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RUIN PROBABILITIES OF THE DISCRETE-TIME RISK MODEL WITH INHOMOGENEOUS CLAIMS

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VILNIAUS UNIVERSITETAS

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BANKROTO TIKIMYBĖS DISKRETAUS LAIKO RIZIKOS MODELIUI SU KELIOMIS SKIRTINGAI PASISKIRSČIUSIOMIS ŽALOMIS

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Agneška Korvel
Vilnius
August 18, 2016

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Notation

$\mathbb{N}_0$ and $\mathbb{N}$ denote the sets of natural numbers, $\mathbb{N}_0 = \{0, 1, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$.

$\mathbb{R}$ denotes the set of real numbers.

$\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

X, Y, Z, Z_i, S denote random variables.

$a_k = \mathbb{P}(X = k),\ k \in \mathbb{N}_0$, denote local probabilities of random variable X.

$b_k = \mathbb{P}(Y = k),\ k \in \mathbb{N}_0$, denote local probabilities of random variable Y.

$c_k = \mathbb{P}(Z = k),\ k \in \mathbb{N}_0$, denote local probabilities of random variable Z.

$z_k = \mathbb{P}(Z = k),\ k \in \mathbb{N}_0$, denote local probabilities of random variable Z.

$h_{ik} = \mathbb{P}(Z_i = k),\ k \in \mathbb{N}_0$, denote local probabilities of random variable Z_i.

$s_k = \mathbb{P}(S = k),\ k \in \mathbb{N}_0$, denote local probabilities of random variable S.

$A(x) = \sum_{k=0}^{[x]} a_k$ denotes a distribution function of random variable X.

$B(x) = \sum_{k=0}^{[x]} b_k$ denotes a distribution function of random variable Y.

$C(x) = \sum_{k=0}^{[x]} c_k$ denotes a distribution function of random variable Z.

$D(x) = \sum_{k=0}^{[x]} s_k$ denotes a distribution function of random variable S.

$F_Z(x) = \sum_{k=0}^{[x]} z_k$ denotes a distribution function of random variable Z.

$H_i(x) = \sum_{k=0}^{[x]} h_{ik}$ denotes a distribution function of random variable Z_i.

Symbol $ab(k),\ k \in \mathbb{N}_0$, where $a_k,\ b_k$ are local probabilities is understood in the convolution sense, e.g., $ab(k) = \sum_{i=0}^{\infty} a_i b_{k-i}$.

Symbol $AB(k)$ is understood in the convolution sense, e.g., $AB(k) = \sum_{i=0}^{\infty} A(i)B(k-i)$.

$\mathbb{E}X$ denotes the mean of a random variable X.

$$I_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

$u \in \mathbb{N}_0$ denotes the initial insurer's surplus.

$W_u(n)$ denotes the insurer's surplus at the time moment $n \in \mathbb{N}_0$.

$\psi(u)$ denotes the ruin probability.

$\varphi(u)$ denotes the survival probability.

T_u denotes the ruin time.

i.i.d. – independent identically distributed.

r.v. – random variable.

d.f. – distribution function.

p.m.f. – probability mass function.

Introduction

Research problem and actuality

The finite-time and ultimate ruin probabilities of the discrete-time risk model with inhomogeneous claims describing insurance business are investigated in this thesis.

Risk theory (collective risk theory, ruin theory) uses mathematical models to describe an insurer's vulnerability to insolvency/ruin. The one of the most popular model is Cramer-Lundberg model, developed in 1903, and its further extensions (S. Andersen model, compound binomical model, etc.). The model describes an insurance company who experiences two opposing cash flows: incoming cash premiums and outgoing claims and also depends on initial surplus. Such model is called *a risk model*. The central object of the risk model is to investigate the probability that the insurer's surplus level eventually falls below zero (making the firm bankrupt). The risk model was widely investigated by the different countries' scientists as De Vylder [10, 11], Gerber, Shiu [17, 19, 38–40], of period for one hundred years. But just recently, in the beginning of 21-st century, the risk model with inhomogeneous claims has been investigated. As insurance companies usually encounter different type of claims, this risk model with better describing of reality, becomes more analyzed risk model. The risk model with inhomogeneous claims can be defined as *multi-risk model* (model with several series of various claims) or *multi-seasonal risk model* (model with several series of various claims, where each claim repeats with the same time interval). Multi-risk model has been investigated by Lu [28, 29], Picard, Lefevre and Coulibaly [34], Wang and Wang [45, 46], where authors have obtained asymptotic formulas, whereas in this thesis we find out the algorithms for calculation of the exact values of the ruin probabilities. There are exist only few works where the algorithms for finding the exact values of the ruin probabilities were obtained. For instance, Raducan, Vernic and Zbaganu in

[35] have obtained the recursive algorithm used to evaluate ruin probability of the continuous time risk model with claims sizes distributed in Erlang's family with different parameters. The algorithms for finding the exact values of the ruin probabilities of the discrete-time risk model with inhomogeneous claims were obtained by Bieliauskienė and Šiaulys in [3], Blaževičius, Bieliauskienė and Šiaulys in [4], Damarackas and Šiaulys in [8]. In this thesis we extend previously obtained results and find out the recursive relations for calculating the exact values of the ruin probabilities for more complicated risk models.

Aims and tasks

The main purpose of the thesis is to obtain recursive formulas for the ruin probabilities calculation of the discrete-time risk model with inhomogeneous claims. In particular, we focus on the following tasks:

- Establishing the minimum requirements under which the ultimate ruin probability of the discrete-time risk model with inhomogeneous claims is equal to one (so-called net profit condition).
- Investigating the behaviour of the ultimate ruin probability of the discrete-time risk model with inhomogeneous claims when initial capital u tends to infinity.
- Obtaining the recursive relations for calculation of the exact values of the finite-time ruin probability for discrete-time any multi-risk model.
- Obtaining the recursive relations for calculation of the exact values of ultimate ruin probability for the discrete-time bi-risk model and risk model with three inhomogeneous claims.
- Obtaining the recursive relations for calculation of the exact values of finite-time and ultimate ruin probability for the discrete-time three-seasonal risk model.
- Testing the obtained algorithms, using software, and introducing the numerical values.

Novelty

All results are new. They extend, generalize and complement the results on finding ruin probabilities of discrete-time risk model with several inhomogeneous claims obtained by other authors. Obtained recursive relations enable fast calculation of the finite-time and ultimate ruin probabilities of the discrete-time multi-risk model and discrete-time three-seasonal model. Moreover, the results are tested using software and numerical values of the recursive formulas are presented. The obtained results have been approved in local and international conferences and exposed in the papers [20, 21].

Conferences

- *The 55th Conference of Lithuanian Mathematical Society*, Vilnius, Lithuania, 26–27 June 2014.
- *The 56th Conference of Lithuanian Mathematical Society*, Kaunas, Lithuania, 16–17 June 2015.
- *Quantitative methods in economics*, SGGW, Warsaw, Poland, 22–23 June 2015.
- *19th International Congress on Insurance: Mathematics and Economics (IME)*, The University of Liverpool, Liverpool, United Kingdom, 24–26 June 2015.

Publications

The main results of the thesis are published in the following papers:

- A. Grigutis, A. Korvel and J. Šiaulys. Ruin probabilities of a discrete-time multi-risk model. *Information technology and control*, 44:367-379, 2015
- A. Grigutis, A. Korvel and J. Šiaulys. Ruin probabilities in the three-seasonal discrete-time risk model. *Modern Stochastics: Theory and Applications*, 2:421-441, 2015.

Defended propositions

- Established net profit condition for the ultimate ruin probabilities calculation of the discrete-time risk model with inhomogeneous claims.
- The ultimate ruin probability of the discrete-time risk model with inhomogeneous claims tends to zero as initial capital u tends to infinity.
- Obtained recursive relations for calculation of the exact values of the finite-time ruin probability of the discrete-time any multi-risk model.
- Obtained recursive relations for calculation of the exact values of the ultimate ruin probability of the discrete-time bi-risk model and risk model with three inhomogeneous claims.
- Obtained recursive relations for calculation of the exact values of the finite-time and ultimate ruin probability of the discrete-time three-seasonal risk model.

Structure of the thesis

Chapter 1 contains the outlines of risk theory and risk model. In this chapter we review the model, present all necessary definitions and the main characteristics.

In Chapter 2 the discrete-time multi-risk model is investigated. Following the proof of Theorem 2.1 the recursive relations are obtained for calculation of the finite-time ruin probabilities of the discrete-time any multi-risk model.

In Chapter 3 the discrete-time bi-risk model and risk model with three inhomogeneous claims are investigated. The meaning of the net profit condition for bi-risk model is shown in Theorem 3.2. Theorems 3.2 and 3.3 provide an algorithm for finding $\psi(0)$ and $\psi(1)$. The recursive procedure to calculate the exact values of ultimate ruin probabilities $\psi(u),\ u \geqslant 2$ of bi-risk model is obtained by Theorems 3.1 and 3.2.
For the risk model with three inhomogeneous claims the meaning of the net profit condition is shown in Theorem 3.4. Theorem 3.5 provide an algorithm for finding the exact values of ultimate ruin probabilities.

In Chapter 4 the multi-seasonal risk model are described. Previously obtained results for finite-time and ultimate ruin probabilities calculation of bi-seasonal risk model are introduced. Theorem 4.3 provides an algorithm for finite-time ruin probabilities calculation of the discrete-time three-seasonal risk model. The recursive relations to calculate the exact values of the ultimate ruin probabilities for three-seasonal risk model are obtained in Theorem 4.4.

Finally, in Chapter 5, numerical examples of the obtained recursive relations are presented. In this Chapter we show how the obtained procedures can be applied for calculation of the finite-time and ultimate ruin probabilities.

Chapter 1

Outline of Risk theory and risk model

In actuarial science and applied probability ruin theory [16] (risk theory, collective risk theory) uses mathematical models to describe an insurer's vulnerability to insolvency/ruin. In such models key quantities of interest are the probability of ruin, distribution of surplus immediately prior to ruin and deficit at time of ruin.

The theoretical foundation of ruin theory, known as the Cramér–Lundberg model (or classical compound Poisson risk model, classical risk process or Poisson risk process) was introduced in beginning of the 20th century by the Swedish actuary Filip Lundberg [30]. Lundberg's work was republished in the 1930s by Harald Cramér [7]. The model describes an insurance company who experiences two opposing cash flows: incoming cash premiums and outgoing claims.

E. Sparre Andersen [41] extended the classical model in 1957 and proposed another model as a generalization of the classical (Poisson) risk theory. Instead of assuming only exponentially distributed independent interclaim times, he introduced a more general number process (so-called *renewal process*, see Definition 1.1) but retained the assumption of independence.

1.1 Sparre Andersen model

Nowadays the Sparre Andersen model is one of the most popular and used models in nonlife insurance mathematics, which describes the evolution of the insurance company's wealth over time which is measured by the assets it holds. The insurer's surplus depends on the initial capital, premium income, and outgoing claims. We will introduce the model and define the main concepts used.

Definition 1.1. *Renewal counting process. Let $\theta_1, \theta_2, \ldots$ be an independent identically distributed (i.i.d.) sequence of nonnegative random variables (r.v.'s). Then the random walk*

$$T_0 = 0,\ T_n = \theta_1 + \ldots + \theta_n, \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

is said to be a renewal sequence and the counting process

$$\Theta(t) = \#\{n \geqslant 1 : T_n \leqslant t\}, \quad t \geqslant 0,$$

is the corresponding renewal counting process.

The sequences T_0, T_1, $T_2, \ldots$ and θ_1, $\theta_2, \ldots$ are also referred as the sequences of the arrival and inter-arrival times of the renewal process Θ, respectively.

Definition 1.2. *Aggregate claim amount process. The total claim amount process or aggregate claim amount process is a process defined by:*

$$S(t) = \sum_{i=1}^{\Theta(t)} Z_i = \sum_{i=1}^{\infty} Z_i I_{[0,t]}(T_i), \quad t \geqslant 0,$$

where Z_1, Z_2, $\ldots$ is a sequel of nonnegative i.i.d. r.v.s and Z_1, Z_2, $\ldots$ and θ_1, θ_2, $\ldots$ are mutually independent.

Definition 1.3. *Surplus process. The process W*

$$W_u(t) = u + ct - S(t), \quad t \geqslant 0 \tag{1.1}$$

is called surplus or balance process. Here $u = W_u(0)$ is the initial surplus, c – premium payment rate and $S(t)$ is the total claim amount process.

In figure 1.1 we can see the behaviour of the surplus process $W_u(t)$.

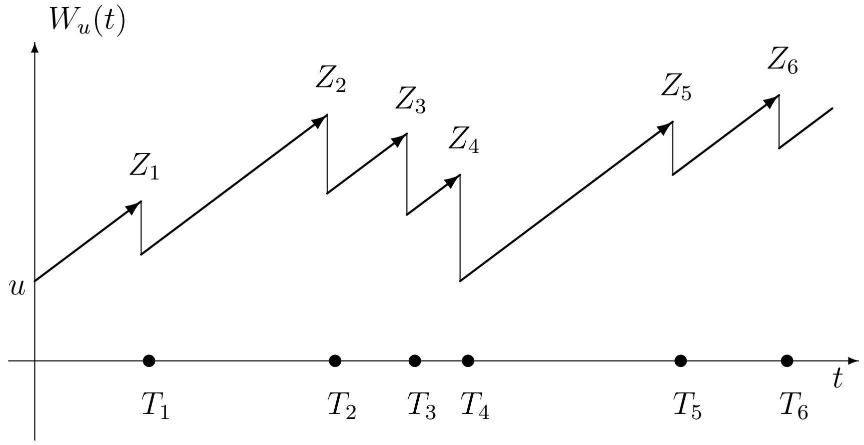

Figure 1.1. Behaviour of the surplus process $W_u(t)$

1.2 Discrete-time risk model

If $u \in \mathbb{N}_0$, $c = 1$, $\theta = 1$, and Z is an integer-valued, then we call the model defined by (1.1) a *discrete-time risk model*. In this case, the behaviour of the process W can be considered only for natural time moments $n \in \mathbb{N}_0$.

Definition 1.4. *We say that insurer's surplus $W_u(n)$ varies according to a discrete-time risk model (homogeneous discrete-time risk model) if for each $n \in \mathbb{N}_0$*

$$W_u(n) = u + n - \sum_{i=1}^{n} Z_i \tag{1.2}$$

and the following restrictions are satisfied:

• *the initial insurer's surplus $u = W_u(0)$ is a nonnegative integer number, i.e. $u \in \mathbb{N}_0$;*

• *claim amounts $Z_1, Z_2, Z_3, ...$ are independent copies of a nonnegative integer-valued random variable Z.*

The claim amount generator Z can be characterized by probability mass function (p.m.f.)

$$z_k = \mathbb{P}(Z = k),\ k \in \mathbb{N}_0$$

or by the cumulative distribution function (c.d.f.)

$$F_Z(x) = \sum_{k=0}^{\lfloor x \rfloor} z_k,\ x \in \mathbb{R}.$$

The discrete-time risk model describe insurer's capital level only in discrete time moments while the classical risk model or Sparre Andersen model are a continuous time models. We can get various generalizations of the discrete-time risk model by weakening the requirements of definition above. For instance, we can suppose that an initial insurer's surplus u is some real nonnegative number in the Eq. (1.2). We can suppose that generating random variable Z is nonnegative real-valued random variable. We can suppose that premium during the time unit is equal to some constant, not necessarily to one. Finally, we can suppose that the random claim amounts $Z_1,\ Z_2,\ Z_3,\ldots$ are independent but not necessarily identically distributed. In the scientific literature, such generalized models are often called just a discrete-time risk models because of a key feature of the model – the surplus calculation in the discrete-time moments.

The homogeneous discrete-time risk model has been extensively investigated by De Vylder and Goovaerts ([10], [11]), Dickson ([13], [14]), Gerber [18], Seal [37], Shiu ([39], [40]), Picard and Lefèvre ([32], [33]), Lefèvre and Loisel [24], Leipus and Šiaulys [25], Tang [43] and other authors.

1.3 Discrete-time multi-risk model

Usually, each insurance company works with several series of various claims. Each series of claims $\{Z_{j1}, Z_{j2}, \ldots\},\ j \in \{1, 2, \ldots, K\}$, can be driven by a specific initial surplus u_j, a specific premium rate c_j, and a specific series of inter-arrival times $\{\theta_{j1}, \theta_{j2}, \ldots\}$.

There are two different ways to consider insurance business in this situation. The first way is to create the so-called *multidimensional renewal risk model*. In this case, we suppose that the insurer's surplus at each moment of time $t \geqslant 0$ is a random vector

$$\left(W_{1,u_1}(n), W_{2,u_2}(n), \ldots, W_{K,u_K}(n)\right),$$

where

$$W_{j,u_j}(n) = u_j + c_j n - \sum_{i=1}^{\Theta_j(n)} Z_{ji}, \quad n \geqslant 0,\ j = 1, 2, \ldots, K,$$

and $\Theta_j(n)$ is the renewal process generated by the r.v. θ_{j1}. Several problems related to the multidimensional renewal risk model were investigated by Collamore [6], Sundt [42], Vernic [44], Denuit et al. [12], Picard et al. [34], Hult et al. [23], Yuen et al. [47], Li et al. [26], Avram et al. [2], Dang et al. [9], Chen et al. [5] and He et al. [22].

The second way to consider insurance business with several series of claims is related to the multi-risk model. We say that the insurer's surplus varies according to the *multi-risk model* if

$$W_u(n) = u + cn - \sum_{j=1}^{K} \sum_{i=1}^{\Theta_j(n)} Z_{ji} \tag{1.3}$$

for all time moments $n \geqslant 0$. Here we suppose that $\{Z_{j1}, Z_{j2}, \ldots\}$ are independent and identically distributed (i.i.d.) r.v.s for each fixed $j = 1, 2, \ldots, K$. The r.v.s $\{\theta_{j1}, \theta_{j2}, \ldots\}$ generating counting renewal processes $\Theta_j(n)$ are also i.i.d. The random claim amounts $\{Z_{j1}, Z_{j2}, \ldots\}_{j=1}^{K}$ and random inter-arrival times $\{\theta_{j1}, \theta_{j2}, \ldots\}_{j=1}^{K}$ are mutually independent. We note that the r.v.s $Z_{11}, Z_{21}, \ldots, Z_{K1}$ and $\theta_{11}, \theta_{21}, \ldots, \theta_{K1}$ in (1.3) may have different distributions.

In the multidimensional risk model, each series of claim amounts has its own dimension, whereas in the multi-risk model, all series of claims are placed in one basket. The multi-risk model was investigated by Wang and Wang ([45, 46]) and by Lu ([28], [29]), where problems related to large deviations of the sum in (1.3) were considered. When all renewal counting processes $\Theta_j(n)$ in (1.3) are generated by degenerate r.v.s, the multi-risk model becomes a discrete-time multi-risk model. For instance, if $K = 3$, $c = 1$, $\theta_{11} = 1$, $\theta_{21} = 2$, and $\theta_{31} = 3$, then from (1.3) it follows that

$$W_u(n) = u + n - \sum_{i=1}^{\lfloor n \rfloor} Z_{1i} - \sum_{i=1}^{\lfloor n/2 \rfloor} Z_{2i} - \sum_{i=1}^{\lfloor n/3 \rfloor} Z_{3i}$$

for all time moments $n \geqslant 0$.

1.4 Discrete-time multi-seasonal model

If r.v.s $Z_1, Z_2, \ldots$ are independent but not necessarily identically distributed, then the model defined by (1.2) is called the *inhomogeneous discrete-time risk model.* The *multi-seasonal risk model* can be described as inhomogeneous risk model with several differently distributed claim amounts which periodically change.

The difference between multi-risk model and multi-seasonal model is such, that in multi-risk model the differently distributed claim amounts repeat with different interclaim time and in multi-seasonal risk model differently distributed claim amounts repeat with the same interclaim time.

1.5 Main characteristics

The ruin time and the ruin probability are the main extremal characteristics of any risk model. In this section we present the main characteristics of the discrete-time risk model, which, essentially, are defined equally for any risk model, investigated in this thesis.

At each moment n the insurer's surplus may remain positive, or become negative, or vanish to zero. The situation, when the capital falls below or is equal to zero, is called *insolvency* or *ruin*.

Definition 1.5. *The ruin time. The first time T_u when insurer's surplus becomes non-positive is called the ruin time, i.e.*

$$T_u = \begin{cases} \inf\ \{n \in \mathbb{N} : W_u(n) \leqslant 0\}, \\ \infty,\ \textit{if}\ W_u(n) > 0\ \textit{for all}\ n \in \mathbb{N}. \end{cases} \tag{1.4}$$

Definition 1.6. *Finite time ruin probability. The probability to ruin by the moment $T \in \mathbb{N}$ is called the finite-time ruin probability*

$$\psi(u,T) = \mathbb{P}(T_u \leqslant T).$$

Definition 1.7. *Ultimate ruin probability. The ultimate ruin probability is defined by*

$$\psi(u) = \mathbb{P}(T_u < \infty).$$

Definition 1.8. *Ultimate survival probability. The ultimate survival probability is defined by*

$$\varphi(u) = 1 - \psi(u) = \mathbb{P}(T_u = \infty).$$

The presented definitions imply that:

$$
\begin{aligned}
\psi(u,T) &= \mathbb{P}\left(\bigcup_{n=1}^{T} \left\{ u + n - \sum_{i=1}^{n} Z_i \leqslant 0 \right\} \right) = \mathbb{P}\left(\max_{1 \leqslant n \leqslant T} \sum_{i=1}^{n} (Z_i - 1) \geqslant u \right), \\
\psi(u) &= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \left\{ u + n - \sum_{i=1}^{n} Z_i \leqslant 0 \right\} \right) = \mathbb{P}\left(\sup_{n \geqslant 1} \sum_{i=1}^{n} (Z_i - 1) \geqslant u \right), \\
\varphi(u) &= \mathbb{P}\left(\bigcap_{n=1}^{\infty} \left\{ u + n - \sum_{i=1}^{n} Z_i > 0 \right\} \right), \\
&\lim_{T \nearrow \infty} \psi(u,T) = \psi(u).
\end{aligned}
$$

Chapter 2

Finite-time ruin probabilities of the discrete-time multi-risk model

In this chapter we present the main obtained recursive relations for the finite-time ruin probabilities calculation of the discrete-time multi-risk model. The method used for obtaining the algorithm is the law of total probability. The same method can be found in [4, 8, 13].

We say that the insurer's surplus W_u *varies according to the discrete-time multi-risk model if, for all time moments* $n \in \mathbb{N}_0$,

$$W_u(n) = u + n - \sum_{i=1}^{K} \sum_{j=1}^{\lfloor n/i \rfloor} Z_{ij}, \tag{2.1}$$

where K *is a fixed natural number,* $u \in \mathbb{N}_0$ *is the insurer's initial surplus, and* $Z_{i1}, Z_{i2}, \ldots$ *are independent copies of an integer valued nonnegative r.v.* Z_i *for each* $i \in \{1, 2, \ldots, K\}$. *In addition, the series of r.v.s* $\{Z_{i1}, Z_{i2}, \ldots\}_{i=1}^{K}$ *are mutually independent.*

Obviously, every discrete-time multi-risk model is generated by the insurer's initial surplus u and collection of r.v.s $Z_1,\ Z_2, \ldots, Z_K$. The claim amount Z_1 occurs at every time moment, Z_2 occurs at every second time moment, and so on.

The nonnegative integer-valued r.v.s $Z_1,\ Z_2, \ldots, Z_K$ generating the multi-risk model can be described by the local probabilities

$$h_{ik} = \mathbb{P}(Z_i = k), \ \ k \in \mathbb{N}_0, \ i = 1, 2, \ldots, K,$$

or by their distribution functions (d.f.)

$$H_i(x) = \sum_{k \leqslant x} h_{ik}, \ \ x \in \mathbb{R}, \ i = 1, 2, \ldots, K.$$

If $K = 2$, then (2.1) implies that

$$W_u(n) = u + n - \sum_{k=1}^{n} X_k - \sum_{l=1}^{\lfloor n/2 \rfloor} Y_l, \ n \in \mathbb{N}_0, \tag{2.2}$$

where $u \in \mathbb{N}_0$, $X_1, X_2, \ldots$ are independent copies of a nonnegative integer valued r.v. $X = Z_1$, and $Y_1, Y_2, \ldots$ are independent copies of an integer valued r.v. $Y = Z_2$.

We call the model defined by (2.2) a *discrete-time bi-risk model.* It is clear that such a model is generated by the insurer's surplus u and two random claim amounts X and Y, where X occurs at every time increment, and Y occurs at every double time increment. In such a case, we use the following notation for the local probabilities and d.f.s of X and Y:

$$\begin{aligned}
a_k &= \mathbb{P}(X = k), \ k \in \mathbb{N}_0; \\
b_l &= \mathbb{P}(Y = l), \ l \in \mathbb{N}_0; \\
A(x) &= \sum_{0 \leqslant k \leqslant \lfloor x \rfloor} a_k, \ \ x \in \mathbb{R}; \\
B(x) &= \sum_{0 \leqslant l \leqslant \lfloor x \rfloor} b_l, \ \ x \in \mathbb{R}.
\end{aligned}$$

Theorem 2.1. *Suppose that r.v.s $Z_1, Z_2, \ldots, Z_K$, $K \geqslant 1$, generate the discrete-time multi-risk model. For all $u, l \in \mathbb{N}_0$, let*

$$\mathcal{D}_{lu}^{K} = \{k_{ij} \in \mathbb{N}_0 : i \in \{1, 2, \ldots, K\}, \ j \in \mathbb{N}, \ \mathcal{B}_{Kl} \leqslant u + l\}$$

and

$$\overline{\mathcal{D}}_{lu}^{K} = \{k_{ij} \in \mathbb{N}_0 : i \in \{1, 2, \ldots, K\}, \ j \in \mathbb{N}, \ \mathcal{B}_{Kl} > u + l\},$$

where

$$\mathcal{B}_{Kl}=\sum_{i=1}^{K}\sum_{j=1}^{\lfloor (l+1)/i\rfloor}k_{ij}=\sum_{j=1}^{l+1}k_{1j}+\sum_{j=1}^{\lfloor (l+1)/2\rfloor}k_{2j}+\cdots+\sum_{j=1}^{\lfloor (l+1)/K\rfloor}k_{Kj}.$$

Then, for all $u\in\mathbb{N}_0$, we have:

$$\begin{aligned}
\psi(u,1)&=\sum_{k_{11}>u}h_{1k_{11}},\\
\psi(u,2)&=\psi(u,1)+\sum_{\substack{k_{11}\leqslant u\\ k_{11}+k_{12}+k_{21}>u+1}}h_{1k_{11}}h_{1k_{12}}h_{2k_{21}},\\
\psi(u,T)&=\psi(u,T-1)+\sum_{\mathcal{D}\subseteq\mathcal{D}^K_{0u}\cap\mathcal{D}^K_{1u}\cap\cdots\cap\mathcal{D}^K_{(T-2)u}\cap\overline{\mathcal{D}^K_{(T-1)u}}}\;\prod_{k_{ij}\in\mathcal{D}}h_{ik_{ij}}
\end{aligned}$$

for all $T\in\{3,4,\ldots,M\}$, where M is the least common multiple of numbers $1,2,\ldots,K$.
If $u\in\mathbb{N}_0$ and $T\geqslant M+1$, then

$$\begin{aligned}
&\psi(u,T)=\psi(u,M)\\
&+\sum_{\mathcal{D}\subseteq\mathcal{D}^K_{0u}\cap\mathcal{D}^K_{1u}\cap\cdots\cap\mathcal{D}^K_{(M-2)u}\cap\mathcal{D}^K_{(M-1)u}}\;\prod_{k_{ij}\in\mathcal{D}}h_{ik_{ij}}\psi(u+M-\mathcal{B}_{K(M-1)},T-M).
\end{aligned}$$

For a larger K, the obtained recursive formulas are quite complex, and numerical application of these formulas requires much resources. Otherwise, when K is relatively small, the formulas of Theorem 2.1 imply a sufficiently simple algorithm to calculate finite-time ruin probabilities. For example, in the bi-risk model, for each $u\in\mathbb{N}_0$, we have that

$$\begin{cases}
\psi(u,1)=\sum\limits_{k>u}a_k,\\
\psi(u,2)=\psi(u,1)+\sum\limits_{\substack{k\leqslant u\\ k+l+m>u+1}}a_ka_lb_m,\\
\psi(u,T)=\psi(u,2)+\sum\limits_{\substack{k\leqslant u\\ k+l+m\leqslant u+1}}\psi(u+2-k-l-m,T-2)\,a_ka_lb_m,\ T\geqslant 3.
\end{cases}\tag{2.3}$$

Theorem 2.1 allows us to calculate the values of $\psi(u,T)$, $u\in\mathbb{N}_0$, $T\in\mathbb{N}$, for an arbitrary discrete-time multi-risk model.

Proof. We prove the assertion only in the particular case $K = 2$. In fact, we prove only the equations given in (2.3). The proof of the general case is similar.
By the definition of the finite-time ruin probability

$$\psi(u,T) = \mathbb{P}\Big(\bigcup_{n=1}^{T} \{ W_u(n) \leqslant 0 \} \Big) \tag{2.4}$$

we have that

$$\psi(u,1) = \mathbb{P}(W_u(1) \leqslant 0) = \mathbb{P}(u + 1 - X_1 \leqslant 0) = \mathbb{P}(X > u) = \sum_{k>u} a_k.$$

Similarly,

$$\begin{aligned}\psi(u,2) &= \mathbb{P}\big(\{W_u(1) \leqslant 0\} \cup \{W_u(2) \leqslant 0\}\big)\\ &= \mathbb{P}\{W_u(1) \leqslant 0\} + \mathbb{P}(\{W_u(2) \leqslant 0\} \cap \{W_u(1) > 0\})\\ &= \psi(u,1) + \sum_{k=0}^{u} \mathbb{P}(X_1 + X_2 + Y_1 \geqslant u + 2, X_1 = k)\end{aligned}$$

by the law of total probability. It is obvious that the second term of the last equality is

$$\sum_{k=0}^{u} \mathbb{P}(k + X_2 + Y_1 \geqslant u + 2)\, a_k = \sum_{\substack{k \leqslant u \\ k+l+m>u+1}} a_k a_l b_m.$$

Consequently, the first two equalities of (2.3) hold. It remains to prove the third one.

If $T \geqslant 3$, then equalities (2.1), (2.4) and the law of total probability imply that

$$
\begin{aligned}
\psi(u,T) &= \mathbb{P}\Big(\bigcup_{n=1}^{T}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j\leqslant 0\Big\}\Big)\\
&= \mathbb{P}\left(u+1-X_1\leqslant 0\right)\\
&+\sum_{k\leqslant u}a_k\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{P}\Big(X_2=l,Y_1=m,\bigcup_{n=2}^{T}\Big\{u+n-k-\sum_{i=2}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j\leqslant 0\Big\}\Big)\\
&= \psi(u,1)+\sum_{k\leqslant u}a_k\sum_{l,m:\ k+l+m>u+1}a_l b_m\\
&+\sum_{k\leqslant u}a_k\sum_{l,m:\ k+l+m\leqslant u+1}a_l b_m\,\mathbb{P}\Big(\bigcup_{n=3}^{T}\Big\{u+n-k-l-m\\
&\qquad -\sum_{i=3}^{n}X_i-\sum_{j=2}^{\lfloor n/2\rfloor}Y_j\leqslant 0\Big\}\Big).
\end{aligned}
$$

The random variables $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) as well as the r.v.s $Y_1, Y_2, \ldots$. Therefore,

$$\sum_{i=3}^{n}X_i\overset{d}{=}\sum_{i=1}^{n-2}X_i\,, \tag{2.5}$$

$$\sum_{j=2}^{\lfloor n/2\rfloor}Y_j\overset{d}{=}\sum_{j=1}^{\lfloor n/2\rfloor-1}Y_j \tag{2.6}$$

for $n\in\{3,4,\ldots\}$. The last relations and the second equality of (2.3) imply that

$$
\begin{aligned}
\psi(u,T) &= \psi(u,2)\\
&+\sum_{\substack{k\leqslant u\\ k+l+m\leqslant u+1}}a_k a_l b_m\,\mathbb{P}\Big(\bigcup_{\tau=1}^{T-2}\Big\{u+2+\tau-k-l-m-\sum_{i=1}^{\tau}X_i-\sum_{j=1}^{\lfloor \tau/2\rfloor}Y_j\leqslant 0\Big\}\Big).
\end{aligned}
$$

Now we see that the last equality of (2.3) follows from expression (2.4), and the particular case of Theorem 2.1 is proved. □

The presented algorithm for calculation of the finite-time ruin probabilities of the multi-risk model returns accurate values. For instance, for the bi-risk model generated by r.v.'s X and Y such that $a_1=\mathbb{P}(X=1)=1/2$, $a_2=\mathbb{P}(X=2)=1/2$, $b_1=\mathbb{P}(Y=1)=0$, $b_2=\mathbb{P}(Y=2)=1/2$, $b_3=\mathbb{P}(Y=3)=1/2$ and for

$u = 1$ we obtain:

$$\begin{cases} \psi(1,1) = 1/2, \\ \psi(1,2) = 1/2 + 1/2 \cdot 1/2 \cdot 1/2 + 1/2 \cdot 1/2 \cdot 1/2 + 1/2 \cdot 1/2 \cdot 1/2 + 1/2 \cdot 1/2 \cdot 1/2 = 1, \\ \psi(1,T) = 1,\ T \geqslant 3. \end{cases}$$

More complicated examples are demonstrated in Chapter 5 (Examples 1, 2, 3, 4).

Chapter 3

Ultimate ruin probabilities of the discrete-time multi-risk model

In this chapter we investigate the discrete-time bi-risk model and multi-risk model with three independent series of claim amounts and present the main obtained recursive relations to calculate the ultimate ruin probabilities. We use the same method as for finding the algorithm for the finite-time ruin probabilities calculation, i.e. the law of total probability.

3.1 Bi-risk model

Bi-risk model is defined in Chapter 2 (see Eq. (2.2)). We use the same notation of claim amounts, its' d.f.'s and local probabilities for bi-risk model as in Chapter 2.

Theorem 3.1. *Let us consider a discrete-time bi-risk model with generating r.v.s X and Y. Then, for all $u \in \mathbb{N}_0$,*

$$\psi(u) = \sum_{k>u} a_k + \sum_{\substack{k \leqslant u \\ k+l+m>u+1}} a_k a_l b_m + \sum_{\substack{k \leqslant u \\ k+l+m \leqslant u+1}} \psi(u+2-k-l-m)\, a_k a_l b_m.$$

Proof. The proof is similar to the proof of Theorem 2.1. Indeed, by (2.4) and the law of total probability we have that

$$\begin{aligned}\psi(u) &= \mathbb{P}\Bigg(\bigcup_{n=1}^{\infty}\{W_u(n) \leqslant 0\}\Bigg)\\ &= \mathbb{P}(W_u(1) \leqslant 0) + \mathbb{P}(W_u(1) > 0,\, W_u(2) \leqslant 0)\\ &\qquad + \mathbb{P}\Bigg(W_u(1) > 0,\, W_u(2) > 0,\, \bigcup_{n=3}^{\infty}\{W_u(n) \leqslant 0\}\Bigg)\\ &= \sum_{k>u} a_k + \sum_{\substack{k\leqslant u\\ k+l+m>u+1}} a_k a_l b_m\end{aligned}$$

$$+ \sum_{\substack{k\leqslant u\\ k+l+m\leqslant u+1}} a_k a_l b_m\, \mathbb{P}\Bigg(\bigcup_{n=3}^{\infty}\Big\{u+n-k-l-m-\sum_{i=3}^{n} X_i - \sum_{j=2}^{\lfloor n/2\rfloor} Y_j \leqslant 0\Big\}\Bigg).$$

To complete the proof, it suffices to observe that the last sum equals

$$\begin{aligned}&\sum_{\substack{k\leqslant u\\ k+l+m\leqslant u+1}} a_k a_l b_m\, \mathbb{P}\Bigg(\bigcup_{\tau=1}^{\infty}\Big\{u+2+\tau-k-l-m-\sum_{i=1}^{\tau} X_i - \sum_{j=1}^{\lfloor \tau/2\rfloor} Y_j \leqslant 0\Big\}\Bigg)\\ &\qquad = \sum_{\substack{k\leqslant u\\ k+l+m\leqslant u+1}} a_k a_l b_m\, \psi(u+2-k-l-m)\end{aligned}$$

due to Eqs. (2.5), (2.6) and definition (2.4). Theorem 2.1 is proved. □

We see from the last theorem that we can calculate the values of $\psi(u)$ for $u \geqslant 2$ if we know $\psi(0)$ and $\psi(1)$. Theorems 3.2 and 3.3 provide an algorithm for finding $\psi(0)$ and $\psi(1)$.

For every $u \in \mathbb{N}_0$, we denote:

$$\begin{aligned}AAB(u) &= \sum_{k+l+m\leqslant u} a_k a_l b_m,\\ \overline{AAB}(u) &= 1 - AAB(u).\end{aligned}$$

Theorem 3.2. *Let us consider a discrete-time bi-risk model with generating r.v.s X and Y for finite means $\mathbb{E}X$ and $\mathbb{E}Y$.*

(i) If $\mu_{X,Y} := \mathbb{E}X + \mathbb{E}Y/2 \geqslant 1$ and the r.v.s X, Y are non-degenerate, then $\psi(u) = 1$ for all $u \in \mathbb{N}_0$.

(ii) If $\mu_{X,Y} < 1$ and $b_0 = 0$, then we have:

$$\psi(0) = 2\mu_{X,Y} - 1,$$
$$\psi(1) = 1 - \frac{2}{AAB(1)}\left(1 - \mu_{X,Y}\right),$$
$$\psi(u) = \frac{1}{AAB\,(1)}\left(\sum_{v=1}^{u-1}\psi(v)\overline{AAB}(u+1-v) + \sum_{v=u+1}^{\infty}\overline{AAB}(v)\right)$$

for all $u \in \{2, 3, \ldots\}$.

Proof. Proof of part (i). Let

$$S_n := \sum_{i=1}^{n} X_i + \sum_{j=1}^{\lfloor n/2 \rfloor} Y_j - n,\ \ n \in \mathbb{N}. \tag{3.1}$$

It follows from (2.4) that, for every $u \in \mathbb{N}$,

$$\begin{aligned}\psi(u) = \mathbb{P}(S_n \geqslant u \text{ for some } n \in \mathbb{N}) &\geqslant \mathbb{P}(S_{2m} \geqslant u \text{ for some } m \in \mathbb{N}) \\ &\geqslant \mathbb{P}(\limsup_{m\to\infty} S_{2m} \geqslant 2u).\end{aligned} \tag{3.2}$$

However,

$$S_{2m} = \sum_{i=1}^{2m} X_i + \sum_{j=1}^{m} Y_j - 2m = \sum_{i=1}^{m} \xi_i$$

for every $m \in \mathbb{N}$, where $\{\xi_1, \xi_2, \ldots\}$ are independent copies of the r.v. $\xi = X_1 + X_2 + Y_1 - 2$.

Since $\mathbb{E}\xi \geqslant 0$ and $\mathbb{P}(\xi = 0) < 1$, we have that

$$\mathbb{P}(\limsup_{m\to\infty} S_{2m} = \infty) = 1 \tag{3.3}$$

(see, for instance, Proposition 7.2.3 in [36]). The obtained relations (3.2) and (3.3) imply part (i) of Theorem 3.2.

Proof of part (ii) consists of several steps.

- First, we prove that the condition $\mu_{X,Y} < 1$ implies

$$\lim_{u\to\infty} \psi(u) = 0. \tag{3.4}$$

According to definition (2.4) we have that, for every $u \in \mathbb{N}$,

$$\begin{aligned}\psi(u) &= \mathbb{P}(S_n \geqslant u \text{ for some } n \in \mathbb{N}) \\ &\leqslant \mathbb{P}\Big(\sup_{m\in\mathbb{N}} S_{2m} \geqslant u\Big) + \mathbb{P}\Big(\sup_{m\in\mathbb{N}} S_{2m+1} \geqslant u\Big),\end{aligned} \tag{3.5}$$

where S_n is defined in (3.1). It is clear that

$$\frac{S_{2m}}{2m} = \frac{1}{2m}\sum_{i=1}^{m}\xi_i,$$

where r.v.s $\xi_1, \xi_2, \dots$ are described in the proof of part (i). Hence, by the strong law of large numbers,

$$\frac{S_{2m}}{2m} \underset{t\to\infty}{\overset{\text{a.s.}}{\to}} \frac{1}{2}\,\mathbb{E}\xi = \mu_{X,Y} - 1 =: -\Delta < 0.$$

Therefore,

$$\mathbb{P}\Big(\sup_{m\geqslant p}\Big|\frac{S_{2m}}{2m} + \Delta\Big| \leqslant \frac{\Delta}{2}\Big) \underset{p\to\infty}{\to} 1. \tag{3.6}$$

If $N \geqslant 2$ and u is positive, then

$$\begin{aligned}&\mathbb{P}\Big(\sup_{m\in\mathbb{N}} S_{2m} < u\Big) \\ &\geqslant \mathbb{P}\Big(\bigcap_{m=1}^{N-1}\{S_{2m} \leqslant \frac{u}{2}\}, \bigcap_{m=N}^{\infty}\{S_{2m} \leqslant \frac{u}{2}\}\Big) \\ &\geqslant \mathbb{P}\Big(\bigcap_{m=1}^{N-1}\{S_{2m} \leqslant \frac{u}{2}\}\Big) + \mathbb{P}\Big(\bigcap_{m=N-1}^{\infty}\{S_{2m} \leqslant \frac{u}{2}\}\Big) - 1 \\ &\geqslant \mathbb{P}\Big(\bigcap_{m=1}^{N-1}\{S_{2m} \leqslant \frac{u}{2}\}\Big) + \mathbb{P}\Big(\sup_{m\geqslant N}\Big|\frac{S_{2m}}{2m} + \Delta\Big| \leqslant \frac{\Delta}{2}\Big) - 1.\end{aligned}$$

This inequality and relation (3.6) imply that

$$\lim_{u\to\infty}\mathbb{P}\Big(\sup_{m\in\mathbb{N}} S_{2m} < u\Big) = 1. \tag{3.7}$$

On the other hand, for all $m \in \mathbb{N}$,

$$\frac{S_{2m+1}}{2m+1} = \frac{2m}{2m+1}\,\frac{1}{2m}\sum_{i=1}^{m}\xi_i + \frac{X_{2m+1} - 1}{2m+1}.$$

Due to the strong law of large numbers,

$$\frac{1}{m}\sum_{i=1}^{m}\xi_i \underset{m\to\infty}{\overset{a.s.}{\to}} 2\mu_{X,Y}, \quad \frac{X_{2m+1}-1}{2m+1} \underset{m\to\infty}{\overset{a.s.}{\to}} 0\,.$$

Therefore,

$$\frac{S_{2m+1}}{2m+1} \underset{m\to\infty}{\overset{a.s.}{\to}} -\Delta\,,$$

and we obtain

$$\lim_{u\to\infty}\mathbb{P}\Big(\sup_{m\in\mathbb{N}} S_{2m+1} < u\Big) = 1 \tag{3.8}$$

using the same procedure as for the sums S_{2m}, $m \in \mathbb{N}$. Eq. (3.4) follows now from estimate (3.5) and Eqs. (3.7), (3.8).

- In this step, we prove that

$$\begin{aligned}
&\psi(0) + a_0 b_0 A(v+1)\psi(1)\\
&= \sum_{u=1}^{v+1}\overline{AAB}(u) + a_0^2 b_0\,\psi(v+2) + \psi(v+1)\\
&\quad - \sum_{u=1}^{v+1}\psi(u)\,\overline{AAB}(v+2-u) + a_0 b_0\,(A(v+1)-a_0)
\end{aligned} \tag{3.9}$$

for all $v \in \mathbb{N}_0$.

From Theorem 3.1 we have that

$$\psi(u) = \overline{AAB}(u+1) + a_{u+1}a_0b_0 + \sum_{\substack{k\leqslant u\\ k+l+m\leqslant u+1}} \psi(u+2-k-l-m)\,a_k a_l b_m$$

for all $u \in \mathbb{N}_0$. Therefore,

$$\sum_{u=0}^{v}\psi(u) = \sum_{u=0}^{v}\overline{AAB}(u+1) + a_0 b_0 \sum_{u=0}^{v} a_{u+1} + \mathcal{S}, \tag{3.10}$$

where $v \in \mathbb{N}_0$, and

$$\mathcal{S} = \sum_{u=0}^{v}\sum_{\substack{k\leqslant u\\ k+l+m\leqslant u+1}} a_k a_l b_m\,\psi(u+2-k-l-m).$$

Changing the order of summation in $\mathcal{S}$, we obtain the following expression of $\mathcal{S}$:

$$\begin{aligned}
&\sum_{u=0}^{v}\sum_{k=0}^{u}\sum_{l=0}^{u+1-k}\sum_{m=0}^{u+1-k-l} a_k a_l b_m \psi(u+2-k-l-m)\\
&=\sum_{k=0}^{v}\sum_{u=k}^{v}\sum_{l=0}^{u+1-k}\sum_{m=0}^{u+1-k-l} a_k a_l b_m \psi(u+2-k-l-m)\\
&=\sum_{k=0}^{v}\sum_{l=1}^{v+1-k}\sum_{u=k+l-1}^{v}\sum_{m=0}^{u+1-k-l} a_k a_l b_m \psi(u+2-k-l-m)\\
&\quad+\sum_{k=0}^{v}\sum_{l=0}^{0}\sum_{u=k}^{v}\sum_{m=0}^{u+1-k-l} a_k a_l b_m \psi(u+2-k-l-m)\\
&=\sum_{k=0}^{v}\sum_{l=1}^{v+1-k}\sum_{u=k+l-1}^{v}\sum_{m=0}^{u+1-k-l} a_k a_l b_m \psi(u+2-k-l-m)\\
&\quad+a_0\sum_{k=0}^{v}\sum_{u=k}^{v}\sum_{m=0}^{u+1-k-l} a_k b_m \psi(u+2-k-m)\\
&=\sum_{k=0}^{v}\sum_{l=1}^{v+1-k}\sum_{m=0}^{v+1-k-l}\sum_{u=k+l+m-1}^{v} a_k a_l b_m \psi(u+2-k-l-m)\\
&\qquad+a_0 b_0\sum_{k=0}^{v}\sum_{u=k}^{v} a_k \psi(u+2-k)\\
&\quad+a_0\sum_{k=0}^{v}\sum_{m=1}^{v+1-k}\sum_{u=k+m-1}^{v} a_k b_m \psi(u+2-k-m)\\
&=\sum_{k=0}^{v}\sum_{l=1}^{v+1-k}\sum_{m=0}^{v+1-k-l}\sum_{r=1}^{v+2-k-l-m} \psi(r) a_k a_l b_m\\
&\quad+a_0 b_0\sum_{k=0}^{v}\sum_{r=2}^{v+2-k} \psi(r) a_k\\
&\qquad+a_0\sum_{k=0}^{v}\sum_{m=1}^{v+1-k}\sum_{r=1}^{v+2-k-m} \psi(r) a_k b_m.
\end{aligned}$$

Now, changing the order of summation in the opposite direction, we get that $\mathcal{S}$ can be written in the following form:

$$\sum_{k=0}^{v}\sum_{l=1}^{v+1-k}\sum_{r=1}^{v+2-k-l}\psi(r)\sum_{m=0}^{v+2-k-l-r}a_k a_l b_m$$
$$+a_0 b_0\sum_{r=2}^{v+2}\psi(r)\sum_{k=0}^{v+2-r}a_k+a_0\sum_{k=0}^{v}\sum_{r=1}^{v+1-k}\psi(r)\sum_{m=1}^{v+2-k-r}a_k b_m$$
$$=\sum_{k=0}^{v}\sum_{r=1}^{v+1-k}\psi(r)\sum_{l=1}^{v+2-k-r}\sum_{m=0}^{v+2-k-l-r}a_k a_l b_m$$

$$+a_0 b_0\sum_{r=2}^{v+2}\psi(r)\sum_{k=0}^{v+2-r}a_k+a_0\sum_{r=1}^{v+1}\psi(r)\sum_{k=0}^{v+1-r}\sum_{m=1}^{v+2-k-r}a_k b_m$$
$$=\sum_{r=1}^{v+1}\psi(r)\sum_{k=0}^{v+1-r}\sum_{l=1}^{v+2-k-r}\sum_{m=0}^{v+2-k-l-r}a_k a_l b_m$$
$$+a_0 b_0\sum_{r=2}^{v+2}\psi(r)\sum_{k=0}^{v+2-r}a_k+a_0\sum_{r=1}^{v+1}\psi(r)\sum_{k=0}^{v+1-r}\sum_{m=1}^{v+2-k-r}a_k b_m$$
$$=\sum_{r=1}^{v+1}\psi(r)AAB(v+2-r)-a_0 b_0 A(v+1)\psi(1)+a_0^2 b_0\psi(v+2).$$

The last expression and Eq. (3.10) immediately imply relation (3.9).

- In this step, we complete the proof of Theorem 3.2. By Eq. (3.4) we have

$$\lim_{v\to\infty}\sum_{u=1}^{v+1}\psi(u)\overline{AAB}(v+2-u)=0.$$

On the other hand,

$$\lim_{v\to\infty}\sum_{u=1}^{v+1}\overline{AAB}(u)=\sum_{u=0}^{\infty}\overline{AAB}(u)-\overline{AAB}(0)=2\mu_{X,Y}-1+a_0^2 b_0$$

because of

$$\sum_{u=0}^{\infty}\overline{AAB}(u)=2\mu_{X,Y}.$$

Therefore,

$$\psi(0)+a_0 b_0\psi(1)=2\mu_{X,Y}+a_0 b_0-1 \qquad (3.11)$$

as $v \to \infty$ in both sides of Eq. (3.9).

If $b_0 = 0$, then $\psi(0) = 2\mu_{X,Y} - 1$, and the first statement of part (ii) follows.

If we set $v = 0$ and $b_0 = 0$ in (3.9), then we get

$$\psi(1) = 1 - \frac{1 - \psi(0)}{AAB(1)} = 1 - \frac{1 - \psi(0)}{a_0^2 b_1},$$

and the second equality of (ii) follows. The third equality of (ii) also follows from (3.9) if $b_0 = 0$. Theorem 3.2 is proved.

□

Theorem 3.3. *Let us consider a discrete-time bi-risk model with generating r.v.s X and Y. Suppose that $a_0 \neq 0$, $b_0 \neq 0$, and $\mu_{X,Y} = \mathbb{E}X + \mathbb{E}Y/2 < 1$. Then*

$$\psi(0) = 1 - 2(\mu_{X,Y} - 1) \lim_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\beta_{n+1} - \beta_n},$$
$$\psi(1) = \frac{1}{a_0 b_0} \left(2\mu_{X,Y} - 1 + a_0 b_0 - \psi(0)\right),$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are two recurrent sequences defined as follows:

$$\beta_0 = 1,\ \beta_1 = -\frac{1}{a_0 b_0},\ \beta_n = \frac{1}{\alpha_0}\left(\beta_{n-2} - \sum_{i=1}^{n-1} \alpha_i \beta_{n-i} - a_{n-1}\right),\ n \in \{2, 3, \ldots\},$$
$$\gamma_0 = 0,\ \gamma_1 = \frac{1}{a_0 b_0},\ \gamma_n = \frac{1}{\alpha_0}\left(\gamma_{n-2} - \sum_{i=1}^{n-1} \alpha_i \gamma_{n-i} + a_{n-1}\right),\ n \in \{2, 3, \ldots\},$$

and $\alpha_r = \sum_{k+l+m=r} a_k a_l b_m$ for $r \in \mathbb{N}_0$.

Proof. Recall that $\alpha_k = \mathbb{P}(X_1 + X_2 + Y_1 = k)$ for all $k \in \mathbb{N}_0$. Let $\varphi(u) = 1 - \psi(u)$ be the survival probability of the discrete-time bi-risk model for the initial insurer's surplus $u \in \mathbb{N}_0$. By definition (2.4), the law of total probability, and Eqs. (2.5), (2.6) we obtain

$$\begin{aligned}
\varphi(u) &= \mathbb{P}\Big(\bigcap_{n=1}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j>0\Big\}\Big)\\
&= \mathbb{P}\Big(\bigcap_{n=2}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j>0\Big\}\Big)\\
&\qquad -\mathbb{P}\Big(\bigcap_{n=2}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j>0\Big\}, X_1\geqslant u+1\Big)\\
&= \sum_{k=0}^{u+1}\mathbb{P}\Big(X_1+X_2+Y_1=k,\ \bigcap_{n=3}^{\infty}\Big\{u+n-\sum_{i=3}^{n}X_i-\sum_{j=2}^{\lfloor n/2\rfloor}Y_j-k>0\Big\}\Big)\\
&\qquad -\mathbb{P}\Big(X_1=u+1,\ X_2=0,\ Y_1=0,\ \bigcap_{n=3}^{\infty}\Big\{u+n-\sum_{i=3}^{n}X_i-\sum_{j=2}^{\lfloor n/2\rfloor}Y_j>0\Big\}\Big)\\
&= \sum_{k=0}^{u+1}\alpha_k\varphi(u+2-k)-a_0b_0a_{u+1}\varphi(1).
\end{aligned}$$

So, for an arbitrary $u\in\mathbb{N}_0$, we have that

$$\varphi(u)=\sum_{k=0}^{u+1}\alpha_{u+1-k}\,\varphi(k+1)-a_0b_0a_{u+1}\varphi(1). \tag{3.12}$$

Let β_n and γ_n be two recurrent sequences defined in Theorem 3.3. Let us prove by induction that

$$\varphi(n)=\beta_n\varphi(0)+2(1-\mu_{X,Y})\gamma_n \tag{3.13}$$

for all $n\geqslant 0$. If $n=0$, then (3.13) is evident. If $n=1$, then relation (3.13) follows from (3.11) because

$$\varphi(1)=-\frac{1}{a_0b_0}\,\varphi(0)+\frac{2(1-\mu_{X,Y})}{a_0b_0}.$$

We now prove that (3.13) is true for $n=N+1$ assuming that it holds for $n\leqslant N$. Substituting $u=N-1$ into Eq. (3.12), we get

$$\varphi(N-1)=\sum_{k=0}^{N}\alpha_{N-k}\,\varphi(k+1)-a_0b_0a_N\varphi(1).$$

Therefore,

$$\varphi(N+1) = \frac{1}{\alpha_0}\Big(\varphi(N-1) - \sum_{k=1}^{N}\alpha_k\varphi(N-k+1) + a_0b_0a_N\varphi(1)\Big),$$

and by the induction hypothesis we have that

$$\begin{aligned}
&\varphi(N+1)\\
&= \frac{1}{\alpha_0}\bigg(\beta_{N-1}\varphi(0) + 2(1-\mu_{X,Y})\,\gamma_{N-1} - \sum_{k=1}^{N}\alpha_k\Big(\beta_{N-k+1}\varphi(0)\\
&\qquad + 2(1-\mu_{X,Y})\gamma_{N-k+1}\Big) + a_0b_0a_N(\beta_1\varphi(0) + 2(1-\mu_{X,Y})\gamma_1)\bigg)\\
&= \varphi(0)\bigg(\frac{1}{\alpha_0}\Big(\beta_{N-1} - \sum_{k=1}^{N}\alpha_k\beta_{N+1-k} + a_0b_0a_N\beta_1\Big)\bigg)\\
&\qquad + 2(1-\mu_{X,Y})\bigg(\frac{1}{\alpha_0}\Big(\gamma_{N-1} - \sum_{k=1}^{N}\alpha_k\gamma_{N+1-k} + a_0b_0a_N\gamma_1\Big)\bigg)\\
&= \beta_{N+1}\varphi(0) + 2(1-\mu_{X,Y})\,\gamma_{N+1}.
\end{aligned}$$

Consequently, Eq. (3.13) holds for all $n \geqslant 0$.

Now we derive both equalities of Theorem 3.3. The sequence $\psi(u), u \in \mathbb{N}_0$, is nonincreasing by (2.4). Therefore, $\varphi(u)$ is nondecreasing with respect to u, and there exists a finite limit $\lim\limits_{u\to\infty}\varphi(u)$. Consequently,

$$\lim_{n\to\infty}\left(\varphi(n+1) - \varphi(n)\right) = 0.$$

From the last equality and relation (3.13) we obtain that

$$\lim_{n\to\infty}\big((\beta_{n+1} - \beta_n)\,\varphi(0) + 2(1-\mu_{X,Y})\,(\gamma_{n+1} - \gamma_n)\big) = 0.$$

Therefore,

$$\varphi(0) = 2(\mu_{X,Y} - 1)\lim_{n\to\infty}\frac{\gamma_{n+1} - \gamma_n}{\beta_{n+1} - \beta_n}, \tag{3.14}$$

provided that

$$\inf_{n\in\mathbb{N}_0}|\,\beta_{n+1} - \beta_n| \geqslant c \tag{3.15}$$

for a positive constant c.

We observe that the statement of Theorem 3.3 follows immediately from (3.14) and (3.11). It remains to show that

$$\begin{cases} \beta_{2k+1} \leqslant \beta_{2k-1}, \\ \beta_{2k} \geqslant \beta_{2k-2}, \end{cases} \tag{3.16}$$

for all $k \in \mathbb{N}$ because (3.15) with $c = 2$ follows from (3.16) by considering odd and even n separately.

It is easy to see that (3.16) is true for $k = 0$. Let us show that it holds for $k = N + 1$ if it does for $k = 1, 2, \ldots, N$. If $k = N + 1$, then

$$\beta_{2N+2} = \frac{1}{\alpha_0}\Big(\beta_{2N} - \sum_{i=1}^{2N+1} \alpha_i \beta_{2N+2-i} - a_{2N+1}\Big).$$

By the induction hypothesis,

$$\begin{aligned} &\beta_{2N+2} \\ \geqslant &\frac{1}{\alpha_0}(\beta_{2N} - \alpha_2\beta_{2N} - \alpha_4\beta_{2N} - \cdots - \alpha_{2N}\beta_{2N} - \alpha_1\beta_1 - \alpha_3\beta_1 - \cdots - \alpha_{2N+1}\beta_1 - a_{2N+1}) \\ = &\frac{1}{\alpha_0}(\beta_{2N}(1 - \alpha_2 - \alpha_4 - \cdots - \alpha_{2N}) - \beta_1(\alpha_1 + \alpha_3 + \cdots + \alpha_{2N+1}) - a_{2N+1}) \\ \geqslant &\frac{1}{\alpha_0}(\beta_{2N}\alpha_0 + \frac{1}{a_0 b_0}\alpha_{2N+1} - a_{2N+1}) \\ \geqslant &\beta_{2N}. \end{aligned} \tag{3.17}$$

Similarly, by the induction hypothesis and the proved estimate (3.17) we have

$$\begin{aligned} &\beta_{2N+3} \\ = &\frac{1}{\alpha_0}\Big(\beta_{2N+1} - \sum_{i=1}^{2N+2} \alpha_i\beta_{2N+3-i} - a_{2N+2}\Big) \\ \leqslant &\frac{1}{\alpha_0}(\beta_{2N+1}(1 + \alpha_2 - \alpha_4 - \cdots - \alpha_{2N+2})) \\ \leqslant &\beta_{2N+1}. \end{aligned} \tag{3.18}$$

Inequalities (3.17) and (3.18) imply that (3.16) holds for all $k \in \mathbb{N}$. This finishes the proof of Theorem 3.3.

□

The obtained algorithm works in a following way: first step should be calculation of $\psi(0)$, then $\psi(1)$ using calculated value of $\psi(0)$ (Theorems 3.2, 3.3). This step should be repeated for every $\psi(u)$, $u \in \{2, 3, \ldots\}$ using recursive formula (Theorems 3.1, 3.2).

Let us consider the bi-risk model generated by r.v.'s X and Y such that $a_0 = \mathbb{P}(X = 0) = 3/4$, $a_1 = \mathbb{P}(X = 1) = 1/8$, $a_2 = \mathbb{P}(X = 2) = 1/8$, $b_1 = \mathbb{P}(Y = 1) = 9/10$, $b_2 = \mathbb{P}(Y = 2) = 1/10$. In this case, net profit condition holds, i.e. $\mu_{X,Y} = \mathbb{E}X + \mathbb{E}Y/2 = 37/40 < 1$. Then, from Theorem 3.2 we obtain:

$$\psi(0) = 2\mu_{X,Y} - 1 = 2 \cdot 37/40 - 1 = 17/20,$$

$$\psi(1) = 1 - \frac{2}{a_0^2 b_1}(1 - \mu_{X,Y}) = 19/27,$$

$$\psi(2) = \frac{1}{a_0^2 b_1}\Bigg(\psi(1)\overline{AAB}(2) + \sum_{v=0}^{\infty} \overline{AAB}(v) - \overline{AAB}(0) - \overline{AAB}(1) - \overline{AAB}(2)\Bigg) = 1195/2187.$$

...etc (see Example 2 in Chapter 5).

Another example is demonstrated in Chapter 5 (Example 1).

3.2 Multi-risk model with three independent series of claim amounts

In this section we present the multi-risk model with three nonidentically distributed independent series of claim amounts where the claims repeat with time periods of one, two and three units accordingly, that is, claim distributions coincide at times $\{1, 2, 3, \ldots\}$, at times $\{2, 4, 6, \ldots\}$ and at times $\{3, 6, 9, \ldots\}$. claims. We present the recursive formulas to calculate the ultimate ruin probabilities.

The discrete-time risk model with three inhomogeneous claims is obtained from Eq. (2.1) with $K = 3$. The more accurate definition is presented below.

Definition 3.1. *We say that the insurer's surplus $W_u(n)$ follows the three claims risk model if*

$$W_u(n) = u + n - \sum_{i=1}^{n} X_i - \sum_{j=1}^{\lfloor n/2 \rfloor} Y_j - \sum_{k=1}^{\lfloor n/3 \rfloor} Z_k \tag{3.19}$$

for each $n \in \mathbb{N}_0$ and the following assumptions hold:

- *the initial insurer's surplus* $u \in \mathbb{N}_0$*,*
- $X_1, X_2, \ldots$ *are independent copies of nonnegative integer-valued r.v.* X*,*
  $Y_1, Y_2, \ldots$ *are independent copies of nonnegative integer-valued r.v.* Y*,*
  $Z_1, Z_2, \ldots$ *are independent copies of nonnegative integer-valued r.v.* Z*.*

Let us define p.m.f.'s and p.d.f.'s by the following equalities

$$a_k = \mathbb{P}(X_1 = k),\ b_k = \mathbb{P}(Y_1 = k),\ c_k = \mathbb{P}(Z_1 = k),\ k \in \mathbb{N}_0,$$
$$A(x) = \sum_{k=0}^{\lfloor x \rfloor} a_k,\ B(x) = \sum_{k=0}^{\lfloor x \rfloor} b_k,\ C(x) = \sum_{k=0}^{\lfloor x \rfloor} c_k,\ x \geqslant 0.$$

Let

$$\mathbb{P}(X_1 + \ldots + X_i + Y_1 + \ldots Y_j + Z_1 + \ldots + Z_k = m) = a^i b^j c^k(m),$$

and

$$\mathbb{P}(X_1 + \ldots + X_i + Y_1 + \ldots Y_j + Z_1 + \ldots + Z_k \leqslant m) = A^i B^j C^k(m),$$

where $i = \overline{0,6}$, $j = \overline{0,3}$, $k = \overline{0,2}$ and $m = 0, 1, 2, \ldots$

- If any of i, j, k equals zero, then we do not have such random variable in convolution.
- If any of i, j, k equals one, we include the corresponding r. v. in convolution only one time.
- $\overline{A^i B^j C^k}(m) = 1 - A^i B^j C^k(m)$.

By the law of total probability we have, that survival probability

$$\begin{aligned}\varphi(u) &= \mathbb{P}\Big(\bigcap_{n=1}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}\Big) \qquad (3.20)\\
&= \mathbb{P}\Big(\bigcap_{n=2}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}\Big)\\
&\qquad - \mathbb{P}\Big(\bigcap_{n=2}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1\geqslant u+1\Big)\\
&= \mathbb{P}\Big(\bigcap_{n=3}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}\Big)\\
&\qquad - \mathbb{P}\Big(\bigcap_{n=3}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+X_2+Y_1\geqslant u+2\Big)\\
&\qquad - \mathbb{P}\Big(\bigcap_{n=3}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1\geqslant u+1\Big)\\
&\quad + \mathbb{P}\Big(\bigcap_{n=3}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+X_2+Y_1\geqslant u+2, X_1\geqslant u+1\Big).\end{aligned}$$

Continuing in the same way we obtain:

$$\begin{aligned}\varphi(u) &= \mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}\Big)\\
&- \mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2\\
&\qquad\qquad + Z_1\geqslant u+5\Big)\\
&- \mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2\\
&\qquad\qquad + Z_1\geqslant u+4\Big)\end{aligned}$$

$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_4+Y_1+Y_2 + Z_1\geqslant u+4, X_1+\ldots+X_5+Y_1+Y_2+Z_1\geqslant u+5\Big)$$

$$-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+X_2+X_3+Y_1+Z_1\geqslant u+3\Big)$$

$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2 + Z_1\geqslant u+5, X_1+X_2+X_3+Y_1+Z_1\geqslant u+3\Big)$$

$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_4+Y_1+Y_2 + Z_1\geqslant u+4, X_1+X_2+X_3+Y_1+Z_1\geqslant u+3\Big)$$

$$-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2 + Z_1\geqslant u+5, X_1+\ldots+X_4+Y_1+Y_2+Z_1\geqslant u+4, X_1+X_2 + X_3+Y_1+Z_1\geqslant u+3\Big)$$

$$-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+X_2+Y_1\geqslant u+2\Big)$$

$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2 + Z_1\geqslant u+5, X_1+X_2+Y_1\geqslant u+2\Big)$$

$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_4+Y_1+Y_2 + Z_1\geqslant u+4, X_1+X_2+Y_1\geqslant u+2\Big)$$

$$-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2 + Z_1\geqslant u+5, X_1+\ldots+X_4+Y_1+Y_2+Z_1\geqslant u+4, X_1+X_2 + Y_1\geqslant u+2\Big)$$

$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+X_2+X_3+Y_1 + Z_1\geqslant u+3, X_1+X_2+Y_1\geqslant u+2\Big)$$

$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2$$
$$+Z_1\geqslant u+5, X_1+\ldots+X_4+Y_1+Y_2+Z_1\geqslant u+4, X_1+X_2$$
$$+X_3+Y_1+Z_1\geqslant u+3, X_1+X_2+Y_1\geqslant u+2\Big)$$
$$-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1\geqslant u+1\Big)$$
$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2$$
$$+Z_1\geqslant u+5, X_1\geqslant u+1\Big)$$
$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_4+Y_1+Y_2$$
$$+Z_1\geqslant u+4, X_1\geqslant u+1\Big)$$
$$-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2$$
$$+Z_1\geqslant u+5, X_1+\ldots+X_4+Y_1+Y_2+Z_1\geqslant u+4, X_1\geqslant u+1\Big)$$
$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+X_2+X_3+Y_1$$
$$+Z_1\geqslant u+3, X_1\geqslant u+1\Big)$$
$$-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2$$
$$+Z_1\geqslant u+5, X_1+X_2+X_3+Y_1+Z_1\geqslant u+3, X_1\geqslant u+1\Big)$$
$$-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_4+Y_1+Y_2$$
$$+Z_1\geqslant u+4, X_1+X_2+X_3+Y_1+Z_1\geqslant u+3, X_1\geqslant u+1\Big)$$
$$+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2$$
$$+Z_1\geqslant u+5, X_1+\ldots+X_4+Y_1+Y_2+Z_1\geqslant u+4, X_1+X_2$$
$$+X_3+Y_1+Z_1\geqslant u+3, X_1\geqslant u+1\Big)$$

$$
\begin{aligned}
&+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+X_2\\
&\qquad +Y_1\geqslant u+2, X_1\geqslant u+1\Big)\\
&-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2\\
&\qquad +Z_1\geqslant u+5, X_1+X_2+Y_1\geqslant u+2, X_1\geqslant u+1\Big)\\
&-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_4+Y_1+Y_2\\
&\qquad +Z_1\geqslant u+4, X_1+X_2+Y_1\geqslant u+2, X_1\geqslant u+1\Big)\\
&+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2\\
&\qquad +Z_1\geqslant u+5, X_1+\ldots+X_4+Y_1+Y_2+Z_1\geqslant u+4, X_1+X_2\\
&\qquad +Y_1\geqslant u+2, X_1\geqslant u+1\Big)\\
&-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+X_2+X_3+Y_1\\
&\qquad +Z_1\geqslant u+3, X_1+X_2+Y_1\geqslant u+2, X_1\geqslant u+1\Big)\\
&+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2\\
&\qquad +Z_1\geqslant u+5, X_1+X_2+X_3+Y_1+Z_1\geqslant u+3, X_1+X_2\\
&\qquad +Y_1\geqslant u+2, X_1\geqslant u+1\Big)\\
&+\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_4+Y_1+Y_2\\
&\qquad +Z_1\geqslant u+4, X_1+X_2+X_3+Y_1+Z_1\geqslant u+3, X_1+X_2\\
&\qquad +Y_1\geqslant u+2, X_1\geqslant u+1\Big)\\
&-\mathbb{P}\Big(\bigcap_{n=6}^{\infty}\Big\{u+n-\sum_{i=1}^{n}X_i-\sum_{j=1}^{\lfloor n/2\rfloor}Y_j-\sum_{k=1}^{\lfloor n/3\rfloor}Z_k>0\Big\}, X_1+\ldots+X_5+Y_1+Y_2\\
&\qquad +Z_1\geqslant u+5, X_1+\ldots+X_4+Y_1+Y_2+Z_1\geqslant u+4, X_1+X_2+X_3+Y_1\\
&\qquad +Z_1\geqslant u+3, X_1+X_2+Y_1\geqslant u+2, X_1\geqslant u+1\Big)
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{k=0}^{u+5} a^6b^3c^2(u+5-k)\varphi(k+1) - \sum_{j=u+1}^{u+5} a_j \sum_{k=0}^{u+5-j} a^5b^3c^2(u+5-j-k)\varphi(k+1) \\
&- \sum_{j=0}^{u} a_j \sum_{i_1=u+1-j}^{u+5-j} ab(i_1) \sum_{k=0}^{u+5-j-i_1} a^4b^2c^2(u+5-j-i_1-k)\varphi(k+1) \\
&- \sum_{j=0}^{u} a_j \sum_{i_1=0}^{u+1-j} ab(i_1) \sum_{i_2=u+3-j-i_1}^{u+5-j-i_1} ac(i_2) \\
&\quad \times \sum_{k=0}^{u+5-j-i_1-i_2} a^3b^2c^2(u+5-j-i_1-i_2-k)\varphi(k+1) \\
&- \sum_{j=0}^{u} a_j \sum_{i_1=0}^{u+1-j} ab(i_1) \sum_{i_2=0}^{u+2-j-i_1} ac(i_2) \sum_{i_3=u+4-j-i_1-i_2}^{u+5-j-i_1-i_2} ab(i_3) \\
&\quad \times \sum_{k=0}^{u+5-j-i_1-i_2-i_3} a^2bc(u+5-j-i_1-i_2-i_3-k)\varphi(k+1) \\
&- \sum_{j=0}^{u} a_j \sum_{i_1=0}^{u+1-j} ab(i_1) \sum_{i_2=0}^{u+2-j-i_1} ac(i_2) \sum_{i_3=0}^{u+3-j-i_1-i_2} ab(i_3) \sum_{i_4=u+5-j-i_1-i_2-i_3}^{u+5-j-i_1-i_2-i_3} a_{i_4} \\
&\quad \times \sum_{k=0}^{u+5-j-i_1-i_2-i_3} abc(u+5-j-i_1-i_2-i_3-i_4-k)\varphi(k+1).
\end{aligned}
$$

From this equality we see that the values of $\varphi(u+6)$ are recursively related to $\varphi(0)$, $\varphi(1)$,..., $\varphi(u+5)$ for $u \geqslant 0$.

Net profit condition

Our first result describes the meaning of the net profit condition in the discrete-time risk model with three inhomogeneous claims.

Theorem 3.4. *Let us consider a three claims model generated by independent r.v.'s. X, Y and Z for finite means $\mathbb{E}X$, $\mathbb{E}Y$ and $\mathbb{E}Z$. If $\mathbb{E}X+\mathbb{E}Y/2+\mathbb{E}Z/3>1$, then $\psi(u)=1$ for each initial surplus $u\in\mathbb{N}_0$. If $\mathbb{E}X+\mathbb{E}Y/2+\mathbb{E}Z/3=1$, then we have the following possible cases:*

- *$\psi(0)=1$ and $\psi(u)=0$ for all $u\in\mathbb{N}$ if $\{a_1=b_0=c_0=1\}$ or $\{a_0=b_2=c_0=1\}$ or $\{a_0=b_0=c_3=1\}$;*
- *$\psi(u)=1$ for all $u\in\mathbb{N}_0=\{0,1,2,\ldots\}$ if $a^6b^3c^2(6)<1$.*

Proof. We rewrite the equality (3.20) in a following form

$$
\begin{aligned}
&\varphi(u) \qquad (3.21)\\
&=\sum_{k=0}^{u+5} a^6b^3c^2(u+5-k)\varphi(k+1)\\
&-\sum_{k=0}^{4}\varphi(k+1)\sum_{j=1}^{5-k} a_{u+j}a^5b^3c^2(5-k-j)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{k=0}^{3}\varphi(k+1)\sum_{j=2}^{5-k}\sum_{i_1\leqslant u} a_{i_1}ab(u+j-i_1)a^4b^2c^2(5-k-j)\\
&-\sum_{k=0}^{2}\varphi(k+1)\sum_{j=3}^{5-k}\sum_{\substack{i_1\leqslant u\\ i_2\leqslant u+1-i_1}} a_{i_1}ab(i_2)ac(u+j-i_1-i_2)a^3b^2c(5-k-j)\\
&-\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\sum_{\substack{i_1\leqslant u\\ i_2\leqslant u+1-i_1\\ i_3\leqslant u+2-i_1-i_2}} a_{i_1}ab(i_2)ac(i_3)ab(u+j-i_1-i_2-i_3)a^2bc(5-k-j)\\
&-\varphi(1)abc(0)\sum_{\substack{i_1\leqslant u\\ i_2\leqslant u+1-i_1\\ i_3\leqslant u+2-i_1-i_2\\ i_4\leqslant u+3-i_1-i_2-i_3}} a_{i_1}ab(i_2)ac(i_3)ab(i_4)a_{u+5-i_1-i_2-i_3-i_4}\\
&:= S_0(u)-S_1(u)-S_2(u)-S_3(u)-S_4(u)-S_5(u).
\end{aligned}
$$

Therefore, for $v\in\mathbb{N}_0$ we have

$$
\sum_{u=0}^{v}\varphi(u)=\sum_{u=0}^{v}\sum_{k=0}^{6}S_k(u). \qquad (3.22)
$$

So, for the sum $S_0(u)$ we obtain

$$\begin{aligned}
\sum_{u=0}^{v} S_0(u) &= \sum_{k=0}^{4}\sum_{u=0}^{v} a^6b^3c^2(u+5-k)\varphi(k+1) \\
&\quad + \sum_{k=5}^{v+5}\sum_{u=k-5}^{v} a^6b^3c^2(u+5-k)\varphi(k+1) \\
&= \sum_{k=0}^{4} \varphi(k+1)\{A^6B^3C^2(v+5-k) - A^6B^3C^2(4-k)\} \\
&\quad + \sum_{k=5}^{v+5} \varphi(k+1)A^6B^3C^2(v+5-k) \\
&= \sum_{k=1}^{5} \varphi(k)\{A^6B^3C^2(v+6-k) - A^6B^3C^2(5-k)\} \\
&\quad - \sum_{k=0}^{5} \varphi(k)A^6B^3C^2(v+6-k) + \sum_{k=0}^{v+6} \varphi(k)A^6B^3C^2(v+6-k) := \tilde{S}_0.
\end{aligned}$$

For the sum $S_1(u)$ we obtain

$$\begin{aligned}
\sum_{u=0}^{v} S_1(u) &= \sum_{k=0}^{4} \varphi(k+1) \sum_{j=0}^{5-k} a^5b^3c^2(5-k-j) \sum_{u=0}^{v} a_{u+j} \\
&= \sum_{k=0}^{4} \varphi(k+1) \sum_{j=0}^{5-k} a^5b^3c^2(5-k-j)A(v+j) := \tilde{S}_1.
\end{aligned}$$

For the sum $S_2(u)$ we obtain

$$\begin{aligned}
\sum_{u=0}^{v} S_2(u) &= \sum_{k=0}^{3} \varphi(k+1) \sum_{u=0}^{v} \sum_{i_1=0}^{u} \sum_{j=2-i_1}^{5-i_1-k} a_{i_1} ab(u+j)\ a^4b^2c^2(5-i_1-j-k) \\
&= \sum_{k=0}^{3} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{j=2-i_1}^{5-i_1-k} \sum_{u=i_1}^{v} ab(u+j)\ a^4b^2c^2(5-i_1-j-k) \\
&= \sum_{k=0}^{3} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{j=2-i_1}^{5-i_1-k} \{AB(j+v) - AB(j-1)\} a^4b^2c^2(5-i_1-j-k) \\
&= \sum_{k=0}^{3} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{j=2}^{5-k} \{AB(v+j-i_1) - AB(j-1)\} a^4b^2c^2(5-k-j) \\
&= \sum_{k=0}^{3} \varphi(k+1) \sum_{j=2}^{5-k} a^4b^2c^2(5-k-j) \sum_{i_1=0}^{v} a_{i_1} \{AB(v+j-i_1) - AB(j-1)\} := \tilde{S}_2.
\end{aligned}$$

For the sum $S_3(u)$ we obtain

$$\begin{aligned}
\sum_{u=0}^{v} S_3(u) &= \sum_{k=0}^{2} \varphi(k+1) \sum_{u=0}^{v} \sum_{i_1=0}^{u} a_{i_1} \sum_{i_2=0}^{u+1-i_1} ab(i_2) \sum_{j=3}^{5-k} ac(u+j-i_1-i_2) a^3b^2c(5-k-j) \\
&= \sum_{k=0}^{2} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{u=i_1}^{v} \sum_{i_2=0}^{u+1-i_1} ab(i_2) \sum_{j=3}^{5-k} ac(u+j-i_1-i_2) a^3b^2c(5-k-j) \\
&= ab(0) \sum_{k=0}^{2} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{u=i_1}^{v} \sum_{j=3}^{5-k} ac(j+u-i_1) a^3b^2c(5-k-j) \\
&\quad + \sum_{k=0}^{2} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{i_2=1}^{v+1-i_1} ab(i_2) \sum_{u=i_2+i_1-1}^{v} \sum_{j=5}^{5-k} ac(j+u-i_1 \\
&\qquad - i_2) a^3b^2c(5-k-j) \\
&= ab(0) \sum_{k=0}^{2} \varphi(k+1) \sum_{j=3}^{5-k} \sum_{i_1=0}^{v} a_{i_1} \{AC(v+j-i_1) \\
&\quad - AC(j-1)\} a^3b^2c(5-k-j) \\
&\quad + \sum_{k=0}^{2} \varphi(k+1) \sum_{j=3}^{5-k} \sum_{i_1=0}^{v} \sum_{i_2=1}^{v+1-i_1} a_{i_1} ab(i_2) \{AC(v+j-i_1-i_2) \\
&\qquad - AC(j-2)\} a^3b^2c(5-k-j) := \tilde{S}_3.
\end{aligned}$$

For the sum $S_4(u)$ we obtain

$$\sum_{u=0}^{v} S_4(u) = \sum_{k=0}^{1} \varphi(k+1) \sum_{u=0}^{v} \sum_{i_1=0}^{u} \sum_{i_2=0}^{u+1-i_1} \sum_{i_3=0}^{u+2-i_1-i_2} \sum_{j=4}^{5-k} a_{i_1} ab(i_2) ac(i_3) ab(u+j-i_1-i_2-i_3) \times a^2bc(5-k-j)$$

$$= \sum_{k=0}^{1} \varphi(k+1) \sum_{i_1=0}^{v} \sum_{u=i_1}^{v} \sum_{i_2=0}^{u+1-i_1} \sum_{i_3=0}^{u+2-i_1-i_2} \sum_{j=4}^{5-k} a_{i_1} ab(i_2) ac(i_3) ab(u+j-i_1-i_2-i_3) \times a^2bc(5-k-j)$$

$$= ab(0) \sum_{k=0}^{1} \varphi(k+1) \sum_{i_1=0}^{v} \sum_{u=i_1}^{v} \sum_{i_3=0}^{u+2-i_1} \sum_{j=0}^{u+2-i_1-i_2} \sum_{j=4}^{5-k} a_{i_1} ac(i_3) ab(u+j-i_1-i_3) \times a^2bc(5-k-j)$$

$$+ \sum_{k=0}^{1} \varphi(k+1) \sum_{i_1=0}^{v} \sum_{i_2=1}^{v+1-i_1} \sum_{u=i_1+i_2-1}^{v} \sum_{i_3=0}^{u+2-i_1-i_2} \sum_{j=4}^{5-k} a_{i_1} ab(i_2) ac(i_3) ab(u+j-i_1-i_3) \times a^2bc(5-k-j)$$

$$= ab(0)ac(0) \sum_{k=0}^{1} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{j=4}^{5-k} \sum_{u=i_1}^{v} ab(u+j-i_1) a^2bc(5-k-j)$$

$$+ ab(0)ac(1) \sum_{k=0}^{1} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{j=4}^{5-k} \sum_{u=i_1}^{v} ab(u+j-1-i_1) a^2bc(5-k-j)$$

$$+ ac(0) \sum_{k=0}^{1} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{i_2=1}^{v+1-i_1} \sum_{j=4}^{5-k} \sum_{u=i_1+i_2-1}^{v} ab(i_2) ab(u+j-1-i_1-i_2) \times a^2bc(5-k-j)$$

$$+ \sum_{k=0}^{1} \varphi(k+1) \sum_{i_1=0}^{v} \sum_{i_2=1}^{v+1-i_1} \sum_{i_3=1}^{v+2-i_1-i_2} \sum_{u=i_1+i_2+i_3-1}^{v} \sum_{j=4}^{5-k} a_{i_1} ab(i_2) ab(u+j-1-i_1-i_2) ac(i_3) \times a^2bc(5-k-j)$$

$$= a^2bc(0) \sum_{k=0}^{1} \varphi(k+1) \sum_{j=4}^{5-k} \sum_{i_1=0}^{v} a_{i_1} (AB(v+j-i_1) - AB(j-1)) \times a^2bc(5-k-j)$$

$$+ ab(0)ac(1) \sum_{k=0}^{1} \varphi(k+1) \sum_{j=4}^{5-k} \sum_{i_1=0}^{v} a_{i_1} (AB(v-1+j-i_1) - AB(j-2)) \times a^2bc(5-k-j)$$

$$+ ab(0) \sum_{k=0}^{1} \varphi(k+1) \sum_{j=4}^{5-k} \sum_{i_1=0}^{v} a_{i_1} \sum_{i_3=2}^{v+2-i_1} ac(i_3) (AB(v+j-i_1-i_3) - AB(j-3)) \times a^2bc(5-k-j)$$

$$+ ab(0) \sum_{k=0}^{1} \varphi(k+1) \sum_{j=4}^{5-k} \sum_{i_1=0}^{v} a_{i_1} \sum_{i_2=1}^{v+1-i_1} ab(i_2) (AB(v+j-i_1-i_2) - AB(j-2))\ a^2bc(5-k-j)$$

$$+ \sum_{k=0}^{1} \varphi(k+1) \sum_{j=4}^{5-k} \sum_{i_1=0}^{v} a_{i_1} \sum_{i_2=1}^{v+1-i_1} ab(i_2) \sum_{i_3=1}^{v+2-i_1-i_2} ac(i_3) (AB(v+j-i_1-i_2-i_3) - AB(j-3))\ a^2bc(5-k-j) := \tilde{S}_4.$$

We rewrite sum $S_5(u)$ in following form

$$\begin{aligned}
S_5(u) =& abc(0)\varphi(1)\sum_{i_1=0}^{u+3}\sum_{i_2=0}^{u+3-i_1}\sum_{i_3=0}^{u+3-i_1-i_2-i_3} a_{i_1}ab(i_2)ac(i_3)ab(i_4)a_{u+5-i_1-i_2-i_3-i_4} \\
&- a^2b^2c(0)\varphi(1)a_2\sum_{i_1=0}^{u+1}\sum_{i_2=0}^{u+1-i_1} a_{i_1}ab(i_2)ac(u+3-i_1-i_2) \\
&\qquad + a^2b^2c(0)\varphi(1)a_2ab(0)ac(2)a_{u+1} \\
&- abc(0)\varphi(1)\sum_{i_1=0}^{u}\sum_{i_2=u+2-i_1}^{u+3-i_1}\sum_{i_3=0}^{u+3-i_1-i_2}\sum_{i_4=0}^{u+3-i_1-i_2-i_3} a_{i_1}ab(i_2)ac(i_3) \\
&\qquad \times ab(i_4)a_{u+5-i_1-i_2-i_3-i_4} \\
&- abc(0)\varphi(1)\sum_{i_1=u+1}^{u+3}\sum_{i_2=0}^{u+3-i_1}\sum_{i_3=0}^{u+3-i_1-i_2}\sum_{i_4=0}^{u+3-i_1-i_2-i_3} a_{i_1}\ ab(i_2)ac(i_3) \\
&\qquad \times ab(i_4)a_{u+5-i_1-i_2-i_3-i_4} \\
=& abc(0)\varphi(1)\sum_{k=0}^{u+3} a^4b^2c(k)a_{u+5-k} - a^2b^2c(0)\varphi(1)a_2\sum_{k=0}^{u+1} a^2b(k)ac(u+3-k) \\
&+ a^3b^3c(0)a_2ac(2)\varphi(1)a_{u+1} \\
&- abc(0)\varphi(1)\sum_{i_1=0}^{u} a_{i_1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1-i_2}\sum_{i_4=0}^{1-i_2-i_3} ab(i_2)ac(i_3)ab(i_4)a_{3-i_2-i_3-i_4} \\
&- abc(0)\varphi(1)\sum_{i_1=0}^{2} a_{i_1}\sum_{i_2=0}^{2-i_1}\sum_{i_3=0}^{2-i_1-i_2}\sum_{i_4=0}^{2-i_1-i_2-i_3} ab(i_2)ac(i_3) \\
&\qquad \times ab(i_4)a_{4-i_1-i_2-i_3-i_4} \\
=& abc(0)\varphi(1)\sum_{k=0}^{u+3} a^4b^2c(k)a_{u+5-k} - a^2b^2c(0)\varphi(1)a_2\sum_{k=0}^{u+1} a^2b(k)\ ac(u+3-k) \\
&+ a^3b^3c(0)a_2ac(2)\varphi(1)a_{u+1} \\
&- abc(0)\varphi(1)\sum_{i_1=0}^{u} a_{i_1}\sum_{i_2=0}^{1} ab(u+2+i_2-i_1)\sum_{k=0}^{1-i_2} a^2bc(k)a_{3-i_2-k} \\
&- abc(0)\varphi(1)\sum_{i_1=0}^{2} a_{u+1+i_1}\sum_{k=0}^{2-i_1} a^3b^2c(k)a_{4-i_1-k}.
\end{aligned}$$

Then,

$$\sum_{u=0}^{v} S_5(u) = abc(0)\varphi(1) \sum_{u=0}^{v} \sum_{k=0}^{u+3} a_{u+5-k} a^4 b^2 c(k) - a^2 b^2 c(0)\varphi(1) a_2 \sum_{u=0}^{v} \sum_{k=0}^{u+1} ac(u+3-k) a^2 b(k) + a^3 b^3 c(0) a_1 ac(2)\varphi(1)(A(v+1) - A(0))$$

$$
\begin{aligned}
&\quad - abc(0)\varphi(1)\sum_{u=0}^{v}\sum_{i_1=0}^{u}\sum_{i_2=0}^{1} a_{i_1} ab(u+2+i_2-i_1)\sum_{k=0}^{1-i_2} a^2bc(k)a_{3-i_2-k} \\
&\quad - abc(0)\varphi(1)\sum_{u=0}^{v}\sum_{i_1=0}^{2} a_{u+1+i_1}\sum_{k=0}^{2-i_1} a^3b^2c(k)a_{4-i_1-k} \\
&= abc(0)\varphi(1)\sum_{k=0}^{2}\sum_{u=0}^{v} a_{u+5-k}a^4b^2c(k) \\
&\quad + abc(0)\varphi(1)\sum_{k=0}^{v+3}\sum_{u=k-3}^{v} a_{u+5-k}a^4b^2c(k) \\
&\quad - a^2b^2c(0)\varphi(1)a_2\sum_{u=0}^{v} ac(u+3)a^2b(0) \\
&\quad - a^2bc(0)\varphi(1)a_2\sum_{k=1}^{v+1}\sum_{u=k-1}^{v} ac(u+3-k)a^2b(k) \\
&\quad + a^3b^3c(0)a_2\ ac(2)\varphi(1)(A(v+1)-A(0)) \\
&\quad - abc(0)\varphi(1)\sum_{i_1=0}^{v} a_{i_1}\sum_{i_2=0}^{1}\sum_{u=i_1}^{v} ab(u+2+i_2-i_1)\sum_{k=0}^{1-i_2} a^2bc(k)a_{3-i_2-k} \\
&\quad - abc(0)\varphi(1)\sum_{i_1=0}^{2}\sum_{u=0}^{v} a_{u+1+i_1}\sum_{k=0}^{2-i_1} a^3b^2c(k)a_{4-i_1-k} \\
&= abc(0)\varphi(1)\sum_{k=0}^{2}(A(v+5-k)-A(4-k))a^4b^2c_k \\
&\quad + abc(0)\varphi(1)\sum_{k=0}^{v+3}(A(v+5-k)-A(1))a^4b^2c(k) \\
&\quad - a^4b^3c(0)\varphi(1)a_2(AC(v+3)-AC(2)) \\
&\quad - a^2b^2c(0)\varphi(1)a_2\sum_{k=1}^{v+1}(AC(v+3-k)-AC(1))a^2b(k) \\
&\quad + a^3b^3c(0)a_2ac(2)\varphi(1)(A(v+1)-A(0)) \\
&\quad - abc(0)\varphi(1)\sum_{i_2=0}^{1}\sum_{k=0}^{1-i_2} a^2bc(k)a_{3-i_2-k}\sum_{i_1=0}^{v} a_{i_1}(AB(v+2+i_2-i_1) \\
&\quad - AB(i_2+1)) - abc(0)\varphi(1)\sum_{i_1=0}^{2}(A(v+1+i_1) \\
&\qquad\qquad - A(i_1))\sum_{k=0}^{2-i_1} a^3b^2c(k)a_{4-i_1-k} := \tilde{S}_5.
\end{aligned}
$$

Substituting $\tilde{S}_0$, $\tilde{S}_1$, $\tilde{S}_2$, ..., $\tilde{S}_5$ into (3.22) we get

$$\sum_{k=0}^{v+6}\varphi(k)\overline{A^6B^3C^2}(v+6-k)-\sum_{k=v+1}^{v+6}\varphi(k) \tag{3.23}$$
$$=\sum_{k=1}^{5}\varphi(k)\{A^6B^3C^2(v+6-k)-A^6B^3C^2(5-k)\}$$

$$-\sum_{k=0}^{5}\varphi(k)A^6B^3C^2(v+6-k)-\sum_{k=1}^{5}\tilde{S}_k.$$

Sequence $\varphi(u), u\in\mathbb{N}_0$, is bounded and non decreasing. So, there exists a limit $\lim\limits_{u\to\infty}\varphi(u):=\varphi(\infty)$. For all $1\leqslant N<v+6$ we have

$$S(v)=\sum_{k=0}^{v+6}\varphi(k)\overline{A^6B^3C^2}(v+6-k)=\sum_{k=0}^{N}\varphi(k)\overline{A^6B^3C^2}(v+6-k)$$
$$+\sum_{k=N+1}^{v+6}\varphi(k)\overline{A^6B^3C^2}(v+6-k)=S_{1N}(v)+S_{2N}(v).$$

$$0\leqslant S_{1N}(v)\leqslant\sum_{k=0}^{v+6}\overline{A^6B^3C^2}(v+6-k)\leqslant\sum_{l=v+6-N}^{\infty}\overline{A^6B^3C^2}(l).$$

$$S_{2N}(v)\leqslant\varphi(\infty)\sum_{k=N+1}^{v+6}\overline{A^6B^3C^2}(v+6-k)\leqslant\varphi(\infty)\sum_{l=0}^{v+6-N-1}\overline{A^6B^3C^2}(l).$$

$$S_{2N}(v)\geqslant\varphi(N+1)\sum_{k=N+1}^{v+6}\overline{A^6B^3C^2}(v+6-k)=\varphi(N+1)\sum_{l=0}^{v+6-N-1}\overline{A^6B^3C^2}(l).$$

Thus,

$$\begin{aligned}\limsup_{v\to\infty} S(v) &\leqslant \limsup_{v\to\infty} S_{1N}(v) + \limsup_{v\to\infty} S_{2N}(v) \qquad (3.24)\\ &\leqslant \limsup_{v\to\infty} \sum_{l=v+6-N}^{\infty} \overline{A^6B^3C^2}(l) + \varphi(\infty) \sum_{l=0}^{v+6-N-1} \overline{A^6B^3C^2}(l)\\ &= 0 + \varphi(\infty)\mathbb{E}(6X+3Y+2Z), \qquad (3.25)\end{aligned}$$

where $\mathbb{E}X$, $\mathbb{E}Y$ and $\mathbb{E}Z$ are finite means.

From the other side, for all $1 \leqslant N < v+6$, we have

$$\begin{aligned}\liminf_{v\to\infty} S(v) &\geqslant \liminf_{v\to\infty} S_{1N}(v) + \liminf_{v\to\infty} S_{2N}(v)\\ &\geqslant \varphi(N+1) \liminf_{v\to\infty} \sum_{l=0}^{v+6-N-1} \overline{A^6B^3C^2}(l) = \varphi(N+1)\mathbb{E}(6X+3Y+2Z).\end{aligned}$$

Thus,

$$\liminf_{v\to\infty} S(v) \geqslant \varphi(\infty)\mathbb{E}(6X+3Y+2Z). \qquad (3.26)$$

From (3.24) and (3.26), we get

$$\begin{aligned}&\lim_{v\to\infty} \sum_{k=0}^{v+6} \varphi(k)\overline{A^6B^3C^2}(v+6-k)\\ &\quad = \varphi(\infty) \lim_{v\to\infty} \sum_{k=0}^{v+6} \overline{A^6B^3C^2}(v+6-k)\\ &\quad = \varphi(\infty)(6\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z). \qquad (3.27)\end{aligned}$$

As $v \to \infty$, the relations (3.23) and (3.27) imply that

$$\begin{aligned}&\varphi(\infty)(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z) \qquad (3.28)\\ &\quad = \varphi(0) + \hat{z}_1\varphi(1) + \hat{z}_2\varphi(2) + \hat{z}_3\varphi(3) + \hat{z}_4\varphi(4) + \hat{z}_5\varphi(5),\end{aligned}$$

where

$$\hat{z}_5 = a^5b^3c^2(0),$$
$$\hat{z}_4 = A^6B^3C^2(1) + \overline{A}(0)\ a^6b^3c^2(1) + \overline{A}(1)\ a^6b^3c^2(0) + \overline{AB}(1)\ a^4b^2c^2(0),$$
$$\hat{z}_3 = A^6B^3C^2(2) + \sum_{i_1=1}^{3} \overline{A}(i_1-1)\ a^5b^3c^2(3-i_1) + \sum_{i_2=2}^{3} \overline{AB}(i_2-1)\ a^4b^2c^2(3-i_2) + a^3b^2c(0)\ \{\overline{AC}(2) + \overline{AB}(0)\ ac(2)\},$$
$$\hat{z}_2 = A^6B^3C^2(3) + \sum_{i_1=1}^{4} \overline{A}(i_1-1)\ a^5b^3c^2(4-i_1) + \sum_{i_2=2}^{4} \overline{AB}(i_2-1)\ a^4b^2c^2(4-i_2) + \sum_{i_3=3}^{4} a^3b^2c(4-i_3)\{\overline{AC}(i_3-1) + \overline{AB}(0)\ ac(i_3-1)\},$$
$$\hat{z}_1 = A^6B^3C^2(4) + \sum_{i_1=1}^{5} A(i_1-1)\ a^5b^3c^2(5-i_1) + \sum_{i_2=2}^{5} \overline{AB}(i_2-1)\ a^4b^2c^2(5-i_2) + \sum_{i_3=3}^{5} a^3b^2c(5-i_3)\{\overline{AC}(i_3-1) + \overline{AB}(0)\ ac(i_3-1)\}$$

$$+ a^2bc(0) \sum_{i_4=4}^{5} \overline{AB}(i_4-1)\ a^2bc(5-i_4)$$
$$+ ab(0)\ ac(1) \sum_{i_4=4}^{5} \overline{AB}(i_4-2)\ a^2bc(5-i_4)$$
$$+ ab(0)\ \overline{AC}(1) \sum_{i_4=4}^{5} \overline{AB}(i_4-3)\ a^2bc(5-i_4)$$
$$+ abc(0) \sum_{k=0}^{2} \overline{A}(4-k)\ a^4b^2c(k) + abc(0)\ \overline{A^2B^2C}(2)\ \overline{A}(1)$$
$$+ a^4b^3c(0)\ a_2\ \overline{AC}(2) + a^2b^2c(0)\ a_2\ \overline{A^2B}(0)\ \overline{AC}(1)$$
$$+ a^3b^3c(0)\ a_2\ ac(2)\ \overline{A}(0) - abc(0) \sum_{i_2=0}^{1} \sum_{k=0}^{1-i_2} abc(k)\ a_{3-i_2-k}\ \overline{AB}(i_2+1)$$
$$- abc(0) \sum_{i_1=0}^{2} \sum_{k=0}^{2-i_1} a^3b^2c(k)\ a_{4-i_1-k}\ \overline{A}(i_1).$$

Now we consider the last equality and examine all possible cases.

(I) If $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 > 1$, then (3.28) implies that $\varphi(\infty) = 0$ because the left side at (3.28) is nonnegative in all cases. So, in this case $\psi(u) = 1$ for all $u \in \{0, 1, 2, \ldots\}$.

(II) If $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 = 1$, then (3.28) implies that

$$\varphi(0) + \hat{z}_1\varphi(1) + \hat{z}_2\varphi(2) + \hat{z}_3\varphi(3) + \hat{z}_4\varphi(4) + \hat{z}_5\varphi(5) = 0.$$

Additionally, in this situation we have that $a^6b^3c^2(6) = 1$ or $a^6b^3c^2(6) < 1$.

(II-A) If $a^6b^3c^2(6) = 1$, then we have

$$\begin{cases} a^6b^3c^2(0) = 0, \\ a^6b^3c^2(1) = 0, \\ a^6b^3c^2(2) = 0, \\ a^6b^3c^2(3) = 0, \\ a^6b^3c^2(4) = 0, \\ a^6b^3c^2(5) = 0, \\ a^6b^3c^2(6) = 1, \\ \varphi(0) + \hat{z}_1\varphi(1) + \hat{z}_2\varphi(2) + \hat{z}_3\varphi(3) + \hat{z}_4\varphi(4) + \hat{z}_5\varphi(5) = 0. \end{cases}$$

Taking into account that all numbers a_k, b_k and c_k are local probabilities for all $k \in \mathbb{N}_0$, the last system implies the following possible cases.

(a) $\{a_1 = b_0 = c_0 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0, u \in \{1, 2, \ldots\}$.

(b) $\{a_0 = b_2 = c_0 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \ldots\}$.

(c) $\{a_0 = b_0 = c_3 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \ldots\}$.

(II-B) If $a^6b^3c^2(6) < 1$ and $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 = 1$ then it is necessary that $a^6b^3c^2(0) \neq 0$ or $a^6b^3c^2(1) \neq 0$ or $a^6b^3c^2(2) \neq 0$ or $a^6b^3c^2(3) \neq 0$ or $a^6b^3c^2(4) \neq 0$

or $a^6b^3c^2(5) \neq 0$. In this situation, it is sufficient to consider the following cases

$$\{a^6b^3c^2(0) \neq 0\}, \quad \{a^6b^3c^2(0) = 0, a^6b^3c^2(1) \neq 0\},$$
$$\{a^6b^3c^2(0) = 0, a^6b^3c^2(1) = 0, a^6b^3c^2(2) \neq 0\},$$
$$\{a^6b^3c^2(0) = 0, a^6b^3c^2(1) = 0, a^6b^3c^2(2) = 0, a^6b^3c^2(3) \neq 0\},$$
$$\{a^6b^3c^2(0) = 0, a^6b^3c^2(1) = 0, a^6b^3c^2(2) = 0, a^6b^3c^2(3) = 0, a^6b^3c^2(4) \neq 0\},$$
$$\{a^6b^3c^2(0) = 0, a^6b^3c^2(1) = 0, a^6b^3c^2(2) = 0, a^6b^3c^2(3) = 0, a^6b^3c^2(4) = 0, a^6b^3c^2(5) \neq 0\}.$$

(d) If $a^6b^3c^2(0) \neq 0$ then (3.28) implies that $\varphi(0) = \varphi(1) = \varphi(2) = \varphi(3) = \varphi(4) = \varphi(5) = 0$, and from (3.20) we obtain $\varphi(u) = 0$ for all $u \in \{6, 7, \ldots\}$. So, $\psi(u) = 1$ if $u \in \{0, 1, \ldots\}$ in the case.

(e) If $a^6b^3c^2(0) = 0$ and $a^6b^3c^2(1) \neq 0$ then (3.28) implies that $\varphi(0) = \varphi(1) = \varphi(2) = \varphi(3) = \varphi(4) = 0$, and from (3.20) we obtain $\varphi(u) = 0$ for all $u \in \{5, 6, \ldots\}$. So, $\psi(u) = 1$ if $u \in \{0, 1, \ldots\}$ in the case.

(f) If $a^6b^3c^2(0) = 0$ and $a^6b^3c^2(1) = 0$ and $a^6b^3c^2(2) \neq 0$ then (3.28) implies that $\varphi(0) = \varphi(1) = \varphi(2) = \varphi(3) = 0$, and from (3.20) we obtain $\varphi(u) = 0$ for all $u \in \{4, 5, \ldots\}$. So, $\psi(u) = 1$ if $u \in \{0, 1, \ldots\}$ in the case.

(g) If $a^6b^3c^2(0) = 0$ and $a^6b^3c^2(1) = 0$ and $a^6b^3c^2(2) = 0$ and $a^6b^3c^2(3) \neq 0$ then (3.28) implies that $\varphi(0) = \varphi(1) = \varphi(2) = 0$, and from (3.20) we obtain $\varphi(u) = 0$ for all $u \in \{3, 4, \ldots\}$. So, $\psi(u) = 1$ if $u \in \{0, 1, \ldots\}$ in the case.

(h) If $a^6b^3c^2(0) = 0$ and $a^6b^3c^2(1) = 0$ and $a^6b^3c^2(2) = 0$ and $a^6b^3c^2(3) = 0$ and $a^6b^3c^2(4) \neq 0$ then (3.28) implies that $\varphi(0) = \varphi(1) = 0$, and from (3.20) we obtain $\varphi(u) = 0$ for all $u \in \{2, 3, \ldots\}$. So, $\psi(u) = 1$ if $u \in \{0, 1, \ldots\}$ in the case.

(i) If $a^6b^3c^2(0) = 0$ and $a^6b^3c^2(1) = 0$ and $a^6b^3c^2(2) = 0$ and $a^6b^3c^2(3) = 0$ and $a^6b^3c^2(4) = 0$ and $a^6b^3c^2(5) \neq 0$ then (3.28) implies that $\varphi(0) = 0$, and from (3.20) we obtain $\varphi(u) = 0$ for all $u \in \{1, 2, \ldots\}$. So, $\psi(u) = 1$ if $u \in \{0, 1, \ldots\}$ in the case.

In all cases, equalities (3.28) and (3.20) imply that $\varphi(u) = 0$ and, so, $\psi(u) = 1$ for all $u \in \mathbb{N}_0$. Theorem 3.4 is proved. □

Recursive formulas

Our second statement proposes a recursive procedure for calculation of the ultimate survival probabilities $\varphi(u) = 1 - \psi(u),\ u \in \mathbb{N}_0$.

Theorem 3.5. *Let us consider a three claims risk model generated by independent* *r.v.'s. X, Y and Z. Suppose that $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 < 1$. Then the following list of statements holds:*

- $\lim\limits_{u\to\infty} \varphi(u) = 1.$
- *If $a_0 \neq 0$, $b_0 \neq 0$ and $c_0 \neq 0$, then*

$$\varphi(n) = \beta_n^0\varphi(0) + \beta_n^1\varphi(1) + \beta_n^2\varphi(2) + \beta_n^3\varphi(3) + \beta_n^4\varphi(4) + \gamma_n(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z),\ n \in N_0 \tag{3.29}$$

where

$$\begin{cases} \beta_0^0 = 1,\ \beta_1^0 = 0,\ \beta_2^0 = 0,\ \beta_3^0 = 0,\ \beta_4^0 = 0,\ \beta_5^0 = -\frac{1}{a^5b^3c^2(0)}, \\ \beta_n^0 = \frac{1}{a^6b^3c^2(0)}\left(\beta_{n-6}^0 - \sum_{k=1}^{n-1} a^6b^3c^2(k)\ \beta_{n-k}^0 - a(n-5)\right), \quad n \geqslant 6; \end{cases}$$

$$\begin{cases} \beta_0^1 = 0,\ \beta_1^1 = 1,\ \beta_2^1 = 0,\ \beta_3^1 = 0,\ \beta_4^1 = 0,\ \beta_5^1 = -\frac{\hat{z}_1}{a^5b^3c^2(0)}, \\ \beta_n^1 = \frac{1}{a^6b^3c^2(0)}\left(\beta_{n-6}^1 - \sum_{k=1}^{n-1} a^6b^3c^2(k)\ \beta_{n-k}^1 + z_1(n-6) - a(n-5)\hat{z}_1\right),\ n \geqslant 6; \end{cases}$$

$$\begin{cases} \beta_0^2 = 0,\ \beta_1^2 = 0,\ \beta_2^2 = 1,\ \beta_3^2 = 0,\ \beta_4^2 = 0,\ \beta_5^2 = -\frac{\hat{z}_2}{a^5b^3c^2(0)}, \\ \beta_n^2 = \frac{1}{a^6b^3c^2(0)}\left(\beta_{n-6}^2 - \sum_{k=1}^{n-1} a^6b^3c^2(k)\ \beta_{n-k}^2 + z_2(n-6) - a(n-5)\hat{z}_2\right),\ n \geqslant 6; \end{cases}$$

$$\begin{cases} \beta_0^3 = 0,\ \beta_1^3 = 0,\ \beta_2^3 = 0,\ \beta_3^3 = 1,\ \beta_4^3 = 0,\ \beta_5^3 = -\frac{\hat{z}_3}{a^5b^3c^2(0)}, \\ \beta_n^3 = \frac{1}{a^6b^3c^2(0)}\left(\beta_{n-6}^3 - \sum_{k=1}^{n-1} a^6b^3c^2(k)\ \beta_{n-k}^3 + z_3(n-6) - a(n-5)\hat{z}_3\right),\ n \geqslant 6; \end{cases}$$

$$\begin{cases} \beta_0^4 = 0,\ \beta_1^4 = 0,\ \beta_2^4 = 0,\ \beta_3^4 = 0,\ \beta_4^4 = 1,\ \beta_5^4 = -\frac{\hat{z}_4}{a^5b^3c^2(0)}, \\ \beta_n^4 = \frac{1}{a^6b^3c^2(0)}\left(\beta_{n-6}^4 - \sum_{k=1}^{n-1} a^6b^3c^2(k)\ \beta_{n-k}^4 + z_4(n-6) - a(n-5)\hat{z}_4\right),\ n \geqslant 6; \end{cases}$$

$$\begin{cases} \gamma_0 = 0,\ \gamma_1 = 0,\ \gamma_2 = 0,\ \gamma_3 = 0,\ \gamma_4 = 0,\ \gamma_5 = \frac{1}{a^5b^3c^2(0)}, \\ \gamma_n = \frac{1}{a^6b^3c^2(0)}\left(\gamma_{n-6} - \sum_{k=1}^{n-1} a^6b^3c^2(k)\ \gamma_{n-k} + a(n-5)\right), \quad n \geqslant 6. \end{cases}$$

Coefficients $z_1(n-6)$, $z_2(n-6)$, $z_3(n-6)$, $z_4(n-6)$ are defined below.

- *If $\{a_0 \neq 0, b_0 = 0, c_0 \neq 0, b_1 \neq 0\}$ then*

$$\varphi(n) = \tilde{\beta}_n^0\varphi(0) + \tilde{\beta}_n^1\varphi(1) + \tilde{\gamma}_n(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z), n \in \mathbb{N}_0, \tag{3.30}$$

where

$$\begin{cases}\tilde{\beta}_0^0=\beta_0^0,\ \tilde{\beta}_1^0=\beta_1^0,\ \tilde{\beta}_2^0=-\frac{1}{a^4b^2c^2(2)},\\ \tilde{\beta}_0^1=\beta_0^1,\ \tilde{\beta}_1^1=\beta_1^1,\ \tilde{\beta}_2^1=-\frac{\hat{z}_1}{a^4b^2c^2(2)},\\ \tilde{\gamma}_0=\gamma_0,\ \tilde{\gamma}_1=\gamma_1,\ \tilde{\gamma}_2=\frac{1}{a^4b^2c^2(2)},\\ \tilde{\beta}_n^0=\frac{1}{a^6b^3c^2(3)}\left(\tilde{\beta}_{n-3}^0-\sum_{k=1}^{n-1}a^6b^3c^2(k+3)\ \tilde{\beta}_{n-k}^0-a(n-2)\right),\ n\geqslant 3,\\ \tilde{\beta}_n^1=\frac{1}{a^6b^3c^2(3)}\left(\tilde{\beta}_{n-3}^1-\sum_{k=1}^{n-1}a^6b^3c^2(k+3)\ \tilde{\beta}_{n-k}^1+z_1(n-3)-a(n-2)\hat{z}_1\right),\ n\geqslant 3,\\ \tilde{\gamma}_n=\frac{1}{a^6b^3c^2(3)}\left(\tilde{\gamma}_{n-3}-\sum_{k=1}^{n-1}a^6b^3c^2(k+3)\ \tilde{\gamma}_{n-k}+a(n-2)\right),\ n\geqslant 3.\end{cases}$$

Coefficient $z_1(n-3)$ *is defined below.*

- *If* $\{a_0\neq 0, b_0\neq 0, c_0=0, c_1\neq 0\}$ *then*

$$\varphi(n)=\breve{\beta}_n^0\varphi(0)+\breve{\beta}_n^1\varphi(1)+\breve{\beta}_n^2\varphi(2)+\breve{\gamma}_n(6-6\mathbb{E}X-3\mathbb{E}Y-2\mathbb{E}Z),\ n\in\mathbb{N}_0,\quad (3.31)$$

where

$$\begin{cases}\breve{\beta}_0^0=\beta_0^0,\ \breve{\beta}_1^0=\beta_1^0,\ \breve{\beta}_2^0=\beta_2^0,\ \breve{\beta}_3^0=-\frac{1}{a^5b^3c^2(2)},\\ \breve{\beta}_0^1=\beta_0^1,\ \breve{\beta}_1^1=\beta_1^1,\ \breve{\beta}_2^1=\beta_2^1,\ \breve{\beta}_3^1=-\frac{\hat{z}_1}{a^5b^3c^2(2)},\\ \breve{\beta}_0^2=\beta_0^2,\ \breve{\beta}_1^2=\beta_1^2,\ \breve{\beta}_2^2=\beta_2^2,\ \breve{\beta}_3^2=-\frac{\hat{z}_2}{a^5b^3c^2(2)},\\ \breve{\gamma}_0=\gamma_0,\ \breve{\gamma}_1=\gamma_1,\ \breve{\gamma}_2=\gamma_2,\ \breve{\gamma}_3=\frac{1}{a^5b^3c^2(2)},\\ \breve{\beta}_n^0=\frac{1}{a^6b^3c^2(2)}\left(\breve{\beta}_{n-4}^0-\sum_{k=1}^{n-1}a^6b^3c^2(k+2)\ \breve{\beta}_{n-k}^0-a(n-3)\right),\ n\geqslant 4,\\ \breve{\beta}_n^1=\frac{1}{a^6b^3c^2(2)}\left(\breve{\beta}_{n-4}^1-\sum_{k=1}^{n-1}a^6b^3c^2(k+2)\ \breve{\beta}_{n-k}^1+z_1(n-4)-a(n-3)\hat{z}_1\right),\ n\geqslant 4,\\ \breve{\beta}_n^2=\frac{1}{a^6b^3c^2(2)}\left(\breve{\beta}_{n-4}^2-\sum_{k=1}^{n-1}a^6b^3c^2(k+2)\ \breve{\beta}_{n-k}^2+z_2(n-4)-a(n-3)\hat{z}_2\right),\ n\geqslant 4,\\ \breve{\gamma}_n=\frac{1}{a^6b^3c^2(2)}\left(\breve{\gamma}_{n-4}-\sum_{k=1}^{n-1}a^6b^3c^2(k+2)\ \breve{\gamma}_{n-k}+a(n-3)\right),\ n\geqslant 4.\end{cases}$$

Coefficients $z_1(n-4)$, $z_2(n-4)$ *are defined below.*

- *If* $\{a_0\neq 0, b_0\neq 0, c_0=c_1=0, c_2\neq 0\}$ *then*

$$\varphi(n)=\hat{\beta}_n^0\varphi(0)+\hat{\gamma}_n(6-6\mathbb{E}X-3\mathbb{E}Y-2\mathbb{E}Z),\ n\in\mathbb{N}_0,\quad (3.32)$$

where

$$\begin{cases} \hat{\beta}_0^0 = \beta_0^0,\ \hat{\beta}_1^0 = -\frac{1}{2a_0^6 b_0^3 c_2^2},\ \hat{\gamma}_0 = \gamma_0,\ \hat{\gamma}_1 = \frac{1}{2a_0^6 b_0^3 c_2^2}, \\ \hat{\beta}_n^0 = \frac{1}{a^6 b^3 c^2(4)} \left(\hat{\beta}_{n-2}^0 - \sum_{k=1}^{n-1} a^6 b^3 c^2 (k+4)\ \hat{\beta}_{n-k}^0 - \frac{a_n}{2a_0} \right),\ n \geqslant 2, \\ \hat{\gamma}_n = \frac{1}{a^6 b^3 c^2(4)} \left(\hat{\gamma}_{n-2} - \sum_{k=1}^{n-1} a^6 b^3 c^2 (k+4)\ \hat{\gamma}_{n-k} + \frac{a_n}{2a_0} \right),\ n \geqslant 2. \end{cases}$$

- *If* $\{a_0 \neq 0, b_0 = 0, c_0 = 0, b_1 \neq 0, c_1 \neq 0\}$ *then*

$$\varphi(0) = 6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z, \tag{3.33}$$
$$\varphi(1) = \frac{\varphi(0)}{a^6 b^3 c^2(5)},$$
$$\varphi(n) = \frac{1}{a^6 b^3 c^2(5)} \Big(\varphi(n-1) - \sum_{k=1}^{u-1} a^6 b^3 c^2 (n+5-k) \varphi(k) \Big),\ n \geqslant 2.$$

Proof. Let we consider the case $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 < 1$. First we prove that $\varphi(\infty) = 1$ or, equivalently, $\psi(\infty) = 0$. Let

$$S_n := \sum_{i=1}^{n} X_i + \sum_{j=1}^{\lfloor n/2 \rfloor} Y_j + \sum_{k=1}^{\lfloor n/3 \rfloor} Z_k - n,\ n \in \mathbb{N}.$$

Hence,

$$S_{6m} = \sum_{i=1}^{6m} X_i + \sum_{j=1}^{3m} Y_j + \sum_{k=1}^{2m} Z_k - 6m = \sum_{i=1}^{m} \xi_i$$

for every $m \in \mathbb{N}$, where $\{\xi_1, \xi_2, \ldots\}$ are independent copies of the r.v. $\xi = X_1 + X_2 + \cdots + X_6 + Y_1 + Y_2 + Y_3 + Z_1 + Z_2 - 6$.

According to the definition

$$\psi(u) = \mathbb{P}(S_n \geqslant u \ \text{ for some } \ n \in \mathbb{N}) \leqslant \mathbb{P}\Big(\sup_{m \in \mathbb{N}} S_{6m} \geqslant u \Big) + \mathbb{P}\Big(\sup_{m \in \mathbb{N}} S_{6m+1} \geqslant u \Big) + \ldots \tag{3.34}$$
$$+ \mathbb{P}\Big(\sup_{m \in \mathbb{N}} S_{6m+5} \geqslant u \Big) = \mathbb{P}\Big(\sup_{m \in \mathbb{N}} S_{6m} \geqslant u \Big) + \sum_{k=1}^{5} \mathbb{P}\Big(\sup_{m \in \mathbb{N}} S_{6m+k} \geqslant u \Big),$$

where $k \in \mathbb{N}$.

It is clear that

$$\frac{S_{6m}}{6m} = \frac{1}{6m} \sum_{i=1}^{m} \xi_i.$$

Hence, the strong law of large numbers implies that

$$\frac{S_{6m}}{6m} \underset{m\to\infty}{\overset{\text{a.s.}}{\to}} \frac{1}{6}\,\mathbb{E}\xi = \frac{1}{6}\,(6\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z - 6) =: -\Delta < 0.$$

Therefore,

$$\mathbb{P}\biggl(\sup_{m\geqslant\tilde{m}}\left|\frac{S_{6m}}{6m} + \Delta\right| \leqslant \frac{\Delta}{2}\biggr) \underset{\tilde{m}\to\infty}{\to} 1. \tag{3.35}$$

If $N \geqslant 2$ and u is positive, then

$$\begin{aligned}
\mathbb{P}\Bigl(\sup_{m\in\mathbb{N}} S_{6m} < u\Bigr) &\geqslant \mathbb{P}\biggl(\Bigl(\bigcap_{m=1}^{N-1}\bigl\{S_{6m} \leqslant \frac{u}{2}\bigr\}\Bigr) \cap \Bigl(\bigcap_{m=N}^{\infty}\bigl\{S_{6m} \leqslant \frac{u}{2}\bigr\}\Bigr)\biggr)\\
&\geqslant \mathbb{P}\biggl(\Bigl(\bigcap_{m=1}^{N-1}\bigl\{S_{6m} \leqslant \frac{u}{2}\bigr\}\Bigr)\biggr) + \mathbb{P}\biggl(\Bigl(\bigcap_{m=N-1}^{\infty}\bigl\{S_{6m} \leqslant \frac{u}{2}\bigr\}\Bigr)\biggr) - 1\\
&\geqslant \mathbb{P}\biggl(\Bigl(\bigcap_{m=1}^{N-1}\bigl\{S_{6m} \leqslant \frac{u}{2}\bigr\}\Bigr)\biggr) + \mathbb{P}\biggl(\sup_{m\geqslant N}\left|\frac{S_{6m}}{6m} + \Delta\right| \leqslant \frac{\Delta}{2}\biggr) - 1.
\end{aligned}$$

This inequality and relation (3.35) imply that

$$\lim_{u\to\infty}\mathbb{P}\Bigl(\sup_{m\in\mathbb{N}} S_{6m} < u\Bigr) = 1. \tag{3.36}$$

On the other hand, for each $p = 1, \ldots, 5,\ p \in \mathbb{N}$ and for all $m \in \mathbb{N}$,

$$\frac{S_{6m+p}}{6m+p} = \frac{6m}{6m+p}\,\frac{1}{6m}\sum_{i=1}^{n}\xi_i + \frac{\sum\limits_{i=1}^{p} X_{6m+i} + \sum\limits_{j=1}^{\lfloor p/2\rfloor} Y_{6m+j} + \sum\limits_{k=1}^{\lfloor p/3\rfloor} Z_{6m+k} - p}{6m+p}.$$

Due to the strong law of large numbers,

$$\frac{1}{m}\sum_{i=1}^{m}\xi_i \underset{m\to\infty}{\overset{a.s.}{\to}} 6\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z,$$

$$\frac{\sum\limits_{i=1}^{p} X_{6m+i} + \sum\limits_{j=1}^{\lfloor p/2\rfloor} Y_{6m+j} + \sum\limits_{k=1}^{\lfloor p/3\rfloor} Z_{6m+k} - p}{6m+p} \underset{m\to\infty}{\overset{a.s.}{\to}} 0 \text{ for each } p = 1, \ldots, 5,\ p \in \mathbb{N}.$$

Therefore, for each $p = 1, \ldots, 5,\ p \in \mathbb{N}$

$$\frac{S_{6m+p}}{6m+p} \underset{m\to\infty}{\overset{a.s.}{\to}} -\Delta\,,$$

and we obtain

$$\lim_{u\to\infty} \mathbb{P}\Big(\sup_{m\in\mathbb{N}} S_{6m+p} < u\Big) = 1 \tag{3.37}$$

using the same procedure as for the sums $S_{6m},\ m \in \mathbb{N}$. Equality $\psi(\infty) = 0$ follows from estimate (3.34) and Eqs. (3.36) and (3.37).

Substituting $\varphi(\infty) = 1$ into (3.28), we get

$$\begin{aligned}(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z)& \\ = \varphi(0) + \hat{z}_1\varphi(1) + \hat{z}_2\varphi(2) + \hat{z}_3\varphi(3) + \hat{z}_4\varphi(4) + \hat{z}_5\varphi(5).&\end{aligned} \tag{3.38}$$

Now we consider formulas (3.20) and (3.38) to get a suitable recursion procedure described in Theorem 3.5.

• First, let $a_0 \neq 0$, $b_0 \neq 0$ and $c_0 \neq 0$ and let the sequences $\beta_n^0,\ \beta_n^1,\ \ldots, \beta_n^4,\ \gamma_n$ be defined in the formulation of Theorem 3.5.

We prove (3.29) by induction. We observe that relation (3.38) implies immediately:

$$\begin{aligned}
\varphi(0) &= \beta_0^0\varphi(0) + \beta_0^1\varphi(1) + \beta_0^2\varphi(2) + \beta_0^3\varphi(3) + \beta_0^4\varphi(4) + \gamma_0(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z),\\
\varphi(1) &= \beta_1^0\varphi(0) + \beta_1^1\varphi(1) + \beta_1^2\varphi(2) + \beta_1^3\varphi(3) + \beta_1^4\varphi(4) + \gamma_1(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z),\\
\varphi(2) &= \beta_2^0\varphi(0) + \beta_2^1\varphi(1) + \beta_2^2\varphi(2) + \beta_2^3\varphi(3) + \beta_2^4\varphi(4) + \gamma_2(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z),\\
\varphi(3) &= \beta_3^0\varphi(0) + \beta_3^1\varphi(1) + \beta_3^2\varphi(2) + \beta_3^3\varphi(3) + \beta_3^4\varphi(4) + \gamma_3(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z),\\
\varphi(4) &= \beta_4^0\varphi(0) + \beta_4^1\varphi(1) + \beta_4^2\varphi(2) + \beta_4^3\varphi(3) + \beta_4^4\varphi(4) + \gamma_4(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z).
\end{aligned}$$

Now suppose that equality (3.29) holds for all $n = 0, 1, \ldots, N$, and we will prove that (3.29) holds for $n = N+1$. First, we rewrite equality (3.20) in following form

$$\varphi(u) = \sum_{k=0}^{u+5} a^{6}b^{3}c^{2}(u+5-k)\varphi(k+1) - \sum_{k=1}^{5} z_k(u)\varphi(k), \tag{3.39}$$

where

$$z_k(u) = \sum_{i=1}^{6-k} \sum_{j=i}^{6-k} \sum_{\bigcap_{l=1}^{i} I_l} a_{k_1}^{\lfloor (4+i)/5 \rfloor} ab^{\lfloor (3+i)/5 \rfloor}(k_2) ac^{\lfloor (2+i)/5 \rfloor}(k_3) ab^{\lfloor (1+i)/5 \rfloor}(k_4) a^{\lfloor i/5 \rfloor}(k_5) \times$$
$$\times\, a^{6-i} b^{3-\lfloor i/2 \rfloor} c^{2-\lfloor i/3 \rfloor}(6-j-k),$$

and I_l is a summation region

$$I_l = \left\{ k_l; 0; u+l-1-\sum_{m=0}^{l-1} k_m \right\} \mathbb{1}\left(\left[\frac{4+i-l}{5} \right] \right)$$
$$\bigcup \left\{ k_l; u+j-\sum_{m=0}^{l-1} k_m; u+j-\sum_{m=0}^{l-1} k_m \right\} \mathbb{1}\left(\left[\frac{5+l-i}{5} \right] \right),$$

According to (3.39) for $u = N-5$ we have

$$\varphi(N-5) = \sum_{k=1}^{N} a^6 b^3 c^2(k) \varphi(N-k+1) + a^6 b^3 c^2(0) \varphi(N+1) - \sum_{k=1}^{5} z_k(N-5) \varphi(k).$$

Therefore,

$$\varphi(N+1) = \frac{1}{a^6 b^3 c^2(0)} \Big(\varphi(N-5) - \sum_{k=1}^{N} a^6 b^3 c^2(k) \varphi(N-k+1) + \sum_{k=1}^{5} z_k(N-5) \varphi(k) \Big),$$

and by the induction hypothesis we get

$$
\begin{aligned}
\varphi(N+1) = \frac{1}{a^6b^3c^2(0)}\Big(&\beta^0_{N-5}\varphi(0) + \beta^1_{N-5}\varphi(1) + \beta^2_{N-5}\varphi(2) + \beta^3_{N-5}\varphi(3) + \beta^4_{N-5}\varphi(4) \\
&+ \gamma_{N-5}(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z) - \sum_{k=1}^{N} a^6b^3c^2(k)\Big(\beta^0_{N-k+1}\varphi(0) + \beta^1_{N-k+1}\varphi(1) \\
&+ \beta^2_{N-k+1}\varphi(2) + \beta^3_{N-k+1}\varphi(3) + \beta^4_{N-k+1}\varphi(4) + \gamma_{N-k+1}(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z)\Big) \\
&+ \sum_{k=1}^{5} z_k(N-5)\varphi(k)\Big) \\
= \varphi(0)\Big(&\frac{1}{a^6b^3c^2(0)}\Big(\beta^0_{N-5} - \sum_{k=1}^{N} a^6b^3c^2(k)\beta^0_{N-k+1}\Big)\Big) \\
&+ \varphi(1)\Big(\frac{1}{a^6b^3c^2(0)}\Big(\beta^1_{N-5} - \sum_{k=1}^{N} a^6b^3c^2(k)\beta^1_{N-k+1}\Big) + z_1(N-5)\Big) \\
&+ \varphi(2)\Big(\frac{1}{a^6b^3c^2(0)}\Big(\beta^2_{N-5} - \sum_{k=1}^{N} a^6b^3c^2(k)\beta^2_{N-k+1}\Big) + z_2(N-5)\Big) \\
&+ \varphi(3)\Big(\frac{1}{a^6b^3c^2(0)}\Big(\beta^3_{N-5} - \sum_{k=1}^{N} a^6b^3c^2(k)\beta^3_{N-k+1}\Big) + z_3(N-5)\Big) \\
&+ \varphi(4)\Big(\frac{1}{a^6b^3c^2(0)}\Big(\beta^4_{N-5} - \sum_{k=1}^{N} a^6b^3c^2(k)\beta^4_{N-k+1}\Big) + z_4(N-5)\Big) \\
&+ (6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z)\Big(\frac{1}{a^6b^3c^2(0)}\Big(\gamma_{N-5} - \sum_{k=1}^{N} a^6b^3c^2(k)\gamma_{N-k+1}\Big)\Big) \\
&+ \varphi(5)\Big(\frac{1}{a^6b^3c^2(0)} z_5(N-5)\Big).
\end{aligned}
\tag{3.40}
$$

Since

$$
\varphi(5) = -\frac{1}{a^6b^3c^2(0)}\Big(\varphi(0) + \hat{z}_1\varphi(1) + \hat{z}_2\varphi(2) + \hat{z}_3\varphi(3) + \hat{z}_4\varphi(4) + 6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z\Big)
$$

due to (3.29) and $z_5(N-5) = a(N-4)a^6b^3c^2(0)$ we obtain from (3.40) that

$$
\begin{aligned}
\varphi(N+1) &= \beta^0_{N+1}\varphi(0) + \beta^1_{N+1}\varphi(1) + \beta^2_{N+1}\varphi(2) + \beta^3_{N+1}\varphi(3) + \beta^4_{N+1}\varphi(4) \\
&\quad + \gamma_{N+1}(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z).
\end{aligned}
$$

Hence, the desired relation (3.29) holds for all $n \in \mathbb{N}_0$ by induction.

• If $\{a_0 \neq 0, b_0 = 0, c_0 \neq 0, b_1 \neq 0\}$ then it follows from equality (3.38) that $\varphi(0)+\hat{z}_1\varphi(1)+\hat{z}_2\varphi(2) = 6-6\mathbb{E}X-3\mathbb{E}Y-2\mathbb{E}Z$. Hence, $\varphi(0) = \tilde{\beta}_0^0\varphi(0)+\tilde{\beta}_0^1\varphi(1)+\tilde{\gamma}_0(6-6\mathbb{E}X-3\mathbb{E}Y-2\mathbb{E}Z)$. This is equality (3.30) for $n = 0$. The validity of (3.30) for other n can be derived from (3.39) using the induction arguments.

• In the case $\{a_0 \neq 0, b_0 \neq 0, c_0 = 0, c_1 \neq 0\}$, then it follows from equality (3.38) that $\varphi(0)+\hat{z}_1\varphi(1)+\hat{z}_2\varphi(2)+\hat{z}_3\varphi(3) = 6-6\mathbb{E}X-3\mathbb{E}Y-2\mathbb{E}Z$. The validity of (3.31) for $n \in \mathbb{N}_0$ can be derived from (3.39) using the induction arguments again.

• In the case $\{a_0 \neq 0, b_0 \neq 0, c_0 = c_1 = 0, c_2 \neq 0\}$, then it follows from equality (3.38) that $\varphi(0) + \hat{z}_1\varphi(1) = 6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z$. The validity of (3.32) for $n \in \mathbb{N}_0$ can be derived from (3.39) using the induction arguments again.

• In the case $\{a_0 \neq 0, b_0 = 0, c_0 = 0, b_1 \neq 0, c_1 \neq 0\}$, it follows immediately from (3.38) that $\varphi(0) = 6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z$. For $n = 0$ we get $\varphi(1) = \varphi(0)/a^6b^3c^2(5)$ from (3.39) and for $n \geqslant 2$ we obtain recursive formula (3.33) from (3.39). □

Theorem 3.5 is proved.

The obtained algorithm works in a following way: first step should be calculation of $\varphi(0)$, then $\varphi(1)$ using calculated value of $\varphi(0)$. This step should be repeated for every $\varphi(u)$, $u \in \{2, 3, \ldots\}$. The ruin probabilities can be found from calculated values of survival probabilities using formula $\varphi(u) = 1 - \psi(u)$.

Let us demonstrate the simplest case, where $b_0 = c_0 = 0$. Consider the risk model generated by r.v.'s X, Y and Z such that $a_0 = \mathbb{P}(X = 0) = 0.9$, $a_1 = \mathbb{P}(X = 1) = 0.05$, $a_2 = \mathbb{P}(X = 2) = 0.05$, $b_1 = \mathbb{P}(Y = 1) = 0.99$, $b_2 = \mathbb{P}(Y = 2) = 0.01$, $c_1 = \mathbb{P}(Z = 1) = 0.99$, $c_2 = \mathbb{P}(Z = 2) = 0.01$. In this case, net profit condition holds, i.e. $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 = 0.991 < 1$. Then, from Eq. (3.33) we obtain:

$$\varphi(0) = 6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z = 0.05,$$

$$\varphi(1) = \frac{\varphi(0)}{a^6b^3c^2(5)} = 0.099,$$

$$\varphi(2) = \frac{1}{a^6b^3c^2(5)}(\varphi(1) - a^6b^3c^2(6)\varphi(1)) = 0.196,$$

$$\ldots etc.$$

A few numerical examples for calculating ruin probabilities are also presented in Chapter 5 (Examples 3, 4).

Chapter 4

Ruin probabilities in the three-seasonal discrete-time risk model

This chapter deals with the discrete-time risk model with three nonidentically distributed claims. We suppose that the claims repeat with time periods of three units, i.e. claim distributions coincide at times $\{1, 4, 7, \ldots\}$, at times $\{2, 5, 8, \ldots\}$ and at times $\{3, 6, 9, \ldots\}$. We present the recursive formulas to calculate the finite-time and ultimate ruin probabilities.

Several formulas and procedures for computing finite-time ruin probability and ultimate ruin probability of discrete-time homogeneous risk model, defined by (1.2), have been proposed in the literature. Here we present some of them having the recursive form.

• *For the homogeneous discrete-time risk model, it holds that (see, for instance, [10, 14, 15]):*

$$\begin{aligned}\psi(u,1) &= 1 - F_Z(u),\ u \in \mathbb{N}_0,\\ \psi(u,T) &= \psi(u,1) + \sum_{k=0}^{u} \psi(u+1-k, T-1) z_k,\ u \in \mathbb{N}_0,\ T \in \{2,3,\ldots\}.\end{aligned}$$

• *If model* (1.2) *is generated by the claim generator Z such that $\mathbb{E}Z < 1$ then* *the ultimate ruin probability is defined by formulas (see, for instance, [14, 15, 40]):*

$$\psi(0) = \mathbb{E}Z, \tag{4.1}$$

$$\psi(u) = \sum_{j=1}^{u-1}(1-F_Z(j))\psi(u-j) + \sum_{j=u}^{\infty}(1-F_Z(j)), u \in \mathbb{N}. \tag{4.2}$$

If the homogeneous discrete-time risk model is generated by Z satisfying condition $\mathbb{E}Z \geqslant 1$, then the net profit condition does not hold and, in such a case, we have that $\psi(u) = 1$ for all $u \in \mathbb{N}_0$ according to the general renewal theory (see, for instance, [31] and references therein).

The formulas presented above enable us to calculate $\psi(u)$ and $\psi(u, T)$ for $u \in \mathbb{N}_0$ and $T \in \mathbb{N}$. Nevertheless, there exist many other methods which allow to calculate or estimate the finite-time and the ultimate ruin probabilities. Some of them can be found in [1, 27, 33].

The assumption for claim amounts $\{Z_1, Z_2, \ldots\}$ to be non-identically distributed random variables is the natural generalization of the homogeneous model. If r.v.'s $\{Z_1, Z_2, \ldots\}$ are independent but not necessarily identically distributed, then the model, defined by equality (1.2) is called the *inhomogeneous discrete-time risk model*. For such model, a recursive procedure for calculation of finite-time ruin probabilities can be found in [3, 4]. For the finite-time ruin probabilities

$$\psi^{(j)}(u,T) = \mathbb{P}\Bigg(\bigcup_{n=1}^{T}\Big\{u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0\Big\}\Bigg),\ j \in \mathbb{N}_0$$

we have the following theorem.

4.1 Review of the bi-seasonal risk model

Theorem 4.1. *Let us consider the inhomogeneous discrete-time risk model defined by Eq. (1.2) in which* $u \in \mathbb{N}_0$, $z_k^{(j)} = \mathbb{P}(Z_{1+j} = k)$, $k, j \in \mathbb{N}_0$, *and* $F_Z^{(j)}(x) = \mathbb{P}(Z_{1+j} \leqslant x)$, $x \in \mathbb{R}$. *Then*

$$\psi^{(j)}(u,1) = 1 - F_Z^{(j)}(u),$$

$$\psi^{(j)}(u,T) = \psi^{(j)}(u,1) + \sum_{k=0}^{u} \psi^{(j+1)}(u+1-k, T-1) z_k^{(j)}$$

for all $u \in \mathbb{N}_0$ *and* $T \in \{2, 3, \ldots\}$.

According to this theorem, we can calculate the finite-time ruin probability $\psi^{(0)}(u, T)$ of the initial model for all $u \in \mathbb{N}_0$ and $T \in \mathbb{N}$. Unfortunately, it is impossible to get formulas similar to the formulas (4.1) and (4.2) for the general case because in the case of nonidentically distributed claims, the future of model behaviour at each time can be completely new. In paper [8], the general discrete-time risk model was restricted to the model with two kinds of claims. In this model, there are two differently distributed claim amounts which periodically change. We call such a model the bi-seasonal discrete-time risk model. In [8] (see Theorem 2.3) the following statement is proved for the calculation of the ultimate ruin probability.

Theorem 4.2. *Let us consider a bi-seasonal discrete-time risk model generated by independent random claim amounts* X *and* Y*, i.e.* $Z_{2k+1} \overset{d}{=} X$ *for* $k \in \{0, 1, \ldots\}$ *and* $Z_{2k} \overset{d}{=} Y$ *for* $k \in \{1, 2, \ldots\}$*. Denote* $S = X + Y$ *and* $x_n = \mathbb{P}(X = n)$*,* $y_n = \mathbb{P}(Y = n)$*,* $s_n = \mathbb{P}(S = n)$ *for* $n \in \mathbb{N}_0 = \{0, 1, \ldots\}$*.*

- *If* $\mathbb{E}X + \mathbb{E}Y < 2$*, then* $\lim_{u\to\infty} \psi(u) = 0$*.*
- *If* $s_0 = x_0 y_0 \neq 0$*, then:*

$$\begin{aligned}\psi(0) &= 1 - (2 - \mathbb{E}S) \lim_{n\to\infty} \frac{b_{n+1} - b_n}{a_n - a_{n+1}},\\ 1 - \psi(u) &= \alpha_u(1 - \psi(0)) + \beta_u(2 - \mathbb{E}S), u \in \mathbb{N},\end{aligned}$$

where $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $n \in \mathbb{N}_0$*, are two sequences of real numbers defined recursively by equalities:*

$$\alpha_0 = 1, \alpha_1 = -\frac{1}{y_0}, \alpha_n = \frac{1}{s_0}\left(\alpha_{n-2} - \sum_{i=1}^{n-1} s_i \alpha_{n-i} - x_{n-1}\right), n \geqslant 2$$

$$\beta_0 = 0, \beta_1 = \frac{1}{y_0}, \beta_n = \frac{1}{s_0}\left(\beta_{n-2} - \sum_{i=1}^{n-1} s_i \beta_{n-i} + x_{n-1}\right), n \geqslant 2.$$

- *If* $s_0 \neq 0$*, then*

$$\begin{aligned}\psi(1) &= 1 - (1 + \psi(0) - \mathbb{E}S)/y_0,\\ \psi(u) &= 1 + \frac{1}{s_0}\left(\psi(u-2) - 1 + \sum_{k=1}^{u-1} s_k(1 - \psi(u-k))\right) - \frac{x_{u-1}(1 - \psi(1))}{x_0},\\ &\qquad u \in \{2, 3, \ldots\}.\end{aligned}$$

- *If* $x_0 = 0$, $y_0 \neq 0$, *then* $s_1 \neq 0$ *and* $\psi(0) = 1$.
- *If* $x_0 \neq 0$, $y_0 = 0$, *then* $s_1 \neq 0$ *and* $\psi(0) = \mathbb{E}S - 1$.
- *If* $s_0 = 0$, *then for* $u \in \mathbb{N}$

$$\psi(u) = 1 - \frac{1}{s_1}\left(1 - \psi(u-1) - \sum_{k=2}^{u} s_k(1 - \psi(u-k+1))\right).$$

4.2 Main results

Below we present the model under consideration.

Definition 4.1. *We say that the insurer's surplus* $W_u(n)$ *follows the three-seasonal risk model if* $W_u(n)$ *is given by equality* (1.2) *for each* $n \in \mathbb{N}_0$ *and the following assumptions hold:*

- *the initial insurer's surplus* $u \in \mathbb{N}_0$,
- *the random claim amounts* $Z_1, Z_2, \ldots$ *are nonnegative integer-valued independent r.v.'s,*
- *for all* $k \in \mathbb{N}_0$ *it holds that* $Z_{3k+1} \overset{d}{=} Z_1$, $Z_{3k+2} \overset{d}{=} Z_2$ *and* $Z_{3k+3} \overset{d}{=} Z_3$.

Let us define p.m.f.'s and p.d.f.'s by the following equalities

$$a_k = \mathbb{P}(Z_1 = k),\ b_k = \mathbb{P}(Z_2 = k),\ c_k = \mathbb{P}(Z_3 = k),\ s_k = \mathbb{P}(S = k),\ k \in \mathbb{N}_0,$$

where $S = Z_1 + Z_2 + Z_3$,

$$A(x) = \sum_{k=0}^{\lfloor x \rfloor} a_k,\ B(x) = \sum_{k=0}^{\lfloor x \rfloor} b_k,\ C(x) = \sum_{k=0}^{\lfloor x \rfloor} c_k,\ D(x) = \sum_{k=0}^{\lfloor x \rfloor} s_k,\ x \geqslant 0.$$

It is not difficult to see that the definitions of ruin time, finite-time ruin probability, ultimate ruin probability and ultimate survival probability remains the same. All expressions of these quantities have no differences from the expressions of the respective quantities of the homogeneous discrete-time risk model. While the procedures to calculate the finite-time or the ultimate probabilities are more complex with respect to respective procedures in the homogeneous or in the bi-seasonal discrete-time risk models.

Our first result follows immediately from Theorem 4.1. The obtained formulas allow us to calculate the finite-time ruin probabilities $\psi(u,T) = \psi^{(0)}(u,T)$ in the three-seasonal discrete-time risk model for all $u \in \mathbb{N}_0$ and all $T \in \mathbb{N}$.

Theorem 4.3. *Let us consider the above three-seasonal discrete-time risk model. For each $u \in \mathbb{N}_0$ we have that*

$$\psi(u,1) = \psi^{(0)}(u,1) = \sum_{k>u} a_k, \ \ \psi^{(1)}(u,1) = \sum_{k>u} b_k, \ \ \psi^{(2)}(u,1) = \sum_{k>u} c_k,$$

and for all $u \in \mathbb{N}_0$ and all $T \in \{2,3,\ldots\}$ we have the following recursive equalities:

$$\begin{aligned}
\psi(u,T) = \psi^{(0)}(u,T) &= \psi^{(0)}(u,1) + \sum_{k=0}^{u} \psi^{(1)}(u+1-k, T-1)a_k, \\
\psi^{(1)}(u,T) &= \psi^{(1)}(u,1) + \sum_{k=0}^{u} \psi^{(2)}(u+1-k, T-1)b_k, \\
\psi^{(2)}(u,T) &= \psi^{(2)}(u,1) + \sum_{k=0}^{u} \psi^{(0)}(u+1-k, T-1)c_k.
\end{aligned}$$

Net profit condition

Our second result describes the meaning of the net profit condition in the three-seasonal discrete-time risk model.

Theorem 4.4. *Let us consider a three-seasonal discrete-time risk model generated by independent random claim amounts Z_1, Z_2 and Z_3. If $\mathbb{E}S > 3$, then $\psi(u) = 1$ for each initial surplus $u \in \mathbb{N}_0$. If $\mathbb{E}S = 3$, then we have the following possible cases:*

- $\psi(0) = \psi(1) = \psi(2) = 1$ *and* $\psi(u) = 0$ *for* $u \in \{3,4,\ldots\}$ *if* $\{a_3 = b_0 = c_0 = 1\}$;
- $\psi(0) = \psi(1) = 1$ *and* $\psi(u) = 0$ *for* $u \in \{2,3,\ldots\}$ *if* $\{a_0 = b_3 = c_0 = 1\}$ *or* $\{a_2 = b_1 = c_0 = 1\}$ *or* $\{a_1 = b_2 = c_0 = 1\}$ *or* $\{a_2 = b_0 = c_1 = 1\}$;
- $\psi(0) = 1$ *and* $\psi(u) = 0$ *for all* $u \in \mathbb{N}$ *if* $\{a_0 = b_0 = c_3 = 1\}$ *or* $\{a_0 = b_2 = c_1 = 1\}$ *or* $\{a_0 = b_1 = c_2 = 1\}$ *or* $\{a_1 = b_0 = c_2 = 1\}$ *or* $\{a_1 = b_1 = c_1 = 1\}$;
- $\psi(u) = 1$ *for all* $u \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ *if* $s_3 < 1$.

Proof. For an arbitrary $u \in \mathbb{N}_0$, we have

$$\begin{aligned}
\varphi(u) &= \mathbb{P}\Big(\bigcap_{n=1}^{\infty}\{u+n-\sum_{i=1}^{n} Z_i > 0\}\Big)\\
&= \mathbb{P}\Big(\bigcap_{n=3}^{\infty}\{u+n-\sum_{i=1}^{n} Z_i > 0\} \cap \{Z_1 \geqslant u+1\} \cap \{Z_1+Z_2 \geqslant u+2\}\Big)\\
&- \mathbb{P}\Big(\bigcap_{n=3}^{\infty}\{u+n-\sum_{i=1}^{n} Z_i > 0\} \cap \{Z_1 \geqslant u+1\}\Big)\\
&- \mathbb{P}\Big(\bigcap_{n=3}^{\infty}\{u+n-\sum_{i=1}^{n} Z_i > 0\} \cap \{Z_1+Z_2 \geqslant u+2\}\Big)\\
&+ \mathbb{P}\Big(\bigcap_{n=3}^{\infty}\{u+n-\sum_{i=1}^{n} Z_i > 0\}\Big) \qquad (4.3)
\end{aligned}$$

Since the model is three-seasonal the last probability in (4.3) can be expressed by sum

$$\begin{aligned}
&\sum_{k=0}^{u+2} s_k \; \mathbb{P}\Big(\bigcap_{n=1}^{\infty}\{u+n+3-k-\sum_{i=1}^{n} Z_i > 0\}\Big)\\
&= \sum_{k=0}^{u+2} s_k \; \varphi(u+3-k), \qquad (4.4)
\end{aligned}$$

where, as above, $s_k = \mathbb{P}(Z_1+Z_2+Z_3 = k)$ for $k \in \mathbb{N}_0$.
The second probability in (4.3) is equal to expression

$$\begin{aligned}
&\sum_{k=u+1}^{\infty} a_k \; \mathbb{P}\Big(\bigcap_{n\geqslant 3}\{u+n-k-Z_2-Z_3-\sum_{i=4}^{n} Z_i > 0\}\Big)\\
&= a_{u+2}\; \mathbb{P}(Z_2+Z_3=0)\; \mathbb{P}\Big(\bigcap_{n=4}^{\infty}\{n-2-\sum_{i=4}^{n} Z_i > 0\}\Big)\\
&\quad + a_{u+1}\; \mathbb{P}(Z_2+Z_3=0)\; \mathbb{P}\Big(\bigcap_{n=4}^{\infty}\{n-1-\sum_{i=4}^{n} Z_i > 0\}\Big)\\
&\qquad + a_{u+1}\; \mathbb{P}(Z_2+Z_3=1)\; \mathbb{P}\Big(\bigcap_{n=4}^{\infty}\{n-2-\sum_{i=4}^{n} Z_i > 0\}\Big)\\
&= a_{u+2}b_0c_0\varphi(1) + a_{u+1}b_0c_0\varphi(2) + a_{u+1}b_0c_1\varphi(1) + a_{u+1}b_1c_0\varphi(1). \qquad (4.5)
\end{aligned}$$

Similarly, the third probability in (4.3) is

$$\mathbb{P}(Z_1+Z_2=u+2)\ \mathbb{P}(Z_3=0)\ \mathbb{P}\biggl(\bigcap_{n=4}^{\infty}\{n-2-\sum_{i=4}^{n}Z_i>0\}\biggr)$$
$$=c_0\varphi(1)\sum_{k=0}^{u+2}a_k b_{u+2-k}, \tag{4.6}$$

and, finally, the first probability in (4.3) is

$$\mathbb{P}(Z_1=u+1)\ \mathbb{P}(Z_2=1)\ \mathbb{P}(Z_3=0)\ \mathbb{P}\biggl(\bigcap_{n=4}^{\infty}\{n-2-\sum_{i=4}^{n}Z_i>0\}\biggr)$$
$$+\mathbb{P}(Z_1=u+2)\ \mathbb{P}(Z_2=0)\ \mathbb{P}(Z_3=0)\ \mathbb{P}\biggl(\bigcap_{n=4}^{\infty}\{n-2-\sum_{i=4}^{n}Z_i>0\}\biggr)$$
$$=a_{u+1}b_1c_0\varphi(1)+a_{u+2}b_0c_0\varphi(1). \tag{4.7}$$

Substituting (4.4)-(4.7) into (4.3) we get that

$$\varphi(u)=\sum_{k=0}^{u+2}s_k\ \varphi(u+3-k)-a_{u+1}b_0c_0\varphi(2)$$
$$-a_{u+1}b_0c_1\varphi(1)-c_0\varphi(1)\sum_{k=0}^{u+2}a_k b_{u+2-k} \tag{4.8}$$

for all $u\in\mathbb{N}_0$.
Therefore, for $v\in\mathbb{N}_0$ we have

$$\sum_{u=0}^{v}\varphi(u)=\sum_{u=0}^{v}\sum_{k=0}^{u+2}s_k\varphi(u+3-k)$$
$$-b_0c_0\varphi(2)(A(v+1)-a_0)-b_0c_1\varphi(1)(A(v+1)-a_0)$$
$$-c_0\varphi(1)\sum_{u=0}^{v}\sum_{k=0}^{u+2}a_k b_{u+2-k}. \tag{4.9}$$

We observe that the sum

$$\sum_{u=0}^{v}\sum_{k=0}^{u+2}a_k b_{u+2-k}$$

can be rewritten in the following form

$$\begin{aligned}
&a_0\sum_{u=0}^{v} b_{u+2} + a_1\sum_{u=0}^{v} b_{u+1} + a_2\sum_{u=0}^{v} b_u + \sum_{k=3}^{v+2} a_k \sum_{u=k-2}^{v} b_{u+2-k}\\
&= a_0(B(v+2) - b_0 - b_1) + a_1(B(v+1) - b_0) + a_2B(v)\\
&\quad + \sum_{k=3}^{v+2} a_k\ B(v+2-k)\\
&\quad = \sum_{k=0}^{v+2} a_k\ B(v+2-k) - a_0b_0 - a_0b_1 - a_1b_0.
\end{aligned} \tag{4.10}$$

While, similarly, the sum

$$\sum_{u=0}^{v}\sum_{k=0}^{u+2} s_k\varphi(u+3-k)$$

is

$$\sum_{k=1}^{v+3}\varphi(k)\ D(v+3-k) - s_0\varphi(1) - s_1\varphi(1) - s_0\varphi(2), \tag{4.11}$$

where

$$D(x) = \sum_{k=0}^{\lfloor x\rfloor} s_k = \sum_{k=0}^{\lfloor x\rfloor} \mathbb{P}(Z_1 + Z_2 + Z_3 = k).$$

Relations (4.9), (4.10) and (4.11) imply that

$$\begin{aligned}
\sum_{k=0}^{v}\varphi(k) &= \sum_{k=1}^{v+3}\varphi(k)\ D(v+3-k)\\
&- b_0c_0\varphi(2)A(v+1) - b_0c_1\varphi(1)A(v+1)\\
&- c_0\varphi(1)\sum_{k=0}^{v+2} a_k\ B(v+2-k).
\end{aligned}$$

Or, equivalently,

$$\sum_{k=0}^{v+3}\varphi(k)(1-D(v+3-k)) = \varphi(v+1)+\varphi(v+2)+\varphi(v+3)$$
$$-\varphi(0)D(v+3)-b_0c_0\varphi(2)A(v+1)-b_0c_1\varphi(1)A(v+1)$$
$$+c_0\varphi(1)\left(\sum_{k=0}^{v+2}a_k\overline{B}(v+2-k)+A(v+2)\right). \tag{4.12}$$

For each $K\in[1,v+2)$ we have

$$\sum_{k=0}^{v+2}a_k\overline{B}(v+2-k)=\sum_{k=0}^{K}a_k\overline{B}(v+2-k)+\sum_{k=K+1}^{v+2}a_k\overline{B}(v+2-k).$$

Therefore

$$\limsup_{v\to\infty}\sum_{k=0}^{v+2}a_k\overline{B}(v+2-k)\leqslant\sum_{k=K+1}^{\infty}a_k$$

for each $K\geqslant 1$, and so

$$\lim_{v\to\infty}\sum_{k=0}^{v+2}a_k\overline{B}(v+2-k)=0. \tag{4.13}$$

Sequence $\varphi(u), u\in\mathbb{N}_0$, is bounded and non decreasing. So, there exists a limit $\lim\limits_{u\to\infty}\varphi(u):=\varphi(\infty)$. Similarly, as in derivation of (4.13) we can get that

$$\limsup_{v\to\infty}\sum_{k=0}^{v+3}(\varphi(\infty)-\varphi(k))(1-D(v+3-k))\leqslant\sup_{k\geqslant K+1}(\varphi(\infty)-\varphi(k))\mathbb{E}S.$$

for each fixed $K\geqslant 1$. Therefore

$$\lim_{v\to\infty}\sum_{k=0}^{v+3}\varphi(k)(1-D(v+3-k))$$
$$=\varphi(\infty)\lim_{v\to\infty}\sum_{k=0}^{v+3}(1-D(k))$$
$$=\varphi(\infty)\mathbb{E}S. \tag{4.14}$$

The relations (4.12), (4.13) and (4.14) imply that

$$\varphi(\infty)(3-\mathbb{E}S)=\varphi(0)+b_0c_0\varphi(2)+b_0c_1\varphi(1)+c_0\varphi(1). \tag{4.15}$$

Now we consider the last equality and examine all possible cases.

(I) If $\mathbb{E}S > 3$, then (4.15) implies that $\varphi(\infty) = 0$ because the left side at (4.15) is nonnegative in all cases. So, in this case $\psi(u) = 1$ for all $u \in \{0, 1, 2, \ldots\}$.

(II) If $\mathbb{E}S = 3$, then (4.15) implies that

$$\varphi(0) + b_0c_0\varphi(2) + b_0c_1\varphi(1) + c_0\varphi(1) = 0.$$

Additionally, in this situation we have that $s_3 = 1$ or $s_3 < 1$.

(II-A) If $s_3 = 1$, then we have

$$\begin{cases} a_0b_0c_0 = 0, \\ a_1b_0c_0 + a_0b_1c_0 + a_0b_0c_1 = 0, \\ a_0b_0c_2 + a_0b_2c_0 + a_2b_0c_0 + a_1b_1c_0 + a_0b_1c_1 + a_1b_0c_1 = 0, \\ a_0b_0c_3 + a_0b_3c_0 + a_3b_0c_0 + a_1b_2c_0 + a_2b_1c_0 \\ \qquad +a_0b_1c_2 + a_0b_2c_1 + a_1b_0c_2 + a_2b_0c_1 + a_1b_1c_1 = 1, \\ \varphi(0) + b_0c_0\varphi(2) + b_0c_1\varphi(1) + c_0\varphi(1) = 0. \end{cases}$$

Taking into account that all numbers a_k, b_k and c_k are local probabilities for all $k \in \mathbb{N}_0$, the last system implies the following possible cases.

(a) $\{a_3 = b_0 = c_0 = 1\}$ and $\varphi(0) = \varphi(1) = \varphi(2) = 0$. In this case, $\psi(0) = \psi(1) = \psi(2) = 1$ and $\psi(u) = 0, u \in \{3, 4, \ldots\}$, because

$$W_u(n) = \begin{cases} u - 2 & \text{if } n \equiv 1 \text{ mod } 3, \\ u - 1 & \text{if } n \equiv 2 \text{ mod } 3, \\ u & \text{if } n \equiv 0 \text{ mod } 3. \end{cases}$$

(b) $\{a_0 = b_3 = c_0 = 1\}$ and $\varphi(0) = \varphi(1) = 0$. In this case, $\psi(0) = \psi(1) = 1$ and $\psi(u) = 0$ for $u \in \{2, 3, \ldots\}$ because

$$W_u(n) = \begin{cases} u + 1 & \text{if } n \equiv 1 \text{ mod } 3, \\ u - 1 & \text{if } n \equiv 2 \text{ mod } 3, \\ u & \text{if } n \equiv 0 \text{ mod } 3. \end{cases}$$

(c) $\{a_0 = b_0 = c_3 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \ldots\}$ because

$$W_u(n) = \begin{cases} u+1 & \text{if } \ n \equiv 1 \bmod 3, \\ u+2 & \text{if } \ n \equiv 2 \bmod 3, \\ u & \text{if } \ n \equiv 0 \bmod 3. \end{cases}$$

(d) $\{a_2 = b_1 = c_0 = 1\}$ and $\varphi(0) = \varphi(1) = 0$. In this case, $\psi(0) = \psi(1) = 1$ and $\psi(u) = 0$ for $u \in \{2, 3, \ldots\}$.

(e) $\{a_1 = b_2 = c_0 = 1\}$ and $\varphi(0) = \varphi(1) = 0$. In this case, $\psi(0) = \psi(1) = 1$ and $\psi(u) = 0$ for $u \in \{2, 3, \ldots\}$.

(f) $\{a_0 = b_2 = c_1 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \ldots\}$.

(g) $\{a_0 = b_1 = c_2 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \ldots\}$.

(h) $\{a_2 = b_0 = c_1 = 1\}$ and $\varphi(0) = \varphi(1) = 0$. In this case, $\psi(0) = \psi(1) = 1$ and $\psi(u) = 0$ for $u \in \{2, 3, \ldots\}$.

(i) $\{a_1 = b_0 = c_2 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \ldots\}$.

(j) $\{a_1 = b_1 = c_1 = 1\}$ and $\varphi(0) = 0$. In this case, $\psi(0) = 1$ and $\psi(u) = 0$ for $u \in \{1, 2, \ldots\}$.

(II-B) If $s_3 < 1$ and $\mathbb{E}S = 3$ then it is necessary that $s_0 \neq 0$ or $s_1 \neq 0$ or $s_2 \neq 0$ because on the contrary $\mathbb{E}S = 3s_3 + 4s_4 + 5s_5 + \ldots > 3(s_3 + s_4 + \ldots) = 3$. In this situation, it is sufficient to consider the following cases

$$\{s_0 \neq 0\}, \quad \{s_0 = 0, s_1 \neq 0\}, \quad \{s_0 = 0, s_1 = 0, s_2 \neq 0\}.$$

(k) If $s_0 = a_0 b_0 c_0 \neq 0$ then (4.15) implies that $\varphi(0) = \varphi(1) = \varphi(2) = 0$, and from (4.8) we obtain $\varphi(u) = 0$ for all $u \in \{3, 4, \ldots\}$. So, $\psi(u) = 1$ if $u \in \{0, 1, \ldots\}$ in the case.

(l) If $s_0 = a_0 b_0 c_0 = 0$ and $s_1 = a_0 b_0 c_1 + a_0 b_1 c_0 + a_1 b_0 c_0 \neq 0$ then there exist the following possible cases

(l-1) $\{a_0 = 0,\ a_1 \neq 0,\ b_0 \neq 0,\ c_0 \neq 0\}$. In this situation, equality (4.15) implies that $\varphi(0) = \varphi(1) = \varphi(2) = 0$, while (4.8) implies that $\varphi(u) = 0$ for all $u \in \{3, 4, \ldots\}$. So, $\psi(u) = 1$ for all $u \in \{0, 1, \ldots\}$ in the case.

(l-2) $\{a_0 \neq 0, b_0 = 0, b_1 \neq 0, c_0 \neq 0\}$. In this situation, equality (4.15) implies that $\varphi(0) = \varphi(1) = 0$, and (4.8) implies that $\varphi(u) = 0$ for all $u \in \{2, 3, \ldots\}$. Therefore $\psi(u) = 1$ for all $u \in \{0, 1, \ldots\}$ again.

(l-3) $\{a_0 \neq 0,\, b_0 \neq 0,\, c_0 = 0,\, c_1 \neq 0\}$.

Equality (4.15) implies that $\varphi(0) = \varphi(1) = 0$, and (4.8) implies that $\varphi(u) = 0$ for all $u \in \{2, 3, \ldots\}$. So, in this case $\psi(u) = 1$ for all $u \in \{0, 1, \ldots\}$.

(m) If $s_0 = a_0b_0c_0 = 0$, $s_1 = a_0b_0c_1 + a_0b_1c_0 + a_1b_0c_0 = 0$ and $s_2 = a_0b_0c_2 + a_0b_2c_0 + a_2b_0c_0 + a_1b_1c_0 + a_1b_0c_1 + a_0b_1c_1 \neq 0$ then there exist the following possible cases.

(m-1) $\{a_0 = 0,\, a_1 = 0,\, a_2 \neq 0,\, b_0 \neq 0,\, c_0 \neq 0\}$;

(m-2) $\{a_0 \neq 0,\, b_0 = 0,\, b_1 = 0,\, b_2 \neq 0,\, c_0 \neq 0\}$;

(m-3) $\{a_0 \neq 0,\, b_0 \neq 0,\, c_0 = 0,\, c_1 = 0,\, c_2 \neq 0\}$;

(m-4) $\{a_0 = 0,\, a_1 \neq 0,\, b_0 = 0,\, b_1 \neq 0,\, c_0 \neq 0\}$;

(m-5) $\{a_0 = 0,\, a_1 \neq 0,\, b_0 \neq 0,\, c_0 = 0,\, c_1 \neq 0\}$;

(m-6) $\{a_0 \neq 0,\, b_0 = 0,\, b_1 \neq 0,\, c_0 = 0,\, c_1 \neq 0\}$.

In all cases, equalities (4.15) and (4.8) imply that $\varphi(u) = 0$ and, so, $\psi(u) = 1$ for all $u \in \mathbb{N}_0$. Theorem 4.4 is proved.

□

Recursive formulas

Our last statement proposes a recursive procedure for calculation of the ultimate survival probabilities $\varphi(u) = 1 - \psi(u), u \in \mathbb{N}_0$.

Theorem 4.5. *Let us consider a three-seasonal discrete-time risk model generated by independent random claim amounts Z_1, Z_2 and Z_3. Denote $S = Z_1 + Z_2 + Z_3$, $s_n = \mathbb{P}(S = n)$ for $n \in \mathbb{N}_0$ and suppose that $\mathbb{E}S < 3$. Then the following list of statements holds.*

- $\lim\limits_{u\to\infty} \varphi(u) = 1.$
- *If $s_0 \neq 0$, then*

$$\varphi(n) = \alpha_n\varphi(0) + \beta_n\varphi(1) + \gamma_n(3 - \mathbb{E}S),\, n \in \mathbb{N}_0, \tag{4.16}$$

where

$$\begin{cases} \alpha_0 = 1,\ \alpha_1 = 0,\ \alpha_2 = -\frac{1}{b_0c_0}, \\ \alpha_n = \frac{1}{s_0}\left(\alpha_{n-3} - \sum\limits_{k=1}^{n-1} s_k\alpha_{n-k} - a_{n-2}\right), \quad n \geqslant 3; \end{cases}$$

$$\begin{cases} \beta_0 = 0,\ \beta_1 = 1,\ \beta_2 = -\frac{c_1}{c_0} - \frac{1}{b_0}, \\ \beta_n = \frac{1}{s_0}\left(\beta_{n-3} - \sum\limits_{k=1}^{n-1} s_k\beta_{n-k} - a_{n-2}c_0 + c_0\sum\limits_{k=0}^{n-1} a_kb_{n-1-k}\right), \quad n \geqslant 3; \end{cases}$$

$$\begin{cases} \gamma_0 = 0, \ \gamma_1 = 0, \ \gamma_2 = \frac{1}{b_0 c_0}, \\ \gamma_n = \frac{1}{s_0}\Big(\gamma_{n-3} - \sum\limits_{k=1}^{n-1} s_k \gamma_{n-k} + a_{n-2}\Big), \quad n \geqslant 3. \end{cases}$$

- *If* $\{a_0 = 0, b_0 \neq 0, c_0 \neq 0, a_1 \neq 0\}$ *then*

$$\begin{cases} \varphi(0) = 0, \\ \varphi(n) = \hat{\beta}_n \varphi(1) + \hat{\gamma}_n (3 - \mathbb{E}S), \ n \in \mathbb{N}, \end{cases} \tag{4.17}$$

where

$$\begin{cases} \hat{\beta}_1 = \beta_1, \ \hat{\beta}_2 = \beta_2, \ \hat{\gamma}_1 = \gamma_1, \ \hat{\gamma}_2 = \gamma_2, \\ \hat{\beta}_n = \frac{1}{s_1}\Big(\hat{\beta}_{n-2} - \sum\limits_{k=2}^{n} s_k \hat{\beta}_{n-k+1} - a_{n-1}c_0 - c_0\varphi(1)\sum\limits_{k=1}^{n} a_k b_{n-k}\Big), \ n \geqslant 3, \\ \hat{\gamma}_n = \frac{1}{s_1}\Big(\hat{\gamma}_{n-2} - \sum\limits_{k=2}^{n} s_k \hat{\gamma}_{n-k+1} + a_{n-1}\Big), \ n \geqslant 3. \end{cases}$$

- *If* $\{a_0 \neq 0, b_0 = 0, c_0 \neq 0, b_1 \neq 0\}$ *then*

$$\varphi(n) = \tilde{\alpha}_n \varphi(0) + \tilde{\gamma}_n (3 - \mathbb{E}S), \ n \in \mathbb{N}, \tag{4.18}$$

where

$$\begin{cases} \tilde{\alpha}_1 = -1/c_0, \ \tilde{\alpha}_2 = c_1/c_0^2 + 1/(a_0 b_1 c_0), \ \tilde{\gamma}_1 = 1/c_0, \ \tilde{\gamma}_2 = -c_1/c_0^2, \\ \tilde{\alpha}_n = \frac{1}{s_1}\Big(\tilde{\alpha}_{n-2} - \sum\limits_{k=2}^{n} s_k \tilde{\alpha}_{n-k+1} - \sum\limits_{k=0}^{n-1} a_k b_{n-k}\Big), \ n \geqslant 3, \\ \tilde{\gamma}_n = \frac{1}{s_1}\Big(\tilde{\gamma}_{n-2} - \sum\limits_{k=2}^{n} s_k \tilde{\gamma}_{n-k+1} + \sum\limits_{k=0}^{n-1} a_k b_{n-k}\Big), \ n \geqslant 3. \end{cases}$$

- *If* $\{a_0 \neq 0, b_0 \neq 0, c_0 = 0, c_1 \neq 0\}$ *then*

$$\varphi(n) = \breve{\alpha}_n \varphi(0) + \breve{\gamma}_n (3 - \mathbb{E}S), \ n \in \mathbb{N}_0, \tag{4.19}$$

where

$$\begin{cases} \breve{\alpha}_0 = 1, \breve{\alpha}_1 = -1/(b_0 c_1), \ \breve{\gamma}_0 = 0, \ \breve{\gamma}_1 = 1/(b_0 c_1), \\ \breve{\alpha}_n = \frac{1}{s_1}\Big(\breve{\alpha}_{n-2} - \sum\limits_{k=2}^{n} s_k \breve{\alpha}_{n-k+1} - \sum\limits_{k=0}^{n-1} a_k b_{n-k}\Big), \ n \geqslant 2, \\ \breve{\gamma}_n = \frac{1}{s_1}\Big(\breve{\gamma}_{n-2} - \sum\limits_{k=2}^{n} s_k \breve{\gamma}_{n-k+1} + \sum\limits_{k=0}^{n-1} a_k b_{n-k}\Big), \ n \geqslant 2. \end{cases}$$

- *If* $\{a_0=0, b_0=0, c_0\neq 0\}$ *then* $\varphi(0)=0$, $\varphi(1)=(3-\mathbb{E}S)/c_0$ *and*

$$\varphi(u+1)=\frac{1}{s_2}\Big((1-s_3)\varphi(u)-\sum_{k=1}^{u-1}\varphi(k)s_{u+3-k}$$
$$+c_0\varphi(1)\sum_{k=0}^{u+2}a_kb_{u+2-k}\Big),\ u\in\mathbb{N}. \tag{4.20}$$

- *If* $\{a_0=0, b_0\neq 0, c_0=0\}$ *then* $\varphi(0)=s_2\varphi(1)$, $\varphi(1)=(3-\mathbb{E}S)/(s_2+b_0c_1)$ *and*

$$\varphi(u+1)=\frac{1}{s_2}\Big((1-s_3)\varphi(u)-\sum_{k=1}^{u-1}\varphi(k)s_{u+3-k}$$
$$+a_{u+1}b_0c_1\varphi(1)\Big),\ u\in\mathbb{N}. \tag{4.21}$$

- *If* $\{a_0\neq 0, b_0=0, c_0=0\}$ *then* $\varphi(0)=3-\mathbb{E}S$, $\varphi(1)=(3-\mathbb{E}S)/s_2$ *and*

$$\varphi(u+1)=\frac{1}{s_2}\Big((1-s_3)\varphi(u)+\sum_{k=1}^{u-1}\varphi(k)s_{u+3-k}\Big),\ u\in\mathbb{N}. \tag{4.22}$$

- *If* $\{a_0=a_1=0, b_0\neq 0, c_0\neq 0\}$ *then* $\varphi(0)=0$, $\varphi(1)=(3-\mathbb{E}S)/(1/a_2+c_0)$ *and the recursion formula* (4.20) *is satisfied.*
- *If* $\{a_0\neq 0, b_0=b_1=0, c_0\neq 0\}$ *then* $\varphi(0)=0$, $\varphi(1)=(3-\mathbb{E}S)/c_0$ *and the same recursion formula* (4.20) *holds.*
- *If* $\{a_0\neq 0, b_0\neq 0, c_0=c_1=0\}$ *then* $\varphi(0)=3-\mathbb{E}S$, $\varphi(1)=(3-\mathbb{E}S)/s_2$ *and the recursion formula* (4.22) *is satisfied.*

We observe that all formulas presented in Theorem 4.5 can be used to calculate numerical values of survival or ruin probabilities for an arbitrary three-seasonal risk model and for an arbitrary initial surplus value u. The algorithms based on the derived relations work quite quickly and return accurate values.

Now we present the proof of the Theorem 4.5. Equality (4.15) from the previous section plays a crucial role in the proof.

Proof. Let we consider the case $\mathbb{E}S<3$. First we prove that $\varphi(\infty)=1$. According to the definition

$$\varphi(\infty)=\lim_{u\to\infty}\mathbb{P}\bigg(\bigcap_{n=1}^{\infty}\Big\{\sum_{i=1}^{n}(Z_i-1)<u\Big\}\bigg)=\lim_{u\to\infty}\mathbb{P}\Big(\sup_{u\geqslant 1}\eta_n<u\Big),$$

where
$$\eta_n = \sum_{i=1}^{n} (Z_i - 1),\ n \in \mathbb{N}.$$
If $n = 3N$, $N \in \mathbb{N}$, then
$$\frac{\eta_n}{n} = \frac{\eta_{3N}}{3N} = \frac{1}{3}\bigg(\frac{1}{N}\sum_{i=1}^{N}(Z_{3i-2} - 1) + \frac{1}{N}\sum_{i=1}^{N}(Z_{3i-1} - 1) + \frac{1}{N}\sum_{i=1}^{N}(Z_{3i} - 1)\bigg).$$
If $n = 3N + 1$, $N \in \mathbb{N}$, then
$$\frac{\eta_n}{n} = \frac{\eta_{3N+1}}{3N+1} = \frac{N+1}{3N+1}\ \frac{1}{N+1}\sum_{i=1}^{N+1}(Z_{3i-2} - 1) + \frac{N}{3N+1}\bigg(\frac{1}{N}\sum_{i=1}^{N}(Z_{3i-1} - 1) + \frac{1}{N}\sum_{i=1}^{N}(Z_{3i} - 1)\bigg).$$
If $n = 3N + 2$, $N \in \mathbb{N}$, then
$$\frac{\eta_n}{n} = \frac{\eta_{3N+2}}{3N+2} = \frac{N}{3N+2}\ \frac{1}{N}\sum_{i=1}^{N}(Z_{3i} - 1) + \frac{N+1}{3N+2}\bigg(\frac{1}{N+1}\sum_{i=1}^{N+1}(Z_{3i-2} - 1) + \frac{1}{N+1}\sum_{i=1}^{N+1}(Z_{3i-1} - 1)\bigg).$$
Hence, the strong law of large numbers implies that
$$\frac{\eta_n}{n} \underset{n\to\infty}{\to} \frac{1}{3}(\mathbb{E}Z_1 - 1 + \mathbb{E}Z_2 - 1 + \mathbb{E}Z_3 - 1) = \frac{\mathbb{E}S - 3}{3}$$
almost surely.

It follows from this that
$$\mathbb{P}\bigg(\sup_{m\geqslant n}\Big|\frac{\eta_m}{m} + \mu\Big| < \frac{\mu}{2}\bigg) \underset{n\to\infty}{\to} 1 \tag{4.23}$$
with $\mu := (\mathbb{E}S - 3)/3 > 0$.

For an arbitrary positive u and arbitrary $N \in \mathbb{N}$, we have

$$\begin{aligned}\mathbb{P}\Big(\sup_{n\geqslant 1}\eta_n < u\Big) &\geqslant \mathbb{P}\Big(\Big(\bigcap_{n=1}^{N}\{\eta_n \leqslant \frac{u}{2}\}\Big)\cap\Big(\bigcap_{n=N+1}^{\infty}\{\eta_n \leqslant \frac{u}{2}\}\Big)\Big)\\ &\geqslant \mathbb{P}\Big(\bigcap_{n=1}^{N}\{\eta_n \leqslant \frac{u}{2}\}\Big) + \mathbb{P}\Big(\bigcap_{n=N+1}^{\infty}\{\eta_n \leqslant 0\}\Big) - 1\\ &\geqslant \mathbb{P}\Big(\bigcap_{n=1}^{N}\{\eta_n \leqslant \frac{u}{2}\}\Big) + \mathbb{P}\Big(\sup_{m\geqslant N+1}\Big|\frac{\eta_m}{m}+\mu\Big| < \frac{\mu}{2}\Big) - 1.\end{aligned}$$

The last inequality implies that

$$\lim_{u\to\infty}\mathbb{P}\Big(\sup_{n\geqslant 1}\eta_n < u\Big) \geqslant \mathbb{P}\Big(\sup_{m\geqslant N+1}\Big|\frac{\eta_m}{m}+\mu\Big| < \frac{\mu}{2}\Big)$$

for an arbitrary $N \in \mathbb{N}$.
Hence, according to (4.23), we have that $\varphi(\infty) = 1$.
Substituting this into (4.15), we get

$$3 - \mathbb{E}S = \varphi(0) + b_0c_0\varphi(2) + b_0c_1\varphi(1) + c_0\varphi(1). \tag{4.24}$$

In addition, equality (4.8) can be rewritten in the following way

$$\begin{aligned}\varphi(u) = \sum_{k=0}^{u+2} s_{u+2-k}\,\varphi(k+1) - a_{u+1}b_0c_0\varphi(2) - a_{u+1}b_0c_1\varphi(1)\\ - c_0\varphi(1)\sum_{k=0}^{u+2} a_k b_{u+2-k}, \quad u \in \mathbb{N}_0.\end{aligned} \tag{4.25}$$

Now we consider the last two formulas to get a suitable recursion procedure described in Theorem 4.5.

- At first let $s_0 = a_0b_0c_0 \neq 0$ and let sequences α_n, β_n, γ_n be defined in the formulation of Theorem 4.5.

We prove (4.16) by induction. We observe that relation (4.24) implies immediately:

$$\begin{aligned}\varphi(0) &= \alpha_0\varphi(0) + \beta_0\varphi(1) + \gamma_0(3 - \mathbb{E}S),\\ \varphi(1) &= \alpha_1\varphi(0) + \beta_1\varphi(1) + \gamma_1(3 - \mathbb{E}S),\\ \varphi(2) &= \alpha_2\varphi(0) + \beta_2\varphi(1) + \gamma_2(3 - \mathbb{E}S).\end{aligned}$$

Now suppose that equality (4.16) holds for all $n = 0, 1, \dots, N-1$, and we will prove that (4.16) holds for $n = N$. By (4.25) we have

$$\begin{aligned}\varphi(N-3) = &\sum_{k=0}^{N-1} s_{N-1-k}\ \varphi(k+1) - a_{N-2}b_0c_0\varphi(2) - a_{N-2}b_0c_1\varphi(1)\\ &- c_0\varphi(1)\sum_{k=0}^{N-1} a_k b_{N-1-k}.\end{aligned}$$

Therefore, using the induction hypothesis, we get

$$\begin{aligned}s_0\varphi(N) &= \varphi(N-3) - \sum_{k=1}^{N-1} s_k\varphi(N-k) + a_{N-2}b_0c_0\varphi(2)\\ &\qquad + a_{N-2}b_0c_1\varphi(1) + c_0\varphi(1)\sum_{k=0}^{N-1} a_k b_{N-1-k}\\ &= \alpha_{N-3}\varphi(0) + \beta_{N-3}\varphi(1) + \gamma_{N-3}(3 - \mathbb{E}S)\\ &\qquad - \sum_{k=1}^{N-1} s_k(\alpha_{N-k}\varphi(0) + \beta_{N-k}\varphi(1)\\ &\qquad + \gamma_{N-k}(3 - \mathbb{E}S)) + a_{N-2}b_0c_0\varphi(2) + a_{N-2}b_0c_1\varphi(1)\\ &\qquad + c_0\varphi(1)\sum_{k=0}^{N-1} a_k b_{N-1-k}. \qquad (4.26)\end{aligned}$$

Since

$$\varphi(2) = -\frac{1}{b_0c_0}\varphi(0) - \frac{c_1}{c_0}\varphi(1) - \frac{1}{b_0}\varphi(1) + \frac{3 - \mathbb{E}S}{b_0c_0}$$

due to (4.24), we obtain from (4.26) that

$$\begin{aligned}\varphi(N) = \varphi(0)\, \frac{1}{s_0}\Big(\alpha_{N-3} - \sum_{k=1}^{N-1} s_k\alpha_{N-k} - a_{N-2}\Big)\\ + \varphi(1)\, \frac{1}{s_0}\Big(\beta_{N-3} - \sum_{k=1}^{N-1} s_k\beta_{N-k} - a_{N-2}c_0 + c_0\sum_{k=0}^{N-1} a_k b_{N-1-k}\Big)\\ + (3-\mathbb{E}S)\, \frac{1}{s_0}\Big(\gamma_{N-3} - \sum_{k=1}^{N-1} s_k\gamma_{N-k} + a_{N-2}\Big)\\ = \alpha_N\varphi(0) + \beta_N\varphi(1) + \gamma_N(3-\mathbb{E}S).\end{aligned}$$

Hence, the desired relation (4.16) holds for all $n \in \mathbb{N}_0$ by induction.

• If $\{a_0 = 0, b_0 \neq 0, c_0 \neq 0, a_1 \neq 0\}$ then $s_0 = 0$ and $s_1 \neq 0$. Equality (4.25) with $u = 0$ implies that $\varphi(0) = 0$. The recursive relation (4.17) can be derived from the basic equalities (4.24) and (4.25) in the same manner as was derived relation (4.16).

• If $\{a_0 \neq 0, b_0 = 0, c_0 \neq 0, b_1 \neq 0\}$ then it follows from equality (4.24) that $3-\mathbb{E}S = \varphi(0)+c_0\varphi(1)$. Hence $\varphi(1) = \tilde{\alpha}_1\varphi(0)+\tilde{\gamma}_1(3-\mathbb{E}S)$. This is equality (4.18) for $n = 1$. The validity of (4.18) for other n can be derived from (4.25) using the induction arguments.

• In the case $\{a_0 \neq 0, b_0 \neq 0, c_0 = 0, c_1 \neq 0\}$, formula (4.19) follows from (4.24) if $n = 1$. For other n formula (4.19) follows from (4.25) using the induction arguments again.

• In the case $\{a_0 = 0, b_0 = 0, c_0 \neq 0\}$, we have that $s_0 = s_1 = 0$ and $s_2 \neq 0$ because of $\mathbb{E}S < 3$. It follows immediately from (4.25) that $\varphi(0) = 0$, whereas from (4.24) it follows that $\varphi(1) = (3-\mathbb{E}S)/s_2$. Finally, the recursive formula (4.20) we can get from (4.25) using the above induction procedure.

• In the case $\{a_0 = 0, b_0 \neq 0, c_0 = 0\}$, similarly as in the previous one we derive that $\varphi(0) = s_2\varphi(1)$ from (4.25), we derive that $3-\mathbb{E}S = \varphi(0)+b_0c_1\varphi(1)$ from (4.24) and the desired formula (4.21) we derive from (4.25) again.

• The case $\{a_0 \neq 0, b_0 = 0, c_0 = 0\}$ is considered fully analogously as the both previous cases. Here we omit details.

We have that $\mathbb{E}S < 3$. So, it remains to study the following possible cases:

$$\{a_0 = a_1 = 0, b_0 \neq 0, c_0 \neq 0\}, \{a_0 \neq 0, b_0 = b_1 = 0, c_0 \neq 0\},$$
$$\{a_0 \neq 0, b_0 \neq 0, c_0 = c_1 = 0\}.$$

In all these cases, the presented recursion relations follow from equality (4.25) and the initial values of survival probability $\varphi(0)$ and $\varphi(1)$ can be obtained using equality (4.24) together with equality (4.25) with $u = 0$ or $u = 1$. Theorem 4.5 is proved. □

A few numerical examples for calculation of the ruin probabilities in the various versions of the three-seasonal risk model are presented in Chapter 5 (Examples 5, 6, 7).

Chapter 5

Numerical examples

In this section, we present examples of computing numerical values of the finite-time ruin probability and the ultimate ruin probability for various discrete-time multi-risk models. All calculations are carried out using software MATHEMATICA.

EXAMPLE 1. Let us consider the bi-risk model generated by r.v.s X and Y having the following simple distributions:

X	0	1	2
$\mathbb{P}$	3/4	1/8	1/8

;

Y	0	1	2
$\mathbb{P}$	1/10	8/10	1/10

.

Using Theorem 2.1, we obtain Table 1 of the values of the function $\psi(u, T)$. The last row of this table shows the values of $\psi(u)$ obtained by Theorems 3.1 and 3.3. Note that the net profit condition is satisfied in this example.

Table 1. Values of functions $\psi(u, T)$ and $\psi(u)$ for the model in Example 1.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	20	30
1	0.25	0.125	0	0	0	0	0	0	0	0	0	0	0
2	0.484	0.258	0.064	0.017	0.002	0	0	0	0	0	0	0	0
3	0.542	0.285	0.087	0.023	0.003	0	0	0	0	0	0	0	0
4	0.606	0.345	0.134	0.046	0.012	0.002	0	0	0	0	0	0	0
5	0.620	0.362	0.147	0.053	0.015	0.004	0.001	0	0	0	0	0	0
6	0.650	0.398	0.180	0.074	0.026	0.008	0.002	0	0	0	0	0	0
7	0.659	0.408	0.190	0.081	0.029	0.009	0.002	0.001	0	0	0	0	0
8	0.677	0.433	0.215	0.098	0.039	0.014	0.005	0.001	0	0	0	0	0
9	0.683	0.440	0.222	0.104	0.043	0.016	0.006	0.002	0	0	0	0	0
10	0.695	0.458	0.241	0.119	0.052	0.022	0.008	0.003	0.001	0	0	0	0
20	0.737	0.522	0.314	0.182	0.101	0.053	0.027	0.013	0.006	0.003	0.001	0	0
30	0.754	0.549	0.348	0.216	0.129	0.075	0.043	0.023	0.012	0.007	0.003	0	0
40	0.762	0.563	0.367	0.235	0.146	0.090	0.054	0.032	0.018	0.010	0.006	0	0
50	0.768	0.572	0.378	0.248	0.158	0.100	0.062	0.038	0.023	0.014	0.008	0	0
60	0.771	0.578	0.386	0.256	0.166	0.107	0.068	0.043	0.027	0.016	0.100	0	0
70	0.773	0.582	0.392	0.262	0.172	0.112	0.073	0.047	0.030	0.019	0.012	0	0
80	0.775	0.584	0.396	0.266	0.176	0.116	0.076	0.049	0.032	0.020	0.013	0	0
90	0.776	0.587	0.399	0.270	0.180	0.119	0.079	0.052	0.034	0.022	0.014	0.0001	0
100	0.777	0.589	0.401	0.272	0.182	0.122	0.081	0.053	0.035	0.023	0.015	0.0001	0
300	0.780	0.594	0.409	0.281	0.191	0.130	0.089	0.061	0.041	0.028	0.019	0.0004	0.00001
∞	0.780	0.594	0.409	0.281	0.191	0.131	0.089	0.061	0.041	0.028	0.019	0.0004	0.00001

Example 2. Let us now consider the bi-risk model generated by a r.v. X as in Example 1 and a r.v. Y such that $\mathbb{P}(Y = 1) = 9/10$ and $\mathbb{P}(Y = 2) = 1/10$. In such a case, we have that $b_0 = \mathbb{P}(Y = 0) = 0$ and $\mu_{X,Y} < 1$. We fill Table 2 by applying Theorems 2.1 and 3.2.

Table 2. Values of functions $\psi(u, T)$ and $\psi(u)$ for the model in Example 2.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	20	30
1	0.25	0.125	0	0	0	0	0	0	0	0	0	0	0
2	0.494	0.269	0.067	0.019	0.002	0	0	0	0	0	0	0	0
3	0.557	0.297	0.092	0.025	0.004	0	0	0	0	0	0	0	0
4	0.630	0.363	0.146	0.050	0.014	0.003	0.001	0	0	0	0	0	0
5	0.644	0.382	0.160	0.059	0.017	0.004	0.001	0	0	0	0	0	0
6	0.678	0.424	0.199	0.084	0.029	0.009	0.002	0.001	0	0	0	0	0
7	0.687	0.436	0.210	0.092	0.034	0.011	0.003	0.001	0	0	0	0	0
8	0.709	0.465	0.240	0.113	0.047	0.017	0.006	0.002	0	0	0	0	0
9	0.714	0.473	0.249	0.120	0.051	0.020	0.007	0.002	0.001	0	0	0	0
10	0.729	0.495	0.272	0.139	0.063	0.027	0.010	0.004	0.001	0	0	0	0
20	0.779	0.575	0.368	0.225	0.130	0.072	0.038	0.019	0.009	0.004	0.002	0	0
30	0.801	0.612	0.416	0.275	0.174	0.107	0.064	0.037	0.020	0.011	0.006	0	0
40	0.813	0.634	0.446	0.307	0.204	0.133	0.085	0.053	0.032	0.019	0.011	0	0
50	0.821	0.648	0.466	0.329	0.226	0.153	0.101	0.066	0.042	0.026	0.016	0	0
60	0.826	0.658	0.480	0.345	0.243	0.168	0.115	0.077	0.051	0.033	0.021	0.0001	0
80	0.833	0.672	0.500	0.368	0.266	0.190	0.135	0.095	0.065	0.045	0.030	0.0003	0
100	0.838	0.680	0.512	0.382	0.281	0.205	0.149	0.107	0.076	0.054	0.038	0.0006	0
300	0.849	0.701	0.542	0.419	0.321	0.246	0.188	0.144	0.110	0.084	0.064	0.0037	0.0002
∞	0.850	0.704	0.546	0.424	0.327	0.252	0.194	0.150	0.116	0.089	0.069	0.0052	0.0004

Example 3. Let now us consider the three claims model generated by r.v.'s

X, Y and Z having the following simple distributions:

X	0	1	2
$\mathbb{P}$	0.92	0.07	0.01

;

Y	1	2
$\mathbb{P}$	0.95	0.05

;

Z	1	2
$\mathbb{P}$	0.85	0.15

.

Using Theorem 2.1 we obtain Table 3 of the values of the function $\psi(u,T)$. The last row of this table shows values of ultimate time function $\psi(u)$ obtained by Theorem 3.5.
Note that $\mathbb{E}Z_1 + \mathbb{E}Z_2/2 + \mathbb{E}Z_3/3 < 1$ in the case under consideration.

Table 3. Values of functions $\psi(u,T)$ and $\psi(u)$ for the model in Example 3.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	15	20
1	0.080	0.010	0	0	0	0	0	0	0	0	0	0	0
2	0.196	0.031	0.003	0	0	0	0	0	0	0	0	0	0
3	0.371	0.084	0.013	0.002	0	0	0	0	0	0	0	0	0
4	0.450	0.128	0.027	0.004	0.001	0	0	0	0	0	0	0	0
5	0.456	0.132	0.028	0.004	0.001	0	0	0	0	0	0	0	0
6	0.624	0.261	0.081	0.019	0.004	0.001	0	0	0	0	0	0	0
7	0.628	0.265	0.082	0.020	0.004	0.001	0	0	0	0	0	0	0
8	0.636	0.273	0.087	0.022	0.004	0.001	0	0	0	0	0	0	0
9	0.656	0.296	0.101	0.028	0.006	0.001	0	0	0	0	0	0	0
10	0.673	0.318	0.115	0.034	0.008	0.002	0	0	0	0	0	0	0
11	0.674	0.319	0.116	0.034	0.009	0.002	0	0	0	0	0	0	0
12	0.722	0.386	0.164	0.058	0.018	0.005	0.001	0	0	0	0	0	0
13	0.724	0.388	0.166	0.059	0.018	0.005	0.001	0	0	0	0	0	0
14	0.727	0.393	0.170	0.062	0.019	0.005	0.001	0	0	0	0	0	0
15	0.736	0.407	0.181	0.068	0.022	0.006	0.002	0	0	0	0	0	0
16	0.744	0.420	0.192	0.075	0.026	0.008	0.002	0.001	0	0	0	0	0
17	0.744	0.421	0.193	0.076	0.026	0.008	0.002	0.001	0	0	0	0	0
18	0.769	0.463	0.232	0.101	0.039	0.013	0.004	0.001	0	0	0	0	0
19	0.770	0.464	0.233	0.102	0.039	0.014	0.004	0.001	0	0	0	0	0
20	0.772	0.468	0.236	0.104	0.040	0.014	0.005	0.001	0	0	0	0	0
30	0.818	0.556	0.331	0.177	0.087	0.039	0.017	0.006	0.002	0.001	0	0	0
40	0.838	0.598	0.381	0.222	0.120	0.061	0.029	0.013	0.005	0.002	0	0	0
50	0.855	0.635	0.428	0.268	0.158	0.087	0.046	0.023	0.011	0.005	0.002	0	0
∞	0.990	0.973	0.955	0.936	0.918	0.900	0.883	0.865	0.849	0.832	0.816	0.739	0.670

Example 4. We say that a r.v. ξ has the Poisson distribution with parameter $\lambda > 0$ ($\xi \sim \Pi(\lambda)$) if $\mathbb{P}(\xi = k) = \mathrm{e}^{-\lambda}\lambda^k/k!$, $k \in \mathbb{N}_0$. Consider the multi-risk model generated by three r.v.s $X \sim \Pi(1/3)$, $Y \sim \Pi(1/4)$, and $Z \sim \Pi(1/5)$. We fill Table 4 of numerical values of the functions $\psi(u,T)$ and $\psi(u)$ using the same theorems as in Example 3.

Table 4. Ruin probabilities for the Poisson model.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	15	20
1	0.283	0.045	0.005	0	0	0	0	0	0	0	0	0	0
2	0.367	0.088	0.017	0.003	0	0	0	0	0	0	0	0	0
3	0.397	0.110	0.025	0.005	0.001	0	0	0	0	0	0	0	0
4	0.416	0.125	0.033	0.008	0.002	0	0	0	0	0	0	0	0
5	0.420	0.129	0.035	0.009	0.002	0	0	0	0	0	0	0	0
6	0.430	0.139	0.040	0.011	0.003	0.001	0	0	0	0	0	0	0
7	0.432	0.140	0.041	0.012	0.003	0.001	0	0	0	0	0	0	0
8	0.434	0.143	0.043	0.012	0.003	0.001	0	0	0	0	0	0	0
9	0.436	0.145	0.044	0.013	0.004	0.001	0	0	0	0	0	0	0
10	0.438	0.146	0.045	0.014	0.004	0.001	0	0	0	0	0	0	0
11	0.438	0.147	0.046	0.014	0.004	0.001	0	0	0	0	0	0	0
12	0.439	0.148	0.047	0.014	0.004	0.001	0	0	0	0	0	0	0
13	0.439	0.149	0.047	0.014	0.004	0.001	0	0	0	0	0	0	0
14	0.440	0.149	0.047	0.015	0.004	0.001	0	0	0	0	0	0	0
15	0.440	0.149	0.047	0.015	0.004	0.001	0	0	0	0	0	0	0
16	0.440	0.150	0.048	0.015	0.005	0.001	0	0	0	0	0	0	0
17	0.440	0.150	0.048	0.015	0.005	0.001	0	0	0	0	0	0	0
18	0.441	0.150	0.048	0.015	0.005	0.001	0	0	0	0	0	0	0
19	0.441	0.150	0.048	0.015	0.005	0.001	0	0	0	0	0	0	0
20	0.441	0.150	0.048	0.015	0.005	0.001	0	0	0	0	0	0	0
30	0.441	0.150	0.048	0.015	0.005	0.001	0	0	0	0	0	0	0
40	0.441	0.150	0.048	0.015	0.005	0.001	0	0	0	0	0	0	0
50	0.441	0.150	0.048	0.015	0.005	0.001	0	0	0	0	0	0	0
∞	0.493	0.226	0.132	0.101	0.092	0.089	0.088	0.087	0.087	0.087	0.087	0.087	0.087

Example 5. Suppose that the three-seasonal discrete-time risk model is generated by r.v.'s

Z_1	0	1	2
$\mathbb{P}$	0.5	0.25	0.25

,

Z_2	0	1	2
$\mathbb{P}$	0.4	0.3	0.3

,

Z_3	0	1	2
$\mathbb{P}$	0.3	0.35	0.35

.

In Table 5, we can find values of the finite-time ruin probability for initial surplus $u \in \{0, 1, \ldots 10, 20\}$ and times $T \in \{1, 2, \ldots, 10, 20\}$ together with values of the ultimate ruin probability for the same u.

Table 5. Values of functions $\psi(u, T)$ and $\psi(u)$ for the model in Example 5.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	20
1	0.5	0.25	0	0	0	0	0	0	0	0	0	0
2	0.65	0.325	0.075	0	0	0	0	0	0	0	0	0
3	0.703	0.404	0.128	0.026	0	0	0	0	0	0	0	0
4	0.733	0.445	0.169	0.046	0.007	0	0	0	0	0	0	0
5	0.751	0.475	0.2	0.066	0.014	0.002	0	0	0	0	0	0
6	0.768	0.503	0.233	0.089	0.026	0.005	0.001	0	0	0	0	0
7	0.779	0.523	0.256	0.106	0.035	0.009	0.002	0	0	0	0	0
8	0.788	0.538	0.275	0.122	0.045	0.014	0.003	0.001	0	0	0	0
9	0.796	0.554	0.295	0.139	0.056	0.019	0.006	0.001	0	0	0	0
10	0.802	0.566	0.310	0.152	0.065	0.024	0.008	0.002	0	0	0	0
20	0.836	0.632	0.402	0.243	0.138	0.075	0.038	0.018	0.008	0.003	0.001	0
∞	0.877	0.722	0.541	0.404	0.301	0.224	0.167	0.125	0.093	0.069	0.052	0.003

Numerical values of the finite-time ruin probability are calculated using the algorithm presented in Theorem 4.3. While values of the ultimate ruin probability are obtained using formulas of Theorem 4.5. Namely, first we observe that $\mathbb{E}S = 2.7$ and $s_0 \neq 0$ in the case. So, Eq. (4.16) holds for an arbitrary $n \in \mathbb{N}_0$. In particular,

$$\begin{cases} \varphi(250) = \alpha_{250}\varphi(0) + \beta_{250}\varphi(1) + 0.3\gamma_{250}, \\ \varphi(251) = \alpha_{251}\varphi(0) + \beta_{251}\varphi(1) + 0.3\gamma_{251}, \end{cases}$$

According to the first statement of Theorem 4.5 we can suppose that $\varphi(250) \approx \varphi(251) \approx 1$. So, we get $\varphi(0)$ and $\varphi(1)$ from the above system after calculating values of $\{\alpha_0, \alpha_1, \ldots \alpha_{251}\}$, $\{\beta_0, \beta_1, \ldots, \beta_{251}\}$ and $\{\gamma_0, \gamma_1, \ldots, \gamma_{251}\}$. Now it remains to use equality (4.16) again to obtain values $\varphi(u) = 1 - \psi(u)$ for initial surplus values $u \in \{2, 3, \ldots, 10, 20\}$.

Example 6. Suppose now that the three-seasonal discrete-time risk model is generated by three Poison distributions: Z_1 with parameter $1/2$, Z_2 with parameter $2/3$ and Z_3 with parameter $4/5$. In Table 6, there are presented values of the finite-time ruin probability for $u \in \{0, 1, \ldots 10, 20\}$, $T \in \{1, 2, \ldots, 10, 20\}$ and values of the ultimate ruin probability for $u \in \{0, 1, \ldots 10, 20\}$. All calculations are made similarly as in the previous example.

Table 6. Ruin probabilities for the Poison model.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	20
1	0.393	0.09	0.014	0.002	0	0	0	0	0	0	0	0
2	0.481	0.152	0.037	0.008	0.001	0	0	0	0	0	0	0
3	0.535	0.205	0.066	0.019	0.005	0.001	0	0	0	0	0	0
4	0.549	0.221	0.075	0.023	0.006	0.002	0	0	0	0	0	0
5	0.562	0.236	0.086	0.028	0.009	0.002	0.001	0	0	0	0	0
6	0.576	0.254	0.099	0.036	0.012	0.004	0.001	0	0	0	0	0
7	0.581	0.26	0.103	0.038	0.013	0.004	0.001	0	0	0	0	0
8	0.585	0.266	0.109	0.042	0.015	0.005	0.002	0.001	0	0	0	0
9	0.591	0.274	0.115	0.046	0.017	0.006	0.002	0.001	0	0	0	0
10	0.593	0.277	0.118	0.048	0.018	0.007	0.002	0.001	0	0	0	0
20	0.605	0.295	0.134	0.059	0.026	0.011	0.005	0.002	0.001	0.0003	0.0001	0
∞	0.609	0.3	0.139	0.064	0.029	0.013	0.006	0.003	0.001	0.001	0.0002	0

EXAMPLE 7. We say that r.v. ξ has the geometric distribution with parameter $p \in (0,1)$ and we denote $\xi \sim \mathcal{G}(p)$ if $\mathbb{P}(\xi = k) = p(1-p)^k, k \in \mathbb{N}_0$. Suppose that the three-seasonal risk model is generated by r.v.'s $Z_1 \sim \mathcal{G}(3/4)$, $Z_2 \sim \mathcal{G}(2/3)$ and $Z_3 \sim \mathcal{G}(1/3)$. In Table 7, we present values of the finite-time and infinite-time ruin probabilities for this geometric model. Values of initial surplus u and times T we left the same as in the previous examples.

Table 7. Ruin probabilities for the geometric model.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	20
1	0.25	0.063	0.016	0.004	0.001	0	0	0	0	0	0	0
2	0.333	0.111	0.037	0.012	0.004	0.001	0	0	0	0	0	0
3	0.556	0.34	0.218	0.143	0.094	0.063	0.042	0.028	0.019	0.012	0.008	0
4	0.566	0.35	0.226	0.149	0.099	0.066	0.044	0.029	0.019	0.013	0.009	0
5	0.576	0.362	0.236	0.156	0.104	0.069	0.046	0.031	0.021	0.014	0.009	0
6	0.653	0.461	0.334	0.243	0.176	0.127	0.091	0.065	0.046	0.033	0.023	0.001
7	0.657	0.466	0.338	0.247	0.18	0.13	0.093	0.067	0.048	0.034	0.024	0.001
8	0.661	0.471	0.344	0.252	0.184	0.133	0.096	0.069	0.049	0.035	0.025	0.001
9	0.703	0.529	0.406	0.312	0.239	0.181	0.137	0.102	0.076	0.056	0.042	0.002
10	0.705	0.532	0.409	0.315	0.241	0.184	0.139	0.104	0.078	0.057	0.042	0.002
20	0.774	0.635	0.528	0.438	0.363	0.298	0.243	0.197	0.159	0.127	0.101	0.008
∞	0.927	0.879	0.84	0.803	0.769	0.736	0.705	0.675	0.647	0.619	0.593	0.385

Conclusions

In this last Chapter, a brief summary of the results obtained is given.

- The net profit condition for the ultimate ruin probabilities of the discrete-time risk model with inhomogeneous claims was established.
- The ultimate ruin probability of the discrete-time risk model with inhomogeneous claims tends to zero as initial capital u tends to infinity.
- The recursive relations for calculation of the the exact values of the finite-time ruin probabilities of the discrete-time any multi-risk model were obtained.
- The recursive relations for calculation of the the exact values of the ultimate ruin probabilities of the bi-risk model and multi-risk model with three inhomogeneous claim amounts were obtained.
- The recursive relations for calculation of the exact values of the finite-time and ultimate ruin probabilities of the three-seasonal risk model were obtained.
- Numerical values of all obtained recursive relations were calculated using software and presented in this thesis.

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