the case b from $b+1$ we obtain the following theorem:

Theorem 3.3. *For $a \neq 1$,*

$$N(a,T) = (g-1)T^2 + o(T)$$

and, for $a = 1$,

$$N(1,T) = (g-1)T^2 - \frac{T}{2\pi}\log N(P_{00}) + o(T).$$

If $a = 0$, then the last theorem is true with a better error term $O(T/\log T)$ (Hejhal [35, Section 2.8, Theorem 8.19]).

The expression

$$\sum_{\tau<\gamma_a\leq T}(\beta_a - \frac{1}{2}) = \sum_{\tau<\gamma_a\leq T}(\beta_a + b) - (\frac{1}{2}+b)\sum_{\tau<\gamma_a\leq T} 1$$

together with Proposition 3.2 and Theorem 3.3 leads to the following statement:

Theorem 3.4. *Let τ be defined by formula* (3.3)*. For $a \neq 1$,*

$$\sum_{\tau<\gamma_a\leq T}(\beta_a - \frac{1}{2}) = -\frac{T}{2\pi}\log|1-a| + o(T),$$

and, for $a = 1$,

$$\sum_{\tau<\gamma_1\leq T}(\beta_1 - \frac{1}{2}) = \frac{T}{4\pi}\log\frac{N(P_{00})}{m_0^2} + o(T).$$

If $a \neq 0$, then we see that many a-values lie off the critical line $\sigma = 1/2$ and are distributed asymmetrically with respect to this line.

Let $N^+(a,\delta,T)$ and $N^-(a,\delta,T)$ be the number of non-trivial a-values in the corresponding regions $\sigma > 1/2+\delta$, $1 < t \leq T$ and $\sigma < 1/2-\delta$, $1 < t \leq T$. Let

$$N_0(a,\delta,T) := N(a,T) - \left(N^-(a,\delta,T) + N^+(a,\delta,T)\right).$$

The next theorem shows that almost all a-values are arbitrarily close to the critical line.

Theorem 3.5. *For*

$$\delta = \frac{(\log\log T)^2}{\log T}$$

we have

$$N^-(a,\delta,T) + N^+(a,\delta,T) \ll \frac{T^2}{\log\log T}$$

and

$$N_0(a,\delta,T) = (g-1)T^2 + O\left(\frac{T^2}{\log\log T}\right).$$

The theorem will be proved in the next section.

The obtained results show a similar distribution of a-values for the Riemann and the Selberg zeta-functions. If we replace $(g-1)T^2$ by $\frac{1}{2\pi}T\log T$ and $o(T)$ by $O(\log T)$ we obtain statements nearly identical to the ones proved by Levinson [61] for the Riemann zeta-function. The difference is that in his proof Levinson used the mean value bound for $\zeta(s)$, while here we use the Lindelöf type bound (3.1).

The next section contains the proofs of Lemma 3.1, Proposition 3.2, and Theorem 3.5.

3.2 Proofs

Proof of Lemma 3.1. In light of the definition (2.3) of $X(s)$, we calculate the following integral

$$\begin{aligned}
&\int_0^{s-\frac{1}{2}} v\tan(\pi v)\,dv = i\int_0^{s-\frac{1}{2}} v\frac{1-e^{2\pi iv}}{1+e^{2\pi iv}}\,dv \\
&= i\int_0^{s-\frac{1}{2}} v\,dv - i\left(\int_0^{it} + \int_{it}^{\sigma-1/2+it}\right)\frac{2v}{1+e^{-2\pi iv}}\,dv \\
&= \frac{i}{2}(s-\frac{1}{2})^2 + 2i\int_0^t \frac{x}{1+e^{2\pi x}}\,dx - 2i\int_0^{\sigma-1/2}\frac{(x+it)}{1+e^{2\pi t-2\pi ix}}\,dx \\
&= \frac{i}{2}(s-\frac{1}{2})^2 + \frac{i}{24} + \\
&+ O\left(\frac{t}{e^{2\pi t}}\right) + O\left(\frac{(\sigma-1/2)^2}{e^{2\pi t}}\right) + O\left(\frac{(\sigma-1/2)t}{e^{2\pi t}}\right) \qquad (t\to\infty).
\end{aligned}$$

Here we used the fact that

$$\int_0^\infty \frac{x}{1+e^{2\pi x}}\,dx = \frac{1}{48}.$$

This proves the lemma. □

Proof of Proposition 3.2. First we consider the case $a \neq 1$. Put

$$G(s) = \frac{Z(s) - a}{1 - a}.$$

Obviously, the zeros of $G(s)$ correspond exactly to the a-values of $Z(s)$. Let

$$c = \log T. \tag{3.4}$$

Recall that $b = b(T)$ is a positive and unboundedly increasing function. Thus for sufficiently large T we have $Z(s) \neq a$, if $\sigma \leq -b$ or $\sigma \geq c$. We assume that $Z(s) \neq a$ on the boundaries of the rectangle R with vertices $c + i\tau'$, $c + iT'$, $-b + iT'$, $-b + i\tau'$, where $T < T' < T + 1/T$ and $\tau < \tau' < \tau + 1$. Applying Littlewood's lemma (Lemma 2.11, also see Titchmarsh [92, Section 9.9]) to the function $G(s)$ on the rectangle R we get

$$\begin{aligned} 2\pi \sum_{\substack{\beta_a > -b \\ \tau' < \gamma_a \leq T'}} (\beta_a + b) &= \int_{\tau'}^{T'} \log |G(-b + it)|\, dt - \int_{\tau'}^{T'} \log |G(c + it)|\, dt - \\ &\quad - \int_{-b}^{c} \arg G(\sigma + i\tau')\, d\sigma + \int_{-b}^{c} \arg G(\sigma + iT')\, d\sigma = \\ &=: \sum_{j=1}^{4} I_j. \end{aligned} \tag{3.5}$$

Here $\arg G(s)$ is defined by continuous variation starting at $s = A + 1$, then along lines connecting $A+1$ with $A+1+it$ and $A+1+it$ with $\sigma+it$, provided that the path does not cross a zero of $G(s)$; if it does, we put

$$\arg G(s) = \lim_{\epsilon \to +0} \arg G(\sigma + it + \epsilon).$$

In light of (3.2) we choose the branch of argument such that $\arg G(s)$ tends to zero as $\sigma \to \infty$. Then $\arg Z(A + 1) = 0$.

By the functional equation (2.2), Lemma 3.1 and formula (3.2) we get

$$\begin{aligned}
I_1 =& \int_{\tau'}^{T'} \log \left| \frac{X(b+it)Z(1+b-it)-a}{1-a} \right| dt \\
=& 4\pi(g-1) \int_{\tau'}^{T'} \left(t(\frac{1}{2}+b) + O\left(\frac{\sigma^2+|\sigma t|}{e^{2\pi t}} \right) \right) dt \\
&+ \int_{\tau'}^{T'} \log \left| Z(1+b-it) - \frac{a}{X(-b+it)} \right| dt - \int_{\tau'}^{T'} \log |1-a| \, dt \\
=& 2\pi(g-1){T'}^2(\frac{1}{2}+b) - T' \log|1-a| + O\left(1 + \frac{T'}{(\min\{N(P_{00}),e\})^b} \right). \quad (3.6)
\end{aligned}$$

Similarly, only simpler,

$$I_2 = O\left(\frac{T'}{N(P_{00})^c} \right) = o(T). \quad (3.7)$$

Now we consider the integral I_4. We split I_4 into two integrals.

$$I_4 = \int_{-b}^{c} \arg G(\sigma + iT') \, d\sigma = \int_{-b}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{c} = I_{41} + I_{42}. \quad (3.8)$$

First we will evaluate I_{42}. In view of formula (3.2) and definition (3.4) we can choose a large fixed positive number c' such that

$$I_{42} = \int_{\frac{1}{2}}^{c} \arg G(\sigma + iT') \, d\sigma = \int_{\frac{1}{2}}^{c'} \arg G(\sigma + iT') \, d\sigma + O(\log T').$$

We will show that, for $\sigma \geq 1/2$,

$$\arg G(\sigma + iT') = o(T'). \quad (3.9)$$

Let c' be large enough, such that $\Re G(s) \neq 0$ for $\sigma > c'$. Suppose that $\Re G(\sigma + iT')$ has N zeros for $1/2 \leq \sigma \leq c'$. Then divide $[1/2, c']$ into at most $N+1$ subintervals in each of which $\Re G(\sigma + iT')$ is of constant sign. Then

$$|\arg G(\sigma + iT')| \leq (N+1)\pi.$$

To estimate N let

$$g(z) := \frac{1}{2} \left(G(z + iT') + \overline{G(\overline{z} + iT')} \right).$$

Then we have $g(\sigma) = \Re G(\sigma + iT')$. Let $n(r)$ denote the number of zeros of $g(z)$ in $|z - c'| \leq r$. Then

$$|\arg G(\sigma + iT')| \leq \left(n(c' - \frac{1}{2}) + 1\right)\pi.$$

By Jensen's theorem (Titchmarsh [91], Section 3.61, also see Theorem 2.14 in this dissertation)

$$\int_0^R \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_0^{2\pi} \log|g(c' + Re^{i\theta})|\,d\theta - \log|g(c')|. \tag{3.10}$$

Let $R = c' - 1/2 + \delta$, $\delta > 0$. We have that, for any $\epsilon > 0$ and $\sigma \geq 1/2$ (see Garunkštis [23], comments below Theorem 5),

$$Z(\sigma + iT') \ll \exp(\epsilon T'). \tag{3.11}$$

Thus, for $\Re(c' + Re^{i\theta}) \geq 1/2$ and large T',

$$\log|g(c' + Re^{i\theta})| < 2\epsilon T'.$$

Then the functional equation (2.2) and Lemma 3.1 give that there is an absolute constant $d > 0$ such that

$$\log|g(c' + Re^{i\theta})| < d(\delta + \epsilon)T'.$$

The length of the arc of the circle $|s - c'| = R$, which is to the left of the critical line $\sigma = 1/2$, is

$$2R \arcsin\frac{\sqrt{2R\delta - \delta^2}}{R} = O(\sqrt{R\delta}) = O(\sqrt{\delta}) \qquad (\delta \to 0).$$

Now we see that the right-hand side of (3.10) is at most

$$O(\epsilon T') + O(\delta^{1/2}(\delta + \epsilon)T').$$

Since

$$\frac{\delta}{R} n(R - \delta) \leq \int_0^R \frac{n(r)}{r}\,dr,$$

we conclude that

$$n(R-\delta) = O\left(\frac{\epsilon}{\delta}T' + \delta^{-\frac{1}{2}}(\delta+\epsilon)T'\right)$$

and taking $\delta = \epsilon^{2/3}$ we obtain that $n(R-\delta) = O(\epsilon^{1/3}T')$. The bound (3.9) is proved. Hence

$$I_{42} = o(T').$$

We return to the integral I_{41} defined by the equality (3.8). To evaluate I_{41} we will use the functional equation (2.2). First we choose the branch of $\arg X(\sigma + iT')$. The functional equation (2.2) and the equality $\arg Z(1/2 - iT') = -\arg Z(1/2 + iT')$ give that

$$\arg X(1/2 + iT') \equiv 2\arg Z(1/2 + iT') \mod 2\pi.$$

We choose

$$\arg X(1/2 + iT') = 2\arg Z(1/2 + iT') \tag{3.12}$$

and we define $\arg X(\sigma + iT')$ by a continuous variation along the segment connecting $1/2 + iT'$ with $\sigma + iT'$. Applying functional equation (2.2) we get

$$\begin{aligned} I_{41} = &\int_{-b}^{\frac{1}{2}} \arg X(\sigma + iT')\, d\sigma + \int_{-b}^{\frac{1}{2}} \arg\left(Z(1-\sigma-iT') - \frac{a}{X(\sigma+iT')}\right) d\sigma \\ &- \int_{-b}^{\frac{1}{2}} \arg(1-a)\, d\sigma = I_{411} + I_{412} + o(\log T'). \end{aligned}$$

It is known (see Hejhal [35, Chapter 2, Completion of the proof of Theorem 8.1] or Randol [73, Theorem 1]) that $\arg Z(1/2 + iT') = O(T'/\log T')$. Then the equality (3.12) gives that

$$\arg X(1/2 + iT') = O\left(\frac{T'}{\log T'}\right). \tag{3.13}$$

Further by Lemma 3.1 we get

$$\arg X(\sigma + iT') = 2\pi(g-1)(\sigma - \frac{1}{2})^2 - T'^2 + 2\pi k + O\left(\frac{\sigma^2 + |\sigma T'|}{e^{2\pi T'}}\right), \tag{3.14}$$

where the constant $2\pi k$, where k is an integer, depends on the branch of $\arg X(\sigma + iT')$. By (3.13) and (3.14) we have that $2\pi k = T'^2 + O(T'/\log T')$.

This and (3.14) give

$$I_{411} = O(b^3) + O\left(\frac{bT'}{\log T'}\right) = o(T).$$

Arguing similarly as for I_{42} we obtain that $I_{412} = o(T)$. Thus $I_4 = o(T)$ and similarly $I_3 = o(\log T)$. By this and (3.5), (3.6), (3.7), Proposition 3.2 follows for $a \neq 1$. If $a = 1$ then the proposition is proved in an analogous way, considering the function

$$G^*(s) = \frac{Z(s) - 1}{m_0/N(P_{00})^s}.$$

This gives Proposition 3.2. □

To prove Theorem 3.5 we will need the following lemma.

Lemma 3.6. *We have*

$$\sum_{\substack{1<\gamma_a\leq T \\ \beta_a>\frac{1}{2}}} (\beta_a - \frac{1}{2}) \ll \frac{T^2 \log\log T}{\log T}.$$

Proof. By Theorem 3.3

$$\begin{aligned} \sum_{\substack{1<\gamma_a\leq T \\ \beta_a>\frac{1}{2}}} \left(\beta_a - \frac{1}{2}\right) = & \sum_{\substack{1<\gamma_a\leq T \\ \beta_a>\frac{1}{2}+\frac{\log\log T}{\log T}}} \left(\beta_a - \frac{1}{2} - \frac{\log\log T}{\log T}\right) \\ & + \sum_{\substack{1<\gamma_a\leq T \\ \beta_a>\frac{1}{2}+\frac{\log\log T}{\log T}}} \frac{\log\log T}{\log T} + \sum_{\substack{1<\gamma_a\leq T \\ \frac{1}{2}<\beta_a\leq\frac{1}{2}+\frac{\log\log T}{\log T}}} \left(\beta_a - \frac{1}{2}\right) \\ = & \sum_{\substack{1<\gamma_a\leq T \\ \beta_a>\frac{1}{2}+\frac{\log\log T}{\log T}}} \left(\beta_a - \frac{1}{2} - \frac{\log\log T}{\log T}\right) + O\left(\frac{T^2\log\log T}{\log T}\right). \end{aligned}$$

Arguing along the same lines as in the proof of Proposition 3.2, only with

$b = -1/2 - \frac{\log\log T}{\log T}$, we obtain

$$\sum_{\substack{1<\gamma_a\leq T \\ \beta_a>\frac{1}{2}+\frac{\log\log T}{\log T}}} \left(\beta_a - \frac{1}{2} - \frac{\log\log T}{\log T}\right) = \int_1^T \log\left|Z\left(\frac{1}{2} + \frac{\log\log T}{\log T} + it\right) - a\right| dt +$$

$$+ O(T) = \int_{\sqrt{T}}^T \log\left|Z\left(\frac{1}{2} + \frac{\log\log T}{\log T} + it\right) - a\right| dt + O(T),$$

where in the last step we used the bound 3.11. By Hejhal [35, Chapter 2, Proposition 10.10 and Theorem 8.1], it follows

$$\log|Z(\sigma+it)| \ll \left(\frac{t}{\log t}\right)^{2\max(0,1-\sigma)} \log t, \quad \text{for} \quad \sigma \geq \frac{1}{2} + \frac{1}{\log\frac{t}{\log t}}.$$

Hence, for sufficiently large T_0, there is a positive constant C such that, for $T > T_0$ and $\sqrt{T} \leq t \leq T$, we have

$$\log\left|Z\left(\frac{1}{2} + \frac{\log\log T}{\log T} + it\right) - a\right| < C\frac{T}{(\log T)^2}.$$

This proves Lemma 3.6. □

Proof of Theorem 3.5. By Lemma 3.6 we have that

$$N^+(a,\delta,T) \leq \frac{\log T}{(\log\log T)^2} \sum_{\substack{1<\gamma_a\leq T \\ \beta_a>\frac{1}{2}+\frac{\log\log T}{\log T}}} (\beta_a - \frac{1}{2}) \ll \frac{T^2}{\log\log T}.$$

Next we consider a bound for

$$N^-(a,\delta,T) = \sum_{\substack{1<\gamma_a\leq T \\ \beta_a<\frac{1}{2}-\delta}} 1.$$

Let $b = b(T)$ satisfy the conditions of Proposition 3.2. If T is sufficiently large then for any a-value $\rho_a = \beta_a + i\gamma_a$ we have that $\beta_a < b$ and thus

$$\sum_{1<\gamma_a\leq T} (\beta_a + b) \leq \sum_{\substack{1<\gamma_a\leq T \\ \beta_a>\frac{1}{2}}} \left(\beta_a - \frac{1}{2}\right) + \left(b + \frac{1}{2}\right) \sum_{\substack{1<\gamma_a\leq T \\ \beta_a\geq\frac{1}{2}-\delta}} 1 + \left(b + \frac{1}{2} - \delta\right) \sum_{\substack{1<\gamma_a\leq T \\ \beta_a<\frac{1}{2}-\delta}} 1.$$

The last inequality together with Lemma 3.6, Proposition 3.2, and Theorem 3.3, yields

$$0 \leq O\left(\frac{T^2 \log\log T}{\log T}\right) - \delta N^-(a, \delta, T).$$

Thus

$$N^-(a, \delta, T) \ll \frac{T^2}{\log\log T}.$$

The above bounds for $N^+(a, \delta, T)$, $N^-(a, \delta, T)$ and Theorem 3.3 give the required formula for $N_0(a, \delta, T)$, which completes the proof.

Chapter 4

Horizontal distribution of the *a*-values of the Selberg zeta-function associated to a finite volume Riemann surface

In this chapter, we obtain results analogous to the results in the previous chapter. The difference is that instead of compact Riemann surfaces we consider finite volume Riemann surfaces. Therefore, this section could be viewed as a generalization of the previous one. Quite a few results overlap. However, there are some significant differences, such as a-value free regions. The main result, which is that almost all the a-values are clustered around the vertical line of the complex plane with the real part $1/2$, is the same.

Here we consider the a-values of the Selberg zeta-functions Z associated to a finite volume Riemann surface. For a discussion of the definition of the Selberg zeta-function associated to a finite volume Riemann surface, see Section 2.3. This section could be viewed as a continuation of the investigations of Garunkštis and Šimėnas [25], where we studied the a-values of the Selberg zeta-functions associated to compact hyperbolic Riemann surfaces. One of the most historically significant articles dealing with the horizontal distribution of the a-values of the Riemann zeta-function is Levinson [61]. Here Levinson proved that the real parts of almost all the a-values of the Riemann zeta-function are clustered around the critical line $\sigma = 1/2$.

In Theorem 4.1, we describe a-value free regions. For $n_1 \geq 1$, where n_1 is the number of cusps of the corresponding Riemann surface M, we define the following region on the left-hand side of the complex plane

$$L := \left\{ \sigma + it : \left| t - \frac{2n_1}{\mathrm{vol}(M)} \log(-\sigma) + \frac{2n_1 - 2\log \mathfrak{g}_1 - 2n_1 \log 2}{\mathrm{vol}(M)} \right| \leq \frac{1}{\log^2(-\sigma)}, \right.$$
$$\left. \sigma < -2, t > 0 \right\}.$$

Here constant $\mathfrak{g}_1$ is defined at the beginning of the next section.

Theorem 4.1. *Let $a \in \mathbb{C}$ and $a \neq 0$. If $n_1 = 0$ then there exist real numbers $\sigma_0 < 0$, $t_0 > 0$, and $\sigma_1 > 1$ such that $Z(s) \neq a$ in the region*

$$\{s \in \mathbb{C} : \sigma < \sigma_0, \textit{and } t > t_0\} \cup \{s \in \mathbb{C} : \sigma > \sigma_1\}.$$

Here $\sigma_0 = \sigma_0(a, M)$, $t_0 = t_0(a, M)$, and $\sigma_1 = \sigma_1(a, M)$.

If $n_1 \geq 1$, then there exist real numbers $\sigma_0 < 0$ and $\sigma_1 > 1$ such that $Z(s) \neq a$ in the region

$$\{s \in \mathbb{C} : s \notin L,\ \sigma < \sigma_0, \textit{and } t > 2\} \cup \{s \in \mathbb{C} : \sigma > \sigma_1\}.$$

Again, here $\sigma_0 = \sigma_0(a, M)$, $t_0 = t_0(a, M)$, and $\sigma_1 = \sigma_1(a, M)$.

Assume

$$\mathrm{vol(M)} < 4\pi n_1. \tag{4.1}$$

Then the number of roots of $Z(s) = a$ in the region $\{s \in L : \sigma' \leq \sigma < \sigma_0\}$ is

$$\frac{\mathrm{vol}(M)}{4\pi} \sigma'^2 + o(\sigma'^2) \qquad (\sigma' \to -\infty). \tag{4.2}$$

In the lower half-plane, the distribution of the a-values can be obtained from the equality $Z(\overline{s}) = \overline{Z(s)}$. In addition, we expect that the region L contains infinitely many zeros for $\mathrm{vol}(M) \geq 4\pi n_1$. For a more precise version of the formula (4.2), see Lemma 4.8 in Section 4.2 below.

Next we discus the condition (4.1). Let the surface M have ℓ inequivalent elliptic points of orders m_1, m_2, ..., m_ℓ. The set of numbers $(g; m_1, m_2, \ldots, m_\ell; n_1)$, where g is the genus of the surface M, is a group invariant. It is called the signature of Γ and satisfies the Gauss-Bonett formula (see Iwaniec [41, Chapter 2])

$$2g - 2 + \sum_{j=1}^{\ell} \left(1 - \frac{1}{m_j}\right) + n_1 = \frac{\mathrm{vol}(M)}{2\pi}.$$

It is known that the positivity of the left side of the last formula guarantees the existence of Γ with the given signature. From this it follows that there are infinitely many groups Γ for which the condition (4.1) is satisfied. By Shimura [83, Section 1.6], we see that the principal congruence group of the level N satisfies the condition (4.1) if and only if $2 \leq N \leq 11$.

Let us fix the numbers σ_0 and σ_1 in Theorem 4.1. We say that a-value is *non-trivial* if it is located in the strip $\sigma_0 \leq \sigma \leq \sigma_1$. We denote a non-trivial a-value of Z by $\rho_a = \beta_a + i\gamma_a$, where $\beta_a,\ \gamma_a \in \mathbb{R}$ are the corresponding real and imaginary parts. Suppose $N(a,T)$ is the number of non-trivial a-values of the Selberg zeta-function in the region $2 < t \leq T$. Let us define

$$N(P_{00}) := \min_{P_0}\{N(P_0)\} \text{ and } m_0 = \#\{\{P_0\} : N(P_0) = N(P_{00})\}. \tag{4.3}$$

The next theorem estimates the number of non-trivial a-values (see Jorgenson and Smajlović [44, formula (88)]).

Theorem 4.2. *We have the following estimate for the number of non-trivial a-values of Z up to T provided $a \neq 1$*

$$N(a,T) = \frac{\mathrm{vol}(M)}{4\pi}T^2 - \frac{n_1}{\pi}T\log T + \frac{n_1 - \log \mathfrak{g}_1 - n_1\log 2}{\pi}T + o(T).$$

If $a = 1$, then

$$\begin{aligned} N(1,T) = &\frac{\mathrm{vol}(M)}{4\pi}T^2 - \frac{n_1}{\pi}T\log T + \frac{n_1 - \log \mathfrak{g}_1 - n_1\log 2}{\pi}T \\ &- \frac{T}{2\pi}\log N(P_{00}) + o(T). \end{aligned}$$

By $N^+(a,\delta,T)$, let us denote the number of a-values of the Selberg zeta-function in the region $1/2+\delta < \sigma$, $1 < t \leq T$, where $\delta = (\log\log T)^2/\log T$. By $N^-(a,\delta,T)$, we denote the a-values in the region $1/2-\delta > \sigma$, $1 < t \leq T$. By $N^0(a,\delta,T)$, we denote the magnitude $N(a,T) - N^+(a,\delta,T) - N^-(a,\delta,T)$. We have the following theorem

Theorem 4.3. *The following estimates hold*

$$N^-(a,\delta,T) + N^+(a,\delta,T) \ll \frac{T^2}{\log\log T}$$

and

$$N^0(a,\delta,T) = \frac{\mathrm{vol}(M)}{4\pi}T^2 + O\left(\frac{T^2}{\log\log T}\right).$$

The following theorem generalizes formula (2.6).

Theorem 4.4. *For non-trivial $\gamma_a < T$ and $a \neq 1$, we have*

$$\sum_{0<\gamma_a\leq T}\left(\frac{1}{2}-\beta_a\right) = \frac{n_1}{4\pi}T\log T - \frac{T}{2\pi}\left(\frac{n_1}{2}+\frac{n_1}{2}\log\pi + \log|d(1)| - \log\mathfrak{g}_1\right)$$
$$-\frac{T}{2\pi}\log|1-a| + o(T),$$

and, for $a = 1$ respectively,

$$\sum_{0<\gamma_a\leq T}\left(\frac{1}{2}-\beta_a\right) = \frac{n_1}{4\pi}T\log T - \frac{T}{2\pi}\left(\frac{n_1}{2}+\frac{n_1}{2}\log\pi + \log|d(1)| - \log\mathfrak{g}_1\right)$$
$$-\frac{T}{2\pi}\log m_0 + \frac{T}{4\pi}\log N(P_{00}) + o(T).$$

An important instrument in the proofs of our results will be the functional equation which is investigated in the next section. In Section 4.2, we prove Theorem 4.1, and Section 4.3 is devoted to the proofs of Theorems 4.2, 4.3, and 4.4.

4.1 The functional equation

Selberg zeta-function satisfies the following functional equation (see Hejhal [36, pp. 499–500], also see Jorgenson and Smajlović [44, Section 2.3])

$$Z(s)\phi(s) = \eta(s)Z(1-s). \tag{4.4}$$

Here Z, ϕ and η all depend on the underlying Riemann surface M. The scattering matrix determinant $\phi(s)$ (Hejhal [36, Definition 3.8, p. 281 and formula (3.32), p. 298]) satisfies the functional equation (Hejhal [36, generalization of Theorem 11.8, p. 296])

$$\phi(s)\phi(1-s) = 1. \tag{4.5}$$

The function ϕ is given by (see Hejhal [36, formula (3.35), p. 299], and Jorgenson and Smajlović [44, Section 1.3])

$$\phi(s) = \pi^{n_1/2} \left(\frac{\Gamma(s - 1/2)}{\Gamma(s)} \right)^{n_1} \sum_{n=1}^{\infty} \frac{d(n)}{\mathfrak{g}_n^{2s}}, \tag{4.6}$$

where n_1 is the number of cusps of the underlying Riemann surface. $d(n)$ and $\mathfrak{g}_\mathfrak{n}$ are sequences of real numbers such that

$$0 < \mathfrak{g}_1 < \mathfrak{g}_2 < \dots.$$

We have (Hejhal [36, generalization of Theorem 12.9, p. 300–301])

$$\phi(s) \ll 1 \qquad (t \to \infty) \tag{4.7}$$

uniformly in $\sigma \geq 1/2$. Function ϕ can be further decomposed into the following factors

$$\phi(s) = K(s)H(s), \tag{4.8}$$

where

$$K(s) = \pi^{n_1/2} \left(\frac{\Gamma(s - 1/2)}{\Gamma(s)} \right)^{n_1} e^{c_1 s + c_2} \tag{4.9}$$

with $c_1 = -2 \log \mathfrak{g}_1$ and $c_2 = \log d(1)$. In addition,

$$H(s) = 1 + \sum_{n=2}^{\infty} \frac{a(n)}{r_n^{2s}} \qquad (\sigma > 1) \tag{4.10}$$

with $r_n = \mathfrak{g}_n/\mathfrak{g}_1 > 1$ and $a(n) = d(n)/d(1)$. The function η is given by (see Hejhal [36, formula (5.10), p. 501] and Jorgenson and Smajlović [44, formula (15)])

$$\eta(s) = \eta(1/2) \exp \left(\int_{1/2}^{s} \frac{\eta'(u)}{\eta(u)} \, du \right) \tag{4.11}$$

with $\eta(1/2) = \pm 1$ and

$$\begin{aligned}\frac{\eta'(s)}{\eta(s)} =&\mathrm{vol}(M)(s-1/2)\tan(\pi(s-1/2)) \\ &-\pi \sum_{\substack{\{R\} \\ 0<\theta(R)<\pi}} \frac{1}{M_R \sin\theta} \frac{\cos((2\theta-\pi)(s-1/2))}{\cos\pi(s-1/2)} \\ &+ 2n_1 \log 2 + n_1 \left(\frac{\Gamma'}{\Gamma}(1/2+s) + \frac{\Gamma'}{\Gamma}(3/2-s) \right).\end{aligned} \tag{4.12}$$

Here $\{R\}$ is the set of inconjugate elliptic elements of Γ such that $0 < \theta(R) < \pi$ is uniquely determined real number such that R is conjugate to the matrix

$$\begin{pmatrix} \cos\theta(R) & -\sin\theta(R) \\ \sin\theta(R) & \cos\theta(R) \end{pmatrix}$$

and M_R is the order of the centralizer of R with respect to Γ.

Lemma 4.5. *For any $\delta > 0$, the factor $\eta(s)$ on the right-hand side of* (4.4) *is*

$$\begin{aligned}\frac{\eta(s)}{\eta(1/2)} &= \exp\left(\frac{\mathrm{vol}(M)i}{2} \left(s - \frac{1}{2}\right)^2 + n_1 \left((2s-1)\log(-s) + (i\pi - 2)s\right) \right) \\ &\times \exp\Bigg(2n_1 s \log 2 - n_1 \log 2 \\ &+ \frac{\mathrm{vol}(M)i}{24} - \pi \sum_{\substack{\{R\} \\ 0<\theta(R)<\pi}} \left(\frac{i}{2M_R \sin^2\theta} + O\left(\frac{1}{e^{2(\pi-\theta)t}} \right) \right) \Bigg) \\ &\times \exp\left(O\left(\frac{|s-1/2|}{e^{2\pi t}} + \frac{1}{|s|} \right) \right) \qquad (|s| \to +\infty)\end{aligned}$$

uniformly in $|\arg(-s)| \leq \pi - \delta$, $t \geq 2$. Here $s = \sigma + it$.

Proof. In light of the formulas (4.11) and (4.12), we write

$$\frac{\eta(s)}{\eta(1/2)} = \exp\left(I_1 - I_2 + \int_0^{s-1/2} 2n_1 \log 2 \, du + n_1 I_3 \right).$$

The reader is warned that in other places of this dissertation, I_n denote different integrals. All formulas in this proof are valid as $|s| \to \infty$, uniformly in $|\arg(-s)| \leq \pi - \delta$, $t \geq 2$. We have

$$\begin{aligned}
\frac{I_1}{\mathrm{vol}(M)} &= \int_0^{s-\frac{1}{2}} v \tan(\pi v)\, dv = i \int_0^{s-\frac{1}{2}} v \frac{1-e^{2\pi i v}}{1+e^{2\pi i v}}\, dv \\
&= i \int_0^{s-\frac{1}{2}} v\, dv - i \left(\int_0^{it} + \int_{it}^{\sigma-1/2+it} \right) \frac{2v}{1+e^{-2\pi i v}}\, dv \\
&= \frac{i}{2}(s-\frac{1}{2})^2 + 2i \int_0^t \frac{x}{1+e^{2\pi x}}\, dx - 2i \int_0^{\sigma-1/2} \frac{x+it}{1+e^{2\pi t - 2\pi i x}}\, dx.
\end{aligned}$$

Formulas

$$\int_0^\infty \frac{x}{1+e^{2\pi x}}\, dx = \frac{1}{48}, \qquad \begin{aligned}\int_0^{\sigma-1/2} \frac{dx}{1+e^{2\pi t-2\pi i x}} &= -\frac{i}{2\pi} \log\left(1+e^{2\pi i x - 2\pi t}\right)\Big|_{x=0}^{\sigma-1/2} \\ &= -\frac{i}{2\pi}\left(\log(1+\exp(2\pi i \sigma - \pi i - 2\pi t)) - \log(1+\exp(-2\pi t))\right)\end{aligned}$$

and

$$\begin{aligned}
2i \int_0^{\sigma-1/2} \frac{x}{1+e^{2\pi t - 2\pi i x}}\, dx &= 2i \int_0^{\sigma-1/2} x\, d\left(\int_1^x \frac{dv}{1+e^{2\pi t - 2\pi i v}} \right) \\
&= O\left(\frac{|\sigma-1/2|}{e^{2\pi t}} \right)
\end{aligned}$$

give that

$$\frac{I_1}{\mathrm{vol}(M)} = \frac{i}{2}\left(s-\frac{1}{2}\right)^2 + \frac{i}{24} + O\left(\frac{t}{e^{2\pi t}}\right) + O\left(\frac{|\sigma-1/2|}{e^{2\pi t}}\right).$$

We turn to the integral I_2.

$$\begin{aligned}
&\int_{1/2}^{s} \frac{\cos((2\theta-\pi)(u-1/2))}{\cos(\pi(u-1/2))}\, du = \int_0^{s-1/2} \frac{\cos((2\theta-\pi)u)}{\cos \pi u}\, du \\
=&\int_0^{it} \frac{e^{i(2\theta-\pi)u} + e^{-i(2\theta-\pi)u}}{e^{i\pi u} + e^{-i\pi u}}\, du + \int_{it}^{\sigma-1/2+it} \frac{e^{i(2\theta-\pi)u} + e^{-i(2\theta-\pi)u}}{e^{i\pi u} + e^{-i\pi u}}\, du \\
=&i \int_0^\infty \frac{e^{-(2\theta-\pi)y} + e^{(2\theta-\pi)y}}{e^{-\pi y} + e^{\pi y}}\, dy - i \int_t^\infty \frac{e^{-(2\theta-\pi)y} + e^{(2\theta-\pi)y}}{e^{-\pi y} + e^{\pi y}}\, dy \\
&+ \int_0^{\sigma-1/2} \frac{e^{i(2\theta-\pi)(x+it)} + e^{-i(2\theta-\pi)(x+it)}}{e^{i\pi(x+it)} + e^{-i\pi(x+it)}}\, dx.
\end{aligned}$$

Let us evaluate the terms in the above formula one-by-one. Using known integral (see Gradshteyn and Ryzhik [30, Formula 3.311 3])

$$\int_{-\infty}^{\infty} \frac{e^{-ax}dx}{1+e^{-x}} = \frac{\pi}{\sin(\pi a)} \qquad (\Re a > 0),$$

we obtain

$$\int_0^\infty \frac{e^{-(2\theta-\pi)y} + e^{(2\theta-\pi)y}}{e^{-\pi y} + e^{\pi y}}\, dy = \frac{1}{2\sin\theta}.$$

Continuing our calculations, we have

$$\int_t^\infty \frac{e^{-(2\theta-\pi)y} + e^{(2\theta-\pi)y}}{e^{-\pi y} + e^{\pi y}}\, dy \leq 2\int_t^\infty e^{(2\theta-2\pi)y}\, dy = \frac{1}{\pi-\theta} e^{2(\theta-\pi)t},$$

and, with slightly more efforts,

$$\int_0^{\sigma-1/2} \frac{e^{i(2\theta-\pi)(x+it)} + e^{-i(2\theta-\pi)(x+it)}}{e^{i\pi(x+it)} + e^{-i\pi(x+it)}}\, dx \ll e^{2(\theta-\pi)t}.$$

In the end, we have the following estimate for the integral I_2

$$I_2 = \pi \sum_{\substack{\{R\} \\ 0<\theta(R)<\pi}} \left(\frac{i}{2M_R \sin^2\theta} + O\left(\frac{1}{e^{2(\pi-\theta)t}}\right)\right).$$

To calculate the logarithmic derivatives of the Gamma functions, we need Stirling's formula (Titchmarsh [91, Section 4.42]). For any constant a,

$$\log\Gamma(s+a) = \left(s+a-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right),$$

as $|s| \to \infty$, uniformly for $-\pi+\delta \leq \arg s \leq \pi - \delta$. Therefore

$$\begin{aligned}
\exp(I_3) &= \exp(\int_{1/2}^{s} \left(\frac{\Gamma'}{\Gamma}(1/2+u) + \frac{\Gamma'}{\Gamma}(3/2-u)\right) du) = \frac{\Gamma(1/2+s)}{\Gamma(3/2-s)} \\
&= \frac{\pi}{(1/2-s)\sin(\pi(1/2-s))\Gamma^2(1/2-s)} \\
&= \frac{2i\pi\exp\left(2s\log(-s) - 2s - \log 2\pi + O(|s|^{-1})\right)}{(1/2-s)(e^{i\pi(1/2-s)} - e^{-i\pi(1/2-s)})} \\
&= \exp\left((2s-1)\log(-s) + (i\pi-2)s\right) + O(e^{-2\pi t} + |s|^{-1}).
\end{aligned}$$

This proves Lemma 4.5. □

From Lemma 4.5 and recalling that $|\eta(1/2)| = 1$, we derive the following formula

$$|\eta(s)| = \exp\left(-\mathrm{vol}(M)\left(\sigma - \frac{1}{2}\right)t + n_1(2\sigma - 1)\log|s|\right) \tag{4.13}$$
$$\times \exp\left(-n_1(2\arg(-s) + \pi)t - 2n_1\sigma + 2n_1\sigma\log 2 - n_1\log 2\right)$$
$$\times \exp\left(O\left(\frac{|s-1/2|}{e^{2\pi t}} + \frac{1}{|s|}\right) + \sum_{\substack{\{R\} \\ 0<\theta(R)<\pi}} O\left(\frac{1}{e^{2(\pi-\theta)t}}\right)\right) \quad (|s| \to \infty)$$

uniformly in $|\arg(-s)| \leq \pi - \delta$, $t \geq 2$. We will often use the fact that $\arg(-s) = \arctan(t/\sigma)$ if $\sigma < 0$.

For $|s| \to \infty$, Stirling's formula also yields

$$K(1-s) =\pi^{n_1/2}\left(\frac{\Gamma(1/2-s)}{\Gamma(1-s)}\right)^{n_1}\exp(c_1 + c_2 - c_1 s) \tag{4.14}$$
$$= \exp\left(-c_1 s - \frac{n_1}{2}\log(-s) + \frac{n_1}{2}\log\pi + c_1 + c_2 + O(|s|^{-1})\right)$$

uniformly in $\arg(-s) \leq \pi - \delta$.

4.2 Trivial a-values. Proof of Theorem 4.1

Theorem 4.1 is derived from the following three lemmas.

Lemma 4.6. *Let $n_1 \geq 1$. Let $a \in \mathbb{C}$ and $a \neq 0$. Then there is $\sigma_0 = \sigma_0(a, M) < 0$ such that $Z(s) \neq a$ in $\sigma \leq \sigma_0$, and t lies in the following region*

$$t \geq \frac{2n_1}{\mathrm{vol}(M)}\log(-\sigma) + \frac{1}{\log^2(-\sigma)} - \frac{2n_1 + c_1 - 2n_1\log 2}{\mathrm{vol}(M)} \tag{4.15}$$

and

$$2 \leq t \leq \frac{2n_1}{\mathrm{vol}(M)}\log(-\sigma) - \frac{1}{\log^2(-\sigma)} - \frac{2n_1 + c_1 - 2n_1\log 2}{\mathrm{vol}(M)}. \tag{4.16}$$

Proof. From the Euler product (2.5) for Z and by the definition (4.3) of m_0 and $N(P_{00})$, we obtain

$$Z(s) = 1 + m_0 N(P_{00})^{-s} + o(N(P_{00})^{-\sigma}) \quad (\sigma \to \infty), \tag{4.17}$$

uniformly in $t \in \mathbb{R}$.

Note that we have

$$Z(s) = \eta(s)\phi(1-s)Z(1-s).$$

By equation (4.17), we obtain $\lim_{\sigma\to-\infty} Z(1-s) = 1$ uniformly for $t \in \mathbb{R}$. Recall that $\phi(1-s) = H(1-s)K(1-s)$ and $\lim_{\sigma\to-\infty} H(1-s) = 1$. To prove the lemma, first we will show that $|\eta(s)K(1-s)| \geq 2|a|$ in the region defined by (4.15). Then we show that $|\eta(s)K(1-s)| \leq |a|/2$ in the region defined by (4.16).

Let s satisfy the relation (4.15). We divide the region defined by this relation into two subregions, which are given by

$$\frac{2n_1}{\mathrm{vol}(M)}\log(-\sigma) + \frac{1}{\log^2(-\sigma)} - \frac{2n_1 + c_1 - 2n_1\log 2}{\mathrm{vol}(M)} \leq t < \frac{3n_1}{\mathrm{vol}(M)}\log(-\sigma) \tag{4.18}$$

and

$$\frac{3n_1}{\mathrm{vol}(M)}\log(-\sigma) \leq t. \tag{4.19}$$

In both (4.18) and (4.19), $-\sigma$ is sufficiently large.

Let us consider the region defined by (4.18). We introduce variable b, satisfying the relation

$$t = \frac{2n_1}{\mathrm{vol}(M)}\log(-\sigma) + \frac{b}{\log^2(-\sigma)} - \frac{2n_1 + c_1 - 2n_1\log 2}{\mathrm{vol}(M)} \tag{4.20}$$

Here $b \geq 1$ and b is such that the inequalities (4.18) hold. Then by (4.13) and (4.14), for $\sigma \to -\infty$,

$$|\eta(s)K(1-s)| = \exp\left(b\frac{\mathrm{vol}(M)(-\sigma)}{\log^2(-\sigma)} + O\left(\log|\sigma| + |\sigma|^{1-4\pi n_1/\mathrm{vol}(M)}\right)\right), \tag{4.21}$$

Therefore, for sufficiently large $-\sigma$,

$$|\eta(s)K(1-s)| \geq \exp\left(\frac{1}{2}\frac{\mathrm{vol}(M)(-\sigma)}{\log^2(-\sigma)}\right) > 2|a|.$$

In the region defined by (4.19), in light of (4.13), (4.14) and for sufficiently large $-\sigma$, we have

$$|\eta(s)K(1-s)| \geq \exp(-\frac{n_1}{2}\sigma\log(-\sigma)) > 2|a|.$$

This proves the first part of the lemma.

Now let us prove the lemma in the region defined by the inequalities (4.16). We aim to show that in this case $|\eta(s)K(1-s)| < |a|/2$. In order to do so, we divide the region defined by (4.16) into two parts

$$\frac{n_1}{\mathrm{vol}(M)}\log(-\sigma) \leq t \leq \frac{2n_1}{\mathrm{vol}(M)}\log(-\sigma) - \frac{1}{\log^2(-\sigma)} - \frac{2n_1 + c_1 - 2n_1\log 2}{\mathrm{vol}(M)} \tag{4.22}$$

and

$$2 \leq t \leq \frac{n_1}{\mathrm{vol}(M)}\log(-\sigma). \tag{4.23}$$

Similarly as in the previous part of this proof, let us introduce the variable b, which is defined exactly as in (4.20), this time with $b \leq -1$ and b is such that the inequalities (4.22) hold. Again by (4.13) and (4.14), for sufficiently large $-\sigma$, we have

$$\begin{aligned}|\eta(s)K(1-s)| &= \exp\left(b\frac{\mathrm{vol}(M)(-\sigma)}{\log^2(-\sigma)} + O\left(\log|\sigma| + |\sigma|^{1-2\pi n_1/\mathrm{vol}(M)}\right)\right)\\ &\leq \exp\left(-\frac{1}{2}\frac{\mathrm{vol}(M)(-\sigma)}{\log^2(-\sigma)}\right) < \frac{|a|}{2}.\end{aligned}$$

As for the second part of our region, defined by the inequalities (4.23), for sufficiently large $-\sigma$, we obtain

$$|\eta(s)K(1-s)| \leq \exp(\frac{n_1}{2}\sigma\log(-\sigma)) \leq \frac{|a|}{2}$$

in the region (4.23). This proves Lemma 4.6. □

Next we investigate a-values located between a-value free regions (4.15) and (4.16) indicated in the previous lemma. We will use Rouché's theorem together with an auxiliary function

$$\begin{aligned}h(s) := \exp\Bigg(&\frac{\mathrm{vol}(M)i}{2}\left(s-\frac{1}{2}\right)^2 + n_1(2s-3/2)\log(-s) \\ &+(i\pi n_1 - 2n_1 + 2n_1\log 2 - c_1)s + C\Bigg),\end{aligned} \tag{4.24}$$

where C is a complex constant. If $-\sigma$ is large, then the function $h(s)$ is the 'main' part of the function $\eta(s)K(1-s)$. Therefore, in view of the functional equation $Z(s) = \eta(s)K(1-s)H(1-s)Z(1-s)$, we have that $h(s)$ is 'similar' to $Z(s)$.

For $\sigma_3 < 0$, define a curve in $\mathbb{C}$ by

$$\ell_{\sigma_3} := \{\sigma + it : |h(\sigma + it)| = 1, \sigma < \sigma_3, t > 2\}.$$

Let $N(\sigma')$ denote the number of zeros of $h(s) - 1$ in $\sigma' \leq \sigma \leq \sigma_3$.

Lemma 4.7. *Let $n_1 \geq 1$. There is $\sigma_3 = \sigma_3(C, M) < 0$ and a real function $t(\sigma) = t(\sigma, C, M)$, such that*

$$\ell_{\sigma_3} = \{\sigma + it(\sigma) : \sigma \leq \sigma_3\},$$

and, for $\sigma \to -\infty$,

$$t(\sigma) = \frac{2n_1}{\mathrm{vol}(M)} \log(-\sigma) - \frac{2n_1 + c_1 - 2n_1 \log 2}{\mathrm{vol}(M)} + O\left(\frac{\log(-\sigma)}{-\sigma}\right). \tag{4.25}$$

Moreover, for $\sigma \to -\infty$,

$$N(\sigma) = \frac{\mathrm{vol}(M)}{4\pi}\left(\sigma - \frac{1}{2}\right)^2 + \frac{n_1}{2}\sigma - \frac{n_1^2 \log^2(-\sigma)}{\pi \mathrm{vol}(M)} + \frac{2n_1^2 \log(-\sigma)}{\pi \mathrm{vol}(M)} + O(1). \tag{4.26}$$

Proof. Let

$$|h(s)| = \exp(f(\sigma, t)),$$

where

$$\begin{aligned} f(\sigma, t) = &- \mathrm{vol}(M)\left(\sigma - \frac{1}{2}\right)t + n_1\left(2\sigma - \frac{3}{2}\right)\log|s| \\ &- n_1(2\arg(-s) + \pi)t - (2n_1 + c_1 - 2n_1 \log 2)\sigma + \Re C. \end{aligned} \tag{4.27}$$

As above, we assume $|\arg(-s)| < \pi$.

The equation $|h(s)| = 1$ is equivalent to $f(\sigma, t) = 0$. The equality

$$f(\sigma, t) = 0$$

implies $n_1 \log(-\sigma)/\mathrm{vol}(M) < t < 3n_1 \log(-\sigma)/\mathrm{vol}(M)$, provided $-\sigma$ is sufficiently large. Thus

$$f(\sigma,t) = -\operatorname{vol}(M)\left(\sigma - \frac{1}{2}\right)t + n_1\left(2\sigma - \frac{3}{2}\right)\log(-\sigma) \\ - n_1\pi t - (2n_1 + c_1 - 2n_1\log 2)\sigma + \Re C + O\left(\frac{\log^2(-\sigma)}{-\sigma}\right) = 0$$

as $\sigma \to -\infty$. From the last formula, it follows that

$$t = \frac{2n_1}{\operatorname{vol}(M)}\log(-\sigma) - \frac{2n_1 + c_1 - 2n_1\log 2}{\operatorname{vol}(M)} + O\left(\frac{\log(-\sigma)}{-\sigma}\right). \tag{4.28}$$

We see that if $n_1\log(-\sigma)/\operatorname{vol}(M) < t < 3n_1\log(-\sigma)/\operatorname{vol}(M)$, then by (4.27) we have

$$\frac{\partial f(\sigma,t)}{\partial t} = \operatorname{vol}(M)\left(\frac{1}{2} - \sigma\right) - n_1\pi + O\left(\frac{\log(-\sigma)}{(-\sigma)}\right) > 0 \tag{4.29}$$

for $-\sigma$ sufficiently large. Note that

$$f(\sigma, \log(-\sigma)/\operatorname{vol}(M)) < 0 < f(\sigma, 3\log(-\sigma)/\operatorname{vol}(M)).$$

Therefore, by the positivity of the derivative (4.29) and by the Implicit Function Theorem, the relation $f(\sigma,t) = 0$ implies the existence of a differentiable function $t(\sigma)$, such that

$$\ell_{\sigma_3} = \{\sigma + it(\sigma) : \sigma \le \sigma_3\},$$

for sufficiently large $-\sigma_3$. We will show that $t = t(\sigma)$ is a decreasing function. Calculating the partial derivative of f with respect to σ yields, for $t = t(\sigma)$,

$$\left.\frac{\partial f(\sigma,t)}{\partial\sigma}\right|_{t=t(\sigma)} = 2n_1 + O\left(\frac{\log(-\sigma)}{(-\sigma)}\right). \tag{4.30}$$

From $f(\sigma,t) = 0$, for $t = t(\sigma)$, we have

$$\frac{\partial f(\sigma,t)}{\partial\sigma} + \frac{\partial f(\sigma,t)}{\partial t}\frac{dt(\sigma)}{d\sigma} = 0.$$

By the last formula together with (4.29), (4.30), and (4.28), for sufficiently large $-\sigma$, we get

$$\frac{dt(\sigma)}{d\sigma} = -\frac{2n_1}{\mathrm{vol}(M)(-\sigma)} + O\left(\frac{\log(-\sigma)}{\sigma^2}\right) < 0.$$

Next we investigate how many zeros the function $h(s) - 1$ has on the curve $\sigma + it(\sigma)$. It is easy to check that such zeros are simple for large $-\sigma$. The function $h(s)$ is analytic and has no zeros for $t > 0$. For s in the upper half-plane, we define $\arg(h(s))$ by continuous variation starting at the point $1/2$, then going to s by any path lying in $t > 0$. We see that

$$\arg\left(h(s)\right) = \frac{\mathrm{vol}(M)}{2}\left(\left(\sigma - \frac{1}{2}\right)^2 - t^2\right) + 2n_1 t\log|s| + n_1(2\sigma - 3/2)\arg(-s) \tag{4.31}$$
$$+ n_1\pi\sigma - (2n_1 + c_1 - 2n_1\log 2)t + \arg(C).$$

On the curve $\sigma + it(\sigma)$, we have

$$\frac{d\arg\left(h(\sigma + it(\sigma))\right)}{d\sigma} = \frac{\partial\arg\left(h(\sigma + it)\right)}{\partial\sigma} + \frac{\partial\arg\left(h(\sigma + it)\right)}{\partial t}\frac{dt(\sigma)}{d\sigma}$$
$$= \mathrm{vol}(M)\left(\sigma - \frac{1}{2}\right) + n_1\pi + O\left(\frac{\log(-\sigma)}{-\sigma}\right) < 0. \tag{4.32}$$

Thus dividing the formula (4.31) by 2π and using the expression (4.28), we obtain the root counting formula (4.26). This proves Lemma 4.7. □

Lemma 4.8. *Let* $n_1 \geq 1$. *Let* $a \in \mathbb{C}$ *and* $a \neq 0$. *Assume that* $\mathrm{vol}(M) < 4\pi n_1$. *Then the number of roots of* $Z(s) = a$ *in the region* $\sigma' \leq \sigma \leq -2$, $t \geq 2$, *and*

$$\left| t - \frac{2n_1}{\mathrm{vol}(M)}\log(-\sigma) + \frac{2n_1 + c_1 - 2n_1\log 2}{\mathrm{vol}(M)} \right| \leq \frac{1}{\log^2(-\sigma)} \tag{4.33}$$

is

$$\frac{\mathrm{vol}(M)}{4\pi}\left(\sigma' - \frac{1}{2}\right)^2 + \frac{n_1}{2}\sigma' - \frac{n_1^2\log^2(-\sigma')}{\pi\mathrm{vol}(M)} + \frac{2n_1^2\log(-\sigma')}{\pi\mathrm{vol}(M)} + O(1).$$

Proof. In this proof, we will use the auxiliary function $h(s)$ defined by formula (4.24) with the constant C such that

$$C = -\pi \sum_{\substack{\{R\} \\ 0<\theta(R)<\pi}} \frac{i}{M_R \sin^2\theta} - n_1 \log 2 + \frac{n_1}{2}\log\pi + c_1 + c_2 + \frac{\mathrm{vol}(M)i}{24} + \log\eta(1/2) - \log a.$$

Using Rouché's theorem, we will show that in the left half-plane the zeros of the functions $Z(s) - a$ and $h(s) - 1$ are distributed very similarly.

By formulas (4.17), (4.10), (4.13), and (4.14), we see that in the region defined by the inequality (4.33), there is an analytic function $v(s) \neq 0$ such that

$$Z(s)/a = \eta(s)\phi(1-s)Z(1-s)/a = h(s)(1+v(s));$$

here we have that

$$|1+v(s)| = \exp\left(O\left(|\sigma|^{1-4\pi n_1/\mathrm{vol}(M)}\right) + \sum_{\substack{\{R\} \\ 0<\theta(R)<\pi}} O\left(|\sigma|^{-4n_1(\pi-\theta)/\mathrm{vol}(M)}\right) \right).$$

Note that in the last equality the term $O\left(|\sigma|^{1-4\pi n_1/\mathrm{vol}(M)}\right)$ comes from the term $O(|s-1/2|e^{-2\pi t})$ in the formula (4.13). Thus, by the condition $\mathrm{vol}(M) < 4\pi n_1$, we see that $v(s) \to 0$, as $\sigma \to -\infty$ and s satisfies the inequality (4.33).

We turn to the construction of the contour for Rouché's theorem. Let σ_3 and ℓ_{σ_3} be from Lemma 4.7. Let $z \in \ell_{\sigma_3}$ be such that $h(z) = 1$ and $-\Re z$ be sufficiently large. By (4.32), we see that there are $z', z'' \in \ell_{\sigma_3}$, $\Re z' < \Re z < \Re z''$, such that $h(z') = h(z'') = -1$, and in the strip $\Re z' \leq \sigma \leq \Re z''$ the equation $h(s) = 1$ has only one solution $s = z$. Next we consider a contour K with vertices at $z' + i/\log^2(-\Re z')$, $z'' + i/\log^2(-\Re z'')$, $z'' - i/\log^2(-\Re z'')$, $z' - i/\log^2(-\Re z')$, where the vertices $z' + i/\log^2(-\Re z')$ and $z' - i/\log^2(-\Re z')$ (also $z'' + i/\log^2(-\Re z'')$ and $z'' - i/\log^2(-\Re z'')$) are connected by vertical lines. The vertices $z' + i/\log^2(-\Re z')$ and $z'' + i/\log^2(-\Re z'')$ are connected by the shifted curve $\ell_{\sigma_3} + i/\log^2(-\sigma)$, similarly the edges $z'' - i/\log^2(-\Re z'')$ and $z' - i/\log^2(-\Re z')$ are connected by the shifted curve $\ell_{\sigma_3} - i/\log^2(-\sigma)$, where $\Re z' \leq \sigma \leq \Re z''$.

We have $Z(s)/a - 1 = h(s) - 1 + h(s)v(s)$. In order to apply Rouché's theorem, we have to show that

$$|h(s) - 1| > |h(s)v(s)| \tag{4.34}$$

on K. Then, inside of K, the functions $Z(s) - a$ and $h(s) - 1$ have the same number of zeros and the statement of the lemma will follow from Lemma 4.7.

We consider the inequality (4.34) on the vertical lines of K. By (4.31) and (4.25), there is a positive constant c such that, for any sufficiently large $-\sigma$,

$$|\arg(h(\sigma + it(\sigma) + i\delta) - \arg(h(\sigma + it(\sigma)))| < \frac{c}{\log(-\sigma)},$$

if $-1/\log^2(-\sigma) \leq \delta \leq 1/\log^2(-\sigma)$. Thus, on the vertical lines of K, the argument of $h(s)$ is almost equal to the argument of -1, i.e.

$$|h(s) - 1| \geq |-|h(s)|e^{ic/\log(-\sigma)} - 1| > |h(s)v(s)|.$$

The inequality (4.34) on the 'horizontal' sides of K follows from

$$\left|h\left(\sigma + it(\sigma) \pm \frac{i}{\log^2(-\sigma)}\right)\right| = \exp\left(\pm\frac{\mathrm{vol}(M)(-\sigma)}{\log^2(-\sigma)} + O(\log(-\sigma))\right),$$

for $\sigma \to -\infty$. This proves Lemma 4.8.

□

Proof of Theorem 4.1. Since $N(P_{00}) > 1$ (see Hejhal [35, Chap. 2, Sect. 2]), there exists σ_1, depending on a, such that $Z(s) \neq a$ for $\sigma \geq \sigma_1$.

If $n_1 = 0$, then the a-value free region on the left hand-side of $\mathbb{C}$ follows from the functional equation (4.4) and from the formulas (4.10), (4.13), (4.14), and (4.17).

The remaining part of Theorem 4.1 follows from Lemmas 4.6 and 4.8. □

4.3 Non-trivial *a*-values. Proofs of Theorems 4.2, 4.3, and 4.4

Theorems 4.2 and 4.3 will be derived from the following proposition

Proposition 4.9. *Suppose a is a complex and b is a real number, $-b < \sigma_0$, where σ_0 is from Theorem 4.1. Then in the case of $a \neq 1$, we have*

$$\sum_{2<\gamma_a\leq T} (\beta_a + b) = \frac{\mathrm{vol}(M)}{4\pi}\left(\frac{1}{2}+b\right)T^2 - \frac{n_1}{2\pi}\left(2b+\frac{3}{2}\right)T\log T \tag{4.35}$$
$$+ \frac{1}{2\pi}\left((2n_1+c_1)b + 3n_1/2 - 2n_1 b\log 2 - n_1\log 2 + \frac{n_1}{2}\log\pi + c_1 + \Re c_2\right)T$$
$$- \frac{T}{2\pi}\log|1-a| + o(T).$$

If $a = 1$, we have

$$\sum_{2<\gamma_a\leq T} (\beta_a + b) = \frac{\mathrm{vol}(M)}{4\pi}\left(\frac{1}{2}+b\right)T^2 - \frac{n_1}{2\pi}\left(2b+\frac{3}{2}\right)T\log T$$
$$+ \frac{1}{2\pi}\left((2n_1+c_1)b + 3n_1/2 - 2n_1 b\log 2 - n_1\log 2 + \frac{n_1}{2}\log\pi + c_1 + \Re c_2\right)T$$
$$- \frac{T}{2\pi}\log m_0 - \frac{bT}{2\pi}\log N(P_{00}) + o(T).$$

The constant $N(P_{00})$ is defined in formula (4.3), n_1 is the number of cusps of the corresponding Riemann surface, $c_1 = -2\log\mathfrak{g}_1$, and $c_2 = \log d(1)$.

Proof. Let $a \neq 1$. Consider the function

$$G(s) = \frac{Z(s) - a}{1 - a}.$$

Note that the zeros of G correspond to the a-values of Z. Let $c \geq \sigma_1 + 1$, which is independent and is defined later. We are interested in the behavior of G in the rectangle R with vertices $c + i\tau'$, $c + iT'$, $-b + iT'$, and $-b + i\tau'$. Here $T < T' < T + 1/T$ and $2 < \tau' < 3$ are such that $G(s) \neq 0$ on the rectangle R. In case our rectangle crosses the region containing trivial a-values, there will be only a finite number of trivial a-values in the rectangle. The 'contribution' of these trivial a-values is captured by the error term.

Applying Littlewood's lemma (see Titchmarsh [92, Section 9.9]) to G on the rectangle R yields (c.f. Garunkštis and Šimėnas [25, equation (3.2)]):

$$2\pi \sum_{\substack{\beta_a > -b \\ \tau' < \gamma_a \le T'}} (\beta_a + b) = \int_{\tau'}^{T'} \log|G(-b+it)|\,dt - \int_{\tau'}^{T'} \log|G(c+it)|\,dt \tag{4.36}$$
$$- \int_{-b}^{c} \arg G(\sigma + i\tau')\,d\sigma + \int_{-b}^{c} \arg G(\sigma + iT')\,d\sigma$$
$$=: \sum_{j=1}^{4} I_j.$$

The value $\arg G(s)$ is defined by continuous variation starting at $s = \sigma_1 + 1$, then going to $\sigma_1 + 1 + it$, and finally to $\sigma + it$, assuming the path does not cross a zero of $G(s)$. In case it does, we set $\arg G(s) = \lim_{\epsilon \to +0} \arg G(\sigma + it + \epsilon)$.

By the functional equation (4.4) and by the decomposition (4.8) of the function $\phi(s)$, we have

$$I_1 = \int_{\tau'}^{T'} \log\left|\frac{\eta(-b+it)\phi(1+b-it)Z(1+b-it) - a}{1-a}\right| dt$$
$$= \int_{\tau'}^{T'} \log|\eta(-b+it)K(1+b-it)|\,dt - \int_{\tau'}^{T'} \log|1-a|\,dt$$
$$+ \int_{\tau'}^{T'} \log|Z(1+b-it)H(1+b-it)|\,dt$$
$$+ \int_{\tau'}^{T'} \log\left|1 - \frac{a}{\eta(-b+it)Z(1+b-it)H(1+b-it)K(1+b-it)}\right| dt.$$

In light of formulas (4.17), (4.10), (4.13), and (4.14), we have that the last integral is $o(T)$. Cauchy's theorem and expressions (4.17), (4.10) yield

$$- \int_{-b+i\tau'}^{-b+iT'} \log Z(1-s)H(1-s)\,ds = \int_{-b+iT'}^{\infty+iT'} \log Z(1-s)H(1-s)\,ds \tag{4.37}$$
$$+ \int_{\infty+i\tau'}^{-b+i\tau'} \log Z(1-s)H(1-s)\,ds = O(1).$$

This, together with formulas (4.13), (4.14), and $\arg(b - it) = -\pi/2 + b/t +$

$O(t^{-3})$, gives

$$\begin{aligned}
I_1 = &\int_{\tau'}^{T'} \left(\mathrm{vol}(M)\left(\frac{1}{2}+b\right)t - n_1\left(2b+\frac{3}{2}\right)\log t\right.\\
&\left.-2n_1 b\log 2 - n_1\log 2 + c_1 b + \frac{n_1}{2}\log\pi + c_1 + \Re c_2 + O\left(\frac{1}{t}\right)\right)\,dt\\
&- T'\log|1-a| + o(T')\\
=&\frac{\mathrm{vol}(M)}{2}\left(\frac{1}{2}+b\right)T'^2 - n_1\left(2b+\frac{3}{2}\right)T'\log T'\\
&+\left((2n_1+c_1)b + 3n_1/2 - 2n_1 b\log 2 - n_1\log 2 + \frac{n_1}{2}\log\pi + c_1 + \Re c_2\right)T'\\
&- T'\log|1-a| + o(T').
\end{aligned}$$

As for I_2, using Cauchy's theorem, we get

$$I_2 = o(T).$$

Let us turn our attention to I_4 in (4.36). We can express it as

$$I_4 = \int_{-b}^{c} \arg G(\sigma + iT')\,d\sigma = \int_{-b}^{1/2} + \int_{1/2}^{c} = I_{41} + I_{42}.$$

Consider I_{42}. For our purposes, we need to show that, for $\sigma \geq 1/2$, we have

$$\arg G(\sigma + iT') = o(T').$$

Based on (4.17), we choose c large enough so that $\Re G(s) \neq 0$ for $\sigma > c$. Suppose that $\Re G(\sigma + iT')$ has N zeros for $1/2 \leq \sigma \leq c$. Divide $[1/2, c]$ into at most $N+1$ intervals in each of which $\Re G(\sigma + iT')$ is of constant sign. We have

$$|\arg G(\sigma + iT')| \leq (N+1)\pi.$$

We can see that our task reduces to estimating N in the above equation. We define an auxiliary function

$$h(z) := \frac{1}{2}(G(z + iT') + \overline{G(\overline{z} + iT')}).$$

Observe that $h(\sigma) = \Re G(\sigma + iT')$. Let $n(r)$ denote the number of zeros of

$h(z)$ in the disc $|z - c'| \leq r$. Then

$$|\arg G(\sigma + iT')| \leq \left(n\left(c' - \frac{1}{2}\right) + 1\right)\pi.$$

Let us now use Jensen's theorem (see Titchmarsh [91, Section 3.61])

$$\int_0^R \frac{n(r)}{r} dr = \frac{1}{2\pi}\int_0^{2\pi} \log|h(c + Re^{i\theta})|\, d\theta - \log|h(c)|. \tag{4.38}$$

Let $R = c - 1/2 + \delta$, $\delta > 0$. From Jorgenson and Smajlović [44, Proposition 6] (c.f. Garunkštis [23, comments below Theorem 5]), following the proofs of Lemmas 12 and 13 in [44], we obtain

$$Z(\sigma + iT') \ll \exp(\epsilon T'), \tag{4.39}$$

where $\sigma \geq 1/2$ and $\epsilon > 0$. It follows that for $\Re(c + Re^{i\theta}) \geq 1/2$ and sufficiently large T', we have

$$\log|h(c + Re^{i\theta})| < 2\epsilon T'.$$

By the functional equations (4.4), (4.5), the bound (4.7) and formula (4.13), there exists an absolute constant $d > 0$, such that

$$\log|h(c + Re^{i\theta})| < d(\delta + \epsilon)T'.$$

By the Pythagorean theorem, the length of the arc of the circle $|s - c| = R$ left of the critical line $\sigma = 1/2$ is given by

$$2R \arcsin \frac{\sqrt{2R\delta - \delta^2}}{R} = O(\sqrt{R\delta}) = O(\delta)\ \ (\delta \to 0).$$

The right-hand side of (4.38) is at most

$$O(\epsilon T') + O(\delta^{1/2}(\delta + \epsilon)T').$$

Since we have

$$\frac{\delta}{R} n(R - \delta) \leq \int_0^R \frac{n(r)}{r}\, dr,$$

it follows

$$n(R - \delta) = O\left(\frac{\epsilon}{\delta}T' + \delta^{-1/2}(\delta + \epsilon)T'\right).$$

By taking $\delta = \epsilon^{2/3}$, we get $n(R-\delta) = O(\epsilon^{1/3}T')$. This gives us

$$I_{42} = o(T).$$

Now we return to I_{41}. By the functional equation (4.4) and the equality $\arg Z(1/2 - iT') = -\arg Z(1/2 + iT')$, the latter following from the fact that the coefficients in the Dirichlet series expansion for Z are real, we get

$$\arg\left(\eta\left(\frac{1}{2}+iT'\right)\phi\left(\frac{1}{2}-iT'\right)\right) \equiv 2\arg Z\left(\frac{1}{2}+iT'\right) \mod 2\pi.$$

We choose the value of $\arg\eta(1/2+iT')\phi(1/2-iT')$ for which we have

$$\arg\left(\eta\left(\frac{1}{2}+iT'\right)\phi\left(\frac{1}{2}-iT'\right)\right) = 2\arg Z\left(\frac{1}{2}+iT'\right).$$

We also choose $0 \leq \arg\phi(1/2 - iT') < 2\pi$. By this, $\arg\left(\eta\left(\frac{1}{2}+iT'\right)\right)$ is fixed, too. For $\sigma \leq 1/2$, we define $\arg(\eta(\sigma+iT'))$ and $\arg(\phi(1-\sigma-iT'))$ by the continuous variation along the segment connecting $1/2+iT'$ with $\sigma+iT'$. If the path crosses a zero of $\phi(1-s)$, we set $\arg\phi(1-s) = \lim_{\epsilon\to+0}\arg\phi(1-\sigma-it-\epsilon)$. By the functional equation (4.4), we get

$$\begin{aligned} I_{41} = &\int_{-b}^{1/2} \arg(\eta(\sigma+iT'))\,d\sigma \\ &+ \int_{-b}^{1/2} \arg\left(Z(1-\sigma-iT')\phi(1-\sigma-iT') - \frac{a}{\eta(\sigma+iT')}\right)d\sigma \\ &- \int_{-b}^{1/2} \arg(1-a)\,d\sigma = I_{411} + I_{412} + o(\log T'). \end{aligned}$$

By analogy with the function G, we have $\arg Z(1/2+it) = o(t)$. From Lemma 4.5, we see that $\arg(\eta(\sigma+iT')) - \arg(\eta(1/2+iT')) \ll 1$, for $-b \leq \sigma \leq 1/2$. Thus $I_{411} = o(T)$.

Arguing similarly as for I_{42} and using the boundedness of $\phi(1-\sigma-iT')$ (see (4.7)), we obtain that $I_{412} = o(T)$. This gives us the proof of equation (4.35) for $a \neq 1$.

Let us consider the case $a = 1$. By analogy with the case $a \neq 1$, we have the function $G^*(s)$ defined by

$$G^*(s) = \frac{Z(s)-1}{m_0/N(P_{00})^s}.$$

Reiterating the argument as in the case of $a = 1$, we get the following estimate

$$\sum_{2<\gamma_a<T}(\beta_a + b) = \mathrm{vol}(M)\frac{1}{4\pi}\left(\frac{1}{2}+b\right)T^2 - \frac{n_1}{2\pi}\left(2b+\frac{3}{2}\right)T\log T \tag{4.40}$$
$$+\frac{1}{2\pi}((2n_1+c_1)b + 3n_1/2 - 2n_1 b\log 2 - n_1\log 2$$
$$+\frac{n_1}{2}\log\pi + c_1 + \Re c_2\Big)T$$
$$-\frac{T}{2\pi}\log m_0 - \frac{bT}{2\pi}\log N(P_{00}) + o(T).$$

This gives the proof of Proposition 4.9. □

Proof of Theorem 4.2. The theorem follows from Proposition 4.9 where we subtract the case b from $b+1$. □

In the proof of Theorem 4.3, we need the following lemma

Lemma 4.10. *We have*

$$\sum_{\substack{1<\gamma_a\leq T\\ \beta_a>1/2}}\left(\beta_a - \frac{1}{2}\right) \ll \frac{T^2\log\log T}{\log T}.$$

Proof. By Theorem 4.2, we have

$$\sum_{\substack{1<\gamma_a\leq T\\ \beta_a>1/2}}\left(\beta_a - \frac{1}{2}\right) = \sum_{\substack{1<\gamma_a\leq T\\ \beta_a>1/2+\frac{\log\log T}{\log T}}}\left(\beta_a - \frac{1}{2} - \frac{\log\log T}{\log T}\right)$$
$$+\sum_{\substack{1<\gamma_a\leq T\\ \beta_a>1/2+\frac{\log\log T}{\log T}}}\frac{\log\log T}{\log T} + \sum_{\substack{1<\gamma_a\leq T\\ 1/2<\beta_a\leq 1/2+\frac{\log\log T}{\log T}}}\left(\beta_a - \frac{1}{2}\right)$$
$$= \sum_{\substack{1<\gamma_a\leq T\\ \beta_a>1/2+\frac{\log\log T}{\log T}}}\left(\beta_a - \frac{1}{2} - \frac{\log\log T}{\log T}\right) + O\left(\frac{T^2\log\log T}{\log T}\right).$$

Applying Littlewood's lemma as in Proposition 4.9 with

$b = -1/2 - \log\log T/\log T$, and using the bound (4.39), we get the following

$$\sum_{\substack{1<\gamma_a\leq T\\ \beta_a>1/2+\frac{\log\log T}{\log T}}} \left(\beta_a - \frac{1}{2} - \frac{\log\log T}{\log T}\right) = \int_1^T \log\left|Z\left(\frac{1}{2} + \frac{\log\log T}{\log T} + it\right) - a\right| dt$$

$$+ O(T) = \int_{\sqrt{T}}^T \log\left|Z\left(\frac{1}{2} + \frac{\log\log T}{\log T} + it\right) - a\right| dt + O(T).$$

Recalling the growth estimate (4.39) for Z completes the proof. □

Proof of Theorem 4.3. By Lemma 4.10, we have

$$N^+(a,\delta,T) \leq \frac{\log T}{(\log\log T)^2} \sum_{\substack{1<\gamma_a\leq T\\ \beta_a>1/2+(\log\log T)^2/\log T}} \left(\beta_a - \frac{1}{2}\right) \ll \frac{T^2}{\log\log T}.$$

Suppose b is as formulated in Proposition 4.9. For sufficiently large T and for any a-value $\rho_a = \beta_a + i\gamma_a$, we have $\beta_a < b$. Consider the following inequality

$$\sum_{1<\gamma\leq T}(\beta_a + b) \leq \sum_{\substack{1<\gamma_a\leq T\\ \beta_a>1/2}} \left(\beta_a + \frac{1}{2}\right) + \left(b + \frac{1}{2}\right) \sum_{\substack{1<\gamma_a\leq T\\ \beta_a\geq 1/2-\delta}} 1 + \left(b + \frac{1}{2} - \delta\right) \sum_{\substack{1<\gamma_a\leq T\\ \beta_a<1/2-\delta}} 1.$$

By Proposition 4.9, Theorem 4.2, and Lemma 4.10, we have

$$0 \leq O\left(\frac{T^2\log\log T}{\log T}\right) - \delta N^-(a,\delta,T).$$

It follows that

$$N^-(a,\delta,T) \ll \frac{T^2}{\log\log T}.$$

This proves the Theorem.

□

Proof of Theorem 4.4. We subtract the value $(b+1/2)N(a,T)$, where $N(a,T)$ is as in Theorem 4.2, from the value $\sum(\beta_a + b)$ in Proposition 4.9. □

Chapter 5

Distribution modulo one of the *a*-values of the Selberg zeta-function associated to a compact Riemann surface

This chapter follows Garunkštis et al. [28]. From the definition (2.1) and the functional equation (2.2), it follows that there are positive constants $A = A(a)$ and $\tau = \tau(a)$ such that $Z(s) \neq a$ for $\sigma \geq A$ and

$$Z(s) \neq a \qquad \text{for} \quad \sigma \leq 1 - A \quad \text{and} \quad |t| \geq \tau$$

(see Garunkštis and Šimėnas [25]). Accordingly an a-point is called *non-trivial* if it lies in the strip $1 - A < \sigma < A$; non-trivial a-points are denoted by $\rho_a = \beta_a + i\gamma_a$. Any a-point inside in the region $\sigma < 1 - A$ and $|t| < \tau$ is called *trivial.* Denote by $N_a(T)$ the number of non-trivial a-points (counted with multiplicities) of $Z(s)$ in the region $\tau < t \leq T$. In Garunkštis and Šimėnas [25], also Chapter 3 of this dissertation, it was proved that, for $a \neq 1$,

$$N_a(T) = (g-1)T^2 + o(T) \tag{5.1}$$

and, for $a = 1$,

$$N_1(T) = (g-1)T^2 - \frac{T}{2\pi} \log N(P_{00}) + o(T),$$

where $N(P_{00}) = \min_{P_0}\{N(P_0)\}$. If $a = 0$, then formula (5.1) is known to hold with the better error term $O(T/\log T)$ (Hejhal [35, §2.8, Theorem 8.19]).

It is known that almost all non-trivial a-points are arbitrary close to the critical line $\sigma = 1/2$. More precisely, let $N_a^-(\delta, T)$ and $N_a^+(\delta, T)$ denote the number of non-trivial a-points of $Z(s)$ lying in the corresponding regions $\sigma < 1/2 - \delta$, $1 < t \leq T$, respectively $\sigma > 1/2 + \delta$, $1 < t \leq T$. Furthermore, define

$$N_a^0(\delta, T) = N_a(T) - (N_a^-(\delta, T) + N_a^+(\delta, T)).$$

Then, for $\delta = (\log\log T)^2/\log T$ we have (Garunkštis and Šimėnas [25, Theorem 3])

$$N_a^-(\delta, T) + N_a^+(\delta, T) \ll \frac{T^2}{\log\log T} \tag{5.2}$$

and

$$N_a^0(\delta, T) = (g-1)T^2 + O\left(\frac{T^2}{\log\log T}\right). \tag{5.3}$$

In Garunkštis [23] the connection between the distribution of a-points and the growth of $Z(s)$ was considered. The value distribution of the Selberg zeta-function associated to the modular group in the sense of universality theorem was investigated in Drungilas et al. [15].

Here we shall prove

Theorem 5.1. *Let $a \in \mathbb{C}$. The imaginary parts of non-trivial a-points of the Selberg zeta-function $Z(s)$ are uniformly distributed modulo one.*

For the Riemann zeta-function it was Rademacher [69] who proved under assumption of the truth of the Riemann hypothesis that the imaginary parts of the non-trivial zeros are uniformly distributed modulo one; Elliott [17] and (independently) Hlawka [37] gave unconditional proofs of this result. Further extensions and generalizations can be found in the articles Akbary and Murty [1], Ford et al. [21], and Fujii [22]; the analogue of Theorem 5.1 has been proved in Steuding [90].

The proof of Theorem 5.1 relies on

Proposition 5.2. *Let x be a fixed positive real number not equal to 1. Then, as $T \to \infty$,*

$$\sum_{0<\gamma\leq T} x^\rho = O(T).$$

Furthermore, we consider the eigenvalues λ_j of the hyperbolic Laplacian Δ on X.

Theorem 5.3. *Let $x = e^{2\pi n}$, $n \in \mathbb{Z}$. The following two statements are equivalent:*

1. *the eigenvalues λ_j are uniformly distributed modulo one.*

2. *the following bounds are valid.*

$$\int_1^T x^{2t/T+it^2} \frac{Z'}{Z}\left(\frac{1}{2}+\frac{1}{T}-it\right) dt = o(T^2) \qquad \text{for } n > 0$$

and

$$\int_1^T x^{-2t/T-it^2} \frac{Z'}{Z}\left(\frac{1}{2}+\frac{1}{T}+it\right) dt = o(T^2) \qquad \text{for } n < 0.$$

In the next section we state lemmas. Theorems 5.1, 5.3, and Proposition 5.2 are proved in Section 5.2.

5.1 Preliminaries

In the proof of Theorem 5.1, we will use Weyl criterion (see Theorem 2.9).

Lemma 5.4. *If $f(s)$ is analytic and $f(s_0) \neq 0$ with*

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M$$

in $\{s : |s - s_0| \leq r\}$ with $M > 1$, then

$$\left|\frac{f'(s)}{f(s)} - \sum_\rho \frac{1}{s-\rho}\right| < C\frac{M}{r}$$

for $|s - s_0| \leq r/4$, where C is some constant and ρ runs through the zeros of $f(s)$ such that $|\rho - s_0| \leq r/2$.

Lemma 5.4 is applied in the proof of the next lemma.

Lemma 5.5. *Let $a \in \mathbb{C}$ and $B, b \geq 1/2$ be fixed. Let T be such that $Z(\sigma + iT) \neq a$ for $1 - b \leq \sigma \leq B$. Then*

$$\int_{1-b}^B \left|\frac{Z'(\sigma+iT)}{Z(\sigma+iT)-a}\right| d\sigma \ll T.$$

Proof. In Lemma 5.4 we choose $s_0 = B + iT$ and $r = 4(B - (1 - b))$. We can take $M = cT$ with some $c > 0$ (see Randol [73, Lemma 2] or Garunkštis and Šimėnas [25, comment above Theorem 5], which is Chapter 3 in this dissertation). Then Lemma 5.4 gives

$$\frac{Z'(s)}{Z(s) - a} = \sum_{|\rho_a - s_0| \le \frac{r}{2}} \frac{1}{s - \rho_a} + O(T), \tag{5.4}$$

for $|s - s_0| \le r/4$. Thus

$$\begin{aligned} &\int_{1-b}^{B} \left| \frac{Z'(\sigma + iT)}{Z(\sigma + iT) - a} \right| d\sigma \le \int_{1-b}^{B} \sum_{|\rho_a - s_0| \le \frac{r}{2}} \left| \frac{1}{\sigma + iT - \rho_a} \right| d\sigma + O(T) \\ &= \sum_{|\rho_a - s_0| \le \frac{r}{2}} \int_{1-b}^{B} \frac{1}{\sqrt{(\sigma - \beta_a)^2 + (T - \gamma_a)^2}} d\sigma + O(T) \\ &= \sum_{|\rho_a - s_0| \le \frac{r}{2}} \left(\log \left(B - \beta_a + \sqrt{(T - \gamma_a)^2 + (B - \beta_a)^2} \right) \right. \\ &\qquad \left. - \log \left(1 - b - \beta_a + \sqrt{(T - \gamma_a)^2 + (1 - b - \beta_a)^2} \right) \right) + O(T) \\ &\ll T \end{aligned}$$

since the disc $|\rho_a - s_0| \le \frac{r}{2}$ contains $O(T)$ many a-points. □

In the following lemma we express the Selberg zeta-function by a general Dirichlet series.

Lemma 5.6. *There is the unbounded sequence $1 < x_2 < x_3 \dots$ of real numbers and real numbers a_n, $n = 2, 3, \dots$, such what*

$$Z(s) = 1 + \sum_{n=2}^{\infty} \frac{a_n}{x_n^s}, \tag{5.5}$$

where the Dirichlet series converges absolutely for $\sigma > 1$.

Proof. Multiplying the Euler product we obtain a formal Dirichlet series

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}) = 1 + \sum_{n=2}^{\infty} \frac{a_n}{x_n^s}.$$

In view of the properties of Dirichlet series (Hardy and Riesz [34, §2.2, Theorem 1]) it is enough to prove that the series (5.5) converges absolutely at $s = \sigma > 1$. For any positive x, we have that

$$1 + \sum_{x_n \le x} \frac{|a_n|}{x_n^\sigma} \le \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 + N(P_0)^{-\sigma-k}).$$

In the last formula, the product converges for $\sigma > 1$ since (Hejhal [35, §1.2, Proposition 2.5])

$$\sum_{\substack{\{P_0\} \\ N(P_0) \le x}} 1 = O(x).$$

This proves the lemma. □

The next lemma is essentially due to Landau [55] and deals with general Dirichlet series. Let $1 = x_1 < x_2 < \dots$ be an unbounded sequence X of real numbers and define

$$S = \{x_{k_1} x_{k_2} \dots x_{k_m} : m \in \mathbb{N}, k_1 \in \mathbb{N}, \dots, k_m \in \mathbb{N}\}$$

as the set of all possible products of elements of the sequence X. Let $1 = y_1 < y_2 < \dots$ be an ordered sequence of all different numbers of S.

Lemma 5.7. *For $n \in \mathbb{N}$ let a_n and b_n be complex numbers such that the general Dirichlet series $A(s) = \sum_n a_n x_n^{-s}$ and $B(s) = \sum_n b_n x_n^{-s}$ converge absolutely in the right half-plane $\sigma > \sigma_0$. If $b_1 \neq 0$, then there exist a real number $\sigma_1 \ge \sigma_0$ and complex numbers c_n, $n = 1, 2, \dots$, such that*

$$\frac{A(s)}{B(s)} = \sum_{n=1}^{\infty} \frac{c_n}{y_n^s}$$

and the series converges absolutely for $\sigma > \sigma_1$.

Proof. Without loss of generality, we assume that $b_1 = 1$. Then there exists $\sigma_1 \ge \sigma_0$ such that $|B(s) - 1| < 1$ for $\sigma > \sigma_1$, and the series of $B(s) - 1$ converges absolutely. Thus there exist complex numbers d_n such that

$$\frac{1}{B(s)} = \sum_{n=0}^{\infty} (-1)^n (B(s) - 1)^n = \sum_{n=1}^{\infty} \frac{d_n}{y_n^s},$$

where the last series converges absolutely for $\sigma > \sigma_1$. Now the lemma follows in view of the absolute convergence of the series for $A(s)$ and $B(s)^{-1}$. □

The following lemma describes the asymptotic behavior of the factor $X(s)$ from the functional equation (2.2).

Lemma 5.8. *For $t \geq 1$,*

$$X(s) = \exp\left(2\pi i(g-1)\left(s-\frac{1}{2}\right)^2 + \frac{\pi i(g-1)}{6} + O\left(\frac{t}{e^{2\pi t}}\right) + O\left(\frac{(\sigma-1/2)^2}{e^{2\pi t}}\right) + O\left(\frac{(\sigma-1/2)t}{e^{2\pi t}}\right)\right) \qquad (t \to \infty)$$

uniformly in σ.

Proof. This is Lemma 3.1 in this dissertation. □

5.2 Proofs

Proof of Proposition 5.2. First, we may assume $a \neq 1$. Let B be a sufficiently large fixed number, such that $B \geq A$, where A is defined in Introduction. Then the strip $1-B \leq \sigma \leq B$ contains all the non-trivial a-points and a finite number of trivial a-points.

Next let T be such that there are no a-points on the line $t = T$. Using the residue theorem and the fact that the logarithmic derivative of $Z(s) - a$ has simple poles at each a-point ρ_a with residue equal to the order of ρ_a, we get

$$\sum_{0<\gamma_a \leq T} x^{\rho_a} = \frac{1}{2\pi i} \int_{\square} x^s \frac{Z'(s)}{Z(s)-a}\, ds + O(1);$$

here □ denotes the counterclockwise oriented rectangular contour with vertices $B+i$, $B+iT$, $1-B+iT$, $1-B+i$. If the line $t=1$ contains a-points, we slightly alter the lower edge of the rectangular contour □.

In order to evaluate the integral, we write

$$\int_{\square} x^s \frac{Z'(s)}{Z(s)-a}\, ds = \left\{\int_{B+i}^{B+iT} + \int_{B+iT}^{1-B+iT} + \int_{1-B+iT}^{1-B+i} + \int_{1-B+i}^{B+i}\right\} x^s \frac{Z'(s)}{Z(s)-a}\, ds = \sum_{j=1}^{4} I_j,$$

say. We shall evaluate each I_j individually.

In view of Lemmas 5.6 and 5.7, we may suppose that the logarithmic derivative of $Z(s)-a$ has an absolutely convergent Dirichlet series expansion for $\sigma > B$, namely

$$\frac{Z'(s)}{Z(s)-a} = \sum_{n=2}^{\infty} \frac{c_n}{y_n^s}.$$

Now we interchange summation and integration on the right-hand side of the rectangle, which gives

$$\begin{aligned} I_1 &= \sum_{n=2}^{\infty} c_n \int_{B+i}^{B+iT} \left(\frac{x}{y_n}\right)^s ds = \sum_{n=2}^{\infty} c_n i \int_1^T \exp((B+it)\log(x/y_n))\, dt \\ &= \sum_{n=2}^{\infty} c_n i \exp(B\log(x/y_n)) \int_1^T \exp(it\log(x/y_n))\, dt. \end{aligned}$$

By

$$\begin{aligned} &\int_1^T \exp(it\log(x/y_n))\, dt \\ &\quad = \begin{cases} T-1 & \text{if } x = y_n, \\ (\exp(iT\log(x/y_n)) - \exp(i\log(x/y_n)))/(i\log(x/y_n)) & \text{otherwise.} \end{cases} \end{aligned}$$

we obtain

$$I_1 = ic(x)T + O(1).$$

Here $c(x)$ equals the Dirichlet coefficient c_n if $x = y_n$ and 0 otherwise.

Next we estimate the integrals along the horizontal segments. Clearly, $I_4 = O(1)$. In view of Lemma 5.5, the contribution of the upper horizontal segment gives

$$I_2 = \int_{1-B}^{B} x^{\sigma+it'} \frac{Z'(\sigma+iT)}{Z(\sigma+iT)-a}\, d\sigma \ll \int_{1-B}^{B} \left|\frac{Z'(\sigma+iT)}{Z(\sigma+iT)-a}\right| d\sigma \ll T.$$

It remains to estimate the integral along the left-hand side:

$$I_3 = O(1) - \int_{1-B+it_0}^{1-B+iT} x^s \frac{Z'(s)}{Z(s)-a}\, ds. \tag{5.6}$$

In view of the expression of $Z(s)$ by a Dirichlet series (Lemma 5.6), we may assume $|Z(1-\sigma-it)| \geq 1/2$ for $\sigma \leq 1-B$ and all t; it follows from Lemma 5.8

above that

$$Z(1-B+it) \gg \exp(t),$$

as $t \to \infty$. Hence there exists t_0 such that the absolute value of $Z(1-B+it)$ is greater than $2|a|$ for $t > t_0$ and we obtain the following expansion into a geometric series:

$$\frac{Z'(s)}{Z(s)-a} = \frac{Z'}{Z}(s)\frac{1}{1-a/Z(s)} = \frac{Z'}{Z}(s)\left(1+\sum_{k=1}^{\infty}\left(\frac{a}{Z(s)}\right)^k\right).$$

Then, in view of the bound $Z'/Z(1-B+it) \ll t$, for $t \to \infty$ (see Randol [72, Lemma 2]), we get

$$\int_{1-B+it_0}^{1-B+iT} x^s \frac{Z'}{Z}(s)\sum_{k=1}^{\infty}\left(\frac{a}{Z(s)}\right)^k ds \ll x^{1-B}T^2\sum_{k=1}^{\infty}\left(\frac{1}{\exp(T)}\right)^k \ll 1.$$

By Hejhal [35, Chapter 2, Proposition 4.2] we have

$$\frac{Z'}{Z}(s) = \sum_{\{P_0\}}\sum_{k=1}^{\infty}\frac{\log(N(P_0))(1-N(P_0)^{-k})^{-1}}{N(P_0)^{ks}}, \tag{5.7}$$

where the series converges absolutely in the half-plane $\sigma > 1$.

Recall that $x \neq 1$. By the functional equation (Lemma 5.8) and (5.7), for the second part of the integral in (5.6) we get

$$\begin{aligned}
-\int_{1-B+it_0}^{1-B+iT} x^s \frac{Z'(s)}{Z(s)}\, ds &= \int_{1-B+it_0}^{1-B+iT} x^s\left(\frac{Z'}{Z}(1-s) - \frac{X'}{X}(s)\right) ds \\
&= -ix^{1-B}\sum_{P_0}\sum_{k=1}^{\infty}\frac{\log(N(P_0))(1-N(P_0)^{-k})^{-1}}{N(P_0)^{kB}}\int_{t_0}^{T}\left(xN(P_0)^k\right)^{it} dt \\
&\quad + ix^{1-B}\int_{t_0}^{T} x^{it}(-4\pi(g-1)t + O(1))\, dt \\
&\ll T.
\end{aligned}$$

Thus $I_3 \ll T$.

So far we have been considering the case $a \neq 1$. Now we consider the case $a = 1$. In the expression of $Z(s)$ by a Dirichlet series (Lemma 5.6), we can suppose that $a_2 \neq 0$. Let us define the function:

$$\ell(s) = x_2^s(Z(s) - 1) = 1 + \sum_{n=3}^{\infty} \frac{a_n}{a_2} \left(\frac{x_2}{x_n}\right)^s.$$

Then the logarithmic derivative of ℓ is given by

$$\frac{\ell'}{\ell}(s) = \log x_2 + \frac{Z'(s)}{Z(s) - 1}.$$

Applying contour integration and the above reasoning to this function proves Proposition 5.2. □

Proof of Theorem 5.1. Our argument follows along the lines of the proof of Theorem 1 in Steuding [90]. We use the property that non-trivial a-values are clustered around the critical line. By formulas (5.2) and (5.3), we have

$$\begin{aligned}\sum_{1<\gamma_a\leq T} \left|\beta_a - \frac{1}{2}\right| &= \left\{\sum_{1<\gamma_a\leq T,\ |\beta_a - 1/2|>\delta} + \sum_{1<\gamma_a\leq T,\ |\beta_a-1/2|\leq\delta}\right\} \left|\beta_a - \frac{1}{2}\right| \\ &\ll \frac{T^2}{\log\log T} + \frac{T^2(\log\log T)^2}{\log T}.\end{aligned}$$

Since the function $Z(s)$ has only a bounded number of non-trivial a-points satisfying $0 < t \leq 1$, we get

$$\sum_{0<\gamma_a\leq T} \left|\beta_a - \frac{1}{2}\right| \ll \frac{T^2}{\log\log T}.$$

Since, for any real number y,

$$|\exp(y) - 1| = \left|\int_0^y \exp(t)\, dt\right| \leq |y| \max\{1, \exp(y)\},$$

we find

$$\begin{aligned}|x^{1/2+i\gamma_a} - x^{\beta_a+i\gamma_a}| &= x^{\beta_a} \left|\exp\left(\left(\frac{1}{2} - \beta_a\right)\log x\right) - 1\right| \\ &\leq \left|\beta_a - \frac{1}{2}\right| |\log x| \max\{x^{\beta_a}, x^{1/2}\}.\end{aligned}$$

Furthermore,

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\le T} |x^{1/2+i\gamma} - x^{\beta_a+i\gamma_a}| \le \frac{X}{N_a(T)} \sum_{0<\gamma_a\le T} \left|\beta_a - \frac{1}{2}\right|,$$

where $X = \max\{x^B, 1\}|\log x|$. Hence,

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\le T} (x^{1/2+i\gamma_a} - x^{\beta_a+i\gamma_a}) \ll \frac{X}{\log\log T}.$$

By Theorem 5.2,

$$\sum_{0<\gamma_a\le T} x^{\beta_a+i\gamma_a} \ll T.$$

Therefore, as $T \to \infty$,

$$\frac{1}{N_a(T)} \sum_{0<\gamma_a\le T} x^{1/2+i\gamma_a} \ll \frac{1}{\log\log T}.$$

Now let $x = z^m$ with some positive $z \neq 1$ and $m \in \mathbb{N}$. It follows from the latter formula that

$$\lim_{T\to\infty} \frac{1}{N_a(T)} \sum_{0<\gamma_a\le T} \exp(im\gamma_a \log z) = 0.$$

By Weyl criterion (Lemma **??**), the sequence of numbers $\gamma_a \log z/2\pi$ is uniformly distributed modulo 1. This proves Theorem 5.1. □

Proof of Theorem 5.3. In view of the Weyl criterion (Lemma **??**) the eigenvalues λ_j are uniformly distributed modulo one if, and only if, for any fixed $n \in \mathbb{Z} \setminus \{0\}$,

$$\sum_{0<\lambda_j\le T^2} x^{i\lambda_j} = o\left(\sum_{0<\lambda_j\le T^2} 1\right),$$

where $x = e^{2\pi n}$. By the relation between eigenvalues and non-trivial zeros (2.4) and by the formula for the number of non-trivial zeros (5.1) it follows that

$$\sum_{0<\lambda_j\le T^2+\frac{1}{4}} 1 = \sum_{0<t_j\le T} 1 = (g-1)T^2 + O\left(\frac{T}{\log T}\right).$$

First we consider the case $x > 1$. If T is not an ordinate of a zero, then

$$\sum_{0<\lambda_j<T^2+\frac{1}{4}} x^{i\lambda_j} = \sum_{0<t_j<T} x^{\frac{i}{4}+it_j^2} = \sum_{-T<-t_j<0} x^{\frac{i}{4}+it_j^2} \tag{5.8}$$
$$= \frac{1}{2\pi i}\int_{\square} x^{\frac{i}{4}+is^2}\frac{Z'(s+\frac{1}{2})}{Z(s+\frac{1}{2})}\,ds + O(1) =: I_1+I_2+I_3+I_4+O(1),$$

where the integration is over the counterclockwise oriented rectangular contour $\square$ in the lower half-plane with vertices $1/T-i$, $-1-i$, $-1-iT$, $1/T-iT$.

Clearly, for the integral on the upper horizontal line segment of $\square$ we have $I_1 \ll 1$.

For the integral I_2 over the left vertical line we use the bound $Z'/Z(-1+iT) \ll T$, $T\to\infty$ (Randol [72, Lemma 2]). Then, in view of $x^{is^2} = x^{-2\sigma t+i(\sigma^2-t^2)}$, we deduce $I_2 \ll 1$.

For the integral I_3 over the lower horizontal line we use once more formula (5.4) and derive

$$\begin{aligned}
I_3 &= \int_{-1-iT}^{1/T-iT} x^{\frac{i}{4}+is^2}\frac{Z'(s+\frac{1}{2})}{Z(s+\frac{1}{2})}\,ds \ll \int_{-1-iT}^{1/T-iT}\left|\frac{Z'(s+\frac{1}{2})}{Z(s+\frac{1}{2})}\right| ds\\
&= \sum_{|\rho-s_0|\le\frac{r}{2}}\int_{-1}^{1/T}\frac{1}{\sqrt{\sigma^2+(T-\gamma)^2}}\,d\sigma + O\left(T\right)\\
&= \sum_{|\rho-s_0|\le\frac{r}{2}}\left(\log\left(\frac{1}{T}+\sqrt{(T-\gamma)^2+\frac{1}{T^2}}\right)\right.\\
&\qquad\left.-\log\left(1+\sqrt{(T-\gamma)^2+1}\right)\right) + O\left(T\right)\\
&\ll T\log T.
\end{aligned}$$

Further, the following equality holds

$$I_4 = ix^{i\sigma^2}\int_{-T}^{-1} x^{-2\sigma t-it^2}\frac{Z'}{Z}\left(\frac{1}{2}+\frac{1}{T}+it\right)\,\mathrm{d}t.$$

This proves the assertion of the theorem in the case $n>0$.

In order to prove the assertion in the case $n\le 0$, we choose the rectangular contour in the upper half-plane with vertices $-1+i$, $1/T+i$, $1/T+iT$, $-1+iT$ in formula (5.8) and proceed as in the previous case. This proves Theorem 5.3. $\square$

Chapter 6

Speiser equivalent for the Riemann hypothesis

This chapter is based on Garunkštis and Šimėnas [26]. In the first part of the 20th century, Speiser [86] studied the relationship between the location of the zeros of the derivative of the Riemann zeta-function and the Riemann hypothesis (RH). His result, achieved by geometric means, is that the RH is equivalent to the absence of non-real zeros of the derivative of the Riemann zeta-function left of the critical line.

Later on, Levinson and Montgomery [62] investigated the relationship between the zeros of the Riemann zeta-function and its derivative analytically. They proved the quantitative version of the Speiser's result, namely, that the Riemann zeta-function and its derivative have approximately the same number of zeros left of the critical line. This result was extended to Dirichlet L-functions with primitive Dirichlet characters (Yıldırım [98]), to the Selberg class (Šleževičienė [84]), to the Selberg zeta-function on a compact Riemann surface (Luo [63], see also Garunkštis [24]). In all these cases, an analog of the RH is expected or, as in the case of the Selberg zeta-function, it is known to be true. Here we ask the question if the Speiser equivalent is valid for zeta-functions for which an analog of the RH is not true. The answer is positive and as an example we consider numerically certain zeta-functions depending on a parameter, which for some values of the parameter violate the RH and for other values they are expected to satisfy the RH. To be more general, we choose to investigate the so called extended Selberg class. This class includes classical zeta-functions, such as the Riemann zeta-function, Dirichlet L-functions for primitive characters, Dedekind zeta-functions. This class also contains linear combinations of certain Dirichlet L-functions for which the RH is not valid (Kaczorowski and Kulas [46]).

A not identically vanishing Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which converges absolutely for $\sigma > 1$ belongs to the *extended Selberg class* $S^{\#}$ if:

(i) (Meromorphic continuation) There exists $k \in \mathbb{N}$ such that $(s-1)^k F(s)$ is an entire function of finite order.

(ii) (Functional equation) $F(s)$ satisfies the functional equation:

$$\Phi(s) = \omega \overline{\Phi(1-\overline{s})}, \tag{6.1}$$

where $\Phi(s) := F(s) Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$, with $Q > 0$, $\lambda_j > 0$, $\Re(\mu_j) \geq 0$ and $|\omega| = 1$.

While the data Q, λ_j, μ_j and ω of the functional equation are not uniquely determined by F, the value $d_F = 2\sum_{j=1}^{r} \lambda_j$ is an invariant. It is called the *degree* of F. More about the (extended) Selberg class see the survey papers of Kaczorowsky and Perelli [45], [48], [47], [66], [65].

For a positive degree, the zeros of $F(s)$ located at the poles of the Gamma functions in the functional equation (6.1), i.e. at $s = -\frac{\mu_j + k}{\lambda_j}$ with $k = 0, 1, 2, \ldots$ and $j = 1, \ldots, r$, are called *trivial zeros*. If the degree is equal to zero, then the functional equation of $F(s)$ has no gamma factors and thus $F(s)$ has no trivial zeros. For any degree, let $\sigma_F \geq 1/2$ be the least real number such that $F(s)$ does not have any zeros in the right-half plane $\sigma > \sigma_F$ (cf. Kaczorowski [45], Section 2.1). Then by the functional equation (6.1) we see that in the left-half plane $\sigma < 1 - \sigma_F$ the function $F(s)$ can have only trivial zeros.

Suppose that $F(s) \in S^{\#}$ is a non-constant function. Next, we consider zero free regions of $F'(s)$ in the left-half plane. If the degree is equal to zero, then $F(s)$ is a Dirichlet polynomial whose structure is well understood (see Kaczorowski and Perelli [48] or formula (6.10) below). It follows that for

$d_F = 0$, there is σ_1 such that $F'(s) \neq 0$ if $\sigma \leq \sigma_1$. For $d_F > 0$, a zero free region of $F'(s)$ is described by the next proposition.

Proposition 6.1. *Let $F(s) \in S^{\#}$ and $d_F > 0$. Then there is $\tau \geq 0$ such that $F'(s) \neq 0$ in $\sigma < 1 - \sigma_F$, $|t| \geq \tau$.*

From the proof of Proposition 6.1, we see that for a given function $F(s)$ the explicit upper bound for τ can be calculated.

In this chapter, T always tends to plus infinity. The main results of this chapter for the functions of the extended Selberg class are the following:

Theorem 6.2. *Let $F(s) \in S^{\#}$, $d_F > 0$ and $\sigma_0 > \sigma_F$. Let τ be the same as in Proposition 6.1. Let $N(T)$ and $N_1(T)$ respectively denote the number of zeros of $F(s)$ and $F'(s)$ in the region $\tau < t < T$, $\sigma < 1/2$. Then*

$$N(T) = N_1(T) + O(\log T).$$

Moreover, if $N(T) < T/(2\sigma_0 - 1) + O(1)$, then there is a monotonic sequence $\{T_j\}$, $T_j \to \infty$, $j \to \infty$ such that

$$N(T_j) - N(T_1) = N_1(T_j) - N_1(T_1).$$

Theorem 6.3. *Let $F(s) \in S^{\#}$ be a non-constant function with $d_F = 0$. Let $N(T)$ and $N_1(T)$ respectively denote the number of zeros of $F(s)$ and $F'(s)$ in the region $0 < t < T$, $\sigma < 1/2$. Then*

$$N(T) = N_1(T) + O(1).$$

It is well known that $\zeta'(1/2+it) \neq 0$ if $\zeta(1/2+it) \neq 0$, see Spira [87, Corollary 3]. Analogous statement is true for the functions from the extended Selberg class.

Proposition 6.4. *Let $F(s) \in S^{\#}$ be a non-constant function. Then there is $\tau \geq 0$ such that, for $t \geq \tau$,*

$$F'(1/2+it) \neq 0 \quad \text{if} \quad F(1/2+it) \neq 0.$$

Moreover, if $d_F = 0$ then $\tau = 0$.

As we have already mentioned, the extended Selberg class contains zeta-functions for which the RH is not true, but all these functions still satisfy functional equations (6.1) of the Riemann type. Can we extend the Speiser equivalent to zeta-functions without such functional equations? Possibly the simplest example of such functions are Dirichlet L-functions with imprimitive characters.

For a Dirichlet character χ, the Dirichlet L-function $L(s,\chi)$ is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Now, suppose χ is an imprimitive Dirichlet character modulo q induced by a primitive Dirichlet character $\chi_1 \bmod q_1$. Then q_1 divides q and

$$\chi(n) = \begin{cases} \chi_1(n) & \text{if } \gcd(n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The following Euler product holds:

$$L(s,\chi) = L(s,\chi_1) \prod_{p|q} (1 - \chi_1(p)p^{-s}). \tag{6.2}$$

We see that $L(s,\chi)$ has zeros on the line $\sigma = 0$ and in this sense the RH is not valid. Also, a functional equation of the Riemann type does not hold for $L(s,\chi)$.

In analogy with Proposition 6.1, we demonstrate the existence of a zero-free region in the left half-plane for $L'(s,\chi)$.

Proposition 6.5. *Let χ be an imprimitive Dirichlet character. Then for any $\sigma_0 > 1$, there exists τ such that $L'(s,\chi)$ does not vanish for $\sigma \leq 1 - \sigma_0$ and $|t| \geq \tau$.*

The next theorem is analogous to Theorem 6.2.

Theorem 6.6. *Let $\chi \bmod q$ be an imprimitive Dirichlet character induced by a primitive character $\chi_1 \bmod q_1$. Let τ be the same as in Proposition 6.5. Let $N(T)$ and $N_1(T)$ denote the number of zeros of respectively $L(s,\chi)$ and*

$L'(s,\chi)$ in the region $\tau < t < T$, $\sigma < 1/2$. Then

$$N_1(T) = N(T) + O(\log T) \geq \frac{\log(q/q_1)}{2\pi} T + O(\log T),$$

where the inequality can be replaced by the equality provided the Riemann hypothesis is valid for $L(s,\chi_1)$.

From the proof we see that Theorem 6.6 can be generalized to zeta-functions from the extended Selberg class, multiplied by some simple function. On the other hand, Hurwitz and Lerch zeta-functions also do not satisfy the Riemann type functional equation, but computations do not confirm the Speiser equivalent in these cases. The proofs of above results follow the proof of Levinson and Montgomery [62].

The next section is devoted to computer calculations. To illustrate Theorem 6.2 "in action," we compute the trajectories of the zeros of a certain zeta-function depending on a parameter as well as the trajectories of the zeros of its derivative. We also calculate several zeros of the derivative of the celebrated Davenport-Heilbronn zeta-function, which is an element of the extended Selberg class and has non-real zeros off the critical line. Section 3 contains the proofs.

6.1 Computations

All computations in this section were done using the program Mathematica. Computations should be regarded as heuristic because their accuracy was not controlled explicitly. We investigate the function of the following form:

$$f(s,\tau) := f_0(s) \cdot (1-\tau) + f_1(s) \cdot \tau,$$

where $\tau \in [0,1]$, $f_0(s) := (1+\sqrt{5}/5^s)\zeta(s)$, $\zeta(s)$ is the Riemann zeta-function, and $f_1(s) := L(s,\psi)$, where $\psi \bmod 5$, $\psi(2) = -1$.

For any τ, the function $f(s,\tau)$ satisfies the functional equation

$$f(s) = 5^{-s+1/2} 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right) f(1-s) \tag{6.3}$$

since both f_0 and f_1 satisfy (6.3). Thus, for any τ, the function $f(s,\tau)$ is an element of the extended Selberg class of degree $d_f = 1$ and Theorem 6.2 for

$f(s,\tau)$ is valid.

It is quite likely that all the non-trivial zeros of $f(s,0)$ and $f(s,1)$ are located on the line $\sigma = 1/2$. However, by the joint universality theorem for Dirichlet L-functions, it follows that for any $0 < \tau < 1$ there are infinitely many zeros of $f(s,\tau)$ in the strip $1/2 < \Re s < 1$ (see Theorem 2 in Kaczorowski and Kulas [46]).

By $f_s^{(v)}(s,\tau)$ we denote that vth partial derivative of f with respect to s:

$$f_s^{(v)}(s,\tau) = \frac{\partial^v}{\partial s^v} f(s,\tau).$$

Suppose that $\rho = \rho(\tau_0)$ is a zero of multiplicity m of $f(s,\tau_0)$ (i.e. $f_s^{(v)}(\rho(\tau_0),\tau_0) = 0$, $v = 0,1,\dots,m-1$, $f_s^{(m)}(\rho(\tau_0),\tau_0) \neq 0$). By Rouché's theorem, we have that for every sufficiently small open disc D with center at ρ in which the function $f(s,\tau_0)$ has no other zeros except for ρ, there exists $\delta = \delta(D) > 0$ such that each function $f(s,\tau)$, where $\tau \in (\tau_0 - \delta, \tau_0 + \delta)$, has exactly m zeros (counted with multiplicities) in the disc D (c.f. Theorem 1 in Balanzario and Sánchez-Ortiz [5] and Lemma 8 in Dubickas et al. [16]). If zero ρ is of multiplicity $m = 1$, then there exists a neighborhood of τ_0 and some function $\rho = \rho(\tau)$, which is continuous at τ_0 and, in addition, satisfies the relation $f(\rho(\tau),\tau) = 0$. This way, we can speak about the continuous zero trajectory $\rho(\tau)$. Similarly, the trajectories of the zeros of the derivative $f_s'(s,\tau)$ are understood.

In view of the functional equation (6.3), for any τ, the non-real zeros of $f(s,\tau)$ are symmetrically distributed with respect to the critical line $\sigma = 1/2$. It follows that provided the RH holds for $f(s,0)$ and if we have its simple zero $\rho_1(0)$, the functional equation forbids the corresponding zero trajectory $\rho_1(\tau)$ from leaving the critical line as τ increases unless it meets the trajectory $\rho_2(\tau)$ of another zero, say at $\tau = \tau'$. Then $\rho_1(\tau') = \rho_2(\tau')$ is a double zero of $f(s,\tau')$. In the case of the zeros of degree two, they can leave the critical line. Moreover, their trajectories are symmetric with respect to the critical line. Let the trajectories $\rho_1(\tau)$ and $\rho_2(\tau)$ meet each other at $\tau = \tau'$ and let them split into two trajectories, which leave the critical line and which are symmetric with respect to this line. For definitiveness, we say that the trajectory $\rho_1(\tau)$ (similarly $\rho_2(\tau)$) at $\tau = \tau'$ turns to the right hand-side as τ increases. Then the trajectories $\rho_1(\tau)$ and $\rho_2(\tau)$ remain defined in the a neighborhood of τ'. Note that this definition makes sense because in all the

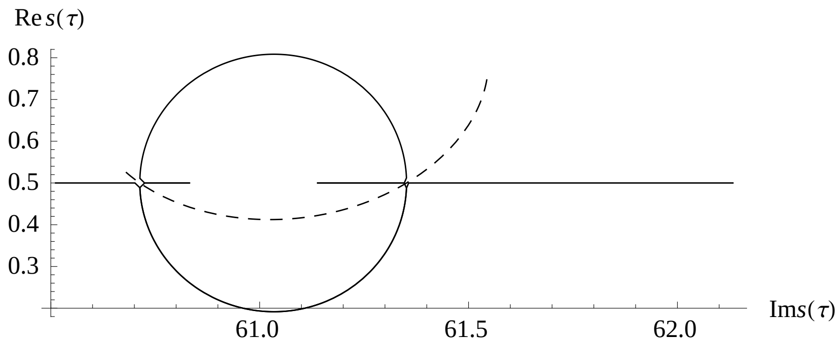

Figure 6.1: Solid black trajectories are zero trajectories $\rho_1(\tau)$ and $\rho_2(\tau)$: $f(\rho_1(\tau),\tau) = 0$, $\rho_1(0) = 0.5 + i60.84$, $\rho_1(1) = 0.5 + i62.13$ and $f(\rho_2(\tau),\tau) = 0$, $\rho_2(0) = 0.5 + i60.51$, $\rho_2(1) = 0.5 + i61.14$. Dashed trajectory is a derivative zero trajectory $q(\tau)$: $f'_s(q(\tau),\tau) = 0$, $q(0) = 0.52 + i60.68$, $q(1) = 0.76 + i61.55$.

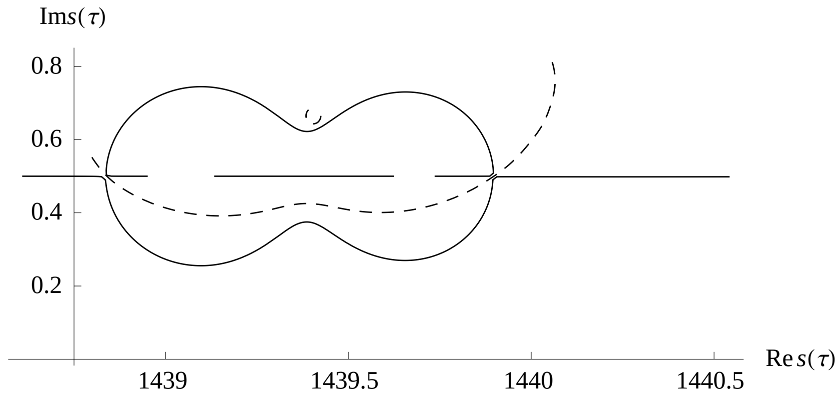

Figure 6.2: Solid black trajectories are zero trajectories with $\rho_1(0) = 0.50 + i1438.61$, $\rho_2(0) = 0.50 + i1438.95$, and $\rho_3(0) = 0.50 + i1439.62$ ($f(\rho_j(\tau),\tau) = 0$, $0 \leq \tau \leq 1$, $j = 1, 2, 3$). Dashed black trajectories are derivative zero trajectories with $q_1(0) = 0.55 + i1438.80$, and $q_2(0) = 0.66 + i1439.42$ ($f'_s(q_j(\tau),\tau) = 0$, $0 \leq \tau \leq 1$ $j = 1, 2$).

calculations in this section, the trajectories $\rho_1(\tau)$ and $\rho_2(\tau)$ move in opposite directions before they meet at $\tau = \tau'$. In addition, the derivative $f'_s(s,\tau')$ must vanish at the meeting point $s = \rho_1(\tau')$. Thus there is a trajectory $q(\tau)$ of a zero of $f'_s(s,\tau)$ such that $q(\tau') = \rho_1(\tau') = \rho_2(\tau')$.

Figures 6.1 and 6.2 are parametric plots of the trajectories of the zeros of $f(s,\tau)$ and its derivative, solid and dotted lines respectively. We see that the trajectory of the derivative crosses the critical line in accordance with Theorem 6.2.

To find the solutions, i.e. the zero trajectories, $\rho(\tau)$ and $q(\tau)$, $0 \leq \tau \leq 1$, of

$$f(\rho(\tau),\tau)=0 \quad \text{and} \quad f'_s(q(\tau),\tau)=0,$$

we solve the differential equations numerically

$$\frac{\partial\rho(\tau)}{\partial\tau}=-\frac{\frac{\partial f(\rho,\tau)}{\partial\tau}}{\frac{\partial f(\rho,\tau)}{\partial\rho}} \quad \text{and} \quad \frac{\partial q(\tau)}{\partial\tau}=-\frac{\frac{\partial^2 f(q,\tau)}{\partial q\partial\tau}}{\frac{\partial^2 f(q,\tau)}{\partial q^2}}.$$

As the initial conditions, some zeros of $f(s,0)=(1+\frac{\sqrt{5}}{5^s})\zeta(s)$ and $f'_s(s,0)$ are used.

In the upper half-plane region with $0<t\leq 1500$ the function $f(s,0)$ has 1452 zeros, all on the critical line. According to our computations, 286 of these zeros leave the critical line and, except several cases similar to Figure 6.2, are similar to Figure 6.1.

Further, we consider the Davenport-Heilbronn zeta-function defined by

$$\ell(s):=\frac{1}{2\cos\alpha}\left(e^{-i\alpha}L(s,\eta)+e^{i\alpha}L(s,\overline{\eta})\right),$$

where η mod 5, $\eta(2)=i$, and $\tan\alpha=\frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}$. The Davenoport-Heilbronn zeta-function satisfies the functional equation:

$$\ell(s)=5^{1/2-s}2(2\pi)^{s-1}\Gamma(1-s)\cos\left(\frac{\pi s}{2}\right)\ell(1-s).$$

The function $\ell(s)$ has zeros with $\sigma>1$ and has infinitely many zeros on the critical line (Titchmarsh [92, Section 10.25]). Similarly to $f(s,\tau)$, the function $\ell(s)$ has infinitely many zeros in $1/2<\sigma<1$. It belongs to the extended Selberg class and hence falls within the class of functions for which Theorem 6.2 holds. It should be noted that the zeros of the Davenport-Heilbronn zeta-function have been subject to much analysis. Spira [88] calculated the location of some of the zeros of the Davenport-Heilbronn zeta-function off the critical line $\sigma=1/2$ in the region $0\leq t\leq 200$ (see also Balanzario and Sánchez-Ortiz [5]). He did not find any zeros of its derivative left of $\sigma=1/2$ in this region although he did find several locations of the zeros of the function itself. This would go against our Theorem 6.2. However, we recalculated the zeros of the derivative of the Davenport-Heilbronn zeta-function and we did find zeros left of the critical line with imaginary parts less than 200.

For $0 \leq t \leq 200$, R. Spira calculated 8 zeros of $\ell(s)$ off the critical line:
$0.80+i\ 85.69$,
$0.65+i\ 114.16$,
$0.57+i\ 166.47$,
$0.72+i\ 176.70$
and claimed that 'no zeros of $\ell'(s)$ were found in $\sigma < 1/2$, $0 \leq t \leq 200$'. We found 4 zeros of $\ell'(s)$ in $\sigma < 1/2$, $0 \leq t \leq 200$:
$0.40+i\ 85.70$,
$0.47+\ i\ 114.15$,
$0.49+i\ 166.47$,
$0.43+i\ 176.70$.

To detect the number of zeros of the function $\ell'(s)$ in some region D, we compute the integral

$$\int_{\partial D} \frac{\ell''(s)}{\ell'(s)}\ \mathrm{d}s,$$

where ∂D is a boundary of D. If there is one zero q in the region D, then this zero is computed by the formula

$$q = \int_{\partial D} s\frac{\ell''(s)}{\ell'(s)}\ \mathrm{d}s.$$

The zeros where checked using Mathematica's command FindRoot.

6.2 Proofs

Proof of Proposition 6.1. Let $\sigma_0 > \sigma_F$. Also, let $\tau' > \tau$ and $\sigma' < 1 - \sigma_0$. Let R and $\overline{R}$ be two rectangles with vertices $1 - \sigma_0 + i\tau$, $1 - \sigma_0 + i\tau'$, $\sigma' + i\tau'$, $\sigma' + i\tau$ and $1 - \sigma_0 - i\tau$, $1 - \sigma_0 - i\tau'$, $\sigma' - i\tau'$, $\sigma' - i\tau$ respectively. Using formula (6.5) below, we will show that there is τ such that for any τ' and any σ' the inequality

$$\Re\frac{F'}{F}(s) < 0 \tag{6.4}$$

holds for $s \in R$ and $s \in \overline{R}$. By the argument principle, it follows that $F'(s)$ and $F(s)$ have the same number of zeros inside of the rectangle R (also in $\overline{R}$). This will prove the proposition since for sufficiently large τ, the function $F(s)$ has no zeros inside of the rectangles R and $\overline{R}$.

By the definition of the extended Selberg class, there is an integer n_F such that the function $G(s)=s^{n_F}(s-1)^{n_F}\Phi(s)$ is an entire function and $G(1)\neq 0$. By the functional equation (6.1), we have that $G(0)\neq 0$. Moreover, $G(s)$ is an entire function of order 1 (see Lemma 3.3 and the comment below the proof of Lemma 3.3 in Smajlović [85]). Applying Hadamard's factorization theorem to the function $G(s)$ analogously as in Šleževičienė [84, Proof of Theorem 3, formula (6)] (see also Smajlović [85, formulas (8), (10)]), we have that

$$\begin{aligned}\Re\frac{F'}{F}(s) = &\sum_{\rho \text{ non-trivial},\ \rho\neq 0,1}\frac{\sigma-\beta}{|s-\rho|^2}-\frac{n_F\sigma}{|s|^2}-\frac{n_F(\sigma-1)}{|s-1|^2}-\log Q \\ &-\Re\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}(\lambda_j s+\mu_j),\end{aligned} \tag{6.5}$$

where the summation is over the non-trivial zeros of $F(s)$ except possible non-trivial zeros or poles at $s=0$ and $s=1$.

Next we will prove inequality (6.4). In view of $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$, we can write

$$\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}(\lambda_j s+\mu_j)=\sum_{j=1}^{r}\left(\lambda_j\frac{\Gamma'}{\Gamma}(1-\lambda_j s-\mu_j)-\lambda_j\cot\left(\pi(\lambda_j s+\mu_j)\right)\right).$$

Recall that $\lambda_j>0$, $j=1,\dots,r$ and $\sum_{j=1}^{r}\lambda_j>0$. Then by the formulas

$$\frac{\Gamma'}{\Gamma}(s)=\log s+O\left(|s|^{-1}\right)\quad(\Re(s)\geq 0,\ |s|\to\infty), \tag{6.6}$$

$$\cot z = 1+O\left(e^{-|\Im z|}\right)\quad(\Im z\to\pm\infty),$$

$\Gamma'/\Gamma(s+1)=\Gamma'/\Gamma(s)+1/s$ and equality (6.5), we have that there is τ such that, for any τ' and σ', inequality (6.4) is true if $s\in R$ and $s\in\overline{R}$. This proves Proposition 6.1. □

Proof of Theorem 6.2. Let

$$R=\left\{s\in\mathbb{C}:\tau<t<T, 1-\sigma_0<\sigma<\frac{1}{2}\right\}.$$

To prove the theorem, it is enough to consider the difference of the number of zeros of $F(s)$ and $F'(s)$ in the region R.

Without loss of generality, we assume that $F(\sigma+iT)\neq 0$ and $F'(\sigma+iT)\neq 0$

for $1-\sigma_0 \leq \sigma \leq 1/2$. We consider the change of $\arg F'/F(s)$ along the appropriately indented boundary R' of the region R. More precisely, upper, left, and lower sides of R' coincide with the upper, left, and lower boundaries of R. To obtain the right-hand side of the contour of R', we take the right-hand side boundary of R and deform it to bypass the zeros of $F(1/2+it)$ by left semicircles with an arbitrarily small radius.

To prove the first part of Theorem 6.2, we will show that the change of $\arg F'/F(s)$ along the contour R' is $O(\log T)$.

By formula (6.5), similarly as in the proof of Proposition 6.1, we have that

$$\Re \frac{F'}{F}(1-\sigma_0+it) < 0,$$

where $\tau \leq t \leq T$.

We switch to the right hand side of R'. For this, we evaluate the terms of equality (6.5). In view of the symmetry of zeros with respect to the critical line, we consider

$$\frac{\sigma-\beta}{|s-\rho|^2} + \frac{\sigma-1+\beta}{|s-1+\overline{\rho}|^2} = -2\left(\frac{1}{2}-\sigma\right)\frac{(t-\gamma)^2+(\sigma-\frac{1}{2})^2-(\frac{1}{2}-\beta)^2}{|s-\rho|^2|s-1+\overline{\rho}|^2}.$$

Let

$$I_1 := 2\sum_{\beta<1/2}\frac{(t-\gamma)^2+(\sigma-\frac{1}{2})^2-(\frac{1}{2}-\beta)^2}{|s-\rho|^2|s-1+\overline{\rho}|^2} + \sum_{\beta=1/2}\frac{1}{|s-\rho|^2}. \tag{6.7}$$

Then

$$I := \sum_{\substack{\rho \text{ non-trivial} \\ \rho\neq 0,1}} \frac{\sigma-\beta}{|s-\rho|^2} = -\left(\frac{1}{2}-\sigma\right) I_1. \tag{6.8}$$

Suppose that $s=1/2+it$ is not a zero of $F(s)$. When $I=0$ (see equation (6.8)), then, similarly as in the proof of Proposition 6.1, we see that $\Re F'/F(s) < 0$. Let $\rho_0 = 1/2+i\gamma_0$ be a zero of $F(s)$. Then I_1 (see formula (6.7)) can be made arbitrarily large as we move along the left semicircle with an arbitrarily small radius and center at ρ_0. This is because the term $1/|s-\rho_0|^2 \to \infty$ as $|s-\rho_0| \to 0$. Hence on the right hand side of R' we again have $\Re F'(s)/F(s) < 0$.

By the Phragmén-Lindelöf principle and the functional equation, we obtain that for any σ' there is $A > 0$ such that

$$F(\sigma + iT) = O(T^A) \tag{6.9}$$

uniformly in $\sigma \geq \sigma'$ (cf. Steuding [89, Theorem 6.8]). By the Cauchy differentiation formula and by the bound (6.9), we have that the bound analogous to (6.9) is true also for $F'(s)$. Then by using Jensen's theorem, it is possible to show that the change of $\arg F(s)$ and $\arg F'(s)$ along the horizontal sides of R' is $O(\log T)$ (cf. Šleževičienė [84, Proof of Theorem 1] or Titchmarsh [92, Section 9.4]). This proves the first part of Theorem 6.2.

We will prove the second part of Theorem 6.2. Suppose there is a monotonic sequence $\{T_j\}$, $T_j \to \infty$, $j \to \infty$ with the property $\Re(F'/F(\sigma+iT_j)) < 0$, here $1 - \sigma_0 < \sigma < 1/2$. Then by the first part of the proof we have that $N(T_j) - N(T_1) = N_1(T_j) - N_1(T_1)$.

Suppose there is no such sequence $\{T_j\}$. Then for sufficiently high t, there is $1 - \sigma_0 \leq \sigma \leq 1/2$ such that $\Re F'/F(s) \geq 0$. Thus $I > 0$ and $I_1 < 0$. Then at least one term in I_1 must be negative. Hence there is a zero $\rho = \beta + i\gamma$ with $1 - \sigma_0 < \beta < 1/2$ such that

$$\left(\frac{1}{2} - \beta\right)^2 > (t - \gamma)^2 + \left(\sigma - \frac{1}{2}\right)^2.$$

It follows that $|t - \gamma| < \sigma_0 - 1/2$. Thus, if for sufficiently high t we divide the imaginary line into intervals of length $2\sigma_0 - 1$, it would follow that for every interval there will be at least one zero whose imaginary part falls into that interval. Since we started with sufficiently high t, it follows that in this case $F(s)$ has more than $T/(2\sigma_0 - 1) + O(1)$ zeros in the region R. This concludes the proof. □

Proof of Theorem 6.3. Let us denote the set of degree zero functions of the extended Selberg class by $S_0^{\#}$. Let $F(s) \in S_0^{\#}$ and let $q = Q^2$, where Q is from functional equation (6.1) of $F(s)$. By Kaczorowski and Perelli [48], we have that q is a positive integer and

$$F(s) = \sum_{n|q} \frac{a_n}{n^s}, \tag{6.10}$$

where

$$a_n = \frac{\omega n}{\sqrt{q}} \overline{a_{\frac{q}{n}}},$$

moreover, if $\sqrt{q} \in \mathbb{N}$ then $a_{\sqrt{q}} = \varepsilon b$ with $b \in \mathbb{R}$, where ε denotes a fixed square root of ω. By the expression (6.10), we see that $Q > 1$ if $F(s)$ is a non-constant function.

The fact that $d_F = 0$ means that there are no Gamma factors in the functional equation. Hence

$$\frac{F'}{F}(s) = -2\log Q - \overline{\frac{F'}{F}(1-\overline{s})}. \tag{6.11}$$

Let σ_1 be a negative real number with large absolute value such that $F(s) \neq 0$ and $F'(s) \neq 0$ for $\sigma \leq \sigma_1$ (see the comment before Proposition 6.1). Let R be a rectangle with vertices $1/2 - \delta$, $1/2 - \delta + iT$, $\sigma_1 + iT$, σ_1, where $\delta > 0$ is as small as we like and it will be chosen later. Without loss of generality, we assume that $F(s) \neq 0$ and $F'(s) \neq 0$ on the boundary of the rectangle R. In addition, we assume that $F(s) \neq 0$ and $F'(s) \neq 0$ in the interior of the rectangle with vertices $1/2$, $1/2 + iT$, $1/2 + iT - \delta$, and $1/2 - \delta$. We can achieve this because the zeros of F form a discrete set. To prove the theorem it is enough to show that the change of $\arg F'/F(s)$ along the rectangle R is $O(1)$ as $T \to \infty$.

By formulas (6.10) and (6.11), it is easy to see that

$$\lim_{\sigma \to -\infty} \frac{F'}{F}(\sigma + it) = -2\log Q.$$

Suppose that s' is on the left-hand side of R and suppose that σ_1 in the definition of the rectangle R is a negative number with large absolute value. Then $\Re F'/F(s') < 0$.

Similarly as in the proof of Theorem 6.2, the change in argument on the horizontal sides is $O(1)$ since $F(s)$ is bounded on vertical strips.

We consider the right-hand side $1/2 - \delta + it$, $0 \leq t \leq T$ of R. By equality (6.11), we see that

$$\Re \frac{F'}{F}\left(\frac{1}{2} + it\right) = -\log Q$$

if $1/2 + it$ is not a zero of $F(s)$. We claim that there is a sufficiently small

$\delta = \delta(T)$ such that, for $0 \leq t \leq T$,

$$\Re\frac{F'}{F}\left(\frac{1}{2} - \delta + it\right) \leq -\frac{\log Q}{2}. \tag{6.12}$$

To prove this inequality, it is enough to consider the case when t is in the neighborhood of a zero $\rho = 1/2 + i\gamma$. We have

$$\frac{F'}{F}(s) = \frac{m}{s-\rho} + m' + O(|s-\rho|),$$

where m is the multiplicity of ρ. Hence taking $s = 1/2 - \delta + it$, we see that

$$\Re\frac{F'}{F}(s) = -\frac{m\delta}{|s-\rho|^2} + \Re(m') + O(|s-\rho|).$$

Thus $\Re m' = -\log Q$. This proves the inequality (6.12). Therefore, the argument change along the right side of the contour is $O(1)$. This gives the proof of Theorem 6.3. □

Proof of Proposition 6.4. Let the degree $d_F > 0$. Assume the contrary, that there is a large number t such that $F'(1/2+it) = 0$ and $F(1/2+it) \neq 0$. Then by Hadamard's type formula (6.5), Gamma function property (6.6), and using the fact that $\sigma = 1/2$ in (6.8), we obtain the contradiction

$$0 = \Re\frac{F'}{F}(1/2+it) < 0.$$

This proves the proposition for $d_F > 0$. If $d_F = 0$ then the proposition follows by formula (6.11). □

Now we provide the proofs for our results for the Dirichlet L-functions with imprimitive characters.

Proof of Proposition 6.5. Let the character χ mod q be induced by a primitive character χ_1. Taking logarithmic derivatives of both sides of equation (6.2) yields:

$$\frac{L'}{L}(s,\chi) = \frac{L'}{L}(s,\chi_1) + \sum_{p|q} \frac{\chi_1(p)p^{-s}\log p}{1-\chi_1(p)p^{-s}}. \tag{6.13}$$

The sum part on the right hand-side is bounded uniformly for $|\sigma| \geq \delta$ with fixed positive δ. The Dirichlet L-function $L(s,\chi_1)$ is an element of the (extended) Selberg class. Thus for $L'/L(s,\chi_1)$, the equation (6.5) is valid with $F(s) = L(s,\chi_1)$. By an argument similar to that of Proposition 6.1, we conclude that there exists τ such that $\Re(L'/L(s,\chi_1))$ is sufficiently small, that is, negative with large absolute value, to make $\Re(L'/L(s,\chi))$ negative with $\sigma < 1-\sigma_0$, $|t| \geq \tau$. Therefore, $L'(s,\chi)$ has no zeros in our region, just as $L(s,\chi)$. □

Proof of Theorem 6.6. Let R and R' be the same as in the proof of Theorem 6.2. Again, without loss of generality, assume that T is such $L(\sigma+iT,\chi)$ and $L'(\sigma+iT,\chi)$ do not vanish for $1-\sigma_0 \leq \sigma \leq 1/2$. Let $L(s,\chi_1)$ be the corresponding Dirichlet L-function with primitive character χ_1. We show that the change of $\arg L'/L(s,\chi)$ along the contour of R' is $O(\log T)$.

As to the left boundary of R', equation (6.13) yields $\Re(L'/L(s,\chi)) < 0$. Let us look at the right boundary of R'. Equations (6.5) and (6.13) give

$$\begin{aligned}\Re\frac{L'}{L}(s,\chi) = &\sum_{\substack{\rho \text{ non-trivial,}\\ \rho\neq 0,1}} \frac{\sigma-\beta}{|s-\rho|^2} - \frac{n_F\sigma}{|s|^2} - \frac{n_F(\sigma-1)}{|s-1|^2} - \\ &- \log Q - \Re\sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j s + \mu_j) + \\ &+ \Re\sum_{p|q} \frac{\chi_1(p)p^{-s}\log p}{1-\chi_1(p)p^{-s}}.\end{aligned}$$

Here the first summation is taken over the non-trivial zeros of $L(s,\chi_1)$. The non-trivial zeros of $L(s,\chi_1)$ occur in symmetric pairs. Let I_1 and I denote the same magnitudes as in the equations (6.7) and (6.8), respectively. Arguing in the same vein as in the proof of Theorem 6.2, we get $L'/L(s,\chi) < 0$ as s moves along the right contour of R' provided we take τ sufficiently large. As to the remaining estimate of the change in argument of $L'/L(s,\chi)$ as we move along the horizontal segments of the contour of R', we argue exactly as in the proof of Theorem 6.2, which yields that the change in argument is $O(\log T)$. This gives us the proof of the theorem. □

Chapter 7

Conclusions

To sum up our results, we have obtained:

- The a-values of the Selberg zeta-functions associated to a compact Riemann surface are mostly clustered around the critical line $\sigma = 1/2$. The same holds for the a-values of the Selberg zeta-function associated to a finite volume Riemann surface.

- The imaginary parts of the a-values of the Selberg zeta-function associated to a compact Riemann surface are uniformly distributed modulo one.

- The number of zeros of the functions belonging to the extended Selberg class is approximately the same as the number of zeros of their derivatives.

Bibliography

[1] Amir Akbary and M. Ram Murty. Uniform distribution of zeros of Dirichlet series. In *Anatomy of integers*, volume 46 of *CRM Proc. Lecture Notes*, pages 143–158. Amer. Math. Soc., Providence, RI, 2008.

[2] Juan Arias de Reyna. X-ray of Riemann zeta-function. *arXiv preprint math/0309433*, 2003.

[3] Ralph Joseph Backlund. Über die Beziehung zwischen Anwachsen und Nullstellen der Zetafunktion. *Öfv. af finska vet. soc. förh.*, 61(9), 1918.

[4] Bhaskar Bagchi. A joint universality theorem for Dirichlet l-functions. *Math. Z.*, 181(3):319–334, 1982.

[5] Eugenio Balanzario and Jorge Sánchez-Ortiz. Zeros of the Davenport-Heilbronn counterexample. *Math. Comp.*, 76(260):2045–2049, 2007.

[6] Bruce C. Berndt. The number of zeros for $\zeta^{(k)}(s)$. *J. London Math. Soc. (2)*, 2(4):577–580, 1970.

[7] Harald Bohr and Börge Jessen. Über die Werteverteilung der Riemannschen Zetafunktion. *Acta Math.*, 58(1):1–55, 1932.

[8] Harald Bohr, Edmund Landau, and John Edensor Littlewood. Sur la fonction $\zeta(s)$ dans la voisinage de la droite $\sigma = 1/2$. *Bull. de l'Acad. royale de Belgique*, pages 3–35, 1913.

[9] Thomas Christ and Justas Kalpokas. Upper bounds for discrete moments of the derivatives of the Riemann zeta-function on the critical line. *Lith. Math. J.*, 52(3):233–248, 2012.

[10] Johann Cigler. Asymptotische Verteilung reeller Zahlen $\bmod 1$. *Monatsh. Math.*, 64:201–225, 1960.

[11] John Brian Conrey. More than two fifths of the zeros of the Riemann zeta function are on the critical line. *J. reine angew. Math*, 399(1): 1–26, 1989.

[12] Harold Davenport and Wiliam Judson LeVeque. Uniform distribution relative to a fixed sequence. *Michigan Math. J.*, 10:315–319, 1963. ISSN 0026-2285.

[13] Harold Davenport, Paul Erdős, and Wiliam Judson LeVeque. On Weyl's criterion for uniform distribution. *Michigan Math. J.*, 10:311–314, 1963. ISSN 0026-2285.

[14] Peter Gustav Lejeune Dirichlet. Beweis eines Satzes über die arithmetische Progression. *Bericht Ak. Wiss. Berlin*, pages 108–110, 1837.

[15] Paulius Drungilas, Ramūnas Garunkštis, and Audrius Kačėnas. Universality of the Selberg zeta-function for the modular group. *Forum Math.*, 25(3):533–564, 2013.

[16] Artūras Dubickas, Ramūnas Garunkštis, Jörn Steuding, and Rasa Steuding. Zeros of the Estermann zeta function. *J. Aust. Math. Soc.*, 94(01):38–49, 2013.

[17] Peter D.T.A. Elliott. The Riemann zeta function and coin tossing. *J. Reine Angew. Math*, 254:100–109, 1972.

[18] Jürgen Elstrodt. Die Selbergsche Spurformel für kompakte Riemannsche Flächen. *Jahresber. Dtsch. Math.-Ver.*, 83:45–77, 1981.

[19] Paul Erdős. Beweis eines Satzes von Tschebyschef. *Acta Litterarum ac Scientiarum, Szeged*, 5:194–198, 1932.

[20] Paul Erdős. On a new method in elementary number theory which leads to an elementary proof of the prime number theorem. *Proc. Natl. Acad. Sci. USA*, 35(7):374, 1949.

[21] Kevin Ford, Kannan Soundararajan, and Alexandru Zaharescu. On the distribution of imaginary parts of zeros of the Riemann zeta function, II. *Math. Ann.*, 343(3):487–505, 2009.

[22] Akio Fujii. On the uniformity of the distribution of the zeros of the Riemann zeta function. *J. Reine Angew. Math*, 302:167–205, 1978.

[23] Ramūnas Garunkštis. On the Backlund equivalent for the Lindelöf hypothesis. *Adv. Stud. Pure Math.*, 49:91–104, 2007.

[24] Ramūnas Garunkštis. Note on zeros of the derivative of the Selberg zeta-function. *Arch. Math.*, 91(3):238–246, 2008.

[25] Ramūnas Garunkštis and Raivydas Šimėnas. The a-values of the Selberg zeta-function. *Lith. Math. J.*, 52(2):145–154, 2012.

[26] Ramūnas Garunkštis and Raivydas Šimėnas. On the Speiser equivalent for the Riemann hypothesis. *Eur. J. Math.*, 1(2):337–350, 2015.

[27] Ramūnas Garunkštis and Jörn Steuding. On the roots of the equation $\zeta(s) = a$. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, pages 1–15. Springer, 2014.

[28] Ramūnas Garunkštis, Jörn Steuding, and Raivydas Šimėnas. The a-points of the Selberg zeta-function are uniformly distributed modulo one. *Illinois J. Math.*, 58(1):207–218, 2014.

[29] Steven Mark Gonek, Stephen J. Lester, and Micah B. Milinovich. A note on simple a-points of l-functions. *Proc. Amer. Math. Soc.*, 140 (12):4097–4103, 2012.

[30] Izrail Solomonovich Gradshteyn and Iosif Moiseevich Ryzhik. *Table of integrals, series, and products.* Academic Press, 2014.

[31] J. P. Gram. Note sur les zéros de la fonction $\zeta(s)$ de Riemann. *Acta Math.*, 27(1):289–304, 1903.

[32] Jacques Hadamard. Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. *Bull. Soc. Math. France*, 24:199–220, 1896.

[33] Godfrey Harold Hardy and John Edensor Littlewood. On lindelöf's hypothesis concerning the Riemann zeta-function. *Proc. Roy. Soc. Ser. A*, pages 403–412, 1923.

[34] Godfrey Harold Hardy and Marcel Riesz. *The general theory of Dirichlet's series.* Courier Corporation, 2013.

[35] Dennis A. Hejhal. *The Selberg Trace Formula for* $\mathrm{PSL}(2,\mathbb{R})$, volume 1. Springer, 1976.

[36] Dennis A. Hejhal. *The Selberg Trace Formula for* $\mathrm{PSL}(2,\mathbb{R})$, volume 2. Springer, 1983.

[37] Edmund Hlawka. Über die Gleichverteilung gewisser Folgen, welche mit den Nullstellen der Zetafunktion zusammenhängen. *Österreich. Akad. Wiss. Math.-Naturwiss. Kl. S.-B. II*, 184(8-10):459–471, 1975.

[38] Edmund Hlawka. *Theorie der Gleichverteilung*. Wissenschaftsverlag, 1979.

[39] J. I. Hutchinson. On the roots of the Riemann zeta function. *Trans. Amer. Math. Soc.*, 27(1):49–60, 1925.

[40] Martin N. Huxley. Scattering matrices for congruence subgroups. In *Modular forms*, pages 141–156. Durham, 1983.

[41] Henryk Iwaniec. *Spectral methods of automorphic forms*, volume 53. American Mathematical Society Providence, 2002.

[42] Med-Taïb Jakhlouti, Kamel Mazhouda, and Jörn Steuding. On the distribution of the a-points of a Selberg class L-function modulo one. *Arch. Math.*, 104(5):419–429, 2015.

[43] Johan Jensen. Sur un nouvel et important théorème de la théorie des fonctions. *Acta Math.*, 22(1):359–364, 1899.

[44] Jay Jorgenson and Lejla Smajlović. On the distribution of zeros of the derivative of Selberg's zeta function associated to finite volume Riemann surfaces. *arXiv preprint arXiv:1302.5928*, 2013.

[45] Jerzy Kaczorowski. Axiomatic theory of L-functions: the Selberg class. In *Analytic Number Theory*, pages 133–209. Springer, 2006.

[46] Jerzy Kaczorowski and Mieczysław Kulas. On the non-trivial zeros off the critical line for L-functions from the extended Selberg class. *Monatsh. Math.*, 150(3):217–232, 2007.

[47] Jerzy Kaczorowski and Alberto Perelli. The Selberg class: a survey. In *Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997)*, pages 953–992. de Gruyter, Berlin, 1999.

[48] Jerzy Kaczorowski and Alberto Perelli. On the structure of the Selberg class, I: $0 \le d \le 1$. *Acta Math.*, 182(2):207–241, 1999.

[49] Justas Kalpokas and Jörn Steuding. On the value-distribution of the Riemann zeta-function on the critical line. *arXiv preprint arXiv:0907.1910*, 2009.

[50] Justas Kalpokas, Maxim A. Korolev, and Jörn Steuding. Negative values of the Riemann zeta function on the critical line. *Mathematika*, 59(02):443–462, 2013.

[51] Dubi Kelmer. On distribution of poles of Eisenstein series and the length spectrum of hyperbolic manifolds. *Int. Math. Res. Not.*, 23: 12319–12344, 2015.

[52] Helge von Koch. Sur la distribution des nombres premiers. *Acta Math.*, 24(1):159–182, 1901.

[53] Lauwerens Kuipers and Harald Niederreiter. *Uniform distribution of sequences*. Courier Corporation, 2012.

[54] Edmund Landau. Über die Anzahl der Gitterpunkte in gewissen Bereichen. *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, 1924:137–150, 1924.

[55] Edmund Landau. Über den Wertevorrat von $\zeta(s)$ in der Halbebene $\sigma > 1$. *Nachr. Ges. Wiss. Göttingen*, 36:81–91, 1933.

[56] Antanas Laurinčikas. The universality of the Lerch zeta-function. *Lith. Math. J.*, 37(3):275–280, 1997.

[57] Antanas Laurinčikas. The universality of zeta-functions. *Acta Appl. Math.*, 78(1-3):251–271, 2003.

[58] Antanas Laurinčikas. *Limit theorems for the Riemann zeta-function*, volume 352. Springer Science & Business Media, 2013.

[59] Wiliam Judson LeVeque. On uniform distribution modulo a subdivision. *Pacific J. Math.*, 3:757–771, 1953. ISSN 0030-8730.

[60] Norman Levinson. More than one third of zeros of Riemann's zeta-function are on $\sigma = 1/2$. *Adv. Math.*, 13(4):383–436, 1974.

[61] Norman Levinson. Almost all roots of $\zeta(s) = a$ are arbitrarily close to $\sigma = 1/2$. *Proc. Natl. Acad. Sci. USA*, 72(4):1322–1324, 1975.

[62] Norman Levinson and Hugh L. Montgomery. Zeros of the derivatives of the Riemann zeta-function. *Acta Math.*, 133(1):49–65, 1974.

[63] Wenzhi Luo. On zeros of the derivative of the Selberg zeta function. *Amer. J. Math.*, 127(5):1141–1151, 2005.

[64] Kohji Matsumoto. Probabilistic value-distribution theory of zeta-functions. *Sugaku Expositions*, 17(1):51–72, 2004.

[65] Alberto Perelli. A survey of the Selberg class of L-functions, part II. *Riv. Math. Univ. Parma*, 3:83–118, 2004.

[66] Alberto Perelli. A survey of the Selberg class of L-functions, part I. *Milan J. Math.*, 73(1):19–52, 2005.

[67] Ralph S. Phillips and Peter Sarnak. On cusp forms for co-finite subgroups of $PSL(2,\mathbb{R})$. *Invent. Math.*, 80(2):339–364, 1985.

[68] Ralph S. Phillips and Peter Sarnak. The Weyl theorem and the deformation of discrete groups. *Comm. Pure Appl. Math.*, 38(6):853–866, 1985.

[69] Hans Rademacher. Fourier analysis in number theory. *Collected Papers of H. Rademacher*, 2:434–459, 1974.

[70] Srinivasa Ramanujan. A proof of Bertrand's postulate. *J. Indian Math. Soc.*, 11(181-182):27, 1919.

[71] Burton Randol. Small eigenvalues of the Laplace operator on compact Riemann surfaces. *Bull. Amer. Math. Soc.*, 80(5):996–1000, 1974.

[72] Burton Randol. On the asymptotic distribution of closed geodesics on compact Riemann surfaces. *Trans. Amer. Math. Soc.*, 233:241–247, 1977.

[73] Burton Randol. The Riemann hypothesis for Selberg's zeta-function and the asymptotic behavior of eigenvalues of the Laplace operator. *Trans. Amer. Math. Soc.*, 236:209–223, 1978.

[74] Bernhard Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse. *Ges. Math. Werke und Wissenschaftlicher Nachlaß*, 2:145–155, 1859.

[75] Walter Rudin. *Real and complex analysis (3rd)*. New York: McGraw-Hill Inc, 1986.

[76] Wolfgang M. Schmidt. Disproof of some conjectures on Diophantine approximations. *Studia Sci. Math. Hungar.*, 4:137–144, 1969. ISSN 0081-6906.

[77] Lowell Schoenfeld. Sharper bounds for the Chebyshev functions $0(x)$ and $w(x)$. ii. *Math. Comp.*, 30:337–360, 1976.

[78] Atle Selberg. *On the zeros of Riemann's zeta-function*. Dybwad in Komm., 1943.

[79] Atle Selberg. An elementary proof of the prime-number theorem. *Ann. of Math.*, pages 305–313, 1949.

[80] Atle Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc.*, 20:47–87, 1956.

[81] Atle Selberg. Old and new conjectures and results about a class of Dirichlet series. In *Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989)*, pages 367–385, 1992.

[82] Atle Selberg. Remarks on the distribution of poles of Eisenstein series. In *Collected works II*, pages 15–47. Springer-Verlag, 1992.

[83] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 1. Princeton University Press, 1971.

[84] Rasa Šleževičienė. Speiser's correspondence between the zeros of a function and its derivative in Selberg's class of Dirichlet series. *Fiz. Mat. Fak. Moksl. Semin. Darb.*, 6:142–153, 2003.

[85] Lejla Smajlović. On Li's criterion for the Riemann hypothesis for the Selberg class. *J. Number Theory*, 130(4):828–851, 2010.

[86] Andreas Speiser. Geometrisches zur Riemannschen zetafunktion. *Math. Ann.*, 110(1):514–521, 1935.

[87] Robert Spira. On the Riemann zeta function. *J. London Math. Soc.*, 44:325–328, 1969.

[88] Robert Spira. Some zeros of the Titchmarsh counterexample. *Math. Comp.*, 63(208):747–748, 1994.

[89] Jörn Steuding. *Value-distribution of L-functions.* Springer, 2007.

[90] Jörn Steuding. The roots of the equation $\zeta(s) = a$ are uniformly distributed modulo one. In *Analytic and Probabilistic Methods in Number Theory, Proceedings of the Fifth International Conference in Honour of J. Kubilius, Palanga*, pages 243–249, 2011.

[91] Edward Charles Titchmarsh. *The Theory of Functions*, volume 80. London, 1939.

[92] Edward Charles Titchmarsh. *The Theory of the Riemann Zeta-Function.* Oxford University Press, 1986.

[93] Timothy Trudgian. On the success and failure of Gram's Law and the Rosser Rule. *Acta Arith.*, 148(3):225–256, 2011.

[94] Charles Jean de la Vallée-Poussin. Recherches analytiques sur la théorie des nombres premiers, i–iii. *Ann. Soc. Sci. Bruxelles*, 20:183–256, 1896.

[95] Sergei Mikhailovitch Voronin. Theorem on the "universality" of the Riemann zeta-function. *Izv. Math.*, 9(3):443–453, 1975.

[96] Hermann Weyl. Über ein Problem aus dem Gebiete der diophantischen Approximation. *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, pages 234–244, 1914.

[97] Hermann Weyl. Über die Gleichverteilung von Zahlen mod. eins. *Math. Ann.*, 77(3):313–352, 1916.

[98] Cem Yalçın Yıldırım. Zeros of derivatives of Dirichlet L-functions. *Turkish J. Math.*, 20:521–534, 1996.

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