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DISCRETE APPROXIMATIONS FOR SUMS OF WEAKLY DEPENDENT VARIABLES

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1 Introduction

Research topic

Limit Theorems for sums of random variables (rv's) play a central role in Probability Theory and have a wide range of applications in Statistics. In this thesis, we consider the sums of Markov dependent integer-valued rv's. The majority of known research in this field is restricted to the scheme of sequences, i.e., when all X_j do not depend on n and $S_n = X_1+X_2+\cdots+X_n = S_{n-1}+X_n$. The Central Limit Theorem (CLT) and its improvements (the Berry-Esseen type estimates and the Edgeworth type expansions) are investigated in [13], [40], [57], [60], [61], [62], [65], [79]. Notably, the main approximation applied to the discrete sum of Markov dependent variables is a continuous Normal distribution. Consequently, only the uniform Kolmogorov metric can be applied. The more general scheme of triangular arrays ($S_n = X_{1,n} + X_{2,n} + \cdots + X_{n,n}$, $S_{n-1} = X_{1,n-1} + \cdots + X_{n-1,n-1}$) is considerably less explored. For this setup Poisson and Compound Poisson (CP) approximations are applied in [50], [55], [89]. However, the majority of CP approximations indirectly assume strong dependency of the transition probabilities on the number of summands n, and are of the trivial order $O(1)$ for the scheme of sequences. Thus, they can not be viewed as complete discrete analogues or the replacements of the Gaussian law. There are very few approximations (mainly related to the Markov binomial (MB) distribution), which are universal in a sense that they can effectively replace the normal approximation for the whole spectrum of the transition probabilities. In other words, the same approximation can be applied when certain transition probabilities converge to zero with the order $O(n^{-1})$ and to the case, when all transition probabilities are absolute constants, see [26], [35], [64]. As far as we know, the case of Poisson type approximations to the symmetric Markov chain was not investigated.

In this thesis, we apply the CP type approximations to the sums $\sum_{k=1}^{n} f(\xi_k)$, where f is integer-valued function and $\xi_1, \xi_2, \cdots, \xi_n$ form a homogeneous Markov chain with the finite state space. The triangular array of Markov chains is used for the case when the value domain of f is $\{0, 1\}$ or $\{-1, 0, 1\}$. In the latter case, a special emphasis is on the symmetry of transition matrix.

Actuality

Discrete analogues of the normal approximation, that hold for stronger metrics and are more precise in the scheme of triangular arrays are of theoretical and practical importance. The initial distribution of Markov dependent variables can be of a very complicated structure. Meanwhile, infinitely divisible CP approximations have explicit structures and are much more convenient for practical calculations, especially when combined with Fourier transforms or recursive algorithms. Research related applications were investigated in [17], [31], [36], [43], [45], [76], [85], [94].

Aims and goals

1. To show that translated Poisson approximation can be preferable to the Normal approximation in the scheme of sequences of Markov dependent integer-valued rv's.

2. To prove Simons-Johnson theorem for the MB distribution and symmetric three-state Markov chain, that is, to demonstrate that convergence to the CP limit holds in stronger than total variation metric.

3. To prove a partial case of the first uniform Kolmogorov theorem for the MB distribution by constructing CP distribution which approximates MB with the accuracy $O(n^{-2/3})$.

4. To construct a CP approximation to symmetric three-state Markov chain with the accuracy comparable to that of accompanying Poisson approximation to the sum of symmetric independent rv's.

5. To construct a second-order CP approximations for symmetric three-state Markov chain.

6. To obtain non-uniform, local and lower bound estimates for CP approximations for symmetric three-state Markov chain.

Novelty

In this thesis, CP approximation is for the first time applied to the sum of symmetric Markov dependent rv's. Its accuracy of approximation is estimated in the total variation, Wasserstein and local metrics. The partial case of the first uniform Kolmogorov theorem is proved for the MB distribution. The Simons-Johnson theorem is proved for MB and symmetric three-state Markov chain. It is demonstrated that translated Poisson distribution can replace the normal law in the analogue of the CLT for the scheme of sequences.

Main results

Translated Poisson approximation to Markov dependent integer-valued rv's distribution is applied (theorem 3.1). MB distribution is approximated by the set of infinitely divisible laws in the total variation and local metrics (theorems 3.2, 3.3). Simons-Johnson theorem for MB distribution and for symmetric three-state Markov chain is obtained (theorems 3.4, 3.9). Upper and lower bound estimates for approximation of a symmetric three-state Markov chain by the accompanying CP law are established in the total variation, local and Wasserstein metrics (theorems 3.5, 3.7). The approximation of symmetric three-state Markov chain is improved by the second-order approximations (theorem 3.6). Non-uniform local estimates are proved (theorem 3.8).

Statements presented for defence

1. The closeness of the MB distribution to the set of all infinitely divisible distributions in total variation is of the order $O(n^{-2/3})$.

2. Symmetry improves the approximation of Markov dependent variables by CP law. The accuracy of approximation for the sum of a three point distribution is of the order $O(n^{-1})$ and is equivalent to the accuracy of CP approximation in the case of independent rv's.

3. The CP approximations used for the symmetric three-state Markov chain are universal in the sense, that they can be applied for the case of triangular arrays (when some of the transition probabilities are of the order $o(1)$) and for the case of sequences (when all transition probabilities are absolute constants).

4. Lower bound estimates in total variation, local and Wasserstein metrics for constructed CP type approximation to symmetric three-state Markov chain confirm that upper bound estimates are of the right order.

5. The Simons-Johnson theorem holds for the MB distribution and the distribution based on a symmetric three-state Markov chain, that is the convergence to a CP limit holds with exponential weights.

6. Second order CP type approximations for symmetric three-state Markov chain improve the rate of accuracy to $O(n^{-2})$.

7. In the scheme of homogeneous Markov chains the translated Poisson approximation has the same accuracy as the Normal approximation, but is structurally more adequate.

Methods

In this thesis, the characteristic function method is used.

Publications

1. J. Šliogerė, V. Čekanavičius. Two limit theorems for Markov binomial distribution. *Lithuanian Math. J.* 55(3), 451–463, 2015.

2. J. Šliogerė, V. Čekanavičius. Approximation of symmetric three-state Markov chain by compound Poisson law. *to appear in Lithuanian Math. J.*

Conferences

1. J. Šliogerė, V. Čekanavičius. The approximation of Markov binomial distribution by negative binomial law. *11th International Vilnius Conference on Probability Theory and Mathematical Statistics : Abstracts of Communication*, p. 273, June 30 – July 4, 2014, Vilnius, Lithuania.

2. J. Šliogerė, The approximation of Markov binomial distribution by compound Poisson distribution with compounding geometric distribution. *8th Nordic Econometric Meeting.* May 28–30, 2015, Helsinki, Finland.

3. J. Šliogerė, V. Čekanavičius. Approximation of symmetric Markov chain by compound Poisson law. *Lietuvos matematikų draugijos LVI konferencija*, June 16 – 17, 2015, Kaunas.

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Structure of the thesis

In Chapter 2, the necessary notation is introduced and an overview of known results is given. Chapter 3 contains formulations of the results. All proofs and auxiliary results are given in Chapter 4. Finally the conclusions and bibliography are presented Chapter 5.

Notation	**Descriptions**
$\mathbb{Z}$	The set of integers.
$\mathbb{N}$	The set of natural numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{D}$	The set of all infinitely divisible laws.
MB	Markov binomial.
CP	Compound Poisson.
rv	Random variable.
$\mathcal{L}(S_n)$	The distribution of S_n.
$\mathrm{Pois}(\lambda)$	Poisson distribution.
$\mathrm{TP}(\mu, \sigma^2)$	Translated Poisson distribution.
$\mathrm{Bi}(n, p)$	Binomial distribution.
$\Phi(x)$	Normal distribution function.
F_n	The distribution of the sum of a three-state Markov chain.
$\|U\|$	The total variation norm.
$\|U\|_\infty$	The local norm.
$\|U\|_W$	The Wasserstein norm.
$\|U\|_K$	The Kolmogorov norm.
I_k, $I = I_0$	The distribution concentrated at an integer $k \in \mathbb{Z}$.
$\widehat{U}(t)$	Fourier-Stieltjes transform.
C, C_i	Positive absolute constants.
θ	Any complex number satisfying $\lvert\theta\rvert \leqslant 1$.
Θ	Any signed measure satisfying $\|\Theta\| \leqslant 1$.
$\lfloor \cdot \rfloor$	The integer part of a number.
$\{\cdot\}$	The fractional part of a number.
$\pi(\cdot)$	The initial distribution of Markov chain.

2 Known results

2.1 Notation

In this thesis we use four metrics. The total variation metric between two probability measures P and Q on $\mathbb{Z}$ is defined in the following way:

$$\|P-Q\| = \sum_{k=-\infty}^{\infty} |P\{k\}-Q\{k\}| = 2\sup_{A\subset\mathbb{Z}} |P(A)-Q(A)|.$$

This definition can be rewritten in an equivalent form:

$$\|P-Q\| \quad = \quad \sup_{f\in\mathcal{F}} \Big| \sum_{k=-\infty}^{\infty} f(k)P\{k\} - \sum_{k=-\infty}^{\infty} f(k)Q\{k\}\Big|.$$

Here supremum is taken over all functions $f:\mathbb{Z}\to\mathbb{R}$ bounded by 1. The Wasserstein metric (also known as the Dudley, Fortet-Mourier or Kantorovich metric) is defined equivalently by

$$\begin{aligned}\|P-Q\|_W \quad &= \quad \sum_{k=-\infty}^{\infty} |P\{(-\infty,k]\} - Q\{(-\infty,k]\}| \\ &= \quad \sup_{f\in\mathcal{F}_1} \Big| \sum_{k=-\infty}^{\infty} f(k)P\{k\} - \sum_{k=-\infty}^{\infty} f(k)Q\{k\}\Big|.\end{aligned}$$

Here supremum is taken over all functions $f:\mathbb{Z}\to\mathbb{R}$ satisfying $\sup_k |f(k)-f(k-1)| \leqslant 1$. The local and Kolmogorov metrics are respectively denoted by

$$\|P-Q\|_\infty = \sup_{k\in\mathbb{Z}} |P\{k\}-Q\{k\}|, \qquad \|P-Q\|_K = \sup_{k\in\mathbb{Z}} |P\{(-\infty,k]\} - Q\{(-\infty,k]\}|.$$

For an additional information about metrics and their properties see [4] and [9].

Further on we use the following notation. Let I_k denote the distribution concentrated at an integer $k\in\mathbb{Z}$ and set $I=I_0$. Let U and V denote two finite signed measures on $\mathbb{Z}$. Products and powers of U and V are understood in the convolution sense, that is, $U*V\{A\} = \sum_{k=-\infty}^{\infty} U\{A-k\}V\{A\}$ for a set $A\subseteq\mathbb{Z}$; further, $V^0=I$. The exponential of V is denoted by $\exp\{V\} = \sum_{j=0}^{\infty} V^{*j}/j!$. Let $\widehat{V}(t)$ denote the Fourier-Stieltjes transform of V. Note that $\widehat{\exp\{V\}}(t) = \exp\{\widehat{V}(t)\}$, $\widehat{I}(t)=1$, $\widehat{I}_a(t)=\mathrm{e}^{\mathrm{i}ta}$. In the proofs we systematically apply the well known relations:

$$\begin{aligned}\|U*V\| \quad &\leqslant \quad \|U\|\|V\|, \qquad \|U*V\|_K \leqslant \|U\|\|V\|_K, \qquad \|U*V\|_\infty \leqslant \|U\|\|V\|_\infty, \\ |\widehat{U}(t)| \quad &\leqslant \quad \|U\|, \qquad \widehat{U*V}(t) = \widehat{U}(t)\widehat{V}(t), \qquad \|\mathrm{e}^U\| \leqslant \mathrm{e}^{\|U\|}.\end{aligned} \tag{1}$$

Let $U(x) = U\{(-\infty,x]\}$. Let for $y\in\mathbb{R}$ and $j\in\mathbb{N}=\{1,2,3,\dots\}$,

$$\binom{y}{j} = \frac{1}{j!}y(y-1)\dots(y-j+1), \qquad \binom{y}{0} = 1.$$

We define by C all positive absolute constants. In some cases to avoid possible confusion, we supply constants C with indices. Indices in all sections has separate numbering. The notation $C(\cdot)$ is reserved for constants depending on the indicated parameter. Further on we denote by θ any complex number satisfying $|\theta|\leqslant 1$. The values of C and θ can vary from line to line, or even within the same line.

In this thesis the main approximation is a CP distribution with the compounding geometric distribution. Suppose that we have a sequence of independent Bernoulli trials. Assume that probability of success in each independent trial is $0 < p \leqslant 1$. If $\mathcal{X}$ is a geometric rv, it counts the number of attempts needed to obtain the first success. The characteristic function and probability of $\mathcal{X}$ are respectively equal to

$$\varkappa(t) = \frac{p\mathrm{e}^{it}}{1-(1-p)\mathrm{e}^{it}}, \qquad \mathrm{P}(\mathcal{X}=k) = (1-p)^{k-1}p, \quad (k=1,2,\cdots).$$

Let υ be a Poisson rv ($\upsilon \sim P(\lambda)$). Then its characteristic function and probability are respectively

$$y(t) = \exp\{\lambda(\mathrm{e}^{it}-1)\}, \qquad \mathrm{P}(\upsilon = k) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad (k=0,1,...).$$

A CP distribution with the compounding geometric distribution corresponds to a random sum of independent geometric rv's $\mathcal{X}_j$ that are also independent of the number of values υ. Let $\mathcal{Z} = \sum_{j=1}^{\upsilon}\mathcal{X}_j$. The characteristic function of $\mathcal{Z}$ is equal to $\exp\{\lambda(\varkappa(t)-1)\}$.

2.2 The CLT for homogeneous Markov chains

The convergence to the normal law in the case of homogeneous Markov chains with finite number of states was proved by Sirazhdinov [79]. In [62], Nagaev considered homogeneous Markov chains with an arbitrary number of states. We recall the result of [62] related to discrete sums. Let $\mathcal{Y}$ be a space of points ω, let $\mathcal{B}_{\mathcal{Y}}$ be the σ-algebra of its subsets, $p(\omega, \mathcal{A})$ be the transition probability function. Let $p(\omega,\mathcal{A})$ satisfy the following condition: there exists a positive integer k_0 such that

$$\sup_{\omega,\tau,\mathcal{A}} |p^{(k_0)}(\omega,\mathcal{A}) - p^{(k_0)}(\tau,\mathcal{A})| = \delta < 1, \quad \mathcal{A} \in \mathcal{B}_{\mathcal{Y}}, \omega, \tau \in \mathcal{Y}, \tag{2}$$

where $p^{(k_0)}(\omega,\mathcal{A})$ is the transition probability function for k_0 steps. As it is noted in [62], if the condition (2) is satisfied, then there exists a stationary distribution $p(\mathcal{A})$ such that

$$\sup_{\omega\in\mathcal{Y},\mathcal{A}\in\mathcal{B}_{\mathcal{Y}}} |p(\mathcal{A}) - p^{(n)}(\omega,\mathcal{A})| \leqslant \delta^{\lfloor n/k_0 \rfloor} < \delta^{-1}\rho^n.$$

Here $\rho = \delta^{1/k_0}$ and $\lfloor\cdot\rfloor$ denotes the integer part of indicated argument.

Let the sequence of rv's $\mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_n, \cdots$ satisfy the following assumptions

$$\begin{aligned} \mathrm{P}(\mathcal{Y}_1 \in \mathcal{A}) &= \pi(\mathcal{A}), \\ \mathrm{P}(\mathcal{Y}_n \in \mathcal{A}) &= \int_{\mathcal{Y}} p^{(n-1)}(\omega,\mathcal{A})\pi(d\omega). \end{aligned}$$

Here $\pi(\cdot)$ denotes the initial distribution.

Let $f(\omega)$ be a real function defined on $\mathcal{Y}$ and let $f(\omega)$ be measurable with respect to $\mathcal{B}_{\mathcal{Y}}$. As noted in [62], if $\pi(\mathcal{A}) = p(\mathcal{A})$ and $\int_{\mathcal{Y}} f^2(\omega)p(d\omega) < \infty$, then there exists

$$\lim_{n\to\infty} \mathrm{E}\left[\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\left(f(\mathcal{Y}_k) - \int_{\mathcal{Y}} f(\tau)p(d\tau)\right)\right]^2 = \sigma^2 > 0.$$

We denote by $F_{n\pi}(x)$ the distribution function of a normed sum

$$\begin{aligned}\mathcal{S}_n &= \frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{n+1}\Big(f(\Upsilon_k)-\int_{\Upsilon} f(\tau)p(d\tau)\Big)\\ &= \frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{n+1} f(\Upsilon_k)-\frac{n+1}{\sigma\sqrt{n}}\int_{\Upsilon} f(\tau)p(d\tau)=\frac{\tilde{\mathcal{S}}_n}{\sigma\sqrt{n}}-\frac{\mathcal{A}}{\sigma\sqrt{n}}.\end{aligned}$$

Here $\tilde{\mathcal{S}}_n = \sum_{k=1}^{n+1} f(\Upsilon_k)$ and $\mathcal{A} = (n+1)\int_{\Upsilon} f(\tau)p(d\tau)$. We also denote by $\Phi(x)$ the standard normal distribution function and by $\tilde{F}_{n\pi}(x)$ the distribution function of the sum $\tilde{\mathcal{S}}_n$. The following conditions were used in [62].

Condition (H_k): there exists a function $g(x)$ of a real variable such that $\lim_{x\to\infty} g(x)=\infty$ and $\sup_\omega \int_X |f(\tau)|^k g(|f(\tau)|)p(\omega,d\tau)<\infty$.

Condition $(\bar{H})$: let Υ be a countable set of states ω_i, which forms a positive class, $f(\omega_i) = m + k_i h$, where k_i is an integer, m is any real number, $h > 0$, and for arbitrary i and j it is possible to find an index k such that $p_{ik} > 0$ and $p_{jk} > 0$ ($p_{ik} = p(\omega_i,\omega_k)$).

In [62] it is proved that if conditions (H_3) and $(\bar{H})$ are fulfilled, the greatest common divisor k_i equals 1, and $\sum_{k=1}^{\infty}|f(\omega_k)|\pi(\omega_k)<\infty$, then

$$F_{n\pi}(x)-\Phi(x)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\Big(\frac{Q_{1\pi}(x)}{\sqrt{n}}+\frac{S_1(x)}{\sqrt{n}}\Big)+o\Big(\frac{1}{\sqrt{n}}\Big) \tag{3}$$

uniformly in x, where

$$\begin{aligned}S_1(x) &= \frac{h}{\sigma}S_0\Big(\frac{(x+m_n)\sigma\sqrt{n}}{h}\Big), \qquad S_0(x)=\lfloor x\rfloor - x+\frac{1}{2}, \qquad m_n=-\frac{\sqrt{n}m}{\sigma};\\ Q_{1\pi}(x) &= \frac{\lambda_{3p}}{6}(1-x^2)+\lambda_\pi, \qquad \lambda_\pi=\frac{1}{\sigma}\sum_{k=1}^{\infty}\mathrm{E}_\pi\tilde{f}(\Upsilon_k),\\ \lambda_{3p} &= \frac{1}{\sigma^3}\Big(\mathrm{E}_p\tilde{f}^3(\Upsilon_1)+3\sum_{k=1}^{\infty}\mathrm{E}_p\tilde{f}^2(\Upsilon_1)\tilde{f}(\Upsilon_{k+1})+3\sum_{k=1}^{\infty}\mathrm{E}_p\tilde{f}(\Upsilon_1)\tilde{f}^2(\Upsilon_{k+1})\\ &\quad +6\sum_{k,j=1}^{\infty}\mathrm{E}_p\tilde{f}(\Upsilon_1)\tilde{f}(\Upsilon_{k+1})\tilde{f}(\Upsilon_{k+j+1})\Big)\\ \tilde{f}(\omega) &= f(\omega)-\int_{\Upsilon} f(\tau)p(d\tau).\end{aligned}$$

Observe that (3) contains $S_1(x)/\sqrt{n}$. Discrete distribution is approximated by a continuous distribution. Therefore, as is well-known from the classical results of limit theorems, this additional member originates. Moreover, the difference of supports means that the normal approximation is trivial in total variation, that is, $\|F_{n\pi}-\Phi\| = O(1)$.

Next we formulate the local estimate from [62]. If conditions (H_k), $(\bar{H})$ and $\sum_{i=1}^{\infty}|f(\omega_i)|^{k-2}\pi(\omega_i)<\infty$ are fulfilled, then

$$\mathrm{P}\Big(\sum_{1}^{n} f(\Upsilon_i)=mn+sh\Big)=\frac{h}{\sigma\sqrt{n}}\cdot\frac{\mathrm{e}^{-z_{ns}/2}}{\sqrt{2\pi}}\Big(1+\frac{1}{\sqrt{n}}\Big(\frac{\lambda_{3p}}{6}(z_{ns}^3-3z_{ns})-z_{ns}\lambda_\pi\Big)\Big)+o(n^{-1/2}). \tag{4}$$

Here $z_{ns} = \Big(m(n+1) + sh - (n+1)\sum_1^\infty f(\omega_i)p(\omega_i)\Big)/\sigma\sqrt{n}$.

Proof of (3) and (4) is based on the difference of characteristic functions in the neighbourhood of zero. It is proved in [62] that if condition (H_k) is fulfilled and $\int_X |f(\omega_i)|^{k-2}\pi(\omega_i) < \infty$, then there exists ∇, such that for $|t| \leqslant \nabla\sqrt{n}$

$$\left|\widehat{F}_{n\pi}(t) - \mathrm{e}^{-t^2/2}\left(1 + \frac{\alpha(\mathrm{i}t)^3}{\sqrt{n}} + \frac{\lambda_\pi(\mathrm{i}t)}{\sqrt{n}}\right)\right| \leqslant \frac{C}{\sqrt{n}}o(1)|t|^3\mathrm{e}^{-t^2/15} + C|t|\rho_1^n, \tag{5}$$

where $\alpha = \lambda_{3p}/6$ and $\rho_1 = 1/2 + 2/3\rho < 1$.

2.3 Poisson type approximations for Markov chains

Unlike the normal approximation, the accuracy of any Poisson approximation can be estimated in total variation. The Poisson type approximations were considered in [37], [38], [82], [98].

In [10] and [72], the sum of Markov dependent variables was approximated by a translated Poisson law and the accuracy in total variation was proved to be $O(n^{-1/2})$. In both papers the Stein method was applied. We will formulate the main result from [72] more precisely. Let $\mathcal{W}$ be a sum of Markov dependent variables with distribution $\mathcal{L}(\mathcal{W})$,

$$\mathrm{E}\mathcal{W} = \mu, \qquad \mathrm{Var}\mathcal{W} = \sigma^2 < \infty. \tag{6}$$

Let the pair of rv's $(\mathcal{W}_1, \mathcal{W}_2)$ be exchangeable, that is, let $\mathcal{L}(\mathcal{W}_1, \mathcal{W}_2) = \mathcal{L}(\mathcal{W}_2, \mathcal{W}_1)$. Moreover, let there exist a positive number $\lambda < 1$ and a rv R_0 such that

$$\mathrm{E}^{\mathcal{W}_1}(\mathcal{W}_2 - \mu) = (1-\lambda)(\mathcal{W}_1 - \mu) + R_0. \tag{7}$$

Here $\mathrm{E}^{\mathcal{W}_1}$ denotes the conditional expectation with respect to $\mathcal{W}_1$. The remainder term R_0 has mean zero, $\mathrm{E}R_0 = 0$. Let, in addition to (7), assume that

$$\mathcal{W}_2 - \mathcal{W}_1 \in \{-1, 0, 1\}. \tag{8}$$

For approximation of $\mathcal{L}(\mathcal{W})$ a translated Poisson distribution is used. The translated Poisson distribution is the Poisson distribution shifted by an integer number. More precisely, $\mathcal{Y}$ has a translated Poisson distribution with parameters μ and σ^2, $\mathcal{L}(\mathcal{Y}) = \mathrm{TP}(\mu, \sigma^2)$, if $\mathcal{L}(\mathcal{Y} - \mu + \sigma^2 + \ell) = \mathrm{Pois}(\sigma^2 + \ell)$ and $\mathrm{Pois}(\sigma^2 + \ell)$ denotes the Poisson distribution with parameter $\sigma^2 + \ell$. Here $\ell = \{\mu - \sigma^2\}$ is the fractional part of $\mu - \sigma^2$. Observe that $\mathrm{TP}(\sigma^2, \sigma^2) = \mathrm{Pois}(\sigma^2)$. Approximating $\mathcal{W}$ by $\mathrm{TP}(\mu, \sigma^2)$, it is possible to fit the mean exactly and for the variance to have $\sigma^2 \leqslant \mathrm{Var}\mathcal{Y} = \sigma^2 + \ell \leqslant \sigma^2 + 1$.

Let $S' = S'(\mathcal{W}_1) = \mathrm{P}[\mathcal{W}_2 = \mathcal{W}_1 + 1|\mathcal{W}_1]$ and $q_{\max} = \max_{k\in\mathbb{Z}}\mathrm{P}[\mathcal{W}_1 = k]$. The main results from [72] states that

$$\begin{aligned}
\|\mathcal{L}(\mathcal{W}_1) - \mathrm{TP}(\mu, \sigma^2)\| &\leqslant \frac{2\sqrt{\mathrm{Var}S'}}{\lambda\sigma^2} + \frac{4\sqrt{\mathrm{Var}R_0}}{\lambda\sigma} + \frac{4}{\sigma^2},\\
\|\mathcal{L}(\mathcal{W}_1) - \mathrm{TP}(\mu, \sigma^2)\|_\infty &\leqslant \frac{2\sqrt{q_{\max}\mathrm{Var}S'}}{\lambda\sigma^2} + \frac{2q_{\max}\sqrt{\mathrm{Var}R_0}}{\lambda\sigma} + \frac{\sqrt{\mathrm{Var}R_0}}{\lambda\sigma} + \frac{2}{\sigma^2}.
\end{aligned}$$

The order of accuracy for the total variation in the scheme of sequences is $O(n^{-1/2})$. Note that the Stein method can not be applied for rv's with negative values, therefore it cannot be used for symmetric distributions. The other drawback is very complicated conditions containing new concepts such as exchangeable pairs with specific properties. The Stein method, CP approximations for general discrete Markov dependent sums and even more general processes were also considered in [7], [11], [18], [29], [32], [38], [41], [42], [46], [54], [66], [70], [71], [91], [93]. The convergence facts related to Markov dependent rv's, were investigated in [1], [14], [16], [30], [51], [52], [59], [63], [74], [81], [90].

2.4 Approximations to MB distribution

Poisson type approximations to the MB distribution were investigated in numerous papers; see, for example, [22], [26], [39], [50], [84], [88] and the references therein. Note that the definition of a MB distribution slightly varies from paper to paper, see [35], [77], and [88]. We choose the definition used in [22], [26]. Let $\xi_0, \xi_1, \dots, \xi_n, \dots$ be a non-stationary Markov chain with the initial distribution $\mathrm{P}(\xi_0 = 1) = p_0$, $\mathrm{P}(\xi_0 = 0) = 1 - p_0$, $p_0 \in [0, 1]$ and transition probabilities

$$\begin{aligned}
&\mathrm{P}(\xi_i = 1 \,|\, \xi_{i-1} = 1) = p_{11}, \qquad &&\mathrm{P}(\xi_i = 0 \,|\, \xi_{i-1} = 1) = p_{10},\\
&\mathrm{P}(\xi_i = 1 \,|\, \xi_{i-1} = 0) = p_{01}, \qquad &&\mathrm{P}(\xi_i = 0 \,|\, \xi_{i-1} = 0) = p_{00},\\
&p_{11} + p_{10} = p_{01} + p_{00} = 1, \qquad &&p_{11}, p_{01} \in (0, 1), \qquad i \in \mathbb{N}.
\end{aligned}$$

In other words, let the matrix of transition probabilities be equal to

$$\mathcal{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}.$$

The distribution $\mathcal{L}(S_n)$ of $S_n = \xi_1 + \dots + \xi_n$ $(n \in \mathbb{N})$ is called a MB distribution. Actually, S_n shows the number of hits in n Markov dependent trials. In some papers, ξ_0 is added to S_n or stationarity of the chain is additionally assumed. For example, Dorbushin [35] assumed that $p_0 = 1$ and considered $S_{n-1} + 1$. Similar problems were also investigated by Koopman [55]. He assumed that the transition probabilities depend on the number of trials n and obtained the limiting CP distribution for S_n.

The MB distribution is a direct generalization of the binomial distribution. Indeed, if $p_{00} = p_{10}$ then the MB distribution becomes the binomial one. It is known, that a suitably normalized binomial distribution has two non-degenerate limit laws – the normal and the Poisson distributions, see [44], [48], [56], [68], [73]. Meanwhile, the MB distribution has seven limit laws, see [35]. We recall some known results related to approximations of MB and binomial laws. We start with the classical Prokhorov-type Poisson approximation to the binomial distribution. Let $p \in (0, 1]$ and let $\mathrm{Pois}(np)$ denote Poisson distribution with the parameter np, that is $\mathrm{Pois}(np) = \exp\{np(I - I_1)\}$. Prokhorov was the first toprove that if $np \geqslant 1$, then the accuracy of approximation is of the order $O(p)$, see [69]. The estimate with the best possible constant

$$\|((1-p)I + pI_1)^n - \mathrm{Pois}(np)\| \leqslant 2\min(p, np^2)$$

was proved in [9].

Markov [58] showed, that for a constant transition probabilities, not containing zero values, MB distribution with appropriate normalization has an asymptotic normal distribution. For MB if the conditions $p_0 = 1$, $p_{10}n \to \infty$, $p_{01}n \to \infty$ and $\max\{np_{00}, np_{11}\} \to \infty$ are fulfilled, then

$$P\Big\{\frac{(S_{n-1}+1) - n\mathcal{M}}{\sqrt{n\mathcal{N}}} < x\Big\} \quad \to \quad \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\,\mathrm{d}t,$$

where $\mathcal{M} = p_{01}/(p_{10}+p_{01})$ and $\mathcal{N} = (p_{10}p_{01}(p_{11}+p_{00}))/(p_{10}+p_{01})^2$, see [35].

As shown by Dobrushin in [35], the normal distribution is just one of the seven possible limit laws for the MB distribution. For example, if $p_{11} \to \tilde{p}$, $np_{01} \to \lambda$ and $p_0 = 1$, then the limit law of $S_{n-1}+1$ is the convolution $G_0 * \exp\{\lambda(G_0 - I)\}$, where G_0 is a geometric distribution with parameter $\tilde{p}$, that is $\hat{G}_0(t) = (1-\tilde{p})\mathrm{e}^{\mathrm{i}t}/(1-\tilde{p}\mathrm{e}^{\mathrm{i}t})$. CP distribution with geometric distribution as a limit law for MB distribution is investigated in [23], [26], [78]. It is proved in [26] that if $p_0 = p_{01}/(p_{10}+p_{01})$ and $\lambda > 0$, then

$$\|\mathcal{L}(S_n) - \exp\{\lambda(G-I)\}\| \quad\leqslant\quad 2|np_{01} - \lambda| + \frac{2p_{01}(1+p_{11}+np_{01}(2-p_{11}))}{p_{10}+p_{01}}.$$

Here G is a geometric distribution, $\hat{G}(t) = p_{10}\mathrm{e}^{\mathrm{i}t}/(1-p_{11}\mathrm{e}^{\mathrm{i}t})$. The accuracy of this estimation is no better than $O(np_{01}^2)$. A similar estimate was obtained in [88]. Note also that numerous papers are devoted to the closeness of CP distribution to MB distribution, see [22], [23], [39], [55], [77], [78], [84]. Other problems related to MB distribution are considered in [3], [24], [34], [49], [75], [86], [92], [95].

2.5 The first uniform Kolmogorov problem

Almost sixty years ago Kolmogorov [53] proved that in the uniform metric the distribution of any sum of independent rv's is uniformly close to the set of all infinitely divisible laws $\mathbb{D}$. Related problems were considered in [2], [5], [21], [29], [33], [71], [96], see also the references therein. The optimal rate of accuracy of approximation was established by Arak and Zaĭtsev, see [5] for a comprehensive history of the problem. For the sum of identically distributed rv's (so-called the first uniform Kolmogorov theorem) Arak proved that

$$C_1 n^{-2/3} \leqslant \sup_{F\in\mathcal{F}} \inf_{\mathcal{D}\in\mathbb{D}} \|F^{*n} - \mathcal{D}\|_K \leqslant C_2 n^{-2/3}. \tag{9}$$

Here supremum is taken over the set of all distributions. There are no assumptions on F and, nevertheless, the order of accuracy is much better than in the famous Berry-Esseen theorem. Moreover, the best infinitely divisible approximation is not directly related to the limit behavior of the approximated sum.

In general, the first uniform Kolmogorov theorem is insoluble in total variation, see [97]. On the other hand, for lattice random variables having sufficient number of moments, some analogue of (9) in total variation holds; see [8] Theorem 4.1. The problem of approximation by the set $\mathbb{D}$ can be narrowed to approximation of the well-known parametric distributions. Then, by the uniform Kolmogorov problem we understand estimation of distribution's uniform closeness over the set of their parameters. For example, for the binomial distribution, the Kolmogorov problem in total variation is completely solved, see [5] and [67], Chapters IV and VIII. Let $\mathrm{Bi}(n,p)$, $p \leqslant 1/2$ denote the binomial distribution. Then

$$C_3\varepsilon_{n,p} \leqslant \inf_{\mathcal{D}\in\mathbb{D}} \|\mathrm{Bi}(n,p) - \mathcal{D}\| \leqslant C_4\varepsilon_{n,p}. \tag{10}$$

Here $\varepsilon_{n,p} = \min\left(np^2, p, \max\left((np)^{-2}, n^{-1}\right)\right)$. The uniform Kolmogorov theorem for the binomial distribution then follows:

$$C_5 n^{-2/3} \leqslant \sup_{p\leqslant 1/2} \inf_{\mathcal{D}\in\mathbb{D}} \|\mathrm{Bi}(n,p) - \mathcal{D}\| \leqslant C_6 n^{-2/3}. \tag{11}$$

Note that the order of accuracy in (11) is the same as in (9) despite the fact that the stronger metric is used.

Problem of the closeness of the sum of dependent rv's to $\mathbb{D}$ is yet unsolved, since the behavior of the sum strongly depends on the nature of dependency. Even the sum of weakly dependent indicator variables can have properties very different from those of the binomial distribution. The MB distribution is a direct generalization of the binomial distribution. Therefore, one can expect some analogue of (10). From the general Theorem 3.1 in [26] it follows that, if $p_{11} \leqslant 1/2$, then

$$\inf_{\mathcal{D}\in\mathbb{D}} \|\mathcal{L}(S_n)-\mathcal{D}\| \leqslant Cp_{01}(p_{11}+p_{01}) \min\Big(1, \frac{1}{\sqrt{np_{01}}}\Big)+C\min(p_{01}, np_{01}^2)+C(p_{11}+p_{01})\mathrm{e}^{-C_7 n}. \quad (12)$$

Here $C_7 = \ln 30/19 = 0.4567\ldots$. Observe that for $p_{01} = O(1)$ the estimate in (12) is of the same trivial order $O(1)$. As follows from Theorem 1.3 in [22], if $p_0 = 1$, $p_{11} \leqslant 1/20$, $p_{01}/(p_{10}+p_{01}) \leqslant 1/30$ and $p_{11} \leqslant p_{01}$, then

$$\inf_{\mathcal{D}\in\mathbb{D}} \|\mathcal{L}(S_n) - \mathcal{D}\| \leqslant C(n^{-1} + (np_{01})^{-2}). \quad (13)$$

Assumption $p_{11} \leqslant p_{01}$ is very restrictive. It completely excludes the case of CP limit, which occurs when $p_{11} \to \tilde{p} > 0$, $np_{01} \to \lambda$.

2.6 The Simons-Johnson theorem

In 1971 Simons and Johnson [80] proved that the convergence of the binomial distribution $\mathrm{Bi}(n, p)$ to the limit Poisson law $\mathrm{Pois}(\lambda)$ can be much stronger than in total variation. More precisely they proved that if $p = \lambda/n$ and $g(x)$ satisfies $\sum_0^\infty g(k)\mathrm{Pois}(\lambda)\{k\} < \infty$, then

$$\sum_{k=0}^{\infty} g(k)|\mathrm{Bi}(n,p)\{k\} - \mathrm{Pois}(\lambda)\{k\}| \to 0, \quad n \to \infty.$$

The above result was then extended to more general cases of independent lattice variables; see, for example, [6], [15], [87] and the references therein. Similar results hold for sums of m-dependent random variables, see [28] and [27].

In [20], it was proved that if $h > 0$ $np_{01} = \lambda + o(1)$, $p_{11} = o(1)$, then

$$\sum_{j=0}^{\infty} \mathrm{e}^{hj}|MB\{j\} - \mathrm{Pois}(\lambda)\{j\}| \to 0, \qquad n \to \infty. \quad (14)$$

Estimate (14) is unsatisfactory in two aspects: a) it contains Poisson limit distribution, instead of the more general CP law; b) there is no estimate of the rate of convergence. Even in the case of independent summands, CP distribution so far has been considered under very restrictive assumptions, see [87]. Notably, Wang emphasized that his result is inapplicable for CP distribution with compounding geometric distribution.

2.7 Infinitely divisible approximations to the sums of symmetric rv's

It is well-known that under quite weak assumptions the suitably normed sum of independent rv's weakly converges to some infinitely divisible rv. The classical improvement of the limit theorem means estimation of the rate of convergence in some metric. CLT and the Berry-Esseen theorem serve as typical examples. In Statistics, the limit distribution is usually viewed as a natural choice for possible approximation. In modern probability theory, the approximation by some distribution means that its closeness to the approximated sum is investigated under much weaker conditions than are needed for the existence of limit distribution with similar structure. For example, in (9) the closeness of the binomial and Poisson distributions is estimated for all values of binomial parameters n and p, though a Poisson limit occurs only if $p = O(1/n)$.

One of the aims of this thesis is to study the effect of symmetry on the accuracy of CP approximation for Markov dependent rv's. For sums of independent rv the symmetry of initial distribution radically improves the accuracy of approximation. One of the most general results for symmetric distributions states that if $\widehat{F}(t) \geqslant 0$, for all $t \in \mathbb{R}$, then

$$\|F^{*n} - \exp\{n(F - I)\}\|_K \leqslant C_8 n^{-1}, \tag{15}$$

see Theorem 5.5 in [5]. Similar result holds for distributions with some probabilistic mass at zero, see [5], Theorem 7.1. In general, estimate (15) does not hold for the total variation norm. On the other hand, if $F\{1\} = F\{-1\} = b$, $F\{0\} = 1 - 2b$, $b \leqslant 1/4$, then for any $n \in \mathbb{N}$

$$\|F^{*n} - \exp\{n(F - I)\}\| \leqslant C_9 \min(nb^2, n^{-1}). \tag{16}$$

Estimate (16) is essentially Presman's [67] result (see also Theorem 2.1 in [5]) rewritten in a different notation. The extension of (16) to the case of distributions satisfying Franken's condition can be found in [83].

3 Results

3.1 Sequences of Markov dependent variables: translated Poisson approximation

In this subsection we use the notation of subsection 2.2. For the sake of convenience, we repeat the main assumptions and expressions. We explore homogeneous Markov chain with arbitrary number of states. Let $\mathcal{Y}$ be a space of points ω, $\mathcal{B}_{\mathcal{Y}}$ be the σ-algebra of its subsets, $p(\omega, \mathcal{A})$ be the transition probability function. The sequence of rv's $\mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_n, \cdots$ is defined as follows

$$\begin{aligned} \mathrm{P}(\mathcal{Y}_1 \in \mathcal{A}) &= \pi(\mathcal{A}), \\ \mathrm{P}(\mathcal{Y}_n \in \mathcal{A}) &= \int_{\mathcal{Y}} p^{(n-1)}(\omega, \mathcal{A})\pi(d\omega). \end{aligned}$$

Let $f(\omega)$ be a real function defined on $\mathcal{Y}$ and measurable with respect to $\mathcal{B}_{\mathcal{Y}}$, $F_{n\pi}(x)$ is the distribution function of the normed sum

$$\mathcal{S}_n = \frac{\tilde{\mathcal{S}}_n}{\sigma\sqrt{n}} - \frac{\mathcal{A}}{\sigma\sqrt{n}}.$$

Here $\tilde{\mathcal{S}}_n = \sum_{k=1}^{n+1} f(\mathcal{Y}_k)$ and $\mathcal{A} = (n+1)\int_{\mathcal{Y}} f(\tau)p(d\tau)$. We denote by $\tilde{F}_{n\pi}(x)$ the distribution function of the sum $\tilde{\mathcal{S}}_n$.

Condition (H_k): there exists a function $g(x)$ of a real variable such that $\lim_{x\to\infty} g(x) = \infty$ and $\sup_\omega \int_X |f(\tau)|^k g(|f(\tau)|)p(\omega, d\tau) < \infty$.

Condition $(\bar{H})$: let $\mathcal{Y}$ be a countable set of states ω_i, which forms a positive class, $f(\omega_i) = m + k_i h$, where k_i is an integer, m is any real number, $h > 0$, and for arbitrary i and j it is possible to find an index k such that $p_{ik} > 0$ and $p_{jk} > 0$ ($p_{ik} = p(\omega_i, \omega_k)$).

Unlike Nagaev [62], who approximated the distribution $F_{n\pi}$ of the centered and normed sum $\mathcal{S}_n$, we approximate the distribution $\tilde{F}_{n\pi}$ of the initial sum $\tilde{\mathcal{S}_n}$. Such approach seems to be more natural for lattice approximation. Let $\mathcal{G}$ be a measure with the following characteristic function:

$$\begin{aligned}\widehat{\mathcal{G}}(t) &= \exp\{\mathrm{i}t\lfloor \mathcal{A} - n\sigma^2 \rfloor + (n\sigma^2 + \{\mathcal{A} - n\sigma^2\})(\mathrm{e}^{\mathrm{i}t} - 1)\} \times \\ &\quad \left[1 + n\sigma^2\left(\sigma\alpha - \frac{1}{6}\right)(\mathrm{e}^{\mathrm{i}t} - 1)^3 + \sigma\lambda_\pi(\mathrm{e}^{\mathrm{i}t} - 1)\right].\end{aligned}$$

We can formulate the main result of this subsection.

Theorem 3.1 *If conditions* (H_k), $(\bar{H})$ *and* $\sum_{i=1}^{\infty} |f(\omega_i)|\pi(\omega_i) < \infty$ *are fulfilled, then, for all* $n = 1, 2, \ldots,$

$$\|\tilde{F}_{n\pi} - \mathcal{G}\|_\infty = o(n^{-1}), \tag{17}$$

$$\|\tilde{F}_{n\pi} - \mathcal{G}\|_K = o(n^{-1/2}). \tag{18}$$

Comparing (17) and (18) respectively with (4) and (3), we see that, the accuracy of approximations remains the same. On the other hand, $\mathcal{G}$ is structurally better constructed, since it is concentrated on the same lattice as $\tilde{F}_{n\pi}$. Consequently, there is no need for an additional summand $S_1(x)/\sqrt{n}$, which appears in (3).

3.2 Triangular arrays: the first uniform Kolmogorov theorem for the MB distribution

In this thesis, we prove an analogue of (13) for all $p_{11} \leqslant 1/4$ and $p_{01} \leqslant 1/30$. In other words, we consider a small p_{01} and not so small p_{11}, which is in accordance with assumptions needed for the CP limit. On the other hand, we allow for p_{01} and p_{11} to be of the constant order, i.e., we also consider the case when CLT applies.

Next we introduce various characteristics of S_n. Let

$$\begin{aligned}
\gamma_1 &= \frac{p_{10}p_{01}}{p_{10}+p_{01}}, \qquad \gamma_2 = -\frac{p_{10}p_{01}^2}{(p_{10}+p_{01})^2}\left(p_{11} + \frac{p_{10}}{p_{10}+p_{01}}\right) - \frac{\gamma_1^2}{2}, \\
\gamma_3 &= \gamma_1^2\left\{\frac{\gamma_1}{3} + \frac{1}{p_{10}(p_{10}+p_{01})}\left\{p_{11}^2 p_{01} + \frac{p_{11}p_{10}(2p_{01}-p_{10})}{p_{10}+p_{01}} + \frac{2p_{01}p_{10}^2}{(p_{10}+p_{01})^2}\right\}\right. \\
&\quad \left. + \frac{p_{01}}{p_{10}+p_{01}}\left(p_{11} + \frac{p_{10}}{p_{10}+p_{01}}\right)\right\}, \\
\varkappa_1 &= \gamma_1\left(\frac{p_{01}-p_{11}}{p_{10}+p_{01}} - p_0\right), \quad \varkappa_2 = p_0\frac{p_{11}p_{10}}{p_{10}+p_{01}}, \\
\gamma &= -6n(\gamma_2+\gamma_3) + \beta, \quad 0 \leqslant \beta < 1, \text{ and } \beta \text{ is such that } \gamma \in \mathbb{Z}, \\
\lambda_1 &= n(\gamma_1 + 4\gamma_2 + 3\gamma_3) - \beta, \quad \lambda_2 = \frac{\beta}{6}, \quad \lambda_{-1} = -n(2\gamma_2 + 3\gamma_3) + \frac{\beta}{3}.
\end{aligned}$$

We also use the following measures:

$$\begin{aligned}
\tilde{G} &= p_{10}I_1 * \sum_{j=0}^{\infty} p_{11}^j I_j \quad \left(\widehat{\tilde{G}}(t) = \frac{p_{10}\mathrm{e}^{\mathrm{i}t}}{1 - p_{11}\mathrm{e}^{\mathrm{i}t}}\right), \quad \mathcal{H} = I + \varkappa_2(\tilde{G} - I), \\
\mathcal{U} &= \tilde{G}^{*\gamma} * \exp\{\lambda_1(\tilde{G} - I) + \lambda_2(\tilde{G}^{*2} - I) + \lambda_{-1}(I_{-1} - I)/p_{10}\}.
\end{aligned}$$

Now we can formulate the main result of this subsection.

Theorem 3.2 *Let $p_{11} \leqslant 1/4$, $p_{01} \leqslant 1/30$, $np_{01} \geqslant 3$. Then*

$$\|\mathcal{L}(S_n) - \mathcal{H} * \exp\{\varkappa_1(\tilde{G} - I)\} * \mathcal{U}\| \leqslant C_1 \max\left(\frac{1}{n}, \frac{1}{(np_{01})^2}\right), \qquad (19)$$

$$\|\mathcal{L}(S_n) - \mathcal{H} * \exp\{\varkappa_1(\tilde{G} - I)\} * \mathcal{U}\|_\infty \leqslant \frac{C_2}{\sqrt{np_{01}}} \max\left(\frac{1}{n}, \frac{1}{(np_{01})^2}\right). \qquad (20)$$

Remark 3.1 *(i) If, in Theorem 3.2, p_{11} and p_{01} are some absolute constants then CLT holds, see [35]. In this case, estimate (19) is of the order $O(n^{-1})$, i.e., much better than $O(n^{-1/2})$, the order that can be expected from the normal approximation.*

(ii) In [26] Lemma 5.3, it was proved that $\mathcal{H} \in \mathbb{D}$. It will be shown below that $\exp\{\varkappa_1(\tilde{G} - I)\} * \mathcal{U} \in \mathbb{D}$ and, therefore, approximation in Theorem 3.2 is infinitely divisible and probabilistic.

In the context of the uniform Kolmogorov problem, assumption $np_{01} \geqslant 3$ is not very restrictive. If $np_{01} < 3$ then the same accuracy as in Theorem 3.2 can be achieved by any infinite divisible distribution. The following version of the uniform Kolmogorov theorem for MB distribution follows from Theorem 3.3.

Theorem 3.3 *There exist absolute positive constants C_3 and C_4 such that, for all $n \in \mathbb{N}$,*

$$\sup_{p_{11} \leqslant 1/4, p_{01} \leqslant 1/30} \inf_{\mathcal{D} \in \mathbb{D}} \|\mathcal{L}(S_n) - \mathcal{D}\| \leqslant C_3 n^{-2/3}, \qquad (21)$$

$$\sup_{p_{11} \leqslant 1/4, p_{01} \leqslant 1/30} \inf_{\mathcal{D} \in \mathbb{D}} \|\mathcal{L}(S_n) - \mathcal{D}\|_\infty \leqslant C_4 n^{-5/6}. \qquad (22)$$

We see that the accuracy of approximation in (21) is the same as in (9) and (11). In comparison to (13), we managed to drop the very restricting assumption $p_{11} \leqslant p_{01}$. The question about possible extension of (21) to the case $p_{01} \to 1$ remains open.

3.3 Triangular arrays: the Simons-Johnson theorem for the MB distribution

The rate of convergence in CLT can be very slow. In subsection 3.2, we proved that the normal limit distribution can be replaced by much more accurate CP approximation. In this subsection, we show that, on the other hand, if MB distribution converges to CP limit, the rate of convergence is very fast and no other approximation is needed.

Next observe that if $p_{11} \to \tilde{p}$, $np_{01} \to \lambda$ then MB distribution converges to CP distribution with compounding geometric distribution, i.e. we have the case that has not been solved even for independent summands. Below we prove that, nevertheless, some version of the Simons-Johnson theorem holds. Assuming that p_{11} and np_{01} are close to their respective limit values $\tilde{p}$ and λ, we also estimate the rate of convergence. Let

$$|p_{11} - \tilde{p}| \leqslant \frac{\tilde{p}}{3}, \quad |np_{01} - \lambda| \leqslant \frac{\lambda}{2}, \quad p_{01}(\mathrm{e}^h + 1)^2 \leqslant \frac{1}{25}, \quad \tilde{p}\mathrm{e}^h \leqslant \frac{1}{4}. \qquad (23)$$

Let $Z = (1 - \tilde{p})I_1 * \sum_{j=0}^\infty \tilde{p}^j I_j$ and $\Psi = (I + p_0\tilde{p}(Z - I)) * \exp\{\lambda(Z - I)\}$.

Theorem 3.4 *Let $h > 0$, $\lambda > 0$ and $0 \leqslant \tilde{p} < 1$ be some absolute constants. If assumptions (23) are satisfied, then*

$$\sum_{k=0}^\infty \mathrm{e}^{hk} |\mathcal{L}(S_n)\{k\} - \Psi\{k\}| \leqslant C_5(|p_{11} - \tilde{p}| + |np_{01} - \lambda| + n^{-1}).$$

Corollary 3.1 *Let $h > 0$, $0 \leqslant \tilde{p} < 1$, $\tilde{p}\mathrm{e}^h \leqslant 1/4$. If $np_{01} \to \lambda$, $p_{11} \to \tilde{p}$, then*

$$\sum_{k=0}^{\infty} \mathrm{e}^{hk} |\mathcal{L}(S_n)\{k\} - \Psi\{k\}| \to 0.$$

Remark 3.2 *(i) It is unclear to what extent assumption $\tilde{p}\mathrm{e}^h \leqslant 1/4$ can be relaxed. In this paper, it plays technical role, i.e., under this condition, the series $\sum_1^\infty (\tilde{p}\exp\{\mathrm{i}t + h\})^j$ converges absolutely.*

(ii) Note that if $p_0 < 1$, then $\Psi \in \mathbb{D}$, see Lemma 5.3 in [26].

(iii) The choice of exponential weights e^{hj} is determined by the method of proof.

3.4 Approximation of a symmetric three-state Markov chain by CP law

In this subsection, our aim is to prove some analogue of (16) for the Markov dependent rv's. We consider a symmetric three state Markov chain. We prove that its sum can be approximated by CP distribution with the accuracy of the order $O(1/n)$. Though limit CP distribution occurs only if some of the transition probabilities are of the order $O(1/n)$, the accuracy of approximation is estimated under much weaker assumptions when all transition probabilities are of the order $O(1)$. Lower bound estimates demonstrate that, under certain assumptions, the order of accuracy is correct.

Let $\xi_0, \xi_1, \dots \xi_n, \dots$ be a non-stationary three state $\{a_1, a_2, a_3\}$ Markov chain. We denote the distribution of $\mathcal{S}_n = f(\xi_1) + \cdots + f(\xi_n)$ $(n \in \mathbb{N})$ by F_n, that is $P(\mathcal{S}_n = m) = F_n\{m\}$ for $m \in \mathbb{Z}$. Here $f(a_1) = -1$, $f(a_2) = 0$, $f(a_3) = 1$. Let the initial distribution be defined by $P(\xi_0 = a_1) = \pi_1$, $P(\xi_0 = a_2) = \pi_2$ and $P(\xi_0 = a_3) = \pi_3$. Transition probabilities are equal to

$$\begin{array}{lll} \mathrm{P}(\xi_i = a_1 \,|\, \xi_{i-1} = a_1) = a, & \mathrm{P}(\xi_i = a_2 \,|\, \xi_{i-1} = a_1) = 1 - 2a, & \mathrm{P}(\xi_i = a_3 \,|\, \xi_{i-1} = a_1) = a, \\ \mathrm{P}(\xi_i = a_1 \,|\, \xi_{i-1} = a_2) = b, & \mathrm{P}(\xi_i = a_2 \,|\, \xi_{i-1} = a_2) = 1 - 2b, & \mathrm{P}(\xi_i = a_3 \,|\, \xi_{i-1} = a_2) = b, \\ \mathrm{P}(\xi_i = a_1 \,|\, \xi_{i-1} = a_3) = a, & \mathrm{P}(\xi_i = a_2 \,|\, \xi_{i-1} = a_3) = 1 - 2a, & \mathrm{P}(\xi_i = a_3 \,|\, \xi_{i-1} = a_3) = a, \end{array}$$
$$a, b \in [0, 0.5].$$

In other words, the matrix of transition probabilities is equal to

$$\begin{pmatrix} a & 1-2a & a \\ b & 1-2b & b \\ a & 1-2a & a \end{pmatrix}.$$

We introduce further notation:

$$\begin{aligned} L &= \frac{1}{2}(I_{-1} + I_1), \qquad X = L - I, \qquad D = (1 - 2(a-b) - 2aX)^{*2} * (I + \Delta), \\ P_1 &= \pi_1 I + \frac{\Lambda_1 - 2aL}{1-2a}\pi_2 + \pi_3 I, \qquad P_2 = \pi_1 I + \frac{\Lambda_2 - 2aL}{1-2a}\pi_2 + \pi_3 I, \\ \Delta &= \frac{8b}{(1+2b)^2} X * \left(\sum_{j=0}^{\infty} \left(\frac{2a}{1+2b} L \right)^{*j} \right)^{*2}, \quad \Delta_1 = \frac{8(b-a)}{(1-2b)^2} L * \left(\sum_{j=0}^{\infty} \left(\frac{-2a}{1-2b} \right)^{j} L^{*j} \right)^{*2}, \end{aligned}$$

$$\begin{aligned}
\Lambda_{1,2} &= \frac{1}{2}\bigg(I+2(a-b)I+2aX \pm (I-2(a-b)I-2aX)*\sum_{j=0}^{\infty}\binom{1/2}{j}\Delta^{*j}\bigg)\\
&= \frac{1}{2}\bigg((1-2b)I+2aL \pm ((1-2b)I+2aL)*\sum_{j=0}^{\infty}\binom{1/2}{j}\Delta_1^{*j}\bigg),\\
W_{1,2} &= \frac{1}{2}\bigg(I \pm \Big(\frac{2aX+(1-2a+2b)I}{(1+2b)^2}\Big)*\Big(\sum_{j=0}^{\infty}\Big(\frac{2a}{1+2b}L\Big)^{*j}\Big)*\sum_{j=0}^{\infty}\binom{-1/2}{j}\Delta^{*j}\bigg),\\
G &= \exp\Big\{\frac{2b(1-2a)}{1-2a+2b}(H-I)\Big\}, \qquad H=(1-2a)L*\sum_{j=0}^{\infty}(2aL)^{*j},\\
M &= \frac{1-2(a-b)}{1+2b}\sum_{j=0}^{\infty}\Big(\frac{2a}{1+2b}L\Big)^{*j}, \qquad E=\Big(1-\frac{2a\pi_2}{1-2a}\Big)I+\frac{2a\pi_2}{1-2a}L,\\
A_1 &= \frac{-2b^2(1-2a)}{(1-2a+2b)^2}\Big((1+2a)I+\frac{2(1-2a)M}{1-2a+2b}\Big)*(H-I)^{*2},\\
E_1 &= \frac{2b(H-I)}{1-2a+2b}\pi_2, \qquad M_1=\frac{-2b(1-2a)}{(1-2a+2b)^2}(2M-I)*(H-I).
\end{aligned}$$

Distribution F_n converges to some CP distribution, when $nb \to \tilde{b}_0$, $a \to \tilde{a}_0$ (for Poisson limit $\tilde{a}_0 = 0$). Therefore, for a CP *limit*, one needs very restrictive assumption $b = O(1/n)$. To what extent this condition can be weakened for some *prelimit* CP approximation? As we prove below, even for $b = O(1)$ good accuracy of approximation can be achieved. All results are obtained under the following condition

$$0 \leqslant a \leqslant \frac{1}{30}, \quad 0 \leqslant b \leqslant \frac{1}{30}. \tag{24}$$

The smallness of constants is determined by the method of proof.

Now we can formulate the main results of this subsection.

Theorem 3.5 *Let condition (24) hold. Then, for all $n = 1, 2, \dots$,*

$$\begin{aligned}
\|F_n - E*M*G^{*n}\| &\leqslant C\bigg(\min\Big\{\frac{1}{n}, b\Big\} + 0.2^n|a-b|\bigg), \qquad (25)\\
\|F_n - E*M*G^{*n}\|_\infty &\leqslant C\bigg(\min\Big\{\frac{1}{n\sqrt{nb}}, b\Big\} + 0.2^n|a-b|\bigg),\\
\|F_n - E*M*G^{*n}\|_W &\leqslant C\bigg(\min\Big\{\frac{\sqrt{nb}}{n}, b\Big\} + 0.2^n|a-b|\bigg).
\end{aligned}$$

Remark 3.3 *Approximation has three distinctive components:*

(i) Compound binomial distribution E is related to the initial distribution of ξ_0 and is not infinitely divisible. However, if we assume that $\pi_2 = 0$, then $E = I$.

(ii) M is compound geometric distribution with the compounding symmetric distribution L. Consequently, M is a special case of CP distributions. Indeed,

$$M = \exp\Big\{\sum_{j=1}^{\infty}\Big(\frac{2a}{1+2b}\Big)^j\frac{L^{*j}-I}{j}\Big\}.$$

(iii) The main part of approximation is n-fold convolution of G. Obviously, G is CP distribution with the compounding distribution H. Observe that H is also a compound geometric distribution with the compounding symmetric distribution L.

Corollary 3.2 *Let condition (24) hold. Then, for all* $n = 1, 2, \ldots,$

$$\|F_n - E * M * G^{*n}\| \leqslant Cn^{-1}. \tag{26}$$

Comparing estimates of (25) and (16) we see, that the latter is more accurate for $b < 1/n$. In principle, it is possible to get estimate nb^2 at the expense of more complicated structure of the approximation. We demonstrate this fact in the following corollary.

Corollary 3.3 *Let condition (24) hold. Then, for all* $n = 1, 2, \ldots,$

$$\|F_n - (E + E_1) * M * (I + M_1) * G^{*n}\| \leqslant C\big(\min(nb^2, n^{-1}) + 0.2^n|a - b|\big).$$

Remark 3.4 *If* $a = b$*, then* F_n *becomes* F^{*n} *considered in (16). Meanwhile* $E * M * G^{*n}$ *differs from* $\exp\{n(F - I)\}$*. The difference occurs because* $E * M * G^{*n}$ *takes into account possible closeness of* F_n *to the limit CP distribution with the compounding geometric law. Similar limit distributions are impossible for independent three-point rv's.*

Next we demonstrate how the accuracy of approximation can be improved by the second-order approximations.

Theorem 3.6 *Let condition (24) hold. Then, for all* $n = 1, 2, \ldots,$

$$\begin{aligned}
\|F_n - (E + E_1) * G^{*n} * (I + nA_1 + M_1)\| &\leqslant C\bigg(\min\left\{\frac{1}{n^2}, b^2\right\} + 0.2^n|a - b|\bigg), \\
\|F_n - (E + E_1) * G^{*n} * (I + nA_1 + M_1)\|_\infty &\leqslant C\bigg(\min\left\{\frac{1}{n^2\sqrt{nb}}, b^2\right\} + 0.2^n|a - b|\bigg), \\
\|F_n - (E + E_1) * G^{*n} * (I + nA_1 + M_1)\|_W &\leqslant C\bigg(\min\left\{\frac{\sqrt{nb}}{n^2}, b^2\right\} + 0.2^n|a - b|\bigg).
\end{aligned}$$

We also discuss to what extent estimates of Theorem 3.5 are unimprovable. The following lower-bound estimates hold.

Theorem 3.7 *Let condition (24) hold,* $nb \geqslant 1$ *and* $\pi_2 = 0$*. Then there exist absolute constants* C_i*,* $(i = 20, \ldots, 25)$ *such that, for all* $n = 1, 2, \ldots,$

$$\begin{aligned}
\|F_n - E * M * G^{*n}\| &\geqslant \|F_n - E * M * G^{*n}\|_K \geqslant \frac{C_{20}}{n}(1 - C_{21}\, 0.2^n|a - b|), \qquad (27) \\
\|F_n - E * M * G^{*n}\|_\infty &\geqslant \frac{C_{22}}{n\sqrt{nb}}(1 - C_{23}\, 0.2^n|a - b|), \\
\|F_n - E * M * G^{*n}\|_W &\geqslant \frac{C_{24}\sqrt{b}}{\sqrt{n}}(1 - C_{25}\, 0.2^n|a - b|).
\end{aligned}$$

Corollary 3.4 *Let condition (24) hold,* $nb \geqslant 1$ *and* $\pi_2 = 0$*. Then there exist absolute constant* C_{26} *such that, for* $n \geqslant C_{26}$*,*

$$\|F_n - E * M * G^{*n}\| \geqslant \frac{C_{26}}{n}.$$

Thus, the order $O(1/n)$ in (25) can not be improved. Next we formuate non-uniform local estimates. They are necessary in the case when we want to find out the accuracy of approximation when m is far from the mean.

Theorem 3.8 *Let $nb \geqslant 1$ and condition (24) hold. Then, for all $m = 0, 1, 2, \dots,$*

$$\begin{aligned} |(F_n - E * M * G^{*n})\{m\}|\Big(1 + \frac{|m|}{\sqrt{nb}}\Big) &\leqslant \frac{C}{n\sqrt{nb}}, \\ |(F_n - E * M * G^{*n})\{(-\infty, m]\}|\Big(1 + \frac{|m|}{\sqrt{nb}}\Big) &\leqslant \frac{C}{n}. \end{aligned}$$

Next we formulate the analogue of Simons-Johnson theorem for three-state Markov chain distribution. First we introduce further notation. Let $0 \leqslant \tilde{a} < 1$ and

$$\mathcal{E} = \pi_1 I + \Big(I - \frac{2\tilde{a}}{1 - 2\tilde{a}} X\Big)\pi_2 + \pi_3 I, \qquad \mathcal{M} = (1 - 2\tilde{a}) \sum_{j=0}^{\infty} (2\tilde{a}L)^{*j},$$

$$B = \exp\Big\{\frac{2\lambda}{n}(L - I) * \sum_{j=0}^{\infty} (2\tilde{a}L)^{*j}\Big\}, \qquad \mathcal{G} = \mathcal{E} * \mathcal{M} * B^{*n}.$$

Let

$$|a - \tilde{a}| \leqslant \frac{\tilde{a}}{10}, \qquad |nb - \lambda| \leqslant \frac{\lambda}{2}, \qquad b(e^h + 1)^2 \leqslant \frac{1}{25}, \qquad \tilde{a}e^h \leqslant \frac{1}{11}. \tag{28}$$

Theorem 3.9 *Let $h > 0$, $\lambda > 0$ and $0 \leqslant \tilde{a} < 1$ be some absolute constants. If assumptions (28) are satisfied, then*

$$\sum_{k=-\infty}^{\infty} e^{h|k|} |F_n\{k\} - \mathcal{G}\{k\}| \leqslant C(|a - \tilde{a}| + |nb - \lambda| + n^{-1}).$$

Corollary 3.5 *Let $h > 0$, $0 \leqslant \tilde{a} < 1$, $\tilde{a}e^h \leqslant 1/4$. If $nb \to \lambda$, $a \to \tilde{a}$, then*

$$\sum_{k=-\infty}^{\infty} e^{h|k|} |F_n\{k\} - \mathcal{G}\{k\}| \to 0.$$

4 Proofs

4.1 Auxiliary results for Theorems 3.1-3.2

Lemma 4.1 *Let $u, v > 0$, $n \in \mathbb{N}$. Then*

$$u^n = v^n + \sum_{j=1}^{s} \binom{n}{j} v^{n-j}(u-v)^j + r_n(s+1). \tag{29}$$

Here

$$r_n(s+1) = \sum_{j=s+1}^{n} \binom{j-1}{s} u^{n-j}(u-v)^{s+1} v^{j-s-1}.$$

Note that

$$\sum_{j=s+1}^{y} \binom{j-1}{s} = \binom{n}{s+1}.$$

The expression (29) is called Bergström's identity and was obtained in [12].

Lemma 4.2 *Let U be a measure concentrated on $\mathbb{Z}$. Then for all $u > 0$*

$$\|U\|_\infty \leqslant \frac{1}{2\pi u} \int_{-\pi u}^{-\pi u} \Big|\widehat{U}\Big(\frac{t}{u}\Big)\Big| \mathrm{d}t, \qquad \|U\|_K \leqslant \frac{1}{4} \int_{-\pi u}^{-\pi u} \frac{1}{|t|} \Big|\widehat{U}\Big(\frac{t}{u}\Big)\Big| \mathrm{d}t.$$

The first Lemma's statement follows from the formula of inversion, the second statement is a version of Tsaregradskii's inequality.

Lemma 4.3 *For $t \in \mathbb{R}$, $\sigma > 0$, $n \in \mathbb{N}$,*

$$\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1 - \frac{\mathrm{i}t}{\sigma\sqrt{n}} = -\frac{t^2}{2\sigma^2 n} + \frac{(\mathrm{i}t)^3}{6\sigma^3 n\sqrt{n}} + \frac{\theta t^4}{24\sigma^4 n^2} = -\frac{t^2}{2\sigma^2 n} + \frac{\theta |t|^3}{6\sigma^3 n\sqrt{n}} = \frac{\theta C t^2}{\sigma^2 n}, \tag{30}$$

$$\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1 = \frac{\theta C |t|}{\sigma\sqrt{n}}, \tag{31}$$

$$\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3 = \frac{(\mathrm{i}t)^3}{\sigma^3 n\sqrt{n}} + \frac{\theta C t^4}{\sigma^4 n^2}. \tag{32}$$

Lemma's statements trivially follow from the Taylor expansion.

Further in this subsection

$$\widehat{\mathcal{G}}_1\Big(\frac{t}{\sigma\sqrt{n}}\Big) = \exp\Big\{\frac{it\mathcal{A}}{\sigma\sqrt{n}} - \frac{t^2}{2}\Big\}\Big(1 + \frac{\alpha(\mathrm{i}t)^3}{\sqrt{n}} + \frac{\lambda_\pi(\mathrm{i}t)}{\sqrt{n}}\Big),$$

$$\widehat{\mathcal{G}}_2\Big(\frac{t}{\sigma\sqrt{n}}\Big) = \exp\Big\{\frac{it\mathcal{A}}{\sigma\sqrt{n}} - \frac{t^2}{2}\Big\}\Big(1 + n\alpha\sigma^3\Big(\exp\Big\{\frac{it}{\sigma\sqrt{n}}\Big\} - 1\Big)^3 + \sigma\lambda_\pi\Big(\exp\Big\{\frac{it}{\sigma\sqrt{n}}\Big\} - 1\Big)\Big).$$

Lemma 4.4 *For $|t| \leqslant \pi\sigma\sqrt{n}$ we have*

$$\Big|\widehat{\mathcal{G}}_1\Big(\frac{t}{\sigma\sqrt{n}}\Big) - \widehat{\mathcal{G}}_2\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big| \leqslant \frac{C}{n}\mathrm{e}^{-0.1t^2}(\alpha + \lambda_\pi)t^2.$$

Proof. From Lemma 4.3 it follows that

$$n\sigma^3\Big(\exp\Big\{\frac{it}{\sigma\sqrt{n}}\Big\} - 1\Big)^3 = n\sigma^3\Big(\frac{(\mathrm{i}t)^3}{\sigma^3 n\sqrt{n}} + \frac{\theta C t^4}{\sigma^4 n^2}\Big) = \frac{(\mathrm{i}t)^3}{\sqrt{n}} + \frac{\theta C t^4}{\sigma n},$$

$$\sigma\Big(\exp\Big\{\frac{it}{\sigma\sqrt{n}}\Big\} - 1\Big) = \sigma\Big(\frac{\mathrm{i}t}{\sigma\sqrt{n}} + \frac{\theta C t^2}{\sigma^2 n}\Big) = \frac{\mathrm{i}t}{\sqrt{n}} + \frac{\theta C t^2}{\sigma n}.$$

Therefore,

$$\begin{aligned}\left|\widehat{\mathcal{G}}_1\left(\frac{t}{\sigma\sqrt{n}}\right)-\widehat{\mathcal{G}}_2\left(\frac{t}{\sigma\sqrt{n}}\right)\right| &= \left|\exp\left\{\frac{it\mathcal{A}}{\sigma\sqrt{n}}-\frac{t^2}{2}\right\}\right|\left|\frac{\alpha(\mathrm{i}t)^3}{\sqrt{n}}-n\alpha\sigma^3\left(\exp\left\{\frac{it}{\sigma\sqrt{n}}\right\}-1\right)^3\right.\\ &\quad\left.+\frac{\lambda_\pi(\mathrm{i}t)}{\sqrt{n}}-\sigma\lambda_\pi\left(\exp\left\{\frac{it}{\sigma\sqrt{n}}\right\}-1\right)\right|\\ &\leqslant Ce^{-t^2/2}\left(\frac{\alpha t^4}{\sigma n}+\frac{\lambda_\pi t^2}{\sigma n}\right)\leqslant \frac{C}{n}t^2e^{-t^2/2}(\alpha t^2+\lambda_\pi)\\ &\leqslant \frac{C}{n}t^2e^{-2t^2/\pi^2}\big(\alpha t^2e^{-0.3t^2}+\lambda_\pi e^{-0.3t^2}\big)\leqslant\frac{C}{n}t^2e^{-0.1t^2}(\alpha+\lambda_\pi).\end{aligned}$$

□

For the next Lemma we introduce the following notation

$$\phi=\exp\left\{-\frac{t^2}{2n}\right\},\qquad \tilde{\phi}=\exp\left\{\sigma^2\left(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right)\right\}.$$

Lemma 4.5 *Let the conditions from Theorem 3.1 are fulfilled. Then, for $|t|\leqslant\pi\sigma\sqrt{n}$ we have*

$$|\phi-\tilde{\phi}|\leqslant Ce^{6\sigma^2}\frac{|t|^3}{\sigma n\sqrt{n}}\exp\left\{-\frac{t^2}{2n}\right\},\tag{33}$$

$$\phi^n=\tilde{\phi}^n-\phi^n\frac{\sigma^2 n}{6}\left(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1\right)^3+\theta C\frac{e^{12\sigma^2}}{\sigma^2 n}(1+\sigma^2)(t^4+t^6)\exp\left\{-\frac{2t^2}{\pi^2}\right\}.\tag{34}$$

Proof. By (30)

$$\tilde{\phi}=\exp\left\{\sigma^2\left(-\frac{t^2}{2\sigma^2 n}+\frac{\theta|t|^3}{6\sigma^3 n\sqrt{n}}\right)\right\}=\exp\left\{-\frac{t^2}{2n}+\frac{\theta|t|^3}{6\sigma n\sqrt{n}}\right\}=\phi\exp\left\{\frac{\theta|t|^3}{6\sigma n\sqrt{n}}\right\}.$$

Since $e^y=1+\theta|y|e^{|y|}$ and $|t|\leqslant\pi\sigma\sqrt{n}$, we obtain

$$\begin{aligned}|\phi-\tilde{\phi}| &= |\phi|\left|1-\exp\left\{\frac{\theta|t|^3}{6\sigma n\sqrt{n}}\right\}\right|\leqslant|\phi|\frac{C|t|^3}{\sigma n\sqrt{n}}\exp\left\{\frac{|t|^3}{6\sigma n\sqrt{n}}\right\}\\ &\leqslant \phi\frac{|t|^3}{\sigma n\sqrt{n}}\exp\left\{\frac{\pi^3\sigma^3}{6\sigma}\right\}\leqslant Ce^{6\sigma^2}\frac{|t|^3}{\sigma n\sqrt{n}}\exp\left\{-\frac{t^2}{2n}\right\}.\end{aligned}$$

Observe that from (32) it follows that

$$\begin{aligned}\frac{(\mathrm{i}t)^3}{6\sigma n\sqrt{n}} &= \frac{\sigma^2}{6}\left(\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right)^3=\frac{\sigma^2}{6}\left[\left(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1\right)^3+\frac{\theta Ct^4}{\sigma^4 n^2}\right]\\ &= \frac{\sigma^2}{6}\left(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1\right)^3+\frac{\theta Ct^4}{\sigma^2 n^2},\end{aligned}$$

$$\begin{aligned}\left|\frac{(\mathrm{i}t)^3}{6\sigma n\sqrt{n}}+\frac{t^4}{24\sigma^2 n^2}\right| &\leqslant \frac{|t|^3}{6\sigma n\sqrt{n}}+\frac{|t|^4}{24\sigma^2 n^2}=\frac{|t|^3}{n\sqrt{n}}\left(\frac{1}{6\sigma}+\frac{|t|}{24\sigma^2\sqrt{n}}\right)\leqslant\frac{|t|^3}{n\sqrt{n}\sigma}\left(\frac{1}{6}+\frac{\pi}{24}\right)\\ &\leqslant \pi^3\sigma^2\left(\frac{1}{6}+\frac{\pi}{24}\right)\leqslant\pi^3\sigma^2(1.131+0.167)\leqslant\sigma^2\pi^3 0.298\leqslant 10\sigma^2.\end{aligned}$$

Once more applying $\mathrm{e}^y = 1 + y + \theta|y|^2\mathrm{e}^{|y|}$ and (30), we obtain

$$\begin{aligned}
\tilde\phi &= \exp\left\{\sigma^2\left(-\frac{t^2}{2\sigma^2 n} + \frac{(\mathrm{i}t)^3}{6\sigma^3 n\sqrt{n}} + \frac{\theta t^4}{24\sigma^4 n^2}\right)\right\} \\
&= \exp\left\{-\frac{t^2}{2n} + \frac{(\mathrm{i}t)^3}{6\sigma n\sqrt{n}} + \frac{\theta t^4}{24\sigma^2 n^2}\right)\right\} \\
&= \phi\exp\left\{\frac{(\mathrm{i}t)^3}{6\sigma n\sqrt{n}} + \frac{\theta t^4}{24\sigma^2 n^2}\right)\right\} = \phi\left(1 + \frac{(\mathrm{i}t)^3}{6\sigma n\sqrt{n}} + \frac{\theta t^4}{24\sigma^2 n^2} + \theta C\frac{t^6}{\sigma^2 n^3}\mathrm{e}^{10\sigma^2}\right) \\
&= \phi\left(1 + \frac{(\mathrm{i}t)^3}{6\sigma n\sqrt{n}} + \theta C\left(\frac{t^4}{\sigma^2 n^2} + \frac{t^4}{\sigma^2 n^2}\cdot\frac{t^2}{n}\right)\mathrm{e}^{10\sigma^2}\right) \\
&= \phi\left(1 + \frac{(\mathrm{i}t)^3}{6\sigma n\sqrt{n}} + \theta C\left(\frac{t^4}{\sigma^2 n^2} + \frac{t^4}{\sigma^2 n^2}\sigma^2\pi^2\right)\mathrm{e}^{10\sigma^2}\right) \\
&= \phi\left(1 + \frac{(\mathrm{i}t)^3}{6\sigma n\sqrt{n}} + \theta C\frac{t^4}{\sigma^2 n^2}(1+\sigma^2)\mathrm{e}^{10\sigma^2}\right) \\
&= \phi + \frac{\phi(\mathrm{i}t)^3}{6\sigma n\sqrt{n}} + \theta C\phi\frac{t^4}{\sigma^2 n^2}(1+\sigma^2)\mathrm{e}^{10\sigma^2} \\
&= \phi + \phi\frac{\sigma^2}{6}\left(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\} - 1\right)^3 + C\theta\phi\frac{t^4}{\sigma^2 n^2}(1+\sigma^2)\mathrm{e}^{10\sigma^2}. \qquad (35)
\end{aligned}$$

Observing that $|\mathrm{e}^{\mathrm{i}t}| = 1$ and using the obvious inequality $|\sin y| \geqslant 2|y|/\pi$ for $|y| \leqslant \pi/2$ and similarly obtained estimate $|\sin(t/(2\sigma\sqrt{n}))| \geqslant |t|/(\pi\sigma\sqrt{n})$ for $|t| \leqslant \pi\sigma\sqrt{n}$, we get

$$\begin{aligned}
\phi^n &= \exp\left\{-\frac{t^2}{2}\right\} \leqslant \exp\left\{-\frac{2t^2}{\pi^2}\right\}, \\
|\tilde\phi^n| &= \left|\exp\left\{\sigma^2 n\left(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\} - 1 - \frac{\mathrm{i}t}{\sigma\sqrt{n}}\right)\right\}\right| = \mathrm{e}^{\sigma^2 n}\left|\exp\left\{\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\} - 1\right\}\right| \\
&= \mathrm{e}^{\sigma^2 n}\exp\left\{\mathrm{Re}\left[\cos\left(\frac{t}{\sigma\sqrt{n}}\right) - 1 + \mathrm{i}\sin\left(\frac{t}{\sigma\sqrt{n}}\right)\right]\right\} \\
&= \exp\left\{-2n\sigma^2\sin^2\left(\frac{t}{2\sigma\sqrt{n}}\right)\right\} \leqslant \exp\left\{-\frac{2n\sigma^2 t^2}{\pi^2\sigma^2 n}\right\} = \mathrm{e}^{-2t^2/\pi^2}
\end{aligned}$$

For the proof of the second Lemma's statement we apply (29), (33), (35) and prove

$$\begin{aligned}
|\phi^n - \tilde\phi^n - n\tilde\phi^{n-1}(\phi - \tilde\phi)| &\leqslant Cn^2\max\{|\phi|, |\tilde\phi|\}^{n-2}|\phi - \tilde\phi|^2 \\
&\leqslant Cn^2\exp\left\{-\frac{2t^2(n-2)}{n\pi^2}\right\}|\phi - \tilde\phi|^2 \\
&\leqslant Cn^2\exp\left\{-\frac{2t^2}{\pi^2} + \frac{4t^2}{n\pi^2}\right\}\frac{t^6}{\sigma^2 n^3}\mathrm{e}^{12\sigma^2} \\
&\leqslant C\frac{t^6}{\sigma^2 n}\mathrm{e}^{12\sigma^2}\exp\left\{-\frac{2t^2}{\pi^2}\right\},
\end{aligned}$$

$$\begin{aligned}
|n\tilde\phi^{n-1}(\phi - \tilde\phi) - n\phi^{n-1}(\phi - \tilde\phi)| &= n|\phi - \tilde\phi||\tilde\phi^{n-1} - \phi^{n-1}| \leqslant C\frac{|t|^3}{\sigma\sqrt{n}}\mathrm{e}^{6\sigma^2}\phi|\tilde\phi^{n-1} - \phi^{n-1}| \\
&\leqslant C\frac{|t|^3}{\sigma\sqrt{n}}\mathrm{e}^{6\sigma^2}\phi(n-1)\max\{|\tilde\phi|, |\phi|\}^{n-2}|\tilde\phi - \phi| \\
&\leqslant C\frac{|t|^3}{\sigma\sqrt{n}}\mathrm{e}^{12\sigma^2}n\exp\left\{-\frac{t^2}{2n} - \frac{2t^2}{\pi^2} + \frac{4t^2}{n\pi^2}\right\}\frac{|t|^3}{\sigma n\sqrt{n}} \\
&\leqslant C\frac{t^6}{\sigma^2 n}\mathrm{e}^{12\sigma^2}\exp\left\{-\frac{2t^2}{\pi^2}\right\},
\end{aligned}$$

$$
\begin{aligned}
n\phi^{n-1}(\phi-\tilde\phi) &= -\phi^n\frac{\sigma^2 n}{6}\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big)^3+C\theta\phi^n\frac{t^4}{\sigma^2 n}(1+\sigma^2)\mathrm{e}^{10\sigma^2}.\\
&= -\phi^n\frac{\sigma^2 n}{6}\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big)^3+C\theta\frac{t^4}{\sigma^2 n}(1+\sigma^2)\mathrm{e}^{10\sigma^2}\exp\Big\{-\frac{t^2}{2}\Big\}.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\Big|\phi^n-\tilde\phi^n+\phi^n\frac{\sigma^2 n}{6}\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big)^3\Big|\\
&\qquad\leqslant|\phi^n-\tilde\phi^n-n\tilde\phi^{n-1}(\phi-\tilde\phi)|+|n\tilde\phi^{n-1}(\phi-\tilde\phi)-n\phi^{n-1}(\phi-\tilde\phi)|\\
&\qquad\quad+\Big|n\phi^{n-1}(\phi-\tilde\phi)+\phi^n\frac{\sigma^2 n}{6}\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big)^3\Big|\\
&\qquad\leqslant C\frac{\mathrm{e}^{12\sigma^2}}{\sigma^2 n}(t^6+t^4(1+\sigma^2))\exp\Big\{-\frac{2t^2}{\pi^2}\Big\}\leqslant C\frac{\mathrm{e}^{12\sigma^2}}{\sigma^2 n}(1+\sigma^2)(t^4+t^6)\exp\Big\{-\frac{2t^2}{\pi^2}\Big\}.
\end{aligned}
$$

Expression (34) follows from the last estimate. □

Lemma 4.6 *For* $|t|\leqslant\pi\sigma\sqrt{n}$ *we have*

$$
\begin{aligned}
\Big|\widehat{\mathcal{G}}_2\Big(\frac{t}{\sigma\sqrt{n}}\Big)-\widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big| &\leqslant C\frac{\mathrm{e}^{-0.1t}}{\sigma^2 n}t^2(1+\alpha)(1+\sigma^2)\mathrm{e}^{12\sigma^2}+\frac{C}{n}\mathrm{e}^{-0.1t^2}t^2\Big(\sigma+\frac{1}{\sigma}\Big)(1+\alpha)\\
&\quad+C\mathrm{e}^{-0.1t^2}\lambda_\pi\frac{t^2}{\sigma n}.
\end{aligned}
$$

Proof. Let $\varrho=\{\mathcal{A}-n\sigma^2\}$. Then $\lfloor\mathcal{A}-n\sigma^2\rfloor=\mathcal{A}-(n\sigma^2+\varrho)$. Using those notations we rewrite $\widehat{\mathcal{G}}(\frac{t}{\sigma\sqrt{n}})$ in the following form

$$
\begin{aligned}
\widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big) &= \exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\lfloor\mathcal{A}-n\sigma^2\rfloor+(n\sigma^2+\{\mathcal{A}-n\sigma^2\})\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big)\Big\}\times\\
&\quad\Big[1+n\sigma^2\Big(\sigma\alpha-\frac{1}{6}\Big)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big)^3+\sigma\lambda_\pi\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big)\Big]\\
&= \exp\Big\{\frac{\mathrm{i}t\mathcal{A}}{\sigma\sqrt{n}}+(n\sigma^2+\varrho)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\Big\}\times\\
&\quad\Big[1+n\sigma^2\Big(\sigma\alpha-\frac{1}{6}\Big)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big)^3+\sigma\lambda_\pi\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big)\Big].
\end{aligned}\tag{36}
$$

Therefore,

$$
\begin{aligned}
\Big|\widehat{\mathcal{G}}_2\Big(\frac{t}{\sigma\sqrt{n}}\Big)-\widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big| &\leqslant\Big|\exp\Big\{\frac{it\mathcal{A}}{\sigma\sqrt{n}}-\frac{t^2}{2}\Big\}\Big(1+n\alpha\sigma^3\Big(\exp\Big\{\frac{it}{\sigma\sqrt{n}}\Big\}-1\Big)^3\Big)\\
&\quad-\exp\Big\{\frac{\mathrm{i}t\mathcal{A}}{\sigma\sqrt{n}}+(n\sigma^2+\varrho)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\Big\}\times\\
&\qquad\Big(1+n\sigma^2\Big(\sigma\alpha-\frac{1}{6}\Big)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big)^3\Big)\Big|\\
&\quad+\sigma\lambda_\pi\Big|\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1\Big|\times\\
&\qquad\Big|\exp\Big\{\frac{it\mathcal{A}}{\sigma\sqrt{n}}-\frac{t^2}{2}\Big\}-\exp\Big\{\frac{it\mathcal{A}}{\sigma\sqrt{n}}+(n\sigma^2+\varrho)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\Big\}\Big|\\
&\qquad\quad=J_1+J_2.
\end{aligned}
$$

Applying $|\sin(t/(2\sigma\sqrt{n}))| \geqslant |t|/(2\sigma\sqrt{n})$ for $|t| \leqslant \pi\sigma\sqrt{n}$, we obtain

$$\begin{aligned}
&\left|\exp\left\{(n\sigma^2+\varrho)\Big(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\right\}\right| = \exp\{n\sigma^2+\varrho\}\left|\exp\left\{\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1\right\}\right| \\
&\qquad = \exp\left\{-2(n\sigma^2+\varrho)\sin^2\left(\frac{t}{2\sigma\sqrt{n}}\right)\right\} \leqslant \exp\left\{-2n\sigma^2\sin^2\left(\frac{t}{2\sigma\sqrt{n}}\right)\right\} \\
&\qquad \leqslant \exp\left\{-\frac{2n\sigma^2t^2}{\pi^2\sigma^2 n}\right\} = \mathrm{e}^{-2t^2/\pi^2} \leqslant \mathrm{e}^{-0.2t^2}.
\end{aligned}$$

Applying (30) and inequality $|\mathrm{e}^z-\mathrm{e}^y| \leqslant \max\{|\mathrm{e}^z|,|\mathrm{e}^y|\}|z-y|$ we get

$$\begin{aligned}
&\left|\exp\left\{-\frac{t^2}{2}\right\}-\exp\left\{(n\sigma^2+\varrho)\Big(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\right\}\right| \\
&\quad \leqslant \max\left\{|\mathrm{e}^{-t^2/2}|,|\mathrm{e}^{-2t^2/\pi^2}|\right\}\left|-\frac{t^2}{2}-(n\sigma^2+\varrho)\Big(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\right| \\
&\quad \leqslant \mathrm{e}^{-2t^2/\pi^2}\left|-\frac{t^2}{2}-n\sigma^2\Big(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)+\varrho\Big(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\right| \\
&\quad \leqslant \mathrm{e}^{-2t^2/\pi^2}\left|-\frac{t^2}{2}-n\sigma^2\Big(-\frac{t^2}{2\sigma^2 n}+\frac{\theta C|t|^3}{\sigma^3 n\sqrt{n}}\Big)+\frac{C\theta t^2}{\sigma^2 n}\right| \\
&\quad \leqslant \mathrm{e}^{-2t^2/\pi^2}C\Big(\frac{|t|^3}{\sigma\sqrt{n}}+\frac{t^2}{\sigma^2 n}\Big).
\end{aligned}$$

Using (31) we obtain

$$\begin{aligned}
J_2 &= \sigma\lambda_\pi\left|\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1\right| \times \\
&\quad \left|\exp\left\{\frac{it\mathcal{A}}{\sigma\sqrt{n}}-\frac{t^2}{2}\right\}-\exp\left\{\frac{\mathrm{i}t\mathcal{A}}{\sigma\sqrt{n}}+(n\sigma^2+\varrho)\Big(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\right\}\right| \\
&= \sigma\lambda_\pi\left|\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1\right| \times \\
&\quad \left|\exp\left\{-\frac{t^2}{2}\right\}-\exp\left\{(n\sigma^2+\varrho)\Big(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\right\}\right| \\
&\leqslant \frac{\lambda_\pi|t|}{\sqrt{n}}\mathrm{e}^{-2t^2/\pi^2}C\Big(\frac{|t|^3}{\sigma\sqrt{n}}+\frac{t^2}{\sigma^2 n}\Big) \leqslant C\mathrm{e}^{-2t^2/\pi^2}\lambda_\pi\left(\frac{t^4}{\sigma n}+\frac{|t|^3}{\sigma^2 n\sqrt{n}}\right) \\
&\leqslant C\mathrm{e}^{-2t^2/\pi^2}\lambda_\pi\frac{t^4+t^2}{\sigma n} \leqslant C\mathrm{e}^{-2t^2/\pi^2}\lambda_\pi\frac{t^2}{\sigma n} \leqslant C\mathrm{e}^{-0.1t^2}\lambda_\pi\frac{t^2}{\sigma n}. \qquad (37)
\end{aligned}$$

Therefore, J_1 can be expressed in the following form

$$\begin{aligned}
J_1 = &\left|\exp\left\{\frac{it\mathcal{A}}{\sigma\sqrt{n}}-\frac{t^2}{2}\right\}\left(1+n\alpha\sigma^3\Big(\exp\left\{\frac{it}{\sigma\sqrt{n}}\right\}-1\Big)^3\right)\right. \\
&\quad -\exp\left\{\frac{\mathrm{i}t\mathcal{A}}{\sigma\sqrt{n}}+(n\sigma^2+\varrho)\Big(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1-\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\right\}\times \\
&\quad \left.\left(1+n\sigma^2\Big(\sigma\alpha-\frac{1}{6}\Big)\Big(\exp\left\{\frac{it}{\sigma\sqrt{n}}\right\}-1\Big)^3\right)\right|
\end{aligned}$$

$$= \Big| \exp\Big\{ -\frac{t^2}{2}\Big\}\Big(1 + n\alpha\sigma^3\Big(\exp\Big\{\frac{it}{\sigma\sqrt{n}}\Big\} - 1\Big)^3\Big)$$
$$- \exp\Big\{(n\sigma^2 + \varrho)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1 - \frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\Big\}\Big(1 + n\sigma^2\Big(\sigma\alpha - \frac{1}{6}\Big)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3\Big)\Big|$$
$$= \Big|\phi^n\Big(1 + n\alpha\sigma^3\Big(\exp\Big\{\frac{it}{\sigma\sqrt{n}}\Big\} - 1\Big)^3\Big)$$
$$-\tilde{\phi}^n \exp\Big\{\varrho\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1 - \frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\Big\}\Big(1 + n\sigma^2\Big(\sigma\alpha - \frac{1}{6}\Big)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3\Big)\Big|$$
$$\leqslant \Big|\phi^n\Big(1 + n\alpha\sigma^3\Big(\exp\Big\{\frac{it}{\sigma\sqrt{n}}\Big\} - 1\Big)^3\Big) - \tilde{\phi}^n\Big(1 + n\sigma^2\Big(\sigma\alpha - \frac{1}{6}\Big)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3\Big)\Big|$$
$$+\Big|\tilde{\phi}^n\Big(1 + n\sigma^2\Big(\sigma\alpha - \frac{1}{6}\Big)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3\Big) \times$$
$$\Big(1 - \exp\Big\{\varrho\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1 - \frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\Big\}\Big)\Big| = J_{11} + J_{12}.$$

Using (30) and (31), for $|t| \leqslant \pi\sigma\sqrt{n}$ we get

$$\Big|1 - \exp\Big\{\varrho\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1 - \frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\Big\}\Big| \leqslant C\Big|\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1 - \frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big| \leqslant \frac{Ct^2}{\sigma^2 n},$$
$$n\sigma^2\Big(\sigma\alpha - \frac{1}{6}\Big)\Big|\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big|^3 \leqslant Cn\sigma^2(\sigma\alpha + 1)\frac{|t|^3}{\sigma^3 n\sqrt{n}} \leqslant \frac{C(\sigma\alpha + 1)}{\sqrt{n}\sigma}|t|^3 \leqslant C(\sigma\alpha + 1)t^2.$$

Applying $|\phi^n| \leqslant \mathrm{e}^{-2t^2/\pi^2}$, $|\tilde{\phi}^n| \leqslant \mathrm{e}^{-2t^2/\pi^2}$ we estimate J_{12}

$$\begin{aligned} J_{12} &= \Big|\tilde{\phi}^n\Big(1 + n\sigma^2\Big(\sigma\alpha - \frac{1}{6}\Big)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3\Big) \times \\ &\quad \Big(1 - \exp\Big\{\varrho\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1 - \frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big)\Big\}\Big)\Big| \\ &\leqslant C\mathrm{e}^{-2t^2/\pi^2}(1 + (\sigma\alpha + 1)t^2)\frac{t^2}{\sigma^2 n} \leqslant \frac{C}{n}\mathrm{e}^{-t^2/\pi^2}t^2\frac{1 + \sigma\alpha}{\sigma} \\ &\leqslant \frac{C}{n}\mathrm{e}^{-t^2/\pi^2}t^2\frac{(1 + \alpha)(1 + \sigma)}{\sigma} \leqslant \frac{C}{n}\mathrm{e}^{-t^2/\pi^2}t^2\frac{2(1 + \sigma^2)}{\sigma}(1 + \alpha) \\ &\leqslant \frac{C}{n}\mathrm{e}^{-0.1t^2}t^2\Big(\sigma + \frac{1}{\sigma}\Big)(1 + \alpha). \end{aligned} \tag{38}$$

Due to (34) we have

$$\phi^n\Big(1 + n\alpha\sigma^3\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3\Big) = \phi^n + \phi^n n\alpha\sigma^3\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3$$
$$= \tilde{\phi}^n - \phi^n\frac{\sigma^2 n}{6}\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3 + \theta C\frac{\mathrm{e}^{12\sigma^2}}{\sigma^2 n}(1 + \sigma^2)(t^4 + t^6)\exp\Big\{-\frac{2t^2}{\pi^2}\Big\}$$
$$+\phi^n n\alpha\sigma^3\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3$$
$$= \tilde{\phi}^n + \phi^n n\sigma^2\Big(\sigma\alpha - \frac{1}{6}\Big)\Big(\exp\Big\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\Big\} - 1\Big)^3 + \theta C\frac{\mathrm{e}^{12\sigma^2}}{\sigma^2 n}(1 + \sigma^2)(t^4 + t^6)\exp\Big\{-\frac{2t^2}{\pi^2}\Big\}.$$

Collecting the last estimate, (31) and (33) we evaluate J_{11}

$$
\begin{aligned}
J_{11} &= \left|\phi^n\left(1+n\alpha\sigma^3\left(\exp\left\{\frac{it}{\sigma\sqrt{n}}\right\}-1\right)^3\right)-\tilde{\phi}^n\left(1+n\sigma^2\left(\sigma\alpha-\frac{1}{6}\right)\left(\exp\left\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\right\}-1\right)^3\right)\right| \\
&\leqslant \left|(\phi^n-\tilde{\phi}^n)\sigma^2 n\left(\sigma\alpha-\frac{1}{6}\right)\left(\exp\left\{\frac{it}{\sigma\sqrt{n}}\right\}-1\right)^3\right| \\
&+C\frac{\mathrm{e}^{12\sigma^2}}{\sigma^2 n}(1+\sigma^2)(t^4+t^6)\exp\left\{-\frac{2t^2}{\pi^2}\right\} \\
&\leqslant n\max\{|\phi|,|\tilde{\phi}|\}^{n-1}|\phi-\tilde{\phi}|\sigma^2 n(\sigma\alpha+1)\frac{|t|^3}{\sigma^3 n\sqrt{n}}+C\frac{\mathrm{e}^{12\sigma^2}}{\sigma^2 n}(1+\sigma^2)(t^4+t^6)\exp\left\{-\frac{2t^2}{\pi^2}\right\} \\
&\leqslant C\frac{t^6}{\sigma^2 n}(\sigma\alpha+1)\mathrm{e}^{6\sigma^2}\exp\left\{-\frac{2t^2(n-1)}{n\pi^2}-\frac{t^2}{2n}\right\}+C\frac{\mathrm{e}^{12\sigma^2}}{\sigma^2 n}(1+\sigma^2)(t^4+t^6)\exp\left\{-\frac{2t^2}{\pi^2}\right\} \\
&\leqslant C\frac{t^6}{\sigma^2 n}(\sigma\alpha+1)\mathrm{e}^{6\sigma^2}\exp\left\{-\frac{2t^2}{\pi^2}\right\}+C\frac{\mathrm{e}^{12\sigma^2}}{\sigma^2 n}(1+\sigma^2)(t^4+t^6)\exp\left\{-\frac{2t^2}{\pi^2}\right\} \\
&\leqslant C\frac{\mathrm{e}^{-0.1t}}{\sigma^2 n}(t^6(\sigma\alpha+1)+(1+\sigma^2)(t^4+t^6))\mathrm{e}^{12\sigma^2} \\
&\leqslant C\frac{\mathrm{e}^{-0.1t}}{\sigma^2 n}t^2(1+\alpha)(1+\sigma^2)\mathrm{e}^{12\sigma^2}.
\end{aligned}
\tag{39}
$$

Combining (38) and (39) with (37) we complete Lemma's proof. □

Lemma 4.7 *Let the conditions from Theorem 3.1 are fulfilled. Then, for* $|t|\leqslant\pi\sigma\sqrt{n}$ *we have*

$$
\begin{aligned}
\left|\mathrm{E}\mathrm{e}^{it\tilde{\mathcal{S}}_n/(\sigma\sqrt{n})}-\widehat{\mathcal{G}}\left(\frac{t}{\sigma\sqrt{n}}\right)\right| &\leqslant \frac{C}{\sqrt{n}}o(1)|t|^3\mathrm{e}^{-t^2/15}+C|t|\rho_1^n \\
&+C\frac{\mathrm{e}^{-0.1t^2}}{n}t^2(1+\alpha+\lambda_\pi)\left(\sigma^2+\frac{1}{\sigma^2}\right)\mathrm{e}^{12\sigma^2}
\end{aligned}
$$

Proof. Using the result of Nagaev (5), we can express

$$
\begin{aligned}
\left|\mathrm{E}\mathrm{e}^{it\tilde{\mathcal{S}}_n/(\sigma\sqrt{n})}-\widehat{\mathcal{G}}_1\left(\frac{t}{\sigma\sqrt{n}}\right)\right| &= \left|\mathrm{E}\exp\left\{\frac{\mathrm{i}t\tilde{\mathcal{S}}_n}{\sigma\sqrt{n}}\right\}-\widehat{\mathcal{G}}_1\left(\frac{t}{\sigma\sqrt{n}}\right)\right|=\left|\exp\left\{\frac{it\mathcal{A}}{\sigma\sqrt{n}}\right\}\right|\times \\
&\left|\mathrm{E}\exp\left\{\frac{\mathrm{i}t\tilde{\mathcal{S}}_n}{\sigma\sqrt{n}}-\frac{it\mathcal{A}}{\sigma\sqrt{n}}\right\}-\mathrm{e}^{-t^2/2}\left(1+\frac{\alpha(\mathrm{i}t)^3}{\sqrt{n}}+\frac{\lambda_\pi(\mathrm{i}t)}{\sqrt{n}}\right)\right| \\
&\leqslant \frac{C}{\sqrt{n}}o(1)|t|^3\mathrm{e}^{-t^2/15}+C|t|\rho_1^n,
\end{aligned}
\tag{40}
$$

Applying the triangle inequality,

$$
\begin{aligned}
\left|\mathrm{E}\mathrm{e}^{it\tilde{\mathcal{S}}_n/(\sigma\sqrt{n})}-\widehat{\mathcal{G}}\left(\frac{t}{\sigma\sqrt{n}}\right)\right| &\leqslant \left|\mathrm{E}\mathrm{e}^{it\tilde{\mathcal{S}}_n/(\sigma\sqrt{n})}-\widehat{\mathcal{G}}_1\left(\frac{t}{\sigma\sqrt{n}}\right)\right| \\
&+\left|\widehat{\mathcal{G}}_1\left(\frac{t}{\sigma\sqrt{n}}\right)-\widehat{\mathcal{G}}_2\left(\frac{t}{\sigma\sqrt{n}}\right)\right|+\left|\widehat{\mathcal{G}}_2\left(\frac{t}{\sigma\sqrt{n}}\right)-\widehat{\mathcal{G}}\left(\frac{t}{\sigma\sqrt{n}}\right)\right|
\end{aligned}
$$

and using (40), Lemma (4.4) and Lemma (4.6) we complete Lemma's proof. □

Lemma 4.8 *Let $p_{11} \leqslant 1/2$, $p_{01} \leqslant 1/30$. Then*

$$\inf_{\mathcal{D}\in\mathbb{D}} \|\mathcal{L}(S_n) - \mathcal{D}\|_\infty \leqslant Cp_{01}(p_{11}+p_{01}) \min\Big(1, \frac{1}{np_{01}}\Big) + C\min\Big(\sqrt{\frac{p_{01}}{n}}, np_{01}^2\Big) + C(p_{11}+p_{01})\mathrm{e}^{-C_0 n}, \tag{41}$$

$$\|\mathcal{L}(S_n) - \mathcal{H} * \exp\{\varkappa_1(\tilde{G}-I)\} * \mathcal{V}\| \leqslant C(p_{11}+p_{01})\Big\{\min\Big(p_{01}, \frac{1}{n}\Big) + \mathrm{e}^{-C_0 n}\Big\}, \tag{42}$$

$$\|\mathcal{L}(S_n) - \mathcal{H} * \exp\{\varkappa_1(\tilde{G}-I)\} * \mathcal{V}\|_\infty \leqslant C(p_{11}+p_{01})\Big\{\min\Big(p_{01}, \frac{1}{n\sqrt{np_{01}}}\Big) + \mathrm{e}^{-C_0 n}\Big\}, \tag{43}$$

where

$$\mathcal{V} = \exp\{n\gamma_1(\tilde{G}-I) + n\gamma_2(\tilde{G}-I)^{*2} + n\gamma_3(\tilde{G}-I)^{*3}\}.$$

Lemma 4.8 follows from Theorem 3.2 and Theorem 3.5 in [26].

Lemma 4.9 *Let $p_{11} \leqslant 1/4$, $p_{01} \leqslant 1/30$, $|t| \leqslant \pi$. Then*

$$Re(\widehat{\tilde{G}}(t) - 1) \leqslant -\frac{2}{1+p_{11}} \sin^2\frac{t}{2} \leqslant -\frac{8}{9\pi^2}t^2, \quad |\widehat{\tilde{G}}(t) - 1| \leqslant \frac{4}{3}|t|.$$

Here $Re\{z\}$ means the real part of the complex number z.

Lemmas statement easily follows from the definition of $\tilde{G}$.

4.2 Proof of Theorems 3.1-3.4

Proof of Theorem 3.1. Due to Lemma 4.7 and using $u = \sigma\sqrt{n}$ in Lemma 4.2, we obtain

$$\begin{aligned}\|\tilde{F}_{n\pi} - \mathcal{G}\|_\infty \leqslant \frac{1}{2\pi\sigma\sqrt{n}}\Bigg(&\int_{|t|\leqslant\nabla\sqrt{n}} \bigg|\mathrm{E}\mathrm{e}^{it\tilde{\mathcal{S}}_n/(\sigma\sqrt{n})} - \widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\bigg|\mathrm{d}t \\ &+ \int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}} \bigg|\mathrm{E}\mathrm{e}^{it\tilde{\mathcal{S}}_n/(\sigma\sqrt{n})} - \widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\bigg|\mathrm{d}t\Bigg)\end{aligned}$$

We estimate each summand separately. Applying $\rho_1^n = o(n^{-k})$, where $k \in \mathbb{R}$, it follows that

$$\begin{aligned}&\int_{|t|\leqslant\nabla\sqrt{n}} \bigg|\mathrm{E}\mathrm{e}^{it\tilde{\mathcal{S}}_n/(\sigma\sqrt{n})} - \widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\bigg|\mathrm{d}t \\ &\quad\leqslant \int_{|t|\leqslant\nabla\sqrt{n}} \bigg(\frac{C}{\sqrt{n}}o(1)|t|^3\mathrm{e}^{-t^2/15} + C|t|\rho_1^n + C\frac{\mathrm{e}^{-0.1t^2}}{n}t^2(1+\alpha+\lambda_\pi)\Big(\sigma^2 + \frac{1}{\sigma^2}\Big)\mathrm{e}^{12\sigma^2}\bigg)\mathrm{d}t \\ &\quad\leqslant \frac{C}{\sqrt{n}}o(1)\cdot 2\int_0^\infty t^3\mathrm{e}^{-t^2/15}\mathrm{d}t + \nabla\sqrt{n}\rho_1^n\int_{|t|\leqslant\nabla\sqrt{n}} 1\mathrm{d}t + \frac{C}{n}\cdot 2\int_0^\infty t^2\mathrm{e}^{-0.1t^2}\mathrm{d}t \\ &\quad\leqslant C\bigg(\frac{o(1)}{\sqrt{n}} + \rho_1^n n + \frac{1}{n}\bigg) \leqslant C\bigg(\frac{o(1)}{\sqrt{n}} + o(n^{-2})n + \frac{1}{n}\bigg) \leqslant o(n^{-1/2}).\end{aligned}$$

Next we use (36) and the inequalities $\exp\{-2\sin^2\frac{t}{\sigma\sqrt{n}}\}\leqslant\exp\{-\frac{Ct^2}{\sigma^2 n}\}$, $|\exp\{\frac{\mathrm{i}t}{\sigma\sqrt{n}}\}-1|\leqslant 2$. Applying $|\mathrm{E}\mathrm{e}^{it\tilde{S}_n}|\leqslant \mathrm{e}^{-Cn}\leqslant\frac{C}{n^k}$ for $\varepsilon\leqslant|t|\leqslant\pi$, where $k\in\mathbb{R}$, and $\varepsilon>0$, we obtain

$$
\begin{aligned}
&\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}\Big|\mathrm{E}\mathrm{e}^{it\tilde{S}_n/(\sigma\sqrt{n})}-\widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big|\mathrm{d}t\\
&\quad\leqslant\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}\Big|\mathrm{E}\mathrm{e}^{it\tilde{S}_n/(\sigma\sqrt{n})}\Big|\mathrm{d}t+\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}\Big|\widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big|\mathrm{d}t\\
&\quad\leqslant\sigma\sqrt{n}\int_{\nabla/\sigma\leqslant|z|\leqslant\pi}\Big|\mathrm{E}\mathrm{e}^{iz\tilde{S}_n}\Big|\mathrm{d}z\\
&\qquad+\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}\exp\Big\{(n\sigma^2+\varrho)\Big(-2\sin^2\frac{t}{\sigma\sqrt{n}}\Big)\Big\}[1+2^3n\sigma^2|\sigma\alpha-1/6|+2\sigma\lambda_\pi]\mathrm{d}t\\
&\quad\leqslant\sigma\sqrt{n}\mathrm{e}^{-Cn}\Big(\pi-\frac{\nabla}{\sigma}\Big)+\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}C\mathrm{e}^{-Ct^2}(1+n)\mathrm{d}t\\
&\quad\leqslant\sigma\sqrt{n}\cdot\frac{C}{n}+C\mathrm{e}^{-C\nabla n}(1+n)(\pi\sigma\sqrt{n}-\nabla\sqrt{n})\leqslant o(n^{-1/2}).
\end{aligned}
$$

Thus we complete the proof of (17). Next we move on the second part of the proof of Lemma. Applying Lemma 4.7 we get

$$
\begin{aligned}
\|\tilde{F}_{n\pi}-\mathcal{G}\|_K\ \leqslant\ &\frac{1}{4}\Big(\int_{|t|\leqslant\nabla\sqrt{n}}\frac{1}{|t|}\Big|\mathrm{E}\mathrm{e}^{it\tilde{S}_n/(\sigma\sqrt{n})}-\widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big|\mathrm{d}t\\
&+\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}\frac{1}{|t|}\Big|\mathrm{E}\mathrm{e}^{it\tilde{S}_n/(\sigma\sqrt{n})}-\widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big|\mathrm{d}t\Big).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{|t|\leqslant\nabla\sqrt{n}}\frac{1}{|t|}\Big|\mathrm{E}\mathrm{e}^{it\tilde{S}_n/(\sigma\sqrt{n})}-\widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big|\mathrm{d}t\\
&\quad\leqslant\int_{|t|\leqslant\nabla\sqrt{n}}\Big(\frac{C}{\sqrt{n}}o(1)|t|^2\mathrm{e}^{-t^2/15}+C\rho_1^n+C\frac{\mathrm{e}^{-0.1t^2}}{n}|t|(1+\alpha+\lambda_\pi)\Big(\sigma^2+\frac{1}{\sigma^2}\Big)\mathrm{e}^{12\sigma^2}\Big)\mathrm{d}t\\
&\quad\leqslant\frac{C}{\sqrt{n}}o(1)\cdot 2\int_0^\infty t^2\mathrm{e}^{-t^2/15}\mathrm{d}t+\frac{1}{\sqrt{n}}+\frac{C}{n}\cdot 2\int_0^\infty t\mathrm{e}^{-0.1t^2}\mathrm{d}t\leqslant o(n^{-1/2}),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}\frac{1}{|t|}\Big|\mathrm{E}\mathrm{e}^{it\tilde{S}_n/(\sigma\sqrt{n})}-\widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big|\mathrm{d}t\\
&\quad\leqslant\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}\frac{1}{|t|}\Big|\mathrm{E}\mathrm{e}^{it\tilde{S}_n/(\sigma\sqrt{n})}\Big|\mathrm{d}t+\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}\frac{1}{|t|}\Big|\widehat{\mathcal{G}}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big|\mathrm{d}t\\
&\quad\leqslant\sigma\sqrt{n}\int_{\nabla/\sigma\leqslant|z|\leqslant\pi}\frac{|\mathrm{E}\mathrm{e}^{iz\tilde{S}_n}|}{z\sigma\sqrt{n}}\mathrm{d}z\\
&\qquad+\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}\frac{1}{|t|}\exp\Big\{(n\sigma^2+\varrho)\Big(-2\sin^2\frac{t}{\sigma\sqrt{n}}\Big)\Big\}[1+2^3n\sigma^2|\sigma\alpha-1/6|+2\sigma\lambda_\pi]\mathrm{d}t\\
&\quad\leqslant\int_{\nabla/\sigma\leqslant|z|\leqslant\pi}\mathrm{e}^{-Cn}\frac{\sigma}{\nabla}\mathrm{d}z+\int_{\nabla\sqrt{n}\leqslant|t|\leqslant\pi\sigma\sqrt{n}}\frac{C}{|t|}\mathrm{e}^{-Ct^2}(1+n)\mathrm{d}t\\
&\quad\leqslant\mathrm{e}^{-Cn}\Big(\pi-\frac{\nabla}{\sigma}\Big)\frac{\sigma}{\nabla}+\frac{C}{\nabla\sqrt{n}}\mathrm{e}^{-C\nabla n}(1+n)(\pi\sigma\sqrt{n}-\nabla\sqrt{n})\leqslant o(n^{-1/2}).
\end{aligned}
$$

Collecting all estimates, we complete the proof of Theorem 3.1. □

Proof of Theorem 3.2. It was already noted that approximation in Theorem 3.2 is infinitely divisible. Next we check that it is indeed some CP *distribution*. Taking into account that $p_{11} \leqslant 1/4$ and $p_{01} \leqslant 1/30$ we observe that

$$\frac{45}{47} \leqslant \frac{p_{10}}{p_{10}+p_{01}} \leqslant 1, \quad \frac{30}{31} \leqslant \frac{1}{p_{10}+p_{01}} \leqslant \frac{4}{3}, \quad \frac{p_{01}}{p_{10}+p_{01}} \leqslant \frac{2}{47}$$

and, therefore,

$$\gamma_1 \leqslant p_{01}, \quad \gamma_1 \geqslant p_{01}\frac{45}{47} \geqslant 0.957 p_{01}, \tag{44}$$

$$\begin{aligned} |\gamma_2| &= \frac{p_{10}p_{01}^2}{(p_{10}+p_{01})^2}\Big(\frac{p_{11}}{2} + \frac{p_{10}}{p_{10}+p_{01}} + \frac{1}{2}\Big) \\ &\leqslant \frac{p_{10}p_{01}^2}{(p_{10}+p_{01})^2}\Big(\frac{1}{8}+\frac{3}{2}\Big) \leqslant \frac{13}{6}p_{01}^2 \leqslant 0.073 p_{01}, \end{aligned} \tag{45}$$

$$|\gamma_2| \geqslant \frac{p_{10}p_{01}^2}{(p_{10}+p_{01})^2}\Big(\frac{45}{47}+\frac{1}{2}\Big) \geqslant 1.457\frac{p_{10}p_{01}^2}{(p_{10}+p_{01})^2} \geqslant 1.395\frac{p_{01}^2}{p_{10}+p_{01}} \geqslant 1.35 p_{01}^2,$$

$$\begin{aligned} |\gamma_3| &\leqslant \gamma_1^2\Big\{\frac{\gamma_1}{3} + \frac{p_{11}^2p_{01}}{p_{10}(p_{10}+p_{01})} + \frac{2p_{11}p_{01}}{(p_{10}+p_{01})^2} + \frac{p_{11}p_{10}}{(p_{10}+p_{01})^2} + \frac{2p_{10}p_{01}}{(p_{10}+p_{01})^3} \\ &\quad + \frac{p_{11}p_{01}}{(p_{10}+p_{01})} + \frac{p_{10}p_{01}}{(p_{10}+p_{01})^2}\Big\} \\ &\leqslant \frac{p_{01}^2}{p_{10}+p_{01}}\Big\{\frac{p_{01}}{3} + \frac{p_{11}^2p_{01}}{p_{10}+p_{01}} + \frac{2p_{11}p_{01}}{p_{10}+p_{01}} + p_{11} + \frac{2p_{01}}{p_{10}+p_{01}} + p_{11}p_{01} + p_{01}\Big\} \\ &\leqslant 0.55p_{01}^2 \leqslant 0.02 p_{01}, \end{aligned} \tag{46}$$

$$|\varkappa_1| \leqslant p_{01}\max\Big(\frac{p_{01}}{p_{10}+p_{01}}, \frac{p_{11}}{p_{10}+p_{01}} + p_0\Big) \leqslant \frac{4}{90}, \quad |\varkappa_2| \leqslant \frac{1}{4}. \tag{47}$$

Taking into account (44)–(47) we obtain $\gamma \geqslant 6n|\gamma_2| - 6n\gamma_3 \geqslant 6n(|\gamma_2|-|\gamma_3|) \geqslant 4.8np_{01}^2 > 0$. Therefore $\tilde{G}^{*\gamma} \in \mathbb{D}$. Indeed,

$$\widehat{\tilde{G}}^{\gamma}(t) = \exp\Big\{\mathrm{i}t\gamma + \gamma\sum_{j=1}^{\infty}\frac{p_{11}^j}{j}(\mathrm{e}^{\mathrm{i}tj}-1)\Big\},$$

that is, $\tilde{G}^{*\gamma}$ is some CP distribution shifted by γ. Next observe that due to assumption $np_{01} \geqslant 3$,

$$\lambda_1 + \varkappa_1 \geqslant n(\gamma_1 - 4|\gamma_2| - 3|\gamma_3|) - \beta - |\varkappa_1| \geqslant 0.605np_{01} - 1.045 > 0.2np_{01} > 0$$

and, therefore, $\exp\{(\lambda_1+\varkappa_1)(\tilde{G}-I)\} \in \mathbb{D}$. Similarly, $\lambda_2 \geqslant 0$,

$$\lambda_{-1} = n(2|\gamma_2| - 3\gamma_3) + \beta/3 \geqslant n(2|\gamma_2| - 3|\gamma_3|) \geqslant 1.05np_{01}^2 > 0$$

and $\exp\{\lambda_2(\tilde{G}^{*2}-I) + \lambda_{-1}(I_{-1}-I)/p_{10}\} \in \mathbb{D}$. By Lemma 5.3 from [26], it follows that $\mathcal{H} \in \mathbb{D}$ and we proved that $\mathcal{H} * \exp\{\varkappa_1(\tilde{G}-I)\} * \mathcal{U} \in \mathbb{D}$.

The rest of this subsection is devoted to the proof of (19) and (20). We show that $\mathcal{U}$ is close to $\mathcal{V}$ and apply the triangle inequality and Lemma 4.8. Observe that, in general, $\mathcal{V}$ is a signed compound Poisson measure, i.e. *not* a distribution.

First we estimate $|\widehat{\mathcal{V}}(t)|$ and $|\widehat{\mathcal{U}}(t)|$. Note that by the properties of characteristic functions $|\widehat{\tilde{G}}(t)-1|^2 \leqslant 2(1 - Re\widehat{\tilde{G}}(t))$, see, for example Theorem 1.7 in [5]. Applying trivial estimate $|\widehat{\tilde{G}}(t)-1| \leqslant |\widehat{\tilde{G}}(t)|+1 \leqslant 2$, estimates (44)–(46) and Lemma 4.9 we obtain

$$\begin{aligned} |\widehat{\mathcal{V}}(t)| &\leqslant \exp\Big\{n\gamma_1(Re\widehat{\tilde{G}}(t)-1) + 2n|\gamma_2|(1-Re\widehat{\tilde{G}}(t)) + 4n|\gamma_3|(1-Re\widehat{\tilde{G}}(t))\Big\} \\ &\leqslant \exp\{-0.731p_{01}(Re\widehat{\tilde{G}}(t)-1)\} \leqslant \exp\{-0.065p_{01}t^2\}. \end{aligned} \tag{48}$$

Taking into account that any characteristic function is by its absolute value less or equal to unity, we similarly prove that

$$|\widehat{\mathcal{U}}(t)| \leqslant \exp\{\lambda_1(Re\widehat{\widetilde{G}}(t)-1)\} \leqslant \exp\{0.605np_{01}(Re\widehat{\widetilde{G}}(t)-1)\} \leqslant \exp\{-0.054np_{01}t^2\}. \tag{49}$$

Next we compare $\ln\widehat{\mathcal{V}}(t)$ and $\ln\widehat{\mathcal{U}}(t)$. Applying the well known expansion

$$\mathrm{e}^{\mathrm{i}y} = 1+(\mathrm{i}y)+\frac{(\mathrm{i}y)^2}{2}+\frac{(\mathrm{i}y)^3}{6}+\theta C|y|^4, \quad y\in\mathbb{R}$$

we prove that, for $|t|\leqslant\pi$,

$$\begin{aligned}
\widehat{\widetilde{G}}^{\gamma}(t) &= \exp\Big\{\gamma\frac{\mathrm{i}t}{p_{10}}+\frac{\gamma p_{11}(\mathrm{i}t)^2}{2p_{10}^2}+\gamma\frac{p_{11}(1+p_{11})(\mathrm{i}t)^3}{6p_{10}^3}+\theta C|\gamma|t^4\Big\},\\
\widehat{\widetilde{G}}(t)-1 &= \frac{\mathrm{i}t}{p_{10}}+\frac{(1+p_{11})(\mathrm{i}t)^2}{2p_{10}^2}+\frac{(1+4p_{11}+p_{11}^2)(\mathrm{i}t)^3}{6p_{10}^3}+\theta Ct^4,\\
(\widehat{\widetilde{G}}(t)-1)^2 &= \frac{(\mathrm{i}t)^2}{p_{10}^2}+\frac{(1+p_{11})(\mathrm{i}t)^3}{p_{10}^3}+\theta Ct^4, \qquad (\widehat{\widetilde{G}}(t)-1)^3=\frac{(\mathrm{i}t)^3}{p_{10}^3}+\theta Ct^4,\\
\widehat{\widetilde{G}}^2(t)-1 &= \frac{2(\mathrm{i}t)}{p_{10}}+\frac{(2+p_{11})(\mathrm{i}t)^2}{p_{10}^2}+\frac{(4+7p_{11}+p_{11}^2)(\mathrm{i}t)^3}{3p_{10}^3}+\theta Ct^4,\\
\mathrm{e}^{-\mathrm{i}t}-1 &= -(\mathrm{i}t)+\frac{(\mathrm{i}t)^2}{2}-\frac{(\mathrm{i}t)^3}{6}+\theta Ct^4.
\end{aligned}$$

Consequently, taking into account (45) and (46), we obtain expansions

$$\ln\widehat{\mathcal{V}}(t) = n\gamma_1(\widehat{\widetilde{G}}(t)-1)+n\gamma_2\frac{(\mathrm{i}t)^2}{p_{10}^2}+n[\gamma_2(1+p_{11})+\gamma_3]\frac{(\mathrm{i}t)^3}{p_{10}^3}+\theta C(np_{01}^2)t^4$$

and

$$\ln\widehat{\mathcal{U}}(t) = n\gamma_1(\widehat{\widetilde{G}}(t)-1)+n\gamma_2\frac{(\mathrm{i}t)^2}{p_{10}^2}+n[\gamma_2(1+p_{11})+\gamma_3]\frac{(\mathrm{i}t)^3}{p_{10}^3}+\theta C(np_{01}^2+1)t^4.$$

Combining the last two expansions with (48), (49) and trivial estimate $y\mathrm{e}^{-y}\leqslant 1$, $y>0$, we prove that

$$\begin{aligned}
|\widehat{\mathcal{V}}(t)-\widehat{\mathcal{U}}(t)| &\leqslant \max(|\widehat{\mathcal{V}}(t)|,|\widehat{\mathcal{U}}(t)|)|\ln\widehat{\mathcal{V}}(t)-\ln\widehat{\mathcal{U}}(t)|\\
&\leqslant \exp\{-0.054np_{01}t^2\}(np_{01}^2+1)t^4\\
&\leqslant C\exp\{-0.05np_{01}t^2\}((n^{-1}+(np_{01})^{-2}).
\end{aligned} \tag{50}$$

Similarly, for $|t|\leqslant\pi$,

$$\begin{aligned}
(\gamma\ln\widehat{\widetilde{G}}(t))' &= \widehat{\widetilde{G}}'(t)\gamma\bigg(1-\frac{\mathrm{i}t}{p_{10}}+\frac{(\mathrm{i}t)^2}{2p_{10}}\bigg)+\theta C|\gamma||t|^3,\\
(\mathrm{e}^{-\mathrm{i}t}-1)' &= \widehat{\widetilde{G}}'(t)\bigg(-1+\frac{2(\mathrm{i}t)}{p_{10}}-\frac{(\mathrm{i}t)^2(1+p_{10})}{p_{10}^2}\bigg)+\theta C|t|^3,\\
(\ln\widehat{\mathcal{V}}(t))' &= \widehat{\widetilde{G}}'(t)\bigg(n\gamma_1+\frac{2n\gamma_2(\mathrm{i}t)}{p_{10}}+\frac{n(\gamma_2(1+p_{11})+3\gamma_3)(\mathrm{i}t)^2}{p_{10}^2}\bigg)+\theta Cnp_{01}^2|t|^3,\\
(\ln\widehat{\mathcal{U}}(t))' &= \widehat{\widetilde{G}}'(t)\bigg(n\gamma_1+\frac{2n\gamma_2(\mathrm{i}t)}{p_{10}}+\frac{n(\gamma_2(1+p_{11})+3\gamma_3)(\mathrm{i}t)^2}{p_{10}^2}\bigg)+\theta C(np_{01}^2+1)|t|^3,
\end{aligned}$$

$$|(\ln\widehat{\mathcal{V}}(t)-\ln\widehat{\mathcal{U}}(t))'| \leqslant C(np_{01}^2+1)|t|^3. \tag{51}$$

Let $v = np_{01}/(p_{10}+p_{01})$. Then

$$\begin{aligned}(\ln\widehat{\mathcal{V}}(t)-\mathrm{i}tv)' &= n\gamma_1\widehat{\widetilde{G}}'(t)-\mathrm{i}v+2n\gamma_2(\widehat{\widetilde{G}}(t)-1)\widehat{\widetilde{G}}'(t)+3n\gamma_3(\widehat{\widetilde{G}}(t)-1)^3\widehat{\widetilde{G}}'(t)\\ &= \frac{np_{01}\mathrm{i}(p_{10}^2\mathrm{e}^{\mathrm{i}t}-1(1-p_{11}\mathrm{e}^{\mathrm{i}t})^2)}{(p_{10}+p_{01})(1-p_{11}\mathrm{e}^{\mathrm{i}t})^2}+\theta Cnp_{01}^2|\widehat{\widetilde{G}}(t)-1| = \theta Cnp_{01}|t|. \quad (52)\end{aligned}$$

We have

$$\begin{aligned}\Big(\widehat{\mathcal{V}}(t)\mathrm{e}^{-\mathrm{i}tv}-\widehat{\mathcal{U}}(t)\mathrm{e}^{-\mathrm{i}tv}\Big)' &= \widehat{\mathcal{V}}(t)\mathrm{e}^{-\mathrm{i}tv}(\ln\widehat{\mathcal{V}}(t)-\mathrm{i}tv)'-\widehat{\mathcal{U}}(t)\mathrm{e}^{-\mathrm{i}tv}(\ln\widehat{\mathcal{U}}(t)-\mathrm{i}t)'\\ &= (\ln\widehat{\mathcal{V}}-\mathrm{i}tv)'\mathrm{e}^{-\mathrm{i}tv}(\widehat{\mathcal{V}}(t)-\widehat{\mathcal{U}}(t))+\widehat{\mathcal{U}}(t)\mathrm{e}^{-\mathrm{i}tv}(\ln\widehat{\mathcal{V}}(t)-\ln\widehat{\mathcal{U}}(t))'.\end{aligned}$$

Therefore, taking into account (51), (52), (50) we obtain

$$\begin{aligned}\Big|\Big(\widehat{\mathcal{V}}(t)\mathrm{e}^{-\mathrm{i}tv}-\widehat{\mathcal{U}}(t)\mathrm{e}^{-\mathrm{i}tv}\Big)'\Big| &\leqslant Cnp_{01}|t|\exp\{-0.05np_{01}t^2\}(n^{-1}+(np_{01})^{-2})\\ &\quad +C\exp\{-0.054np_{01}t^2\}(np_{01}^2+1)|t|^3\\ &\leqslant C\sqrt{np_{01}}(n^{-1}+(np_{01})^{-2})\exp\{-0.04np_{01}t^2\}. \quad (53)\end{aligned}$$

Applying Lemma 4.11 below with $U=\mathcal{V}-\mathcal{U}$, $u=\sqrt{np_{01}}$, $v=np_{01}/(p_{10}+p_{01})$ and taking into account (50) and (53) we prove that

$$\|\mathcal{V}-\mathcal{U}\|\leqslant C(n^{-1}+(np_{01})^{-2}). \quad (54)$$

From (69) and (50) it follows that

$$\|\mathcal{V}-\mathcal{U}\|_\infty\leqslant C(np_{01})^{-1/2}((n^{-1}+(np_{01})^{-2}). \quad (55)$$

We recall that $\mathcal{H}$ is distribution and, therefore, $\|\mathcal{H}\|=1$. By the properties of total variation norm and (54) and (55)

$$\begin{aligned}\|\mathcal{H}*\exp\{\varkappa_1(\tilde{G}-I)\}*(\mathcal{V}-\mathcal{U})\| &\leqslant \|\mathcal{H}\|\|\exp\{\varkappa_1(\tilde{G}-I)\}\|\|\mathcal{V}-\mathcal{U}\|,\\ &\leqslant \exp\{|\varkappa_1|(\|\tilde{G}\|+\|I\|)\}\|\mathcal{V}-\mathcal{U}\|\\ &\leqslant \exp\{8/90\}\|\mathcal{V}-\mathcal{U}\|\\ &\leqslant C(n^{-1}+(np_{01})^{-2}), \quad (56)\\ \|\mathcal{H}*\exp\{\varkappa_1(\tilde{G}-I)\}*(\mathcal{V}-\mathcal{U})\|_\infty &\leqslant \|\mathcal{H}\|\|\exp\{\varkappa_1(\tilde{G}-I)\}\|\|\mathcal{V}-\mathcal{U}\|_\infty\\ &\leqslant C(np_{01})^{-1/2}((n^{-1}+(np_{01})^{-2}). \quad (57)\end{aligned}$$

To complete the proof of Theorem 3.2 one need to use the triangle inequality and estimates (42), (43), (56) and (57). □

Proof of Theorem 3.3. Let $np_{01}\leqslant 3$. Then observing that total variation norm of any distribution equals 1, we obtain

$$\|\mathcal{L}(S_n)-\mathcal{D}\|\leqslant\|\mathcal{L}(S_n)\|+\|\mathcal{D}\|=2\leqslant\frac{18}{(np_{01})^2}.$$

Similarly, we prove that $\|\mathcal{L}(S_n)-\mathcal{D}\|_\infty\leqslant C(np_{01})^{-5/2}$. Combining Theorem 3.2 with (12) and (41) (see Lemma 4.10 below) we complete the proof of Theorem. □

Proof of Theorem 3.4. Observe that

$$\sum_{j=0}^{\infty} \mathrm{e}^{hj}|\mathcal{L}(S_n)\{j\} - \Psi\{j\}| = \|\mathcal{L}_h(S_n) - \Psi_h\|,$$

where $\mathcal{L}_h(S_n)\{j\} = \mathrm{e}^{hj}\mathcal{L}(S_n)\{j\}$ and $\Psi_h\{j\} = \mathrm{e}^{hj}\Psi\{j\}$. Therefore, we will switch to conjugate measures F_{nh} and E_h and apply properties of the total variation norm. Further on we denote by Θ any signed measure satisfying $\|\Theta\| \leqslant 1$. The expression of Θ can vary from line to line, or even within the same line.

Recalling that all products and powers of measures are understood in the convolution sense, we introduce the following auxiliary measures

$$\begin{aligned}
Y_h &= (\mathrm{e}^h I_1 - I) * \sum_{j=0}^{\infty}(p_{11}\mathrm{e}^h)^j I_j, \quad K_h = \sum_{j=0}^{\infty}(p_{11}\mathrm{e}^h I_1 - p_{01}I - 2\gamma_1 Y_h)^{*j},\\
\Pi_h &= \frac{4p_{01}^2}{(p_{10}+p_{01})^2} Y_h^{*2} * (p_{10}^2 I + p(p_{10}+p_{01})(I - p_{11}\mathrm{e}^h I_1)) * K_h^{*2},\\
\tilde{\Lambda}_{2h} &= p_{11}\mathrm{e}^h I_1 - p_{01}I + (I - \tilde{\Lambda}_{1h}),\\
\tilde{\Lambda}_{1h} &= I + \gamma_1 Y_h + \frac{1}{2}[(1+p_{01})I - p\mathrm{e}^h I_1 + 2\gamma_1 Y_h] * \sum_{j=1}^{\infty}\binom{0.5}{j}(-1)^j \Pi_h^{*j},\\
\tilde{W}_{1h,2h} &= \frac{1}{2}\Big\{ I \pm [(p_{10}+p_{01})I + p_{11}(\mathrm{e}^h I_1 - I)] * K_h * \sum_{j=0}^{\infty}\binom{-0.5}{j}(-1)^j \Pi_h^{*j} \Big\}\\
&\quad \pm (1-p_0)(p_{01}-p_{11})(\mathrm{e}^h I_1 - I) * K_h * \sum_{j=0}^{\infty}\binom{-0.5}{j}(-1)^j \Pi_h^{*j},\\
A_h &= \sum_{j=0}^{\infty}(\tilde{p}\mathrm{e}^h)^j I_j, \quad \Xi_h = I + p_0\tilde{p}(\mathrm{e}^h I_1 - I) * A_h, \quad \tilde{M}_h = \exp\{(\lambda/n)(\mathrm{e}^h I_1 - I) * A_h\}.
\end{aligned}$$

Here

$$\binom{\pm 0.5}{k} = \frac{1}{k!}(\pm 0.5)(\pm 0.5 - 1)\dots(\pm 0.5 - k + 1), \qquad \binom{\pm 0.5}{0} = 1.$$

Observe that $\widehat{\mathcal{L}_h(S_n)}(t)$ can be obtained from $\widehat{\mathcal{L}(S_n)}(t)$ by replacing $\mathrm{e}^{\mathrm{i}t}$ by $\mathrm{e}^{\mathrm{i}t+h}$. Therefore, taking into account expressions (32)–(35) on page 1127 in [26], we prove that $\mathcal{L}_h(S_n) = \tilde{\Lambda}_{1h}^{*n} * \tilde{W}_{1h} + \tilde{\Lambda}_{2h}^{*n} * \tilde{W}_{2h}$. Similarly $\Psi_h = \Xi_h * \tilde{M}_h^{*n}$. Therefore,

$$\|\mathcal{L}_h(S_n) - \Psi_h\| \leqslant \|\tilde{\Lambda}_{1h}^{*n} - \tilde{M}_h^{*n}\|\|\tilde{W}_{1h}\| + \|\tilde{M}_h^{*n}\|\|\tilde{W}_{1h} - \Xi_h\| + \|\tilde{\Lambda}_{2h}\|^n\|\tilde{W}_{2h}\|. \tag{58}$$

From (23) it follows that

$$p_{11}\mathrm{e}^h \leqslant \frac{4}{3}\tilde{p}\mathrm{e}^h \leqslant \frac{1}{3}, \quad p_{01}(\mathrm{e}^h+1) \leqslant 0.02, \quad p_{01} \leqslant 0.01, \quad p_{01} \leqslant Cn^{-1}, \quad \frac{p_{11}}{p_{10}+p_{01}} \leqslant \frac{1}{2}. \tag{59}$$

Applying (59) we consequently prove that

$$
\begin{aligned}
\|Y_h\| &\leqslant (\mathrm{e}^h+1)\sum_{j=0}^{\infty}(p_{11}\mathrm{e}^h)^j \leqslant (\mathrm{e}^h+1)\sum_{j=0}^{\infty}\left(\frac{1}{3}\right)^j = \frac{3}{2}(\mathrm{e}^h+1),\\
p_{01}\|Y_h\| &\leqslant \frac{3}{2}p_{01}(\mathrm{e}^h+1) \leqslant 0.03,\\
\|K_h\| &\leqslant \sum_{j=0}^{\infty}(p_{11}\mathrm{e}^h+p_{01}+2p_{01}\|Y_h\|)^j \leqslant \sum_{j=0}^{\infty}\left(\frac{1}{3}+0.01+0.06\right)^j \leqslant 1.68,\\
\|\Pi_h\| &\leqslant 4p_{01}^2\frac{9(\mathrm{e}^h+1)^2}{4}\left(\frac{p_{10}^2}{(p_{10}+p_{01})^2}+\frac{p_{11}}{p_{10}+p_{01}}\Big(\frac{1}{3}+1\Big)\right)(1.68)^2\\
&\leqslant \frac{9p_{01}}{25}\cdot\frac{5}{3}(1.68)^2 \leqslant 1.694p_{01} \leqslant 0.017,\\
\|\Pi_h\| &\leqslant Cp_{01}^2 \leqslant Cn^{-2},\\
\|\tilde{\Lambda}_{1h}-I\| &\leqslant p_{01}\|Y_h\| + \frac{1}{2}(1+p_{01}+p_{11}\mathrm{e}^h+2p_{01}\|Y_h\|)\sum_{j=1}^{\infty}\|\Pi_h\|^j\\
&\leqslant \frac{3p_{01}(\mathrm{e}^h+1)}{2} + 0.5(1+0.01+\frac{1}{3}+0.06)\|\Pi_h\|\sum_{j=1}^{\infty}(0.017)^{j-1}\\
&\leqslant \frac{3p_{01}(\mathrm{e}^h+1)}{2} + 0.702\cdot 1.694\frac{p_{01}}{0.983} \leqslant p_{01}(1.5(\mathrm{e}^h+1)+1.21) \leqslant 0.043,\\
\|\tilde{\Lambda}_{2h}\| &\leqslant p_{11}\mathrm{e}^h+p_{01}+\|I-\tilde{\Lambda}_{1h}\| \leqslant \frac{1}{3}+0.01+0.043 \leqslant 0.4,\\
\|\tilde{\Lambda}_{1h}\| &\leqslant 1+Cp_{01} \leqslant \exp\{Cp_{01}\} \leqslant \exp\{Cn^{-1}\},\\
\|\tilde{W}_{1h,2h}\| &\leqslant \frac{1}{2}\left\{1+[(p_{10}+p_{01})+p_{11}(\mathrm{e}^h+1)]\|K_h\|\sum_{j=0}^{\infty}\|\Pi_h\|^j\right\}\\
&\quad +|p_{01}-p_{11}|(\mathrm{e}^h+1)\|K_h\|\sum_{j=0}^{\infty}\|\Pi_h\|^j\\
&\leqslant 0.5(1+[1+0.01+0.67]\cdot 1.68\cdot 1.018)+(0.02+0.67)\cdot 1.68\cdot 1.018 \leqslant C,\\
\|A_h\| &\leqslant \sum_{j=0}^{\infty}\left(\frac{1}{4}\right)^j = \frac{4}{3}, \quad \|\Xi_h\| \leqslant 1+\tilde{p}(\mathrm{e}^h+1)\|A_h\| \leqslant C,\\
\|\tilde{M}_h\| &\leqslant \exp\left\{\frac{\lambda}{n}(\mathrm{e}^h+1)\|A_h\|\right\} \leqslant \exp\{Cn^{-1}\}.
\end{aligned}
$$

Finally, observe that

$$
\|\tilde{\Lambda}_{1h}^{*n}-\tilde{M}_h^{*n}\| \leqslant n\max(\|\tilde{\Lambda}_{1h}\|^{n-1},\|\tilde{M}_h\|^{n-1})\|\tilde{\Lambda}_{1h}-\tilde{M}_h\| \leqslant Cn\|\tilde{\Lambda}_{1h}-\tilde{M}_h\|.
$$

Substituting the above estimates into (58) we obtain

$$
\|\mathcal{L}_h(S_n)-\Psi_h\| \leqslant C(n\|\tilde{\Lambda}_{1h}-\tilde{M}_h\|+\|\tilde{W}_{1h}-\Xi_h\|+(0.4)^n). \tag{60}
$$

Taking into account (23) we see that

$$
\begin{aligned}
\tilde{M}_h &= I+\frac{\lambda}{n}(\mathrm{e}^hI_1-I)*A_h+\Theta Cn^{-2}\\
\tilde{\Lambda}_{1h} &= I+\gamma_1Y_h+\Theta C\|\Pi_h\| = I+\left(p_{01}-\frac{p_{01}^2}{p_{10}+p_{01}}\right)Y_h+\Theta Cp_{01}^2 = I+p_{01}Y_h+\Theta Cn^{-2}.
\end{aligned}
$$

Therefore

$$\begin{aligned}
\|Y_h - (\mathrm{e}^h I_1 - I) * A_h\| &\leqslant (\mathrm{e}^h+1)\sum_{j=1}^{\infty} |(p_{11}\mathrm{e}^h)^j - (\tilde{p}\mathrm{e}^h)^j| \\
&\leqslant (\mathrm{e}^h+1)\sum_{j=1}^{\infty} j\max((p_{11}\mathrm{e}^h),(\tilde{p}\mathrm{e}^h))^{j-1}|p_{11}\mathrm{e}^h - \tilde{p}\mathrm{e}^h| \\
&\leqslant \mathrm{e}^h(\mathrm{e}^h+1)|p_{11}-\tilde{p}|\sum_{j=1}^{\infty} j\left(\frac{1}{3}\right)^{j-1} \leqslant C|p_{11}-\tilde{p}|
\end{aligned}$$

and

$$\begin{aligned}
\|\tilde{\Lambda}_{1h} - \tilde{M}_h\| &\leqslant \left|p_{01} - \frac{\lambda}{n}\right|\|Y_h\| + \frac{\lambda}{n}\|Y_h - (\mathrm{e}^h I_1 - I) * A_h\| + \frac{C}{n^2} \\
&\leqslant \frac{C}{n}\big\{|np_{01}-\lambda| + |p_{11}-\tilde{p}| + n^{-1}\big\}. \qquad (61)
\end{aligned}$$

Next observe that

$$\begin{aligned}
\|K_h - A_h\| &\leqslant \sum_{j=1}^{\infty} \|(p_{11}\mathrm{e}^h I_1 - p_{01}I - 2\gamma_1 Y_h)^{*j} - (\tilde{p}\mathrm{e}^h I_1)^{*j}\| \\
&\leqslant \|p_{11}\mathrm{e}^h I_1 - p_{01}I - 2\gamma_1 Y_h - \tilde{p}\mathrm{e}^h I_1\| \sum_{j=1}^{\infty} j(0.4034)^{j-1} \\
&\leqslant C(|p_{11}-\tilde{p}| + p_{01}) \leqslant C(|p_{11}-\tilde{p}| + n^{-1}), \\
\tilde{W}_{1h} &= \frac{1}{2}\{I + [p_{10}I + p_{11}(\mathrm{e}^h I_1 - I)] * K_h\} - (1-p_0)p_{11}(\mathrm{e}^h I_1 - I) * K_h + \Theta C n^{-1} \\
&= \frac{1}{2}\{I + [p_{10}I + p_{11}(\mathrm{e}^h I_1 - I)] * A_h\} - (1-p_0)p_{11}(\mathrm{e}^h I_1 - I) * A_h \\
&+ \Theta C(|p_{11}-\tilde{p}| + n^{-1}).
\end{aligned}$$

Noting that $A_h * (I - \tilde{p}\mathrm{e}^h I_1) = I$, we next obtain

$$\begin{aligned}
\|\tilde{W}_{1h} - \Xi_h\| &\leqslant \frac{1}{2}\|p_{10}A_h - I - p_{11}(\mathrm{e}^h I_1 - I) * A_h\| + C(|p_{11}-\tilde{p}| + n^{-1}) \\
&= \frac{1}{2}\|A_h - p_{11}\mathrm{e}^h I_1 * A_h \pm \tilde{p}\mathrm{e}^h I_1 * A_h\| + C(|p_{11}-\tilde{p}| + n^{-1}) \\
&\leqslant C(|p_{11}-\tilde{p}| + n^{-1}). \qquad (62)
\end{aligned}$$

Substituting estimates (61) and (62) into (60) we complete the proof of Theorem 3.4. □

4.3 Auxiliary results for Theorems 3.5-3.8

Characteristic functions and Fourier-Stieltjes transforms of measures from subsection 3.4 are equal to

$$\widehat{L}(t)=\frac{1}{2}(e^{-it}+e^{it}),\quad \widehat{X}(t)=\widehat{L}(t)-1,\quad \widehat{D}(t)=(1-2(a-b)-2a\widehat{X}(t))^2(1+\widehat{\Delta}(t)),$$
$$\widehat{P}_1(t)=\pi_1+\frac{\widehat{\Lambda}_1(t)-2a\widehat{L}(t)}{1-2a}\pi_2+\pi_3,\qquad \widehat{P}_2(t)=\pi_1+\frac{\widehat{\Lambda}_2(t)-2a\widehat{L}(t)}{1-2a}\pi_2+\pi_3,$$
$$\widehat{\Delta}(t)=\frac{8b\widehat{X}(t)}{(1-2(a-b)-2a\widehat{X}(t))^2},\qquad \widehat{\Delta}_1(t)=\frac{8(b-a)\widehat{L}(t)}{(1-2b+2a\widehat{L}(t))^2},$$
$$\widehat{\Lambda}_{1,2}(t)=\frac{1}{2}\Big(1+2(a-b)+2a\widehat{X}(t)\pm(1-2(a-b)-2a\widehat{X}(t))\sqrt{1+\widehat{\Delta}(t)}\Big),$$
$$\widehat{W}_{1,2}(t)=\frac{1}{2}\Big(1\pm(2a\cos t-(1-2b)+2(1-2a))\widehat{D}^{-1/2}(t)\Big),$$
$$\widehat{G}(t)=\exp\Big\{\frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)\Big\},\qquad \widehat{H}(t)=\frac{(1-2a)\widehat{L}(t)}{1-2a\widehat{L}(t)},$$
$$\widehat{M}(t)=\frac{1-2(a-b)}{1-2(a-b)-2a\widehat{X}(t)},\qquad \widehat{E}(t)=\pi_1+\Big(1-\frac{2a}{1-2a}\widehat{X}(t)\Big)\pi_2+\pi_3,$$
$$\widehat{A}_1(t)=\frac{-2b^2(1-2a)}{(1-2a+2b)^2}\bigg(1+2a+\frac{2(1-2a)\widehat{M}(t)}{1-2a+2b}\bigg)(\widehat{H}(t)-1)^2,$$
$$\widehat{E}_1(t)=\frac{2b(\widehat{H}-1)}{1-2a+2b}\pi_2,\qquad \widehat{M}_1(t)=\frac{-2b(1-2a)}{(1-2a+2b)^2}(2\widehat{M}(t)-1)(\widehat{H}(t)-1).$$

For the sake of brevity further on we write x instead of $\widehat{X}(t)$. We also notice, that $x=-2\sin^2\frac{t}{2}=\cos t-1$. For measure U concentrated on integers we define transformation

$$\mathcal{V}(U)\{k\}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\widehat{U}(t)(\mathrm{e}^{-\mathrm{i}t}-1)\mathrm{e}^{-\mathrm{i}tk}}{4\sin^2(t/2)}\,\mathrm{d}t=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\widehat{U}(t)\mathrm{e}^{-\mathrm{i}tk}}{(\mathrm{e}^{\mathrm{i}t}-1)}\,\mathrm{d}t.$$

Obviously, finite $\mathcal{V}(U)$ does not always exist. On the other hand, if $U=\sum_{k=1}^n\tilde{a}_k(I_1-I)^{*k}$, for some absolutely convergent series $\tilde{a}_k$, then $\mathcal{V}(U)$ is finite measure. If we consider two lattice distributions having equal means and finite factorial moments of all orders, then transformation $\mathcal{V}(\cdot)$ can be applied to their difference. Since $\|U*(I_1-I)\|_W=\|U\|$, we will write $\|U\|_W=\|\mathcal{V}(U)\|$, if $\mathcal{V}(U)$ is finite measure.

Lemma 4.10 *Let condition (24) be satisfied. Then the characteristic function $\widehat{F}_n$ can be expressed in the following way*

$$\widehat{F}_n(t)\;=\;\widehat{P}_1(t)\widehat{\Lambda}_1^n(t)\widehat{W}_1(t)+\widehat{P}_2(t)\widehat{\Lambda}_2^n(t)\widehat{W}_2(t).\tag{63}$$

Proof. The characteristic function $\widehat{F}_n$ can be written as follows

$$\widehat{F}_n(t)\;=\;\big(\pi_1,\ \pi_2,\ \pi_3\big)\big(\widehat{\Lambda}_1^n(t)\overrightarrow{y}_1\overrightarrow{z}_1^T+\widehat{\Lambda}_2^n(t)\overrightarrow{y}_2\overrightarrow{z}_2^T\big)\big(1,\ 1,\ 1\big)^T.\tag{64}$$

The expression (63) is known as Perron's formula. Similar expression was used for Markov binomial distribution; see, for example, [25]. $\widehat{\Lambda}_j(t)$ are eigenvalues of the following matrix:

$$\tilde{P}(t)=\begin{pmatrix}a\mathrm{e}^{-\mathrm{i}t} & 1-2a & a\mathrm{e}^{\mathrm{i}t}\\ b\mathrm{e}^{-\mathrm{i}t} & 1-2b & b\mathrm{e}^{\mathrm{i}t}\\ a\mathrm{e}^{-\mathrm{i}t} & 1-2a & a\mathrm{e}^{\mathrm{i}t}\end{pmatrix}.$$

Here and below $j=1,2$. Note that the third eigenvalue of $\tilde{P}(t)$ is equal to zero. Let $\overrightarrow{y}_j=$

$(y_{j,1},\ y_{j,2},\ y_{j,3})$, $\overrightarrow{z}_j = (z_{j,1},\ z_{j,2},\ z_{j,3})$ be corresponding eigenvectors:

$$\begin{cases} \tilde{P}(t)\overrightarrow{y}_j = \widehat{\Lambda}_j(t)\overrightarrow{y}_j, \\ \overrightarrow{z}_j^T \tilde{P}(t) = \widehat{\Lambda}_j(t)\overrightarrow{z}_j^T, \\ \overrightarrow{z}_j^T \overrightarrow{y}_j = 1. \end{cases} \tag{65}$$

It is not difficult to establish that eigenvalues $\widehat{\Lambda}_j(t)$ satisfy

$$\begin{aligned} &\widehat{\Lambda}_j^2(t) + (2b - 2a\cos t - 1)\widehat{\Lambda}_j(t) + (2a - 2b)\cos t \\ &\quad = \widehat{\Lambda}_j^2(t) + (-1 - 2(a-b) - 2ax)\widehat{\Lambda}_j(t) + (2a - 2b)(1 + x) = 0. \end{aligned} \tag{66}$$

The discriminant of equation (66) can be written in the following way

$$\begin{aligned} D &= (1 - 2b + 2a\widehat{L}(t))^2 - 8(a-b)\widehat{L}(t) \\ &= (1 - 2b + 2a\widehat{L}(t))^2(1 + \widehat{\Delta}_1(t)) \\ &= (1 - 2(a-b) - 2ax)^2 + 8bx \\ &= (1 - 2(a-b) - 2ax)^2(1 + \widehat{\Delta}(t)). \end{aligned}$$

Therefore,

$$\begin{aligned} \widehat{\Lambda}_j(t) &= \frac{1}{2}\Big\{\big(1 - 2b + 2\widehat{L}(t)\big) \pm \big(1 - 2b + 2a\widehat{L}(t)\big)\sqrt{1 + \widehat{\Delta}_1(t)}\Big\} \\ &= \frac{1}{2}\Big\{1 + 2(a-b) + 2ax \pm (1 - 2(a-b) - 2ax)\sqrt{1 + \widehat{\Delta}(t)}\Big\}. \end{aligned}$$

Next we calculate eigenvectors of $\tilde{P}(t)$. We have

$$\begin{pmatrix} a\mathrm{e}^{-\mathrm{i}t} & 1-2a & a\mathrm{e}^{\mathrm{i}t} \\ b\mathrm{e}^{-\mathrm{i}t} & 1-2b & b\mathrm{e}^{\mathrm{i}t} \\ a\mathrm{e}^{-\mathrm{i}t} & 1-2a & a\mathrm{e}^{\mathrm{i}t} \end{pmatrix} \begin{pmatrix} y_{j,1} \\ y_{j,2} \\ y_{j,3} \end{pmatrix} = \widehat{\Lambda}_j(t) \begin{pmatrix} y_{j,1} \\ y_{j,2} \\ y_{j,3} \end{pmatrix}. \tag{67}$$

It is evident that $y_{j,1} = y_{j,3}$. We recall that $\mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i}\sin t$, $\mathrm{e}^{-\mathrm{i}t} = \cos t - \mathrm{i}\sin t$. The equation (67) then can be reduced to

$$\begin{cases} (2a\cos t - \widehat{\Lambda}_j(t))y_{j,1} + (1 - 2a)y_{j,2} = 0 \\ 2b\cos t y_{j,1} + (1 - 2b - \widehat{\Lambda}_j(t))y_{j,2} = 0. \end{cases}$$

Both equations of the system are equivalent, hence

$$\overrightarrow{y}_j = y_{j,1}\left(1,\ \frac{\widehat{\Lambda}_j(t) - 2a\cos t}{1 - 2a},\ 1\right)^T.$$

Similarly,

$$\overrightarrow{z}_j^T = z_{j,1}\mathrm{e}^{\mathrm{i}t}\left(\mathrm{e}^{-\mathrm{i}t},\ \frac{(1-2a)2\mathrm{e}^{\mathrm{i}t}\cos t}{\widehat{\Lambda}_j(t) - (1 - 2b)},\ \mathrm{e}^{\mathrm{i}t}\right).$$

From the third equation of the system (65) we get

$$\begin{aligned}\overrightarrow{z}_j^T\overrightarrow{y}_j &= y_{j,1}z_{j,1}\mathrm{e}^{\mathrm{i}t}\left(\mathrm{e}^{-\mathrm{i}t}+\frac{2\cos t(\widehat{\Lambda}_j(t)-2a\cos t)}{\widehat{\Lambda}_j(t)-(1-2b)}+\mathrm{e}^{\mathrm{i}t}\right)\\ &= y_{j,1}z_{j,1}\mathrm{e}^{\mathrm{i}t}2\cos t\left(1+\frac{\widehat{\Lambda}_j(t)-2a\cos t}{\widehat{\Lambda}_j(t)-(1-2b)}\right)=1,\end{aligned}$$

and, therefore,

$$y_{j,1}z_{j,1} = \left(\mathrm{e}^{\mathrm{i}t}2\cos t\left(1+\frac{\widehat{\Lambda}_j(t)-2a\cos t}{\widehat{\Lambda}_j(t)-(1-2b)}\right)\right)^{-1}.$$

Finally, observe that

$$\begin{aligned}\overrightarrow{y}_1\overrightarrow{z}_1^T(1,\ 1,\ 1)^T &= \left(1,\ \frac{\widehat{\Lambda}_1(t)-2a\cos t}{1-2a},\ 1\right)^T y_{1,1}z_{1,1}\mathrm{e}^{\mathrm{i}t}\left(2\cos t+\frac{(1-2a)2\cos t}{\widehat{\Lambda}_1(t)-(1-2b)}\right)\\ &= \left(1,\ \frac{\widehat{\Lambda}_1(t)-2a\cos t}{1-2a},\ 1\right)^T \frac{\widehat{\Lambda}_1(t)+2b-2a}{2\widehat{\Lambda}_1(t)-1+2b-2a\cos t}\\ &= \left(1,\ \frac{\widehat{\Lambda}_1(t)-2a\cos t}{1-2a},\ 1\right)^T \frac{1}{2}\left(1+\frac{2a\cos t-1+2b-4a}{\widehat{D}(t)^{1/2}}\right)\\ &= \left(1,\ \frac{\widehat{\Lambda}_1(t)-2a\cos t}{1-2a},\ 1\right)^T \widehat{W}_1(t). \qquad (68)\end{aligned}$$

Similarly,

$$\overrightarrow{y}_2\overrightarrow{z}_2^T(1,\ 1,\ 1)^T = \left(1,\ \frac{\widehat{\Lambda}_2(t)-2a\cos t}{1-2a},\ 1\right)^T \widehat{W}_2(t).$$

Substituting (68), (69) into (64) we get (63). □

Lemma 4.11 *Let U be a measure concentrated on $\mathbb{Z}$. Then*

$$\|U\|_\infty \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|\widehat{U}(t)|\mathrm{d}t, \qquad (69)$$

If, in addition, $\sum_{-\infty}^{\infty}|j||U\{j\}|<\infty$*, then for all* $v\in\mathbb{R}$ *and* $u>0$

$$\|U\| \leqslant (1+u\pi)^{1/2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\widehat{U}(t)|^2+\frac{1}{u^2}|\big(\mathrm{e}^{-\mathrm{i}tv}\widehat{U}(t)\big)'|^2\mathrm{d}t\right)^{1/2}. \qquad (70)$$

$$|U\{j\}|\left(1+\frac{|j-v|}{u}\right) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|\widehat{U}(t)|+\frac{1}{u}|(\mathrm{e}^{-\mathrm{i}tv}\widehat{U}(t))'|\mathrm{d}t, \qquad (71)$$

$$|U\{(-\infty,j]\}|\left(1+\frac{|j-v|}{u}\right) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\widehat{U}(t)}{\mathrm{e}^{\mathrm{i}t}-1}\right|+\frac{1}{u}\left|\left(\frac{\mathrm{e}^{-\mathrm{i}tv}\widehat{U}(t)}{\mathrm{e}^{\mathrm{i}t}-1}\right)'\right|\mathrm{d}t. \qquad (72)$$

The estimate (69) follows directly from the formula of inversion, the estimate (70) can be found, for example, in [83]. Estimates (71) and (72) are respectively the inversion formula and Tsaregradskyi's inequality combined with formula (23) from [19]. The following lemma is a partial case of more general result from [83].

Lemma 4.12 *Let U be a measure concentrated on $\mathbb{Z}$, $v \in \mathbb{R}$, $u \geqslant 1$. Then,*

$$\|U\|_K \geqslant C\left|\int_{-\pi}^{\pi} \mathrm{e}^{-t^2/2}\widehat{U}\left(\frac{t}{u}\right)\mathrm{e}^{-\mathrm{i}tv}\mathrm{d}t\right|, \quad \|U\|_\infty \geqslant \frac{C}{u}\left|\int_{-\pi}^{\pi} \mathrm{e}^{-t^2/2}\widehat{U}\left(\frac{t}{u}\right)\mathrm{e}^{-\mathrm{i}tv}\mathrm{d}t\right|.$$

The following trivial Lemma plays technical role in subsequent estimates.

Lemma 4.13 *Let condition (24) be satisfied. Then*

$$0.93 \leqslant 1-2a \leqslant |1-2a+2b-2ax| \leqslant 1+2b+2a \leqslant 1.14, \quad (73)$$

$$0.93 \leqslant 1-2a \leqslant |1-2a+2b| = 1-2a+2b \leqslant 1+2b \leqslant 1.07, \quad (74)$$

$$0.93 \leqslant 1-2a \leqslant |1-2ax-2a| = 1-2a\cos t \leqslant 1+2a \leqslant 1.07, \quad (75)$$

$$0.8 \leqslant 1-4a+2a\cos t \leqslant |1-2a+2b+2ax| \leqslant 1-4a+2b+2a \leqslant 1.07. \quad (76)$$

Proof. It is not difficult to check, that

$$\begin{aligned}
|1-2a+2b-2ax| &= |1-2a+2b-2a(\cos t-1)| = |1+2b-2a\cos t| \\
&\leqslant 1+2b+2a, \\
|1-2a+2b-2ax| &= |1+2b-2a\cos t| \geqslant 1+2b-2a|\cos t| \geqslant 1-2a|\cos t| \geqslant 1-2a, \\
|1-2ax-2a| &= |1-2a\cos t| \geqslant 1-2a|\cos t| \geqslant 1-2a, \\
|1-2a+2b+2ax| &= |1-2a+2b+2a(\cos t-1)| = |1-4a+2b+2a\cos t| \\
&\leqslant 1-4a+2b+2a, \\
|1-2a+2b+2ax| &\geqslant |1-4a+2\cos t| \geqslant 1-6a \geqslant 0.8.
\end{aligned}$$

□

Lemma 4.14 *Let condition (24) be satisfied, $|t| \leqslant \pi$. Then*

$$|\widehat{\Lambda}_1(t)| \leqslant \mathrm{e}^{bx}, \qquad |\widehat{G}(t)| \leqslant \mathrm{e}^{1.6bx}. \quad (77)$$

Proof. We can write $\widehat{\Lambda}_1(t)$ in the following way

$$\begin{aligned}
\widehat{\Lambda}_1(t) &= \frac{1}{2}\Bigg\{1+2(a-b)+2ax+(1-2(a-b)-2ax) \\
&\quad \times\left[1+\frac{1}{2}\widehat{\Delta}(t)-0.125\widehat{\Delta}^2(t)+\sum_{j=3}^{\infty}\binom{1/2}{j}\widehat{\Delta}^j(t)\right]\Bigg\} \\
&= 1+\frac{1}{4}(1-2(a-b)-2ax)\widehat{\Delta}(t)-0.0625(1-2(a-b)-2ax)\widehat{\Delta}^2(t) \\
&\quad +\frac{1}{2}(1-2(a-b)-2ax)\sum_{j=3}^{\infty}\binom{1/2}{j}\widehat{\Delta}^j(t). \quad (78)
\end{aligned}$$

Applying Lemma 4.13 we estimate each summand separately. Note that $x \leqslant 0$.

$$|\widehat{\Delta}(t)| = \left|\frac{8bx}{(1-2a+2b)-2ax)^2}\right| \leqslant \frac{8b|x|}{(0.93)^2} \leqslant 9.3b|x| \leqslant \frac{9.3\cdot 2}{30} \leqslant 0.62, \quad (79)$$

$$\left|1+\frac{1}{4}(1-2a+2b-2ax)\widehat{\Delta}(t)\right| = 1+\frac{2bx}{1-2a\cos t+2b}$$
$$\leqslant 1+\frac{2bx}{1+2a+2b} \leqslant 1+\frac{2bx}{1.14} \leqslant 1+1.75bx,$$

$$\left|0.0625(1-2(a-b)-2ax)\widehat{\Delta}^2(t)\right| = \left|\frac{0.0625(8bx)^2}{(1-2(a-b)-2ax)^3}\right| \leqslant \frac{4b^2x^2}{(0.93)^3} \leqslant 5b^2x^2, \qquad ,$$

$$\frac{1}{2}|1-2(a-b)-2ax|\sum_{j=3}^{\infty}\left|\binom{1/2}{j}\widehat{\Delta}^j(t)\right| \leqslant \frac{1}{2}|1-2(a-b)-2ax|\frac{1}{16}|\widehat{\Delta}(t)|^3\sum_{j=3}^{\infty}0.62^{j-3}$$
$$\leqslant \frac{8^3b^3|x|^3}{32\cdot 0.38|1-2(a-b)-2ax|^5} \leqslant \frac{8^3b^2x^2\cdot 2}{32\cdot 0.38\cdot 30\cdot 0.93^5} \leqslant 4.1b^2x^2. \tag{80}$$

Collecting all estimates we obtain

$$\begin{aligned}|\widehat{\Lambda}_1(t)| &\leqslant \left|1+\frac{1}{4}(1-2(a-b)-2ax)\widehat{\Delta}(t)\right| + 0.0625(1-2(a-b)-2ax)|\widehat{\Delta}(t)|^2 \\ &\quad +\frac{1}{2}(1-2(a-b)-2ax)\sum_{j=3}^{\infty}\binom{1/2}{j}|\widehat{\Delta}(t)|^j \\ &\leqslant 1+1.75bx+5b^2x^2+4.1b^2x^2 \leqslant 1+1.75bx-0.61bx \leqslant 1+bx \leqslant \mathrm{e}^{bx}.\end{aligned}$$

Observe that $\widehat{G}(t)$ can be written in the following way

$$\widehat{G}(t) = \exp\left\{\frac{2b(1-2a)}{(1-2a+2b)}\left(\frac{(1-2a)\widehat{L}(t)}{1-2a\widehat{L}(t)}-1\right)\right\} = \exp\left\{\frac{2b(1-2a)}{(1-2a+2b)}\left(\frac{\widehat{L}(t)-1}{1-2a\widehat{L}(t)}\right)\right\}$$
$$= \exp\left\{\frac{2b(1-2a)}{(1-2a+2b}\left(\frac{x}{1-2a(x+1)}\right)\right\} = \exp\left\{\frac{2bx(1-2a)}{(1-2a+2b)(1-2ax-2a)}\right\}.$$

Combining estimates (74) and (75) with the definition of $\widehat{G}(t)$ we prove that

$$|\widehat{G}(t)| \leqslant \exp\left\{\frac{2bx\cdot 0.93}{1.07^2}\right\} \leqslant e^{1.6bx}.$$

□

Next we expand $\widehat{\Lambda}_1(t)$ in powers of $\widehat{H}(t)-1$.

Lemma 4.15 *Let condition (24) be satisfied, $|t|\leqslant\pi$. Then*

$$\begin{aligned}\widehat{\Lambda}_1(t) &= 1+\frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)-\frac{8ab^2(1-2a)}{(1-2a+2b)^2}(\widehat{H}(t)-1)^2 \\ &\quad -\frac{4b^2(1-2a)^2\widehat{M}(t)}{(1-2a+2b)^3}(\widehat{H}(t)-1)^2+\theta Cb^3|x|^3 \\ &= 1+\frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)+\theta Cb^2x^2,\end{aligned} \tag{81}$$

$$
\begin{aligned}
\widehat{\Lambda}_1'(t) &= \frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)' - \frac{8ab^2(1-2a)}{(1-2a+2b)^2}2(\widehat{H}(t)-1)\widehat{H}'(t) \\
&\quad -\frac{4b^2(1-2a)^2}{(1-2a+2b)^3}(\widehat{M}(t)(\widehat{H}(t)-1)^2)' + \theta C b^3 x^2|x'| \\
&= \frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)' + \theta C b^2|xx'|. \qquad (82)
\end{aligned}
$$

Proof. From Lemma 4.13 and (80) it follows that

$$
\frac{1}{2}|1-2(a-b)-2ax|\sum_{j=3}^{\infty}\Big|\binom{1/2}{j}\widehat{\Delta}^j(t)\Big| \leqslant \frac{8^3b^3|x|^3}{32\cdot 0.38\cdot 0.93^5} \leqslant Cb^3|x|^3,
$$

Collecting it and (78) we obtain

$$
\widehat{\Lambda}_1(t) = 1 + \frac{2bx}{1-2(a-b)-2ax} - \frac{4b^2x^2}{(1-2(a-b)-2ax)^3} + \theta C b^3|x|^3. \qquad (83)
$$

It is not difficult to check that

$$
\begin{aligned}
\frac{1}{1-2(a-b)-2ax} &= \frac{1-2a}{(1-2a+2b)(1-2ax-2a)} \\
&\quad -\frac{4abx}{(1-2(a-b)-2ax)(1-2a+2b)(1-2ax-2a)} \\
&= \frac{1-2a}{(1-2a+2b)(1-2ax-2a)} + \theta C b|x|. \qquad (84)
\end{aligned}
$$

From (84) it follows that

$$
\begin{aligned}
&\frac{4abx}{(1-2(a-b)-2ax)(1-2a+2b)(1-2ax-2a)} \\
&\quad = \frac{4abx}{(1-2a+2b)(1-2ax-2a)}\left(\frac{1-2a}{(1-2a+2b)(1-2ax-2a)} + \theta C b|x|\right) \\
&\quad = \frac{4abx(1-2a)}{(1-2a+2b)^2(1-2ax-2a)^2} + \theta C b^2x^2. \qquad (85)
\end{aligned}
$$

Applying (85) and (84) we get

$$
\frac{1}{1-2(a-b)-2ax} = \frac{1-2a}{(1-2a+2b)(1-2ax-2a)} - \frac{4abx(1-2a)}{(1-2a+2b)^2(1-2ax-2a)^2} + \theta C b^2x^2.
$$

Observe that

$$
\frac{1}{1-2(a-b)-2ax} = \frac{\widehat{M}(t)}{1-2a+2b}, \qquad \frac{x}{1-2ax-2a} = \widehat{H}(t)-1. \qquad (86)
$$

Substituting these expressions into (83) we obtain

$$\begin{aligned}
\widehat{\Lambda}_1(t) &= 1+\frac{2bx(1-2a)}{(1-2a+2b)(1-2ax-2a)}-\frac{8ab^2x^2(1-2a)}{(1-2a+2b)^2(1-2ax-2a)^2}\\
&\quad-\frac{4b^2x^2(1-2a)^2\widehat{M}(t)}{(1-2a+b)^3(1-2ax-2a)^2}+\theta Cb^3|x|^3\\
&= 1+\frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)-\frac{8ab^2(1-2a)}{(1-2a+2b)^2}(\widehat{H}(t)-1)^2\\
&\quad-\frac{4b^2(1-2a)^2\widehat{M}(t)}{(1-2a+2b)^3}(\widehat{H}(t)-1)^2+\theta Cb^3|x|^3.
\end{aligned}$$

Expansion for $\widehat{\Lambda}_1'(t)$ is proved similarly. Here we present only the proof of the shorter version of expansion. From Lemma 4.13 the following estimate of $|\widehat{\Delta}'(t)|$ follows

$$\begin{aligned}
|\widehat{\Delta}'(t)| &= \left|\frac{8bx'(1-2(a-b)-2ax)^2+32abxx'(1-2(a-b)-2ax)}{(1-2(a-b)-2ax)^4}\right|\\
&\leqslant \left|\frac{8bx'(1.14)^2}{(0.93)^4}\right|+\left|\frac{32abxx'\cdot 1.14}{(0.93)^4}\right|\leqslant Cb|x'|. \qquad (87)
\end{aligned}$$

We can write $\widehat{\Lambda}_1'(t)=Z_1+Z_2$. Here

$$Z_1=\left(\frac{1}{4}(1-2(a-b)-2ax)\widehat{\Delta}(t)\right)'=\left(\frac{2bx}{1-2(a-b)-2ax}\right)'=\frac{2bx'(1+2b-2a)}{(1-2(a-b)-2ax)^2}.$$

Applying Lemma 4.13 and (87), we obtain the following estimate of Z_2

$$\begin{aligned}
|Z_2| &= \left|\left(\frac{1}{2}(1-2(a-b)-2ax)\sum_{j=2}^{\infty}\binom{1/2}{j}\widehat{\Delta}^j(t)\right)'\right|\\
&\leqslant a|x'|\sum_{j=2}^{\infty}\left|\binom{1/2}{j}\right||\Delta(t)|^j+\frac{1.14}{2}\left|\left(\sum_{j=2}^{\infty}\binom{1/2}{j}\widehat{\Delta}^j(t)\right)'\right|\\
&\leqslant a|x'||\widehat{\Delta}(t)|^2\sum_{j=2}^{\infty}0.62^{j-2}+C|\widehat{\Delta}'(t)||\Delta(t)|\sum_{j=2}^{\infty}\left|\binom{1/2}{j}\right|j\,0.62^{j-2}\\
&\leqslant C|x'|b^2x^2+Cb^2|x'||x|\leqslant Cb^2|xx'|.
\end{aligned}$$

Therefore

$$\widehat{\Lambda}_1'(t)=\frac{2bx'(1+2b-2a)}{(1-2(a-b)-2ax)^2}+\theta Cb^2|xx'|. \qquad (88)$$

Taking into account Lemma 4.13 we prove that

$$\begin{aligned}
&\left|\frac{2bx'(1+2b-2a)}{(1-2(a-b)-2ax)^2}-\frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)'\right|\\
&\quad= \left|\frac{2bx'(1+2b-2a)}{(1-2(a-b)-2ax)^2}-\frac{2bx'(1-2a)^2}{(1-2a+2b)(1-2a-2ax)^2}\right|\\
&\quad= \left|\frac{2bx'\big((1+2b-2a)^2(1-2a-2ax)^2-(1-2a)^2(1-2(a-b)-2ax)^2\big)}{(1-2(a-b)-2ax)^2(1-2a+2b)(1-2a-2ax)^2}\right|\\
&\quad= \frac{|2bx'||8bx||2a^2x(b-2a)-(1+2b-2a)(1-2a)a|}{|1-2(a-b)-2ax|^2|1-2a+2b||1-2a-2ax|^2}\\
&\quad\leqslant \frac{16}{(0.93)^5}b^2|xx'|\big(|4a^2(b-2a)|+|(1+2b-2a)(1-2a)a|\big)\leqslant Cb^2|xx'|.
\end{aligned}$$

From the last estimate and (88) the estimate (82) easily follows. □

Lemma 4.16 *Let condition (24) be satisfied,* $|t| \leqslant \pi$*. Then*

$$\begin{aligned}
\widehat{G}(t) &= 1+\frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)+\frac{2b^2(1-2a)^2}{(1-2a+2b)^2}(\widehat{H}(t)-1)^2+\theta Cb^3|x|^3\\
&= 1+\frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)+\theta Cb^2x^2,\\
\widehat{G}'(t) &= \frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)'+\frac{2b^2(1-2a)^2}{(1-2a+2b)^2}2(\widehat{H}(t)-1)\widehat{H}'(t)+\theta Cb^3x^2|x'|\\
&= \frac{2b(1-2a)}{1-2a+2b}(\widehat{H}(t)-1)'+\theta Cb^2|xx'|.
\end{aligned}$$

Proof. Let $\kappa=\ln\widehat{G}(t)$. Then it is not difficult to check, that

$$|\kappa| \;=\; \left|\frac{2bx(1-2a)}{(1-2a+2b)(1-2ax-2a)}\right| \leqslant \left|\frac{2bx(1-2a)}{(0.93)^2}\right| = \left|\frac{2bx}{(0.93)^2}\right| \leqslant Cb|x|$$

and, therefore $\mathrm{e}^{|\kappa|}\leqslant \exp\{Cb|x|\}\leqslant C$. Lemma's statement then follows from expansions $e^{\kappa}=1+\kappa+0.5\kappa^2+\theta|\kappa|^3\mathrm{e}^{|\kappa|}$ and $(\mathrm{e}^{\kappa})'=\kappa'\mathrm{e}^{\kappa}=\kappa'(1+\kappa+\theta\kappa^2\mathrm{e}^{|\kappa|})=\kappa'+\kappa\kappa'+\theta|\kappa'|\kappa^2$. □

Lemma 4.17 *Let condition (24) be satisfied. Then, for all* $n=1,2,\dots,$ *and* $|t|\leqslant\pi$

$$|\widehat{\Lambda}_1^n(t)-\widehat{G}^n(t)| \leqslant Cnb^2x^2\,\mathrm{e}^{nbx} \leqslant C\min(nb^2,n^{-1})\,\mathrm{e}^{0.5nbx}, \tag{89}$$

$$|(\widehat{\Lambda}_1^n(t)-\widehat{G}^n(t))'| \leqslant Cnb^2|xx'|\,\mathrm{e}^{0.6nbx} \leqslant C\min(nb^2,\sqrt{nb}n^{-1})\,\mathrm{e}^{0.5nbx}, \tag{90}$$

$$|\Lambda_1^n(t)-\widehat{G}^n(t)(1+\widehat{n}A_1(t))| \leqslant Cnb^3|x|^3\,\mathrm{e}^{0.6nbx} \leqslant C\min(nb^3,n^{-2})\,\mathrm{e}^{0.5nbx}, \tag{91}$$

$$|(\Lambda_1^n(t)-\widehat{G}^n(t)(1+\widehat{n}A_1(t)))'| \leqslant Cnb^3x^2|x'|\,\mathrm{e}^{0.6nbx} \leqslant C\min(nb^3,\sqrt{nb}n^{-2})\,\mathrm{e}^{0.5nbx}. \tag{92}$$

Proof. In what follows we frequently apply trivial estimates

$$\mathrm{e}^{-bx}\leqslant \mathrm{e}^2, \qquad |x'|\,\mathrm{e}^{0.1nbx}\leqslant\frac{C}{\sqrt{nb}}, \qquad |x|^k\mathrm{e}^{0.1nbx}\leqslant\frac{C(k)}{n^kb^k}, \quad k>0. \tag{93}$$

Due to Lemma 4.15 and Lemma 4.16

$$|\widehat{\Lambda}_1(t)-\widehat{G}(t)|\leqslant Cb^2x^2, \qquad |\widehat{\Lambda}_1'(t)-\widehat{G}'(t)|\leqslant Cb^2|xx'|. \tag{94}$$

From (94) and the first estimate in (93) we easily obtain (89) and (90) for $n=1$.

$$\begin{aligned}
|\widehat{\Lambda}_1(t)-\widehat{G}(t)| &\leqslant e^2Cb^2x^2\mathrm{e}^{bx}\leqslant Cb^2x^2\mathrm{e}^{bx},\\
|\widehat{\Lambda}_1'(t)-\widehat{G}'(t)| &\leqslant e^2Cb^2|xx'|\mathrm{e}^{bx}\leqslant Cb^2|xx'|\mathrm{e}^{0.6bx}.
\end{aligned}$$

Let $n\geqslant 2$. We evaluate separately when $nb\leqslant 1$ and $nb>1$. Combining Lemma 4.14 with (94) and applying (93) we prove that

$$\begin{aligned}
|\widehat{\Lambda}_1^n(t)-\widehat{G}^n(t)| &\leqslant n\max\left\{|\widehat{\Lambda}_1(t)|^{n-1},|\widehat{G}(t)|^{n-1}\right\}|\widehat{\Lambda}_1(t)-\widehat{G}(t)|\leqslant n\mathrm{e}^{(n-1)bx}Cb^2x^2\\
&\leqslant \mathrm{e}^2Cn\mathrm{e}^{nbx}b^2x^2\leqslant Cn\mathrm{e}^{0.5nbx}b^2x^2\mathrm{e}^{0.1nbx}\leqslant C\min(nb^2,n^{-1})\,\mathrm{e}^{0.5nbx}.
\end{aligned} \tag{95}$$

Observe that due to Lemma 4.16 and Lemma 4.13

$$|\widehat{G}(t)'|\leqslant\left|\frac{2bx'(1-2a)^2}{(1-2a+2b)(1-2ax-2a)^2}\right|+Cb^2|xx'|\leqslant Cb|x'|.$$

Therefore, similarly to the proof of (95)

$$\begin{aligned}
|(\widehat{\Lambda}_1^n(t)-\widehat{G}^n(t))'| &= |n\widehat{\Lambda}_1^{n-1}(t)\widehat{\Lambda}_1'(t)-n\widehat{G}^{n-1}(t)\widehat{G}'(t)|\\
&= |n\widehat{\Lambda}_1^{n-1}(t)(\widehat{\Lambda}_1'(t)-\widehat{G}'(t))+n\widehat{G}'(t)(\widehat{\Lambda}_1^{n-1}(t)-\widehat{G}^{n-1}(t))|\\
&\leqslant n|\widehat{\Lambda}_1(t)|^{n-1}|\widehat{\Lambda}_1'(t)-\widehat{G}'(t)|+n|\widehat{G}'(t)||\widehat{\Lambda}_1^{n-1}(t)-\widehat{G}^{n-1}(t)|\\
&\leqslant Cn\mathrm{e}^{(n-1)bx}b^2|xx'|+Cnb|x'|(n-1)b^2x^2\mathrm{e}^{(n-2)bx}\\
&\leqslant Cnb^2|xx'|\mathrm{e}^{nbx}+Cnb^2|xx'|\mathrm{e}^{0.9nbx}nb|x|\mathrm{e}^{0.1nbx}\leqslant Cnb^2|xx'|\mathrm{e}^{0.6nbx}.
\end{aligned}$$

Separating two cases when $nb\leqslant 1$ and $nb>1$ we obtain

$$|(\widehat{\Lambda}_1^n(t)-\widehat{G}^n(t))'|\leqslant Cnb^2\mathrm{e}^{0.5nbx}|x|^{3/2}\mathrm{e}^{0.1nbx}\leqslant .C\min(nb^2,\sqrt{nb}n^{-1})\,\mathrm{e}^{0.5nbx}$$

Estimates (89) and (90) are proved. Next we consider the second-order estimates. From Lemmas 4.15 and 4.16 it follows that

$$\widehat{\Lambda}_1(t)-\widehat{G}(t) = \widehat{A}_1(t)+\theta Cb^3|x|^3=\theta Cb^2x^2,\qquad 1-\widehat{G}(t)=\theta Cb|x|. \tag{96}$$

It can be easily proved that $|\widehat{A}_1(t)|\leqslant Cb^2x^2$. Let $n=1$. Then

$$\begin{aligned}
|\widehat{\Lambda}_1(t)-\widehat{G}(t)(1+\widehat{A}_1(t))| &\leqslant |\widehat{\Lambda}_1(t)-\widehat{G}(t)-\widehat{A}_1(t)|+|\widehat{A}_1(t)||1-\widehat{G}(t)|\\
&\leqslant Cb^3|x|^3\leqslant \mathrm{e}^{1.2}Cb^3|x|^3\mathrm{e}^{0.6bx}\leqslant \mathrm{e}^{1.2}Cb^3|x|^3\mathrm{e}^{0.6bx}.
\end{aligned}$$

Let $n\geqslant 2$. Then

$$\begin{aligned}
&|\widehat{\Lambda}_1^n(t)-\widehat{G}^n(t)(1+n\widehat{A}_1(t))|\leqslant|\widehat{\Lambda}_1^n(t)-\widehat{G}^n(t)-nG^{n-1}(t)(\widehat{\Lambda}_1(t)-\widehat{G}(t))|\\
&\quad+|n(\widehat{G}^{n-1}(t)-\widehat{G}^n(t))(\widehat{\Lambda}_1(t)-\widehat{G}(t))|+|n\widehat{G}^n(t)(\widehat{\Lambda}_1(t)-\widehat{G}(t)-\widehat{A}_1(t))|.
\end{aligned}\tag{97}$$

Applying Lemma 4.14, (96), (93) and Bergström identity [12], we prove that

$$\begin{aligned}
&|\widehat{\Lambda}_1^n(t)-\widehat{G}^n(t)-nG^{n-1}(t)(\widehat{\Lambda}_1(t)-\widehat{G}(t))|\\
&\qquad\leqslant \frac{n(n-1)}{2}|\widehat{\Lambda}_1(t)-\widehat{G}(t)|^2\big(\max\{|\widehat{\Lambda}_1(t)|,|\widehat{G}(t)|\}\big)^{n-2}\leqslant Cn^2\mathrm{e}^{nbx}b^4x^4\\
&\qquad\leqslant Cnb|x|\mathrm{e}^{0.1nbx}nb^3|x|^3\mathrm{e}^{0.1nbx}\mathrm{e}^{0.5nbx}\leqslant C\min(nb^3,n^{-2})\mathrm{e}^{0.5nbx},\\
&|n(\widehat{G}^{n-1}(t)-\widehat{G}^n(t))(\widehat{\Lambda}_1(t)-\widehat{G}(t))|=n|\widehat{G}^{n-1}(t)||1-\widehat{G}(t)||\widehat{\Lambda}_1(t)-\widehat{G}(t)|\\
&\qquad\leqslant Cn\mathrm{e}^{1.6nbx}b^3|x|^3\leqslant Cnb^3|x|^3\mathrm{e}^{0.1nbx}\mathrm{e}^{0.5nbx}\leqslant C\min(nb^3,n^{-2})\mathrm{e}^{0.5nbx},\\
&|n\widehat{G}^n(t)(\widehat{\Lambda}_1(t)-\widehat{G}(t)-\widehat{A}_1(t))|\leqslant n|\widehat{G}^n(t)||(\widehat{\Lambda}_1(t)-\widehat{G}(t)-\widehat{A}_1(t))|\\
&\qquad\leqslant Cn\mathrm{e}^{1.6nbx}b^3|x|^3\leqslant C\min(nb^3,n^{-2})\mathrm{e}^{0.5nbx}.
\end{aligned}$$

Substituting these estimates into (97) we prove (91). The proof of (92) is very similar and, therefore, omitted. □

We can estimate now the closeness of Λ_1^n and G^n.

Lemma 4.18 *Let condition (24) be satisfied. Then, for all $n = 1, 2, \dots$,*

$$\|\Lambda_1^{*n} - G^{*n}\| \leqslant C \min\left\{\frac{1}{n}, nb^2\right\}, \qquad \|\Lambda_1^{*n} - G^{*n} * (I + nA_1)\| \leqslant C \min\left\{\frac{1}{n^2}, nb^3\right\}, \tag{98}$$

$$\|\Lambda_1^{*n} - G^{*n}\|_\infty \leqslant C \min\left\{\frac{1}{n\sqrt{nb}}, nb^2\right\}, \quad \|\Lambda_1^{*n} - G^{*n} * (I + nA_1)\|_\infty \leqslant C \min\left\{\frac{1}{n^2\sqrt{nb}}, nb^3\right\}, \tag{99}$$

$$\|\Lambda_1^{*n} - G^{*n}\|_W \leqslant C \min\left\{\frac{\sqrt{nb}}{n}, nb^2\right\}, \quad \|\Lambda_1^{*n} - G^{*n} * (I + nA_1)\|_W \leqslant C \min\left\{\frac{\sqrt{nb}}{n^2}, nb^3\right\}. \tag{100}$$

Proof. Obsereve that, for $|t| \leqslant \pi$, $k > 0$,

$$\frac{|t|}{\pi} \leqslant |\sin(t/2)| \leqslant \frac{t}{2}, \qquad \int_0^\pi |\sin(t/2)|^k \mathrm{e}^{-Cnb\sin^2(t/2)} \mathrm{d}t \leqslant C(k) \min(1, (nb)^{-(k+1)/2}). \tag{101}$$

Local estimates (99) follow directly from (89), (91), (69) and (101). Suppose, that $nb \leqslant 1$, then

$$\begin{aligned}
\|\Lambda_1^{*n} - G^{*n}\|_\infty &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t)| dt \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} Cnb^2 e^{0.5nbx} dt \leqslant Cnb^2, \\
\|\Lambda_1^{*n} - G_1^{*n} * (I + nA_1)\|_\infty &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} Cnb^3 e^{0.5nbx} dt \leqslant Cnb^3.
\end{aligned}$$

When $nb > 1$, then

$$\begin{aligned}
\|\Lambda_1^{*n} - G^{*n}\|_\infty &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} Cn^{-1} e^{0.5nbx} dt \leqslant \frac{C}{n}\int_{-\pi}^{\pi} e^{0.5nbx} dt \leqslant \frac{C}{n\sqrt{nb}}, \\
\|\Lambda_1^{*n} - G_1^{*n} * (I + nA_1)\|_\infty &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} Cn^{-2} e^{0.5nbx} dt \leqslant \frac{C}{n^2\sqrt{nb}}.
\end{aligned}$$

Estimates in total variation follow from estimates of Lemma 4.17 and (70) applied with $v = 0$ and $u = 1$ for $nb \leqslant 1$, that is

$$\begin{aligned}
\|\Lambda_1^{*n} - G^{*n}\| &\leqslant (1+\pi)^{1/2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} (Cnb^2 \mathrm{e}^{0.5nbx})^2 dt\right)^{1/2} \leqslant C(n^2b^4)^{1/2} \leqslant Cnb^2, \\
\|\Lambda_1^{*n} - G_1^{*n} * (I + nA_1)\| &\leqslant (1+\pi)^{1/2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} (Cnb^3 \mathrm{e}^{0.5nbx})^2 dt\right)^{1/2} \leqslant Cnb^3.
\end{aligned}$$

Using $v = 0$ and $u = \sqrt{nb}$ for $nb > 1$ we obtain

$$\begin{aligned}
\|\Lambda_1^{*n} - G^{*n}\| &\leqslant \left(1 + \pi\sqrt{nb}\right)^{1/2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(Cn^{-1}e^{0.5nbx}\right)^2 + \frac{1}{nb}\left(C\sqrt{nb}n^{-1}e^{0.5nbx}\right)^2 dt\right)^{1/2} \\
&\leqslant \left(2\pi\sqrt{nb}\right)^{1/2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} Cn^{-2}e^{nbx} dt\right)^{1/2} \leqslant \frac{C\sqrt[4]{nb}}{n}\left(\int_{-\pi}^{\pi} e^{nbx} dt\right)^{1/2} \\
&\leqslant \frac{C\sqrt[4]{nb}}{n}\left(\frac{C}{\sqrt{nb}}\right)^{1/2} < \frac{C}{n},
\end{aligned}$$

$$\|\Lambda_1^{*n} - G_1^{*n} * (I + nA_1)\| \leqslant \big(1 + \pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{C}{n^2}e^{0.5nbx}\Big)^2 + \frac{1}{nb}\Big(\frac{C\sqrt{nb}}{n^2}e^{0.5nbx}\Big)^2 dt\bigg)^{1/2}$$

$$\leqslant \big(2\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{C}{n^4}e^{nbx}dt\bigg)^{1/2} \leqslant \frac{C\sqrt[4]{nb}}{n^2}\Big(\frac{C}{\sqrt{nb}}\Big)^{1/2} < \frac{C}{n^2}.$$

For the proof of (100) let us note that $\|\Lambda_1^{*n} - G^{*n}\|_W = \|\mathcal{V}(\Lambda_1^{*n} - G^{*n})\|$, $\|\Lambda_1^{*n} - G^{*n} * (I + nA_1)\|_W = \|\mathcal{V}(\Lambda_1^{*n} - G^{*n} * (I + nA_1))\|$ and due to Lemma 4.17

$$|\widehat{\mathcal{V}}(\Lambda_1^{*n} - G^{*n})(t)| = \frac{|\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t)|}{|\mathrm{e}^{\mathrm{i}t} - 1|} \leqslant \frac{Cnb^2x^2\mathrm{e}^{nbx}}{|2\sin(t/2)|} \leqslant Cnb^2|x||\sin(t/2)|\mathrm{e}^{nbx}$$

$$\leqslant Cnb^2|x|^{3/2}\mathrm{e}^{nbx} \leqslant C\min(nb^2, \sqrt{b/n})\mathrm{e}^{0.5nbx}.$$

Similarly,

$$|(\widehat{\mathcal{V}}(\Lambda_1^{*n} - G^{*n})(t))'| \leqslant \frac{|(\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t))'|}{|\mathrm{e}^{\mathrm{i}t} - 1|} + \frac{|\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t)|}{|\mathrm{e}^{\mathrm{i}t} - 1|^2}$$

$$\leqslant \frac{Cnb^2|xx'|\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|} + \frac{Cnb^2x^2\mathrm{e}^{nbx}}{|2\sin(t/2)|^2}$$

$$\leqslant Cnb^2|x|\mathrm{e}^{0.6nbx} \leqslant C\min(nb^2, b)\mathrm{e}^{0.5nbx},$$

$$|\widehat{\mathcal{V}}(\Lambda_1^{*n} - G^{*n} * (I + nA_1))(t)| = \frac{|\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t)(1 + n\widehat{A}_1(t))|}{|\mathrm{e}^{\mathrm{i}t} - 1|} \leqslant \frac{Cnb^3|x|^3\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|}$$

$$\leqslant Cnb^3x^2|\sin(t/2)|\mathrm{e}^{0.6nbx} \leqslant Cnb^3|x|^{2.5}\mathrm{e}^{0.6nbx} \leqslant C\min(nb^3, \sqrt{b/n}n^{-1})\mathrm{e}^{0.5nbx},$$

$$|(\widehat{\mathcal{V}}(\Lambda_1^{*n} - G^{*n} * (I + nA_1))(t))'|$$

$$\leqslant \frac{|(\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t)(1 + n\widehat{A}_1(t)))'|}{|\mathrm{e}^{\mathrm{i}t} - 1|} + \frac{|\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t)(1 + n\widehat{A}_1(t))|}{|\mathrm{e}^{\mathrm{i}t} - 1|^2}$$

$$\leqslant \frac{Cnb^3x^2|x'|\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|} + \frac{Cnb^3|x|^3\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|^2} \leqslant Cnb^3x^2\mathrm{e}^{0.6nbx} \leqslant C\min(nb^3, bn^{-1})\mathrm{e}^{0.5nbx}.$$

It remains to apply (70) with $v = 0$ and $u = 1$ when $nb \leqslant 1$.

$$\|(\Lambda_1^{*n} - G^{*n})\|_W \leqslant \big(1 + \pi\big)^{1/2}\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi}(Cnb^2\mathrm{e}^{0.5nbx})^2dt\bigg)^{1/2} \leqslant Cnb^2,$$

$$\|(\Lambda_1^{*n} - G^{*n} * (I + nA_1))\|_W \leqslant \big(1 + \pi\big)^{1/2}\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi}(Cnb^3\mathrm{e}^{0.5nbx})^2dt\bigg)^{1/2} \leqslant Cnb^3.$$

We also use $v=0$ and $u=\sqrt{nb}$ for $nb>1$.

$$\begin{aligned}\|(\Lambda_1^{*n}-G^{*n})\|_W &\leqslant \big(1+\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\big(C\sqrt{b/n}e^{0.5nbx}\big)^2+\frac{1}{nb}(Cbe^{0.5nbx})^2dt\bigg)^{1/2}\\ &\leqslant \big(2\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{Cb}{n}e^{nbx}dt\bigg)^{1/2}\leqslant C\sqrt[4]{nb}\Big(\frac{Cb}{n}\Big)^{1/2}\bigg(\int\limits_{-\pi}^{\pi}e^{nbx}dt\bigg)^{1/2}\\ &\leqslant C\sqrt[4]{nb}\Big(\frac{Cb}{n}\Big)^{1/2}\Big(\frac{C}{\sqrt{nb}}\Big)^{1/2}\leqslant\frac{C\sqrt{nb}}{n},\end{aligned}$$

$$\begin{aligned}\|(\Lambda_1^{*n}-G^{*n}*(I+nA_1))\|_W &\leqslant \big(1+\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\big(C\sqrt{b/n}n^{-1}e^{0.5nbx}\big)^2\\ &+\frac{1}{nb}(Cbn^{-1}e^{0.5nbx})^2dt\bigg)^{1/2}\\ &\leqslant \big(2\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{Cb}{n^3}e^{nbx}dt\bigg)^{1/2}\leqslant C\sqrt[4]{nb}\Big(\frac{Cb}{n^3}\Big)^{1/2}\Big(\frac{C}{\sqrt{nb}}\Big)^{1/2}\leqslant\frac{C\sqrt{nb}}{n^2}.\end{aligned}$$

□

Lemma 4.19 *Let condition (24) hold. Then, for all* $n=1,2,\dots,$

$$\|\Lambda_1^{*n}*(P_1-E)\|\leqslant C\pi_2\min\Big\{\frac{1}{n},b\Big\},\quad \|\Lambda_1^{*n}*(P_1-E-E_1)\|\leqslant C\pi_2\min\Big\{\frac{1}{n^2},b^2\Big\},$$
$$\|\Lambda_1^{*n}*(P_1-E)\|_\infty\leqslant C\pi_2\min\Big\{\frac{1}{n\sqrt{nb}},b\Big\},\|\Lambda_1^{*n}*(P_1-E-E_1)\|_\infty\leqslant C\pi_2\min\Big\{\frac{1}{n^2\sqrt{nb}},b^2\Big\},$$
$$\|\Lambda_1^{*n}*(P_1-E)\|_W\leqslant C\pi_2\min\Big\{\frac{\sqrt{nb}}{n},b\Big\},\quad \|\Lambda_1^{*n}*(P_1-E-E_1)\|_W\leqslant C\pi_2\min\Big\{\frac{\sqrt{nb}}{n^2},b^2\Big\}.$$

Proof. Due to Lemma 4.15,

$$\widehat{\Lambda}_1(t) = 1+\frac{2bx}{1-2a+2b-2ax}+\theta Cb^2x=1+\theta Cbx.$$

$$|\widehat{\Lambda}_1'(t)| \leqslant \bigg|\frac{2bx'(1-2a)^2}{(1-2a+2b)(1-2ax-2a)^2}\bigg|+\theta Cb^2|xx'|\leqslant Cb|x'|+Cb^2|xx'|\leqslant Cb|x'|.$$

and

$$\frac{\widehat{\Lambda}_1(t)-2ax-2a}{1-2a}=1-\frac{2ax}{1-2a}+\frac{2b(\widehat{H}(t)-1)}{1-2a+2b}+\theta Cb^2x^2.$$

Therefore,

$$\begin{aligned}|\widehat{P}_1(t)-\widehat{E}(t)| &=\bigg|\pi_1+\frac{\widehat{\Lambda}_1(t)-2a(x+1)}{1-2a}\pi_2+\pi_3-\pi_1-\Big(1-\frac{2a}{1-2a}x\Big)\pi_2-\pi_3\bigg|\\ &=\bigg|\pi_2\Big(\frac{\widehat{\Lambda}_1(t)-2ax-2a}{1-2a}-\frac{1-2a-2ax}{1-2a}\Big)\bigg|=\bigg|\pi_2\Big(\frac{1+\theta Cbx-1}{1-2a}\Big)\bigg|\leqslant C\pi_2b|x|,\end{aligned}$$

$$\begin{aligned}|P_1'(t)-E'(t)| &= \left|\pi_2\Big(\frac{\Lambda_1(t)-2ax-2a}{1-2a}\Big)' - \pi_2\Big(\frac{1-2a-2ax}{1-2a}\Big)'\right| \\ &= \left|\frac{\pi_2}{1-2a}\big(\Lambda_1'(t)-2ax'\big)+\frac{2\pi_2 ax'}{1-2a}\right| = \frac{\pi_2}{1-2a}|\widehat{\Lambda}_1'(t)| \leqslant C\pi_2 b|x'|.\end{aligned}$$

$$|\widehat{P}_1(t)-\widehat{E}(t)-\widehat{E}_1(t)| = \left|\pi_2\Big(\frac{\widehat{\Lambda}_1(t)-2ax-2a}{1-2a} - \frac{1-2a-2ax}{1-2a} - \frac{2b(\widehat{H}(t)-1)}{1-2a+2b}\Big)\right| \leqslant C\pi_2 b^2x^2.$$

Combining the last estimates with (77) and (93) we obtain

$$\begin{aligned}|\widehat{\Lambda}_1(t)|^n|\widehat{P}_1(t)-\widehat{E}(t)| &\leqslant C\pi_2\mathrm{e}^{nbx}b|x| \leqslant C\pi_2\mathrm{e}^{0.5nbx}\mathrm{e}^{0.1nbx}b|x| \\ &\leqslant C\min(b,n^{-1})\pi_2\mathrm{e}^{0.5nbx}, \qquad (102)\\ |(\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t)-\widehat{E}(t)))'| &\leqslant n|\widehat{\Lambda}_1(t)|^{n-1}|\widehat{\Lambda}_1'(t)||\widehat{P}_1(t)-\widehat{E}(t)| + |\widehat{\Lambda}_1(t)|^n|\widehat{P}_1'(t)-\widehat{E}'(t)| \\ &\leqslant C\pi_2 nb^2e^{nbx}|xx'| + C\pi_2 be^{nbx}|x'| \\ &= C\pi_2 be^{0.9nbx}|x'|nb|x|e^{0.1nbx} + C\pi_2 be^{nbx}|x'| \\ &\leqslant C\pi_2 be^{0.6nbx}|x'| \leqslant C\pi_2\min(b,\sqrt{b/n}),\mathrm{e}^{0.5nbx}, \qquad (103)\end{aligned}$$

$$|\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t)-\widehat{E}(t)-\widehat{E}_1(t))| \leqslant C\pi_2\mathrm{e}^{nbx}b^2x^2 \leqslant C\pi_2\min(b^2,n^{-2})\mathrm{e}^{0.5nbx}.$$

Similarly,

$$|(\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t)-\widehat{E}(t)-\widehat{E}_1(t)))'| \leqslant C\pi_2\mathrm{e}^{0.6nbx}b^2|xx'| \leqslant C\pi_2\min(b^2,\sqrt{b/n}n^{-1})\mathrm{e}^{0.5nbx}.$$

The rest of the proof is very similar to the proof of Lemma 4.18. Suppose, that $nb \leqslant 1$ and we apply $v=0$ and $u=1$ in Lemma 4.11. Then $\|\Lambda_1^{*n}*(P_1-E)\|_\infty \leqslant C\pi_2 b$, $\|\Lambda_1^{*n}*(P_1-E-E_1)\|_\infty \leqslant C\pi_2 b^2$, $\|\Lambda_1^{*n}*(P_1-E)\| \leqslant C\pi_2 b$ and $\|\Lambda_1^{*n}*(P_1-E-E_1)\| \leqslant C\pi_2 b^2$. Suppose, that $nb>1$. Applying $v=0$ and $u=\sqrt{nb}$ we obtain

$$\|\Lambda_1^{*n}*(P_1-E)\|_\infty \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} C\pi_2 n^{-1}e^{0.5nbx}dt \leqslant \frac{C\pi_2}{n\sqrt{nb}},\quad \|\Lambda_1^{*n}*(P_1-E-E_1)\|_\infty \leqslant \frac{C\pi_2}{n^2\sqrt{nb}},$$

$$\begin{aligned}\|\Lambda_1^{*n}*(P_1-E)\| &\leqslant \big(1+\pi\sqrt{nb}\big)^{1/2}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(C\pi_2 n^{-1}e^{0.5nbx}\big)^2 + \frac{1}{nb}\big(C\pi_2\sqrt{b/n}e^{0.5nbx}\big)^2dt\Big)^{1/2} \\ &\leqslant \big(2\pi\sqrt{nb}\big)^{1/2}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}C\pi_2 n^{-2}e^{nbx}dt\Big)^{1/2} \leqslant \frac{C\pi_2}{n}.\end{aligned}$$

Similarly,

$$\|\Lambda_1^{*n}*(P_1-E-E_1)\| \leqslant \frac{C\pi_2}{n^2}.$$

Similar to the proof of Lemma 4.18, we obtain

$$\begin{aligned}|\widehat{\mathcal{V}}(\Lambda_1^{*n}*(P_1-E))(t)| &= \frac{|\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t)-\widehat{E}(t))|}{|\mathrm{e}^{\mathrm{i}t}-1|} \leqslant \frac{C\pi_2 b|x|\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|} \leqslant C\pi_2 b|\sin(t/2)|\mathrm{e}^{0.6nbx} \\ &\leqslant C\pi_2 b|x|^{1/2}\mathrm{e}^{0.6nbx} \leqslant C\pi_2\min(b,\sqrt{b/n})\mathrm{e}^{0.5nbx},\end{aligned}$$

$$
\begin{aligned}
|(\widehat{\mathcal{V}}(\Lambda_1^{*n} * (P_1 - E))(t))'| \quad &\leqslant \quad \frac{|(\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t) - \widehat{E}(t)))'|}{|\mathrm{e}^{\mathrm{i}t} - 1|} + \frac{|\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t) - \widehat{E}(t))|}{|\mathrm{e}^{\mathrm{i}t} - 1|^2} \\
&\leqslant \quad \frac{C\pi_2 b|x'|\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|} + \frac{C\pi_2 b|x|\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|^2} \leqslant C\pi_2 b\mathrm{e}^{0.5nbx},
\end{aligned}
$$

$$
\begin{aligned}
|\widehat{\mathcal{V}}(\Lambda_1^{*n} * (P_1 - E - E_1))(t)| &= \frac{|\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t) - \widehat{E}(t) - \widehat{E}_1(t))|}{|\mathrm{e}^{\mathrm{i}t} - 1|} \leqslant \frac{C\pi_2 b^2 x^2 \mathrm{e}^{nbx}}{|2\sin(t/2)|} \\
&\leqslant C\pi_2 b^2 |x||\sin(t/2)|\mathrm{e}^{nbx} \leqslant C\pi_2 b^2 |x|^{3/2}\mathrm{e}^{0.6nbx} \leqslant C\pi_2 \min(b^2, \sqrt{b/n}n^{-1})\mathrm{e}^{0.5nbx},
\end{aligned}
$$

$$
\begin{aligned}
&|(\widehat{\mathcal{V}}(\Lambda_1^{*n} * (P_1 - E - E_1))(t))'| \\
&\quad\leqslant \frac{|(\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t) - \widehat{E}(t) - \widehat{E}_1(t)))'|}{|\mathrm{e}^{\mathrm{i}t} - 1|} + \frac{|\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t) - \widehat{E}(t) - \widehat{E}_1(t))|}{|\mathrm{e}^{\mathrm{i}t} - 1|^2} \\
&\quad\leqslant \frac{C\pi_2 b^2 |xx'|\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|} + \frac{C\pi_2 b^2 x^2 \mathrm{e}^{nbx}}{|2\sin(t/2)|^2} \leqslant C\pi_2 b^2 |x|\mathrm{e}^{0.6nbx} \leqslant C\pi_2 \min(b^2, bn^{-1})\mathrm{e}^{0.5nbx}.
\end{aligned}
$$

We apply (70) with $v = 0$ and $u = 1$ when $nb \leqslant 1$. Therefore, $\|\Lambda_1^{*n} * (P_1 - E)\|_W \leqslant C\pi_2 b$ and $\|\Lambda_1^{*n} * (P_1 - E - E_1)\|_W \leqslant C\pi_2 b^2$. When $nb > 1$, we use $v = 0$ and $u = \sqrt{nb}$.

$$
\begin{aligned}
\|\Lambda_1^{*n} * (P_1 - E)\|_W &\leqslant \big(1 + \pi\sqrt{nb}\big)^{1/2} \bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi} \big(C\pi_2\sqrt{b/n}e^{0.5nbx}\big)^2 + \frac{1}{nb}(C\pi_2 b e^{0.5nbx})^2 dt\bigg)^{1/2} \\
&\leqslant \big(2\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{C\pi_2^2 b}{n} e^{nbx} dt\bigg)^{1/2} \leqslant \big(2\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{C\pi_2^2 b}{n\sqrt{nb}}\bigg)^{1/2} \leqslant \frac{C\pi_2\sqrt{nb}}{n}.
\end{aligned}
$$

Similarly we have

$$
\|\Lambda_1^{*n} * (P_1 - E - E_1)\|_W \leqslant \frac{C\pi_2\sqrt{nb}}{n^2}.
$$

□

Lemma 4.20 *Let condition (24) hold. Then, for all* $n = 1, 2, \dots,$

$$
\begin{aligned}
&\|(W_1 - M) * \Lambda_1^{*n}\| \leqslant C\min\left\{\frac{1}{n}, b\right\}, \qquad \|(W_1 - M * (I + M_1)) * \Lambda_1^{*n}\| \leqslant C\min\left\{\frac{1}{n^2}, b^2\right\}, \\
&\|(W_1 - M) * \Lambda_1^{*n}\|_\infty \leqslant C\min\left\{\frac{1}{n\sqrt{nb}}, b\right\}, \|(W_1 - M * (I + M_1)) * \Lambda_1^{*n}\|_\infty \leqslant C\min\left\{\frac{1}{n^2\sqrt{nb}}, b^2\right\}, \\
&\|(W_1 - M) * \Lambda_1^{*n}\|_W \leqslant C\min\left\{\frac{\sqrt{nb}}{n}, b\right\}, \ \|(W_1 - M * (I + M_1)) * \Lambda_1^{*n}\|_W \leqslant C\min\left\{\frac{\sqrt{nb}}{n^2}, b^2\right\}.
\end{aligned}
$$

Proof. From (79) it follows that $1 + \widehat{\Delta}(t) \geqslant 1 - |\widehat{\Delta}(t)| \geqslant 0.38$. Therefore,

$$
\begin{aligned}
\bigg|(1 - 2(a - b) - 2ax)^{-1}\bigg(\frac{1}{(1 + \widehat{\Delta}(t))^{1/2}} - 1\bigg)\bigg| &\leqslant C\bigg|\frac{1 - (1 + \widehat{\Delta}(t))^{1/2}}{(1 + \widehat{\Delta}(t))^{1/2}}\bigg| \\
= \quad C\bigg|&\frac{\widehat{\Delta}(t)}{(1 + \widehat{\Delta}(t))^{1/2}\big(1 + (1 + \widehat{\Delta}(t))^{1/2}\big)}\bigg| \leqslant Cb|x|.
\end{aligned}
$$

and

$$\begin{aligned}
\widehat{D}^{-1/2}(t) &= (1-2(a-b)-2ax)^{-1}(1+\widehat{\Delta}(t))^{-1/2}\\
&= (1-2(a-b)-2ax)^{-1} + (1-2(a-b)-2ax)^{-1}\left(\frac{1}{(1+\widehat{\Delta}(t))^{1/2}}-1\right)\\
&= (1-2(a-b)-2ax)^{-1} + \theta Cb|x|. \qquad (104)
\end{aligned}$$

Therefore,

$$\widehat{W}_1(t) = \frac{1}{2}\left(1+\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\right)+\theta Cb|x|$$

and

$$\begin{aligned}
|\widehat{W}_1(t)-\widehat{M}(t)| &= \left|\frac{1}{2}\left(1+\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\right)+\theta Cb|x| - \frac{1-2(a-b)}{1-2(a-b)-2ax}\right|\\
&= \left|\frac{1-2(a-b)}{1-2(a-b)-2ax}+\theta Cb|x| - \frac{1-2(a-b)}{1-2(a-b)-2ax}\right| \leqslant Cb|x|.
\end{aligned}$$

Combining the last estimate with Lemma 4.14 we prove that

$$|\widehat{W}_1(t)-\widehat{M}(t)||\widehat{\Lambda}_1(t)^n| \leqslant C\mathrm{e}^{nbx}b|x| \leqslant C\mathrm{e}^{0.5nbx}\mathrm{e}^{0.1nbx}b|x| \leqslant C\min(b,n^{-1})\mathrm{e}^{0.5nbx}. \qquad (105)$$

Applying Lemma (4.13), (79) and (87) and we estimate the following members

$$\begin{aligned}
&(1+\widehat{\Delta}(t))^{-1/2} = \sum_{j=0}^{\infty}\binom{-1/2}{j}\widehat{\Delta}^j(t) = 1+\theta Cb|x|,\\
&\frac{1}{4}\left|\left(\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\right)\right|\left|\frac{1}{(1+\widehat{\Delta}(t))^{3/2}}\right||\widehat{\Delta}'(t)| \leqslant Cb|x'|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\widehat{W}_1'(t) &= \frac{1}{2}\left(1+\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\cdot\frac{1}{(1+\widehat{\Delta}(t))^{1/2}}\right)'\\
&= \frac{1}{2}\left(\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\right)'\cdot\frac{1}{(1+\widehat{\Delta}(t))^{1/2}} - \frac{1}{4}\left(\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\right)\cdot\frac{\widehat{\Delta}'(t)}{(1+\widehat{\Delta}(t))^{3/2}}\\
&= \frac{1}{2}\left(\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\right)'(1+\theta Cb|x|)+\theta Cb|x'| = \frac{1}{2}\left(\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\right)'+\theta Cb|x'|\\
&= \frac{1}{2}\left(1+\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\right)'+\theta Cb|x'| = \left(\frac{1-2(a-b)}{1-2(a-b)-2ax}\right)'+\theta Cb|x'|
\end{aligned}$$

and, therefore,

$$|\widehat{W}_1'(t)-M'(t)| = \left|\left(\frac{1-2(a-b)}{1-2(a-b)-2ax}\right)'+\theta Cb|x'| - \left(\frac{1-2(a-b)}{1-2(a-b)-2ax}\right)'\right| \leqslant Cb|x'|.$$

Observe that, due to (93) and estimates in above,

$$\begin{aligned}
|((\widehat{W}_1(t)-\widehat{M}(t))\widehat{\Lambda}_1(t)^n)'| &\leqslant |\widehat{W}_1'(t)-M'(t)||\widehat{\Lambda}_1^n(t)| + |\widehat{W}_1(t)-\widehat{M}(t)|n|\widehat{\Lambda}_1(t)|^{n-1}|\widehat{\Lambda}_1'(t)|\\
&\leqslant Cb\mathrm{e}^{nbx}|x'| + C\mathrm{e}^2nb^2\mathrm{e}^{nbx}|xx'| \leqslant C\min(b^2,\sqrt{b/n})\mathrm{e}^{0.5nbx}. \qquad (106)
\end{aligned}$$

Observe, that

$$\frac{1}{1-2(a-b)-2ax}=\frac{1-2a}{(1-2a+2b)(1-2ax-2a)}+\theta Cbx$$

and

$$(1+\widehat{\Delta}(t))^{-1/2}=1-\frac{4bx}{(1-2a+2b-2ax)^2}+\theta Cb^2x^2,\quad \widehat{M}(t)-1=\frac{2ax}{1-2a+2b-2ax}.$$

Applying the last expression we write $\widehat{W}_1(t)$ in the following way

$$\begin{aligned}
\widehat{W}_1(t)&=\frac{1}{2}\bigg(1+\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\bigg(1-\frac{4bx}{(1-2(a-b)-2ax)^2}\bigg)\bigg)+\theta Cb^2x^2\\
&=\frac{1-2a+2b}{1-2(a-b)-2ax}-\frac{2bx(1-2(a-b)-2ax)+8abx^2}{(1-2(a-b)-2ax)^3}+\theta Cb^2x^2\\
&=\widehat{M}(t)-\frac{2bx}{(1-2(a-b)-2ax)^2}-\frac{4bx(\widehat{M}(t)-1)}{(1-2(a-b)-2ax)^2}+\theta Cb^2x^2\\
&=\widehat{M}(t)-\frac{2bx(2\widehat{M}(t)-1)}{(1-2(a-b)-2ax)^2}+\theta Cb^2x^2\\
&=\widehat{M}(t)-\frac{2b(1-2a)(\widehat{H}-1)(2\widehat{M}(t)-1)}{(1-2a+2b)(1-2(a-b)-2a)}+\theta Cb^2x^2\\
&=\widehat{M}(t)-\widehat{M}(t)\frac{2b(1-2a)(2\widehat{M}(t)-1)(\widehat{H}(t)-1)}{(1-2a+2b)^2}+\theta Cb^2x^2\\
&=\widehat{M}(t)(1+\widehat{M}_1(t))+\theta Cb^2x^2. \qquad (107)
\end{aligned}$$

From (107) and Lemma 4.14 it follows that

$$|\widehat{W}_1(t)-\widehat{M}(t)(1+\widehat{M}_1(t))||\widehat{\Lambda}_1|^n\leqslant C\mathrm{e}^{0.5nbx}\mathrm{e}^{0.5nbx}b^2x^2\leqslant C\min(b^2,n^{-2})\mathrm{e}^{0.5nbx}.$$

Similarly it can be proved that

$$|((\widehat{W}_1(t)-\widehat{M}(t)(1+\widehat{M}_1(t))\widehat{\Lambda}_1^n)'|\leqslant C\min(b^2,\sqrt{nb}n^{-2})\mathrm{e}^{0.5nbx}.$$

The rest of the proof is very similar to the proof of Lemma 4.18 again. Suppose, that $nb\leqslant 1$ and we apply $v=0$ and $u=1$ in Lemma 4.11. Then $\|(W_1-M)*\Lambda_1^{*n}\|_\infty\leqslant Cb$, $\|(W_1-M*(I+M_1))*\Lambda_1^{*n}\|_\infty\leqslant Cb^2$, $\|(W_1-M)*\Lambda_1^{*n}\|\leqslant Cb$ and $\|(W_1-M*(I+M_1))*\Lambda_1^{*n}\|\leqslant Cb^2$. Suppose, that $nb>1$. Applying $v=0$ and $u=\sqrt{nb}$ we obtain

$$\|(W_1-M)*\Lambda_1^{*n}\|_\infty\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}Cn^{-1}e^{0.5nbx}dt\leqslant\frac{C}{n\sqrt{nb}},$$

$$\|(W_1-M*(I+M_1))*\Lambda_1^{*n}\|_\infty\leqslant\frac{C}{n^2\sqrt{nb}},$$

$$\begin{aligned}
\|(W_1-M)*\Lambda_1^{*n}\| &\leqslant \big(1+\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(Cn^{-1}e^{0.5nbx}\big)^2+\frac{1}{nb}\big(C\sqrt{b/n}e^{0.5nbx}\big)^2dt\bigg)^{1/2}\\
&\leqslant \big(2\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi}Cn^{-2}e^{nbx}dt\bigg)^{1/2}\leqslant\frac{C}{n}.
\end{aligned}$$

Similarly,

$$\|(W_1 - M*(I+M_1))*\Lambda_1^{*n}\| \leqslant \frac{C}{n^2}.$$

Similar to the proof of Lemma 4.18, we obtain

$$\begin{aligned}|\widehat{\mathcal{V}}((W_1-M)*\Lambda_1^{*n})(t)| &= \frac{|\widehat{W}_1(t)-\widehat{M}(t)||\widehat{\Lambda}_1(t)^n|}{|\mathrm{e}^{\mathrm{i}t}-1|} \leqslant \frac{Cb|x|\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|} \leqslant Cb|\sin(t/2)|\mathrm{e}^{0.6nbx}\\ &\leqslant Cb|x|^{1/2}\mathrm{e}^{0.6nbx} \leqslant C\min(b,\sqrt{b/n})\mathrm{e}^{0.5nbx},\end{aligned}$$

$$\begin{aligned}|(\widehat{\mathcal{V}}((W_1-M)*\Lambda_1^{*n})(t))'| &\leqslant \frac{|((\widehat{W}_1(t)-\widehat{M}(t))\widehat{\Lambda}_1(t)^n)'|}{|\mathrm{e}^{\mathrm{i}t}-1|} + \frac{|\widehat{W}_1(t)-\widehat{M}(t)||\widehat{\Lambda}_1(t)^n|}{|\mathrm{e}^{\mathrm{i}t}-1|^2}\\ &\leqslant \frac{Cb|x'|\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|} + \frac{Cb|x|\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|^2} \leqslant Cb\mathrm{e}^{0.5nbx},\end{aligned}$$

$$\begin{aligned}|\widehat{\mathcal{V}}((W_1-M*(I+M_1))*\Lambda_1^{*n})(t)| &= \frac{|\widehat{W}_1(t)-\widehat{M}(t)(1+\widehat{M}_1(t))||\widehat{\Lambda}_1|^n}{|\mathrm{e}^{\mathrm{i}t}-1|} \leqslant \frac{Cb^2x^2\mathrm{e}^{nbx}}{|2\sin(t/2)|}\\ &\leqslant Cb^2|x||\sin(t/2)|\mathrm{e}^{nbx} \leqslant Cb^2|x|^{3/2}\mathrm{e}^{0.6nbx} \leqslant C\min(b^2,\sqrt{b/n}n^{-1})\mathrm{e}^{0.5nbx},\end{aligned}$$

$$\begin{aligned}|(\widehat{\mathcal{V}}((W_1-M*(I+M_1))*\Lambda_1^{*n})(t))'| &\leqslant \frac{|((\widehat{W}_1(t)-\widehat{M}(t)(1+\widehat{M}_1(t))\widehat{\Lambda}_1^n)'|}{|\mathrm{e}^{\mathrm{i}t}-1|}\\ &+\frac{|\widehat{W}_1(t)-\widehat{M}(t)(1+\widehat{M}_1(t))||\widehat{\Lambda}_1|^n}{|\mathrm{e}^{\mathrm{i}t}-1|^2}\\ &\leqslant \frac{Cb^2|xx'|\mathrm{e}^{0.6nbx}}{|2\sin(t/2)|} + \frac{Cb^2x^2\mathrm{e}^{nbx}}{|2\sin(t/2)|^2} \leqslant Cb^2|x|\mathrm{e}^{0.6nbx} \leqslant C\min(b^2,bn^{-1})\mathrm{e}^{0.5nbx}.\end{aligned}$$

We apply (70) with $v=0$ and $u=1$ when $nb \leqslant 1$. Therefore, $\|(W_1-M)*\Lambda_1^{*n}\|_W \leqslant Cb$ and $\|(W_1-M*(I+M_1))*\Lambda_1^{*n}\|_W \leqslant Cb^2$. When $nb>1$, we use $v=0$ and $u=\sqrt{nb}$.

$$\begin{aligned}\|(W_1-M)*\Lambda_1^{*n}\|_W &\leqslant \big(1+\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(C\sqrt{b/n}e^{0.5nbx}\big)^2 + \frac{1}{nb}(Cbe^{0.5nbx})^2dt\bigg)^{1/2}\\ &\leqslant \big(2\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{Cb}{n}e^{nbx}dt\bigg)^{1/2} \leqslant \big(2\pi\sqrt{nb}\big)^{1/2}\bigg(\frac{Cb}{n\sqrt{nb}}\bigg)^{1/2} \leqslant \frac{C\sqrt{nb}}{n}.\end{aligned}$$

Similarly we have

$$\|(W_1-M*(I+M_1))*\Lambda_1^{*n}\|_W \leqslant \frac{C\sqrt{nb}}{n^2}.$$

Thus we complete the proof of Lemma 4.20. □

Lemma 4.21 *Let condition (24) hold. Then, for all $n=1,2,\dots$,*

$$\|nA_1*M_1*G^{*n}\| \leqslant C\min\left\{\frac{1}{n^2},nb^3\right\}, \qquad \|nA_1*M_1*G^{*n}\|_\infty \leqslant C\min\left\{\frac{1}{n^2\sqrt{nb}},nb^3\right\},$$

$$\|nA_1*M_1*G^{*n}\|_W \leqslant C\min\left\{\frac{\sqrt{nb}}{n^2},nb^3\right\}.$$

Proof. From (86), Lemma 4.13 and Lemma 4.14 it follows that

$$
\begin{aligned}
|\widehat{A}_1(t)| &= \Big|\frac{2b^2(1-2a)}{(1-2a+2b)^2}\Big(1+4a+\frac{2(1-2a)}{1-2(a-b)-2ax}\Big)\frac{x^2}{1-2ax-2a}\Big| \\
&\leqslant \Big|\frac{2}{(0.93)^3}\Big(1+\frac{4}{30}+\frac{2}{0.93}\Big)\Big|b^2x^2 \leqslant Cb^2x^2,
\end{aligned}
$$

$$
|\widehat{M}_1(t)| \leqslant \Big|\frac{2b(1-2a)}{(1-2a+2b)^2}\cdot\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\cdot\frac{x}{1-2ax-2a}\Big| \leqslant \frac{2\cdot 1.07}{(0.93)^4}b|x| \leqslant Cb|x|.
$$

Combining these estimates with (77) we prove that

$$
|n\widehat{A}_1(t)\widehat{M}_1(t)\widehat{G}^n(t)| \leqslant Cn\mathrm{e}^{1.6nbx}b^3|x|^3 \leqslant C\min(nb^3, n^{-2})\mathrm{e}^{0.5nbx}.
$$

Similarly,

$$
|(n\widehat{A}_1(t)\widehat{M}_1(t)\widehat{G}^n(t))'| \leqslant C\min(nb^3, n^{-2}\sqrt{nb})\mathrm{e}^{0.5nbx}.
$$

For local estimates and estimates in total variation it suffices to apply Lemma 4.11 with $u=\max(1,\sqrt{nb})$ and $v=0$. For estimates in the Wasserstein metric, before applying Lemma 4.11, one should divide estimates of the Fourier- Stieljes transforms by $|\mathrm{e}^{\mathrm{i}t}-1|$. □

Lemma 4.22 *Let condition (24) hold. Then, for all $n=1,2,\dots$,*

$$
\|\Lambda_2\| \leqslant 3.1|a-b|, \qquad \|\Lambda_2\|^n \leqslant 15.5|a-b|(0.2)^n.
$$

Proof. Applying (1) and (24) we prove

$$
\begin{aligned}
\|\Delta_1\| &\leqslant \frac{8|a-b|}{(1-2b)^2}\Big(\sum_{j=0}^{\infty}\Big(\frac{2a}{1-2b}\Big)^j\Big)^2 \leqslant 8|a-b|\Big(\frac{15}{14}\Big)^2\Big(\sum_{j=0}^{\infty}(14)^{-j}\Big)^2 \\
&\leqslant 10.66|a-b| \leqslant 10.66\cdot\frac{1}{30} \leqslant 0.36, \\
\|\Lambda_2\| &\leqslant \frac{1}{2}\|(1-2b)I+2aL\|\sum_{j=1}^{\infty}\Big|\binom{1/2}{j}\Big|\|\Delta_1\|^j \leqslant \frac{10.66|a-b|}{2}\Big(\frac{1}{2}+\frac{1}{8}\sum_{j=2}^{\infty}0.36^{j-1}\Big) \\
&\leqslant 3.1|a-b| \leqslant 3.1/30 \leqslant 0.2, \\
\|\Lambda_2\|^n &\leqslant 3.1|a-b|(0.2)^{n-1} \leqslant 15.5|a-b|0.2^n.
\end{aligned}
$$

Lemma 4.23 *Let condition (24) hold. Then, for all $n=1,2,\dots$,*

$$
\max(\|M\|,\|M_1\|,\|E\|,\|E_1\|,\|P_1\|,\|P_2\|) \leqslant C.
$$

Proof. Observe that M, L, E, H and G^{*n} are distributions. Therefore, $\|M\|=\|L\|=\|E\|=\|H\|=\|G^{*n}\|=1$. From (81) it follows that $\|\Lambda_1\| \leqslant \|G\|+\|\Lambda_1-G\| \leqslant 2$. Therefore, applying Lemma's and 4.22 and 4.13 we prove that

$$
\begin{aligned}
\|P_1\| &\leqslant \pi_1+\frac{1}{1-2a}\Big(\|\Lambda_1\|+2a\Big)\pi_2+\pi_3 \leqslant \pi_1+\frac{1}{0.93}\Big(2+\frac{2}{30}\Big)\pi_2+\pi_3 \leqslant C, \\
\|P_2\| &\leqslant \pi_1+\frac{1}{1-2a}\Big(\|\Lambda_2\|+2a\Big)\pi_2+\pi_3 \leqslant \pi_1+\frac{1}{0.93}\Big(0.2+\frac{2}{30}\Big)\pi_2+\pi_3 \leqslant C.
\end{aligned}
$$

Similarly,

$$\|M_1\| \leqslant \frac{(2\|M\|+1)(\|H\|+1)}{0.93^2} \leqslant \frac{6}{0.93^2}, \quad \|E_1\| \leqslant \frac{2}{0.93}.$$

□

Lemma 4.24 *Let condition (24) hold. Then, for all* $n = 1, 2, \ldots,$

$$\|W_{1,2}\| \leqslant C, \qquad \|W_2\|_W \leqslant C.$$

Proof. From Lemma 4.13 and the fact that $\|L\| = \|I\| = 1$ it follows that

$$\begin{aligned} \|\Delta\| &\leqslant \frac{8b(\|L\|+\|I\|)}{1-2b}\Big\|\sum_{j=0}^{\infty}\Big(\frac{2a}{1+2b}L\Big)^{*j}\Big\|^2 \leqslant \frac{16b}{1-2b}\Big(\sum_{j=0}^{\infty}\Big(\frac{2a}{1+2b}\Big)^j\Big)^2 \\ &= \frac{16b}{1-2b}\Big(1-\frac{2a}{1+2b}\Big)^{-2} \leqslant \frac{16}{30}\Big(1-\frac{2}{30}\Big)^{-2} \leqslant 0.62. \end{aligned}$$

Therefore,

$$\begin{aligned} \|W_{1,2}\| &\leqslant \frac{1}{2}\Big(1+\Big(\frac{4a+(1+2b)}{(1+2b)^2}\Big)\Big(\sum_{j=0}^{\infty}\Big(\frac{2a}{1+2b}\Big)^j\Big)^2\sum_{j=0}^{\infty}\binom{-1/2}{j}\|\Delta\|^j\Big) \\ &\leqslant \frac{1}{2}\Big(1+\Big(\frac{6}{30}+1\Big)\Big(1-\frac{2a}{1+2b}\Big)^{-2}\frac{1}{1-0.62}\Big) \\ &\leqslant \frac{1}{2}\Big(1+\frac{36}{30}\Big(1-\frac{2}{30}\Big)^{-2}\cdot 2.64\Big) \leqslant 2.4 \leqslant C. \end{aligned}$$

For the second estimate observe that, due to (104) and Lemma 4.13, we have

$$\begin{aligned} \widehat{W}_2(t) &= \frac{1}{2}\Big(1-\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\Big)+\theta Cb|x| \\ &= -\frac{2ax}{1-2(a-b)-2ax}+\theta Cb|x| = \theta C|x|, \\ |\widehat{W}_2(t)| &\leqslant C|x| \leqslant C\sin^2\frac{t}{2} \leqslant C\Big|\sin\frac{t}{2}\Big| \leqslant C|\mathrm{e}^{\mathrm{i}t}-1|. \end{aligned} \tag{108}$$

Therefore, we obtain

$$\frac{|\widehat{W}_2(t)|}{|\mathrm{e}^{\mathrm{i}t}-1|} \leqslant C.$$

Similarly,

$$\begin{aligned} \widehat{W}_2'(t) &= -\frac{1}{2}\Big(\frac{1-2(a-b)+2ax}{1-2(a-b)-2ax}\Big)'+\theta Cb|x'| \\ &= -\frac{2ax'(1-4(a-b))}{(1-2(a-b)-2ax)^2}+\theta Cb|x'| = \theta C|x'|, \\ |\widehat{W}_2'(t)| &\leqslant C|x'| \leqslant C\Big|\sin\frac{t}{2}\Big| \leqslant C|\mathrm{e}^{\mathrm{i}t}-1| \end{aligned}$$

and we have

$$\frac{|\widehat{W}_2'(t)|}{|\mathrm{e}^{\mathrm{i}t}-1|} \leqslant C.$$

Consequently,

$$\Big|\Big(\frac{\widehat{W}_2(t)}{\mathrm{e}^{\mathrm{i}t}-1}\Big)'\Big| \leqslant \Big|\frac{\widehat{W}_2'(t)}{\mathrm{e}^{\mathrm{i}t}-1}\Big| + \Big|\frac{\widehat{W}_2(t)}{(\mathrm{e}^{\mathrm{i}t}-1)^2}\Big| \leqslant C.$$

We recall that $\|W_2\|_W = \|\mathcal{V}(W_2)\|$. Therefore, it remains to apply Lemma 4.11 with $v = 0$ and $u = 1$. That is,

$$\|W_2\|_W \leqslant (1+\pi)^{1/2}\bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi} C^2 dt\bigg)^{1/2} \leqslant C.$$

□

4.4 Proof of Theorems 3.5-3.9

Proof of Theorem 3.5. We have

$$\begin{aligned}\|F_n - E*M*G^{*n}\|_W &\leqslant \|P_1*\Lambda_1^{*n}*W_1 - E*M*G^{*n}\|_W + \|P_2*\Lambda_2^{*n}*W_2\|_W \\ &\leqslant \|P_1*(W_1-M)*\Lambda_1^{*n}\|_W + \|M*(P_1*\Lambda_1^{*n} - E*G^{*n})\|_W + \|P_2*\Lambda_2^{*n}*W_2\|_W \\ &\leqslant \|M*\Lambda_1*(P_1-E)\|_W + \|M*E*(\Lambda_1^{*n}-G^{*n})\|_W + \|P_1*(W_1-M)*\Lambda_1^{*n}\|_W \\ &+ \|P_2*\Lambda_2^{*n}*W_2\|_W.\end{aligned}$$

Observe, that

$$\|P_2*\Lambda_2^{*n}*W_2\|_W = \|P_2*\Lambda_2^{*n}*\mathcal{V}(W_2)\| \leqslant \|P_2\|\|\Lambda_2\|^n\|\mathcal{V}(W_2)\| = \|P_2\|\|\Lambda_2\|^n\|W_2\|_W.$$

Arguing similarly for other summands we arrive at

$$\begin{aligned}\|F_n - E*M*G^{*n}\|_W &\leqslant \|M\|\|\Lambda_1^{*n}*(P_1-E)\|_W + \|M\|\|E\|\|\Lambda_1^{*n} - G^{*n}\|_W \\ &+ \|P_1\|\|(W_1-M)*\Lambda_1^{*n}\|_W + \|P_2\|\|W_2\|_W\|\Lambda_2\|^n. \end{aligned} \tag{109}$$

It remains to apply Lemma's 4.18 − 4.24.

$$\begin{aligned}\|F_n - E*M*G^{*n}\|_W &\leqslant C\bigg(\min\bigg\{\frac{\sqrt{nb}}{n}, nb^2\bigg\} + (\pi_2+1)\min\bigg\{\frac{\sqrt{nb}}{n}, b\bigg\} + 0.2^n|a-b|\bigg) \\ &\leqslant C\bigg(\min\bigg\{\frac{\sqrt{nb}}{n}, b\bigg\} + 0.2^n|a-b|\bigg).\end{aligned}$$

For the total variation and local norms one should replace, in (109), respectively $\|\cdot\|_W$ by $\|\cdot\|$ and $\|\cdot\|_\infty$.

$$\begin{aligned}\|F_n - E*M*G^{*n}\| &\leqslant C\bigg(\min\bigg\{\frac{1}{n}, nb^2\bigg\} + (\pi_2+1)\min\bigg\{\frac{1}{n}, b\bigg\} + 0.2^n|a-b|\bigg) \\ &\leqslant C\bigg(\min\bigg\{\frac{1}{n}, b\bigg\} + 0.2^n|a-b|\bigg),\end{aligned}$$

$$\begin{aligned}\|F_n - E*M*G^{*n}\|_\infty &\leqslant C\bigg(\min\bigg\{\frac{1}{n\sqrt{nb}}, nb^2\bigg\} + (\pi_2+1)\min\bigg\{\frac{1}{n\sqrt{nb}}, b\bigg\} + 0.2^n|a-b|\bigg) \\ &\leqslant C\bigg(\min\bigg\{\frac{1}{n\sqrt{nb}}, b\bigg\} + 0.2^n|a-b|\bigg).\end{aligned}$$

□

Proof of Theorem 3.6. Arguing similarly to the proof of Theorem 3.5 we prove that

$$
\begin{aligned}
&\|F_n - (E+E_1)*M*G^{*n}*(I+nA_1+M_1)\|_W \\
&\quad = \|P_1*\Lambda_1^{*n}*W_1 + P_2*\Lambda_2^{*n}*W_2 - (E+E_1)*M*G^{*n}*(I+nA_1+M_1)\|_W \\
&\quad = \|P_1*\Lambda_1^{*n}*W_1 + P_2*\Lambda_2^{*n}*W_2 - (E+E_1)*M*G^{*n}*(I+nA_1+M_1) \\
&\qquad +(E+E_1)*(\Lambda_1^{*n}*W_1 - \Lambda_1^{*n}*W_1 + \Lambda_1^{*n}*M*(I+M_1) - \Lambda_1^{*n}*M*(I+M_1) \\
&\qquad +G^{*n}*M*(I+nA_1)*(I+M_1) - G^{*n}*M*(I+nA_1)*(I+M_1))\|_W \\
&\quad \leqslant \|\Lambda_1^{*n}*(P_1-E-E_1)\|_W\|W_1\| + \|E+E_1\|\|(W_1 - M*(I+M_1))*\Lambda_1^{*n}\|_W \\
&\qquad +\|E+E_1\|\|M\|\|I+M_1\|\|\Lambda_1^{*n} - G^{*n}*(I+nA_1)\|_W \\
&\qquad +\|E+E_1\|\|M\|\|nA_1*M_1*G^{*n}\|_W + \|P_2\|\|\Lambda_2\|^n\|W_2\|_W \\
&\quad \leqslant C\bigg((\pi_2+1)\min\left\{\frac{\sqrt{nb}}{n^2}, b^2\right\} + \min\left\{\frac{\sqrt{nb}}{n^2}, nb^3\right\} + 0.2^n|a-b|\bigg) \\
&\quad \leqslant C\bigg(\min\left\{\frac{\sqrt{nb}}{n^2}, b^2\right\} + 0.2^n|a-b|\bigg).
\end{aligned}
$$

Similar inequalities hold for the total variation and local norms. It remains to apply Lemmas 4.18 – 4.24.

$$
\begin{aligned}
\|F_n - (E+E_1)*M*G^{*n}*(I+nA_1+M_1)\|_\infty &\leqslant C\bigg((\pi_2+1)\min\left\{\frac{1}{n^2\sqrt{nb}}, b^2\right\} \\
+\min\left\{\frac{1}{n^2\sqrt{nb}}, nb^3\right\} + 0.2^n|a-b|\bigg) &\leqslant C\bigg(\min\left\{\frac{1}{n^2\sqrt{nb}}, b^2\right\} + 0.2^n|a-b|\bigg),
\end{aligned}
$$

$$
\begin{aligned}
\|F_n - (E+E_1)*M*G^{*n}*(I+nA_1+M_1)\| &\leqslant C\bigg((\pi_2+1)\min\left\{\frac{1}{n^2}, b^2\right\} \\
+\min\left\{\frac{1}{n^2}, nb^3\right\} + 0.2^n|a-b|\bigg) &\leqslant C\bigg(\min\left\{\frac{1}{n^2}, b^2\right\} + 0.2^n|a-b|\bigg).
\end{aligned}
$$

□

Proof of Corollary 3.3. The estimate follows from Lemmas 4.18 – 4.24 and estimate

$$
\begin{aligned}
\|F_n - (E+E_1)*M*(I+M_1)*G^{*n}\| &\leqslant \|P_2\|\|\Lambda_2\|^n\|W_2\| + \|(P_1-E-E_1)*\Lambda_1^{*n}\|\|W_1\| \\
&+ \|(E+E_1)\|\|\Lambda_1^{*n}*(W_1 - M*(I+M_1))\| + \|E+E_1\|\|M*(I+M_1)\|\|\Lambda_1^{*n} - G^{*n}\| \\
&\leqslant C\bigg((\pi_2+1)\min\left\{\frac{1}{n^2}, b^2\right\} + \min\left\{\frac{1}{n}, nb^2\right\} + 0.2^n|a-b|\bigg) \\
&\leqslant C\bigg(\min\{nb^2, n^{-1}\} + 0.2^n|a-b|\bigg).
\end{aligned}
$$

□

Proof of Theorem 3.7. Let us estimate the following expressions. We apply $x = \cos t - 1 = -\frac{t^2}{2} + \theta C t^4$, (86) and

$$
2\widehat{M}(t) - 1 = \frac{1 - 2(a-b) + 2ax}{1 - 2(a-b) - 2ax}.
$$

Therefore,

$$\begin{aligned}\widehat{M}(t)\widehat{M}_1(t) &= \frac{-2b(1-2a)(1-2(a-b)+2ax)x}{(1-2(a-b)-2ax)^2(1-2a+2b)(1-2ax-2a)}\\ &= \frac{b(1-2a)(1-2(a-b)+2ax)t^2}{(1-2(a-b)-2ax)^2(1-2a+2b)(1-2ax-2a)}+\theta\tilde{C}_1bt^4\\ &\geqslant \frac{0.93\cdot 0.8bt^2}{1.14(1.07)^2}+\theta\tilde{C}bt^4\geqslant 0.57bt^2+\theta\tilde{C}bt^4.\end{aligned}$$

Using $|\sin y|\leqslant|y|,\ \ |e^y-1|\leqslant|y|e^{|y|},\ \ y\in\mathbb{R}$, we estimate

$$|\widehat{M}(t)\widehat{M}_1(t)|=\Big|\frac{4b(1-2a)(1-2(a-b)+2ax)\sin^2\frac{t}{2}}{(1-2(a-b)-2ax)^2(1-2a+2b)(1-2ax-2a)}\Big|\leqslant Cbt^2$$

and

$$|\widehat{G}^n(t)-1|\leqslant\Big|\frac{2nb(1-2a)}{1-2a+2b}\cdot\frac{2\sin^2\frac{t}{2}}{1-2a\cos t}\Big|e^{3.2b\sin^2\frac{t}{2}}\leqslant Cnbt^2e^{0.8nbt^2}.$$

We apply Lemma 4.12 with $v=0$, $u=\sqrt{hnb}$, $U=F_n-M*G^{*n}$, where $h>1$ will be chosen later. Let $\tilde{t}=t/u$. We use the following expression.

$$\begin{aligned}\|F_n-M*G^{*n}\|_K &= \|\Lambda_1^{*n}*W_1+\Lambda_2^{*n}*W_2-M*G^{*n}+W_1*G^{*n}-W_1*G^{*n}\\ &+\ M*M_1*G^{*n}-M*M_1*G^{*n}\|_K\\ &=\ \|M*M_1+\Lambda_2^{*n}*W_2+(\Lambda_1^{*n}-G^{*n})*W_1+G^{*n}*(W_1-M*(I+M_1))\\ &+\ M*M_1*(G^{*n}-1)\|_K\end{aligned}$$

Then, taking into account, Lemmas 4.22, 4.23, 4.24 and (89), (107), (108) we obtain

$$\begin{aligned}\|F_n-M*G^{*n}\|_K &\geqslant C_{11}\Big|\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}(\widehat{F}_n(\tilde{t})-\widehat{M}(\tilde{t})\widehat{G}^n(\tilde{t}))\mathrm{d}t\Big|\\ &\geqslant\ 2C_{11}\Big|\int_0^{\infty}\mathrm{e}^{-t^2/2}\widehat{M}(\tilde{t})\widehat{M}_1(\tilde{t})\mathrm{d}t\Big|-C_{11}\|\Lambda_2\|^n\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}|\widehat{W}_2(\tilde{t})|\mathrm{d}t\\ &\quad -C_{11}\|W_1\|\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}|\widehat{\Lambda}_1^n(\tilde{t})-\widehat{G}^n(\tilde{t})|\mathrm{d}t\\ &\quad -C_{11}\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}|\widehat{W}_1(\tilde{t})-\widehat{M}(\tilde{t})(1+\widehat{M}_1(\tilde{t}))|\mathrm{d}t\\ &\quad -C_{11}\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}|\widehat{M}(\tilde{t})\widehat{M}_1(\tilde{t})||\widehat{G}^n(\tilde{t})-1|\mathrm{d}t\\ &\geqslant\ V_1-V_2-V_3-V_4-V_5,\end{aligned}$$

where

$$V_1\geqslant 2C_{11}\Big|\int_{-\infty}^{\infty}e^{-t^2/2}(0.57b\tilde{t}^2+\tilde{C}b\tilde{t}^4)dt\Big|\geqslant\frac{1.14C_{11}}{hn}\int_0^{\infty}\mathrm{e}^{-t^2/2}\mathrm{d}t,$$

$$V_2\leqslant C_{12}0.2^n|a-b|\int_{-\infty}^{\infty}e^{-t^2/2}\tilde{t}^2dt\leqslant C_{12}0.2^n|a-b|\int_{-\infty}^{\infty}e^{-t^2/2}\frac{t^2}{hnb}dt\leqslant C_{12}0.2^n\frac{|a-b|}{hnb},$$

$$V_3\leqslant C_{13}\int_{-\infty}^{\infty}e^{-t^2/2}nb^2\tilde{t}^4dt\leqslant C_{13}\int_{-\infty}^{\infty}e^{-t^2/2}\frac{t^4}{h^2n}dt\leqslant\frac{C_{13}}{h^2n},$$

$$V_4\leqslant C_{14}\int_{-\infty}^{\infty}e^{-t^2/2}b^2\tilde{t}^4dt\leqslant C_{14}\int_{-\infty}^{\infty}e^{-t^2/2}\frac{t^4}{h^2n^2}dt\leqslant\frac{C_{14}}{h^2n^2},$$

$$V_5\leqslant C_{15}\int_{-\infty}^{\infty}e^{-t^2/2}e^{0.8nb\tilde{t}^2}nb^2\tilde{t}^4dt=C_{15}\int_{-\infty}^{\infty}e^{-t^2/2}e^{0.8t^2/h}\frac{t^4}{h^2n}dt\leqslant\frac{C_{15}}{h^2n}.$$

Therefore,

$$\begin{aligned}\|F_n - M*G^{*n}\|_K &\geqslant \frac{1.14C_{11}}{hn}\int_0^\infty \mathrm{e}^{-t^2/2}\mathrm{d}t - C_{12}0.2^n\frac{|a-b|}{hnb} - \frac{C_{13}}{h^2n} - \frac{C_{14}}{h^2n^2} - \frac{C_{15}}{h^2n}\\ &\geqslant \frac{C_{16}}{hn}\Big(1 - C_{17}0.2^n|a-b| - \frac{C_{18}}{h}\Big).\end{aligned}$$

If $h = 2C_{18}$, then the last estimate becomes (27). Local estimate is proved similarly. For the Wasserstein metric note that

$$\begin{aligned}\|F_n - M*G^{*n}\|_W &= \|\mathcal{V}(F_n - M*G^{*n})\| \geqslant \|\mathcal{V}(F_n - M*G^{*n})\|\\ &\geqslant C_{11}\Big|\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}\frac{(\widehat{F}_n(\tilde{t}) - \widehat{M}(\tilde{t})\widehat{G}^n(\tilde{t}))}{\mathrm{e}^{\mathrm{i}\tilde{t}}-1}\mathrm{d}t\Big|.\end{aligned}$$

□

Proof of Theorem 3.8. From Lemma 4.22, Lemma 4.23 and Lemma 4.24 it follows that for all $n = 1, 2, \dots,$

$$\max(|\widehat{M}(t)|, |\widehat{M_1}(t)|, |\widehat{E}(t)|, |\widehat{E_1}(t)|, |\widehat{P_1}(t)|, |\widehat{P_2}(t)|, |\widehat{W_1}(t)|, |\widehat{W_2}(t)|) \leqslant C, \qquad |\widehat{\Lambda}_2(t)|^n \leqslant 0.2^n.$$

Using Lemma 4.13 we can obtain that

$$|\widehat{M}'(t)| = \Big|\frac{2ax'(1+2a+2b)}{(1-2a+2b-2ax)^2}\Big| \leqslant \frac{0.07\cdot 1.07}{0.93^2}|x'| \leqslant C,$$

$$|\widehat{E}'(t)| = \frac{2a\pi_2}{1-2a}|x'| \leqslant \frac{0.07\pi_2}{0.93}|x'| \leqslant C,$$

Using (88) we have

$$\begin{aligned}&\widehat{\Lambda}_1'(t) = \frac{2bx'(1+2b-2a)}{(1-2a+2b-2ax)^2} + \theta Cb^2|xx'|,\\ &\widehat{\Lambda}_1(t) + \widehat{\Lambda}_2'(t) = \frac{4bx'(1+2b-2a)}{(1-2a+2b-2ax)^2} + \theta Cb^2|xx'|,\\ &|\widehat{\Lambda}_2'(t)| \leqslant \frac{2b|x'|(1+2b-2a)}{(1-2a+2b-2ax)^2} + Cb^2|xx'| \leqslant \frac{0.07\cdot 1.07}{0.93^2}|x'| + Cb^2|xx'| \leqslant C,\\ &|\widehat{P}_2'(t)| \leqslant \frac{\pi_2}{1-2a}|\widehat{\Lambda}_2'(t)| + \frac{2a\pi_2}{1-2a}|x'| \leqslant C, \quad |\widehat{P}_1'(t)| \leqslant \frac{\pi_2}{1-2a}|\widehat{\Lambda}_1'(t)| + \frac{2a\pi_2}{1-2a}|x'| \leqslant C.\end{aligned}$$

Therefore, according to (102), (103), (89), (90), (105), (106) it follows, that

$$\begin{aligned}|\widehat{M}(t)\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t) - \widehat{E}(t))| &\leqslant Cn^{-1}\pi_2 e^{0.5nbx},\\ |(\widehat{M}(t)\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t) - \widehat{E}(t)))'| &= |\widehat{M}'(t)||\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t) - \widehat{E}(t))| + |\widehat{M}(t)||(\widehat{\Lambda}_1^n(t)(\widehat{P}_1(t) - \widehat{E}(t)))'|\\ &\leqslant Cn^{-1}\pi_2 e^{0.5nbx} + C\pi_2\sqrt{b/n}e^{0.5nbx} \leqslant C\pi_2\sqrt{b/n}e^{0.5nbx},\end{aligned}$$

$$\begin{aligned}|\widehat{M}(t)\widehat{E}(t)(\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t))| &\leqslant Cn^{-1}e^{0.5nbx},\\ |(\widehat{M}(t)\widehat{E}(t)(\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t)))'| &\leqslant |(\widehat{M}(t)\widehat{E}(t))'||\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t)| + |\widehat{M}(t)\widehat{E}(t)||(\widehat{\Lambda}_1^n(t) - \widehat{G}^n(t))'|\\ &\leqslant Cn^{-1}e^{0.5nbx} + C\sqrt{b/n}e^{0.5nbx} \leqslant C\sqrt{b/n}e^{0.5nbx},\end{aligned}$$

$$\begin{aligned}
|\widehat{P}_1(t)(\widehat{W}_1(t)-\widehat{M}(t))\widehat{\Lambda}_1^n(t)| &\leqslant Cn^{-1}e^{0.5nbx},\\
|(\widehat{P}_1(t)(\widehat{W}_1(t)-\widehat{M}(t))\widehat{\Lambda}_1^n(t))'| &\leqslant |\widehat{P}_1'(t)||(\widehat{W}_1(t)-\widehat{M}(t))\widehat{\Lambda}_1^n(t)|+|\widehat{P}_1(t)||((\widehat{W}_1(t)-\widehat{M}(t))\widehat{\Lambda}_1^n(t))'|\\
&\leqslant Cn^{-1}e^{0.5nbx}+C\sqrt{b/n}e^{0.5nbx}\leqslant C\sqrt{b/n}e^{0.5nbx},\\
|\widehat{\Lambda}_2^n(t)\widehat{P}_2(t)\widehat{W}_2(t)| &\leqslant C0.2^n\leqslant \frac{C}{n\sqrt{nb}},\\
|(\widehat{\Lambda}_2^n(t)\widehat{P}_2(t)\widehat{W}_2(t))'| &\leqslant |n\widehat{\Lambda}_2^{n-1}(t)\widehat{\Lambda}_2'(t)\widehat{P}_2(t)\widehat{W}_2(t)|+|\widehat{\Lambda}_2^n(t)\widehat{P}_2'(t)\widehat{W}_2(t)|\\
&\quad+|\widehat{\Lambda}_2^n(t)\widehat{P}_2(t)\widehat{W}_2'(t)|\\
&\leqslant C(n0.2^{n-1}+0.2^n)\leqslant Cn^{-1}.
\end{aligned}$$

We also obtain

$$\begin{aligned}
|(F_n-E*M*G^{*n})\{m\}|\bigg(1+\frac{|m|}{\sqrt{nb}}\bigg) &\leqslant (|(M*\Lambda_1^{*n}*(P_1-E))\{m\}|\\
&\quad+|(M*E*(\Lambda_1^{*n}-G^{*n}))\{m\}|\\
&\quad+|(P_1*(W_1-M)*\Lambda_1^{*n})\{m\}|\\
&\quad+|(P_2*\Lambda_2^{*n}*W_2)\{m\}|)\bigg(1+\frac{|m|}{\sqrt{nb}}\bigg).
\end{aligned}$$

Non-uniform estimate follows from estimate of Lemma 4.11, (71) applied with $v=0$, and $u=\sqrt{nb}$.

$$\begin{aligned}
|(F_n-E*M*G^{*n})\{m\}|\bigg(1+\frac{|m|}{\sqrt{nb}}\bigg) &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}C\bigg(\frac{e^{0.5nbx}}{n}+\frac{1}{n\sqrt{nb}}\bigg)+\frac{C}{\sqrt{nb}}\bigg(\sqrt{\frac{b}{n}}e^{0.5nbx}+\frac{1}{n}\bigg)\mathrm{d}t\\
&\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}C\bigg(\frac{e^{0.5nbx}}{n}+\frac{1}{n\sqrt{nb}}\bigg)\mathrm{d}t\leqslant \frac{C}{n\sqrt{nb}}.
\end{aligned}$$

Further we move on the second part of proof of Lemma. Similarly we notice, that

$$\begin{aligned}
&|\widehat{M}(t)||\widehat{\mathcal{V}}(\Lambda_1^{*n}*(P_1-E))(t)|\leqslant C\pi_2\sqrt{b/n}e^{0.5nbx},\ \ |(\widehat{M}(t)\widehat{\mathcal{V}}(\Lambda_1^{*n}*(P_1-E))(t))'|\leqslant C\pi_2be^{0.5nbx},\\
&|\widehat{M}(t)||\widehat{E}(t)||\widehat{\mathcal{V}}(\Lambda_1^{*n}-G^{*n})(t)|\leqslant C\sqrt{b/n}e^{0.5nbx},\quad |(\widehat{M}(t)\widehat{E}(t)\widehat{\mathcal{V}}(\Lambda_1^{*n}-G^{*n})(t))'|\leqslant Cbe^{0.5nbx},\\
&|\widehat{P}_1(t)||\widehat{\mathcal{V}}((W_1-M)*\Lambda_1^{*n})(t)|\leqslant C\sqrt{b/n}e^{0.5nbx},\quad |(\widehat{P}_1(t)\widehat{\mathcal{V}}((W_1-M)*\Lambda_1^{*n})(t))'|\leqslant Cbe^{0.5nbx},\\
&|\widehat{W}_2(t)|\leqslant C\sin^2\frac{t}{2}\leqslant C(\mathrm{e}^{\mathrm{i}t}-1)^2\leqslant C|\mathrm{e}^{\mathrm{i}t}-1|,\quad |\widehat{W}_2'(t)|\leqslant C|\mathrm{e}^{\mathrm{i}t}-1|,
\end{aligned}$$

$$\begin{aligned}
\bigg|\frac{\widehat{\Lambda}_2^n(t)\widehat{P_2}(t)\widehat{W}_2(t)}{\mathrm{e}^{\mathrm{i}t}-1}\bigg| &\leqslant C0.2^n\leqslant\frac{C}{n}\\
\bigg|\bigg(\frac{\widehat{\Lambda}_2^n(t)\widehat{P_2}(t)\widehat{W}_2(t)}{\mathrm{e}^{\mathrm{i}t}-1}\bigg)'\bigg| &\leqslant \bigg|\frac{(\widehat{\Lambda}_2^n(t)\widehat{P_2}(t)\widehat{W}_2(t))'}{\mathrm{e}^{\mathrm{i}t}-1}\bigg|+\bigg|\frac{\widehat{\Lambda}_2^n(t)\widehat{P_2}(t)\widehat{W}_2(t)}{(\mathrm{e}^{\mathrm{i}t}-1)}\bigg|\\
&\leqslant \frac{C|\mathrm{e}^{\mathrm{i}t}-1|(n0.2^{n-1}+0.2^n)}{|\mathrm{e}^{\mathrm{i}t}-1|}+\frac{C0.2^n(\mathrm{e}^{\mathrm{i}t}-1)^2}{(\mathrm{e}^{\mathrm{i}t}-1)^2}\leqslant\frac{C}{n}.
\end{aligned}$$

The second non-uniform estimate follows from estimate of Lemma 4.11, (72) applied with $v=0$

and $u = \sqrt{nb}$ again.

$$
\begin{aligned}
&|(F_n - E * M * G^{*n})\{[-\infty, m]\}|\left(1 + \frac{|m|}{\sqrt{nb}}\right)\\
&\qquad\qquad\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} C\left(\sqrt{\frac{b}{n}}e^{0.5nbx} + \frac{1}{n}\right) + \frac{C}{\sqrt{nb}}\left(be^{0.5nbx} + \frac{1}{n}\right)\mathrm{d}t\\
&\qquad\qquad\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} C\left(\sqrt{\frac{b}{n}}e^{0.5nbx} + \frac{1}{n}\right) + \frac{C}{\sqrt{nb}}\left(be^{0.5nbx} + \frac{1}{n}\right)\mathrm{d}t\\
&\qquad\qquad\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} C\left(\sqrt{\frac{b}{n}}e^{0.5nbx} + \frac{1}{n} + \frac{be^{0.5nbx}}{\sqrt{nb}}\right)\mathrm{d}t \leqslant \frac{C}{n}.
\end{aligned}
$$

Proof of Theorem 3.9. Observe that, due to symmetry of distributions,

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \mathrm{e}^{h|k|}|F_n\{k\} - \mathcal{G}\{k\}| &= |F_n\{0\} - \mathcal{G}\{0\}| + 2\sum_{k=1}^{\infty}\mathrm{e}^{hk}|F_n\{k\} - \mathcal{G}\{k\}|\\
&\leqslant 2\sum_{k=0}^{\infty}\mathrm{e}^{hk}|F_n\{k\} - \mathcal{G}\{k\}| \leqslant 2\sum_{k=-\infty}^{\infty}\mathrm{e}^{hk}|F_n\{k\} - \mathcal{G}\{k\}|\\
&= 2\|F_{nh} - \mathcal{G}_h\|,
\end{aligned}
$$

where $F_{nh}\{k\} = \mathrm{e}^{hk}F_n\{k\}$ and $\mathcal{G}_h\{k\} = \mathrm{e}^{hk}\mathcal{G}\{k\}$. Smilarly $F_{nh} = P_{1h} * \Lambda_{1h}^{*n} * W_{1h} + P_{2h} * \Lambda_{2h}^{*n} * W_{2h}$. Therefore,

$$
\begin{aligned}
\|F_{nh} - \mathcal{G}_h\| &= \|P_{1h} * \Lambda_{1h}^{*n} * W_{1h} + P_{2h} * \Lambda_{2h}^{*n} * W_{2h} - \mathcal{E}_h * \mathcal{M}_h * B_h^{*n}\\
&\quad + P_{1h} * B_h^{*n} * W_{1h} - P_{1h} * B_h^{*n} * W_{1h}\|\\
&\leqslant \|\Lambda_{1h}^{*n} - B_h^{*n}\|\|P_{1h}\|\|W_{1h}\| + \|B_h^{*n}\|\|P_{1h} * W_{1h} - \mathcal{E}_h * \mathcal{M}_h\| + \|P_{2h}\|\|\Lambda_{2h}\|^n\|W_{2h}\|\\
&\leqslant \|\Lambda_{1h}^{*n} - B_h^{*n}\|\|P_{1h}\|\|W_{1h}\| + \|B_h^{n}\|\|P_{1h} * W_{1h} - \mathcal{E}_h * \mathcal{M}_h + \mathcal{E}_h * W_{1h} - \mathcal{E}_h * W_{1h}\|\\
&\quad + \|P_{2h}\|\|\Lambda_{2h}\|^n\|W_{2h}\|\\
&\leqslant \|\Lambda_{1h}^{*n} - B_h^{*n}\|\|P_{1h}\|\|W_{1h}\| + \|B_h^{*n}\|\|W_{1h}\|\|P_{1h} - \mathcal{E}_h\| + \|B_h^{*n}\|\|\mathcal{E}_h\|\|W_{1h} - \mathcal{M}_h\|\\
&\quad + \|P_{2h}\|\|\Lambda_{2h}\|^n\|W_{2h}\| \qquad (110)
\end{aligned}
$$

From (28) it follows that

$$
a\mathrm{e}^h \leqslant \frac{11}{10}\tilde{a}\mathrm{e}^h \leqslant \frac{1}{10}, \quad b(\mathrm{e}^h + 1) \leqslant 0.02, \quad b \leqslant 0.01, \quad b \leqslant Cn^{-1}.
$$

We consequently prove that

$$
\begin{aligned}
\|L_h\| &\leqslant \frac{1}{2}(e^{-h}+e^h) \leqslant \frac{1}{2}(e^h+1),\\
\|X_n\| &\leqslant \|L_h\|+1 \leqslant \frac{1}{2}(e^h+1)+1,\\
b\|X_n\| &\leqslant b\Big(\frac{1}{2}(e^h+1)+1\Big) \leqslant \frac{1}{2}\cdot 0.02+0.01=0.02,\\
\sum_{j=0}^{\infty}\left(\frac{2a}{1+2b}\|L_h\|\right)^j &\leqslant \sum_{j=0}^{\infty}(ae^h+a)^j \leqslant \sum_{j=0}^{\infty}0.2^j=1.25,\\
\|\Delta_h\| &\leqslant \frac{8b}{1-2b}\|X_h\|\Big(\sum_{j=0}^{\infty}\left(\frac{2a}{1+2b}\|L_h\|\right)^j\Big)^2 \leqslant \frac{0.16\cdot 1.57}{0.98} \leqslant 0.3,\\
\|\Delta_h\| &\leqslant Cb \leqslant Cn^{-1},\\
\sum_{j=0}^{\infty}(2\tilde{a}\|L_h\|)^j &\leqslant \sum_{j=0}^{\infty}(\tilde{a}e^h+\tilde{a})^j \leqslant \sum_{j=0}^{\infty}0.19^j \leqslant 1.24,\\
\|B_h\| &= \exp\Big\{\frac{2\lambda}{n}\|X_h\|\sum_{j=0}^{\infty}(2\tilde{a}\|L_h\|)^j\Big\} \leqslant \exp\{Cn^{-1}\},\\
\|W_{1h,2h}\| &\leqslant \frac{1}{2}\left(1+(2a\|X_h\|+1+2b)\sum_{j=0}^{\infty}\left(\frac{2a}{1+2b}\|L_h\|\right)^j\sum_{j=0}^{\infty}\|\Delta_h\|^j\right) \leqslant C.
\end{aligned}
$$

Λ_{1h} could be expressed by

$$
\Lambda_{1h} = I+\frac{1}{4}(I-2(a-b)I-2aX_h)*\Delta_h+\Theta*C\Delta_h^{*2}.
$$

Therefore,

$$
\begin{aligned}
\|\Lambda_{1h}\| &\leqslant 1+Cb \leqslant \exp\{Cb\} \leqslant \exp\{Cn^{-1}\},\\
\|P_{1h}\| &\leqslant \pi_1+\frac{1}{1-2a}\big(\|\Lambda_{1h}\|+2a\|L_h\|\big)\pi_2+\pi_3 \leqslant \pi_1+\frac{1}{0.8}(1+Cb+0.2)\pi_2+\pi_3 \leqslant C,\\
\|\mathcal{E}_h\| &\leqslant \pi_1+\left(1+\frac{2\tilde{a}\|X_h\|}{1-2\tilde{a}}\right)\pi_2+\pi_3 \leqslant \pi_1+\left(1+\frac{\tilde{a}e^h+3\tilde{a}}{1-2\tilde{a}}\right)\pi_2+\pi_3\\
&\leqslant \pi_1+\left(\frac{0.37}{1-0.19}\right)\pi_2+\pi_3 \leqslant C,\\
\|\Lambda_{2h}\| &\leqslant \frac{1}{2}\Big\|4aL_h-4b-(I+2b-2aL_h)*\sum_{j=1}^{\infty}\binom{1/2}{j}\Delta^{*j}\Big\|\\
&\leqslant \frac{1}{2}\left(2a(e^h+1)+4b+(1+2b+a(e^h+1))0.5\sum_{j=1}^{\infty}0.3^j\right)\\
&\leqslant \frac{1}{2}\left(0.44+\frac{1.22\cdot 0.5\cdot 0.3}{1-0.3}\right) \leqslant 0.36,\\
\|P_{2h}\| &\leqslant \pi_1+\frac{1}{1-2a}\big(\|\Lambda_{2h}\|+2a\|L_h\|\big)\pi_2+\pi_3 \leqslant \pi_1+\frac{1}{0.8}(0.36+0.2)\pi_2+\pi_3 \leqslant C.
\end{aligned}
$$

Finally, observe that

$$
\|\Lambda_{1h}^{*n}-B_h^{*n}\| \leqslant n\max(\|\Lambda_{1h}\|^{n-1},\|B_h\|^{n-1})\|\Lambda_{1h}-B_h\| \leqslant Cn\|\Lambda_{1h}-B_h\|.
$$

Substituting the above estimates into (110) we obtain

$$\|F_{nh}-\mathcal{G}_h\| \leqslant C(n\|\Lambda_{1h}-B_h\|+\|P_{1h}-\mathcal{E}_h\|+\|W_{1h}-\mathcal{M}_h\|+(0.36)^n). \tag{111}$$

Taking into account (28) we see that

$$B_h = I+\frac{2\lambda}{n}(L_h-I)*\sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j}+\Theta C n^{-2}$$

and it can be proved that

$$[(1+2b)I-2aL_h]*\Bigg(\sum_{j=0}^{\infty}\Big(\frac{2aL_h}{1+2b}\Big)^{*j}\Bigg)^{*2}=(1+2b)\sum_{j=0}^{\infty}\Big(\frac{2aL_h}{1+2b}\Big)^{*j}.$$

Using the last equality we have

$$\begin{aligned}\Lambda_{1h} &= I+\frac{2bX_h}{1-2b}*[(1+2b)I-2aL_h]*\Bigg(\sum_{j=0}^{\infty}\Big(\frac{2aL_h}{1+2b}\Big)^{*j}\Bigg)^{*2}+\Theta C n^{-2}\\ &= I+\frac{2bX_h(1+2b)}{1-2b}*\sum_{j=0}^{\infty}\Big(\frac{2aL_h}{1+2b}\Big)^{*j}+\Theta C n^{-2}\\ &= I+2bX_h*\sum_{j=0}^{\infty}\Big(\frac{2aL_h}{1+2b}\Big)^{*j}+\Theta C n^{-2}.\end{aligned}$$

Therefore

$$\begin{aligned}\Big\|\sum_{j=0}^{\infty}\Big(\frac{2aL_h}{1+2b}\Big)^{*j}-\sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j}\Big\| &\leqslant \sum_{j=0}^{\infty}\|L_h\|^j\Big|\Big(\frac{2a}{1+2b}\Big)^j-(2\tilde{a})^j\Big|\\ &\leqslant \sum_{j=0}^{\infty}\Big(\frac{1}{2}(e^h+1)\Big)^j j\max\Big(\frac{2a}{1+2b},2\tilde{a}\Big)^{j-1}\Big(\frac{2|a-\tilde{a}|}{1+2b}+\frac{4\tilde{a}b}{1+2b}\Big)\\ &\leqslant (e^h+1)\Big(\frac{|a-\tilde{a}|}{1+2b}+\frac{2\tilde{a}b}{1+2b}\Big)\sum_{j=0}^{\infty}j(ae^h+a)^{j-1}\\ &\leqslant (e^h+1)\Big(\frac{|a-\tilde{a}|}{1+2b}+\frac{2\tilde{a}b}{1+2b}\Big)\sum_{j=0}^{\infty}j(0.2)^{j-1}\leqslant C(|a-\tilde{a}|+n^{-1}).\end{aligned} \tag{112}$$

We obtain

$$\Lambda_{1h}-B_h=2bX_h*\sum_{j=0}^{\infty}\Big(\frac{2aL_h}{1+2b}\Big)^{*j}-\frac{2\lambda}{n}X_h*\sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j}\pm\frac{2\lambda}{n}X_h*\sum_{j=0}^{\infty}\Big(\frac{2aL_h}{1+2b}\Big)^{*j}+\Theta C n^{-2},$$

$$\begin{aligned}\|\Lambda_{1h}-B_h\| &\leqslant 2\Big|b-\frac{\lambda}{n}\Big|\|X_h\|\Big\|\sum_{j=0}^{\infty}\Big(\frac{2aL_h}{1+2b}\Big)^{*j}\Big\|\\ &+ \frac{2\lambda}{n}\|X_h\|\Big\|\sum_{j=0}^{\infty}\Big(\frac{2aL_h}{1+2b}\Big)^{*j}-\sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j}\Big\|+Cn^{-2}\\ &\leqslant \frac{C}{n}\big(|nb-\lambda|+|a-\tilde{a}|+n^{-1}\big).\end{aligned} \tag{113}$$

Using (112) we obtain

$$
\begin{aligned}
W_{1h} &= \frac{1}{2}\Big(I + \Big(\frac{2aX_h - 2a + I + 2b}{(1+2b)^2}\Big) * \sum_{j=0}^{\infty}\Big(\frac{2a}{1+2b}L_h\Big)^{*j}(1 + \Theta * C\Delta_h)\Big) \\
&= \frac{1}{2}\Big(I + \Big(\frac{2aX_h - 2a + I + 2b}{(1+2b)^2}\Big) * \sum_{j=0}^{\infty}\Big(\frac{2a}{1+2b}L_h\Big)^{*j}\Big) + \Theta C n^{-1} \\
&= \frac{1}{2}\Big(I + \Big(\frac{2aX_h - 2a + I + 2b}{(1+2b)^2}\Big) * \sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j}\Big) + \Theta C(|a - \tilde{a}| + n^{-1}).
\end{aligned}
$$

Noting that $\sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j} - 2\tilde{a}L_h * \sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j} = I$, we next obtain

$$
\begin{aligned}
\|W_{1h} - \mathcal{M}_h\| &\leqslant \Big\|\frac{1}{2}I + (aX_h - a - \frac{1}{2}I + 2\tilde{a}) * \sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j}\Big\| + C(|a - \tilde{a}| + n^{-1}) \\
&\leqslant \Big\|\frac{1}{2}\sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j} - \tilde{a}(X_h + I) * \sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j} + (aX_h - a - \frac{1}{2}I + 2\tilde{a}) * \sum_{j=0}^{\infty}(2\tilde{a}L_h)^{*j}\Big\| \\
&\quad + C(|a - \tilde{a}| + n^{-1}) \\
&\leqslant |a - \tilde{a}|\|L_h\|\sum_{j=0}^{\infty}(2\tilde{a}\|L_h\|)^{j} + C(|a - \tilde{a}| + n^{-1}) \\
&\leqslant C(|a - \tilde{a}| + n^{-1}).
\end{aligned}
$$

Next observe that

$$
\begin{aligned}
\|P_{1h} - \mathcal{E}_h\| &= \Big\|\frac{\Lambda_{1h} - 2a(X_h + I)}{1 - 2a} - I + \frac{2\tilde{a}X_h}{1 - 2\tilde{a}}\Big\|\pi_2 \\
&= \Big\|\frac{(\Lambda_{1h} - 2aX_h - 2aI)(1 - 2\tilde{a}) - (1 - 2a)(1 - 2\tilde{a})I + 2\tilde{a}X_h(1 - 2a)}{(1 - 2\tilde{a})(1 - 2a)}\Big\|\pi_2 \\
&= \Big\|\frac{\Lambda_{1h} - 2aX_h - 2\tilde{a}\Lambda_{1h} - I + 2\tilde{a}I + 2\tilde{a}X_h}{(1 - 2\tilde{a})(1 - 2a)}\Big\|\pi_2 \\
&\leqslant \frac{(1 - 2\tilde{a})\|\Lambda_{1h} - I\| + 2\|X_h\||\tilde{a} - a|}{(1 - 2\tilde{a})(1 - 2a)}\pi_2 \leqslant Cb + C|\tilde{a} - a| \\
&\leqslant C(|\tilde{a} - a| + n^{-1}).
\end{aligned} \tag{114}
$$

Substituting estimates (113), (105), (114) into (111) we complete the proof. □

5 Conclusions

Nagaev [62] estimated the accuracy of normal approximation for homogeneous Markov chain with an finite number of states. In this thesis a translated Poisson approximation is constructed for the same Markov chain. In comparison to the normal approximation the translated Poisson approximation has the following advantages:

- it is concentrated on the same lattice as the initial distribution;
- unlike the normal case, no additional smoothing summands are needed;
- the accuracy of a translated Poisson approximation can be estimated in the total variation metric.

The best infinitely divisible approximation is constructed for MB distribution. It is shown that

- the accuracy of the approximation is of the order $O(n^{-1})$ in total variation, i.e. much better than $O(n^{-1/2})$, the order that can be expected from the normal approximation;
- the first uniform Kolmogorov theorem for the MB distribution has the same accuracy as for the case for binomial distribution.

The Simons-Johnson theorem is proved for the MB distribution and for distribution of sum of symmetric three-state Markov chain. It is proved that

- in both cases constructed approximation is a CP distribution with the compounding geometric distribution;
- the convergence holds for sums with exponential weights.

It is proved the limit law of symmetric three-state Markov chain is a CP distribution with the compounding geometric distribution. It is also proved that

- the approximation is of the order $O(n^{-1})$, i.e. the same order as for the case of symmetrized binomial rv;
- the second order CP approximation improves the rate of convergence and is of the order $O(n^{-2})$;
- in some cases the lower-bound estimates are also of the order $O(n^{-1})$;
- non-uniform estimates are constructed.

Ideas for possible future research:

- to relate a symmetric three-state Markov chain and its approximations to some trinomial models in econometrics;
- to adapt Markov chains and its approximations for solving parking problems;
- to apply the approximations for sum of Markov dependent Bernoulli trials in matching DNA sequences, in weather or stock market, where the number of successes can be important for testing trends.

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Appendix

We will prove recursive formulas for the main approximations of the MB and three point Markovian distributions. For the sake of convenience the notation of this appendix is not related to the rest of thesis.

1. The CP distribution with the compounding geometric law

Let the moment generating function of the measure μ is equal to

$$\sum_{k=0}^{\infty} \mu_k \mathrm{e}^{kt} = \exp\Big\{\frac{\lambda(\mathrm{e}^t - 1)}{1 - p\mathrm{e}^t}\Big\}.$$

Calculating the derivative with respect to t we arrive at

$$\sum_{k=0}^{\infty} k\mu_k \mathrm{e}^{kt} = \lambda \frac{(1-p)\mathrm{e}^t}{(1-p\mathrm{e}^t)^2} \exp\Big\{\frac{\lambda(\mathrm{e}^t - 1)}{1 - p\mathrm{e}^t}\Big\} = \lambda \frac{(1-p)\mathrm{e}^t}{(1-p\mathrm{e}^t)^2} \sum_{k=0}^{\infty} \mu_k \mathrm{e}^{tk}$$

or, equivalently,

$$\lambda(1-p)\mathrm{e}^t \sum_{k=0}^{\infty} \mu_k \mathrm{e}^{kt} = (1-p\mathrm{e}^t)^2 \sum_{k=0}^{\infty} k\mu_k \mathrm{e}^{kt}.$$

Assuming, for the sake of brevity, that $\mu_k = 0$, for $k < 0$, and changing the order of summation we get

$$\lambda(1-p) \sum_{k=1}^{\infty} \mu_{k-1} \mathrm{e}^{kt} = \sum_{k=1}^{\infty} [k\mu_k - 2p(k-1)\mu_{k-1} + p^2(k-2)\mu_{k-2}] \mathrm{e}^{kt}$$

or

$$\lambda(1-p)\mu_{k-1} = k\mu_k - 2p(k-1)\mu_{k-1} + p^2(k-2)\mu_{k-2}, \quad k = 1, 2, \dots$$

Note that in $\mu_0 = \mathrm{e}^{-\lambda}$.

2. The symmetric CP distribution

Let the moment generating function of the measure μ is equal to

$$\sum_{k=0}^{\infty} \mu_k (\mathrm{e}^{kt} + \mathrm{e}^{-kt}) = \exp\Big\{\frac{\lambda(\mathrm{e}^t + \mathrm{e}^{-t} - 2)}{1 - p(\mathrm{e}^t + \mathrm{e}^{-t})}\Big\}.$$

Calculating the derivative with respect to t we arrive at

$$\begin{aligned} \sum_{k=0}^{\infty} k\mu_k (\mathrm{e}^{kt} - \mathrm{e}^{-kt}) &= \frac{\lambda(1-2p)(\mathrm{e}^t - \mathrm{e}^{-t})}{(1 - p(\mathrm{e}^t + \mathrm{e}^{-t}))^2} \exp\Big\{\frac{\lambda(\mathrm{e}^t + \mathrm{e}^{-t} - 2)}{1 - p(\mathrm{e}^t + \mathrm{e}^{-t})}\Big\} \\ &= \frac{\lambda(1-2p)(\mathrm{e}^t - \mathrm{e}^{-t})}{(1 - p(\mathrm{e}^t + \mathrm{e}^{-t}))^2} \sum_{k=0}^{\infty} \mu_k (\mathrm{e}^{kt} + \mathrm{e}^{-kt}) \end{aligned}$$

or, equivalently,

$$\lambda(1-2p)(\mathrm{e}^t - \mathrm{e}^{-t}) \sum_{k=0}^{\infty} \mu_k (\mathrm{e}^{kt} + \mathrm{e}^{-kt}) = (1 - p(\mathrm{e}^t + \mathrm{e}^{-t}))^2 \sum_{k=0}^{\infty} k\mu_k (\mathrm{e}^{kt} - \mathrm{e}^{-kt}). \tag{115}$$

Let $b_k = \mathrm{e}^{kt} - \mathrm{e}^{-kt}$. Then

$$(\mathrm{e}^t - \mathrm{e}^{-t})(\mathrm{e}^{kt} + \mathrm{e}^{-kt}) = b_{k+1} - b_{k-1}, \quad (\mathrm{e}^t + \mathrm{e}^{-t})(\mathrm{e}^{kt} - \mathrm{e}^{-kt}) = b_{k+1} + b_{k-1}$$

and we rewrite (115) as

$$\lambda(1-2p)\sum_{k=0}^{\infty}\mu_k(b_{k+1} - b_{k-1}) = \sum_{k=0}^{\infty} k\mu_k[(1+2a^2)b_k - 2a(b_{k+1} + b_{k-1}) + p^2(b_{k+2} + b_{k-2})].$$

Assuming that $\mu_k = 0$, for $k < 0$, and changing the order of summation we obtain

$$\begin{aligned} b_1\lambda(1-2p)(2\mu_0 + \mu_2) + b_2\lambda(1-2p)(\mu_1 - \mu_3) + \sum_{k=3}^{\infty} b_k(\mu_{k-1} - \mu_{k+1})\lambda(1-2p) \\ = \quad b_1[\mu_1(1+p^2) - 4p\mu_2 + 3p^2\mu_3] + b_2[-2p\mu_1 + 2\mu_2(1+2p^2) - 6p\mu_3 + 4p^2\mu_4] \\ + \quad \sum_{k=3}^{\infty} b_k[k\mu_k(1+2p^2) - 2p((k-1)\mu_{k-1} + (k+1)\mu_{k+1}) + p^2((k-2)\mu_{k-2} + (k+2)\mu_{k+2}).] \end{aligned}$$

Therefore, for $k = 2, 3, \ldots,$

$$\lambda(1-2p)(\mu_{k-1} - \mu_{k+1}) = k\mu_k(1+2p^2) - 2p((k-1)\mu_{k-1} + (k+1)\mu_{k+1}) + p^2((k-2)\mu_{k-2} + (k+2)\mu_{k+2})$$

and

$$\lambda(1-2p)(2\mu_0 - \mu_1) = \mu_1(1+p^2) - 4p\mu_2 + 3p^2\mu_3.$$

Observing that, if the distribution G has a moment generating function

$$\sum_{k=0}^{\infty}\mu_k(\mathrm{e}^{kt} + \mathrm{e}^{-kt}),$$

then $G\{0\} = 2\mu_0$, $G\{k\} = \mu_k$, $k = 1, 2, \ldots,$ we finally get, for $k = 1, 2, \ldots,$

$$\begin{aligned} \lambda(1-2p)(G\{k-1\} - G\{k+1\}) \quad &= \quad kG\{k\}(1+2p^2) - 2p((k-1)G\{k-1\} + (k+1)G\{k+1\}) \\ &\quad + p^2((k-2)G\{k-2\} + (k+2)G\{k+2\}). \end{aligned}$$