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LIMIT THEOREMS FOR H-VARIATION OF GAUSSIAN PROCESSES

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VILNIAUS UNIVERSITETAS

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Raimondas Malukas
Vilnius
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Contents

Notation

$\mathbb{N}_0$ and $\mathbb{N}$ denote the sets of natural numbers, $\mathbb{N}_0 = \{0, 1, \dots\}$ and $\mathbb{N} = \{1, 2, \dots\}$.
$\mathbb{Z}$ denotes the set of integers, $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$.
$\mathbb{R}$ denotes the set of real numbers.
$[x]$ denotes the integer part of a real number x.
$\lceil x \rceil$ denotes the ceiling of a real number x, i.e. the smallest integer not less than x.
Z denotes a standard normal random variable.
W denotes a standard Brownian motion.
$\Rightarrow$ denotes weak convergence. For a sequence of random variables $(X_n)_{n \in \mathbb{N}_0}$, $X_n \Rightarrow X_0$ means that X_n converges to X_0 in distribution.
$\mathcal{D}[0, 1]$ denotes the space of càdlàg functions on $[0, 1]$.

Introduction

Let $T > 0$ and $G := \{G(t) : t \in [0,T]\}$ be a mean zero Gaussian stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with an incremental standard deviation function $\sigma_G(s,t) := (\mathbf{E}[G(t) - G(s)]^2)^{1/2}$, $s,t \in [0,T]$. Let $(m_n)_{n\in\mathbb{N}}$ be an increasing and unbounded sequence of positive integers. For each integer $n \in \mathbb{N}$, $t_i^n := iT/m_n,\ i = 0,\ldots,m_n$, are equally spaced points of $[0,T]$ making its regular partition, and its mesh $\Delta_n := T/m_n = t_i^n - t_{i-1}^n \to 0$ as $n \to \infty$. Given a function $H : \mathbb{R} \to \mathbb{R}$ we define

$$V_n := V(G,H,m_n) := \sum_{i=1}^{m_n} H\left(\frac{G(t_i^n) - G(t_{i-1}^n)}{\sigma_G(t_i^n, t_{i-1}^n)}\right), \tag{1}$$

called the *H-variation of G*. When $H(x) = |x|^r,\ x \in \mathbb{R}$, for some $r > 0$, V_n is usually called the *r-th power variation of G*. We denote it $V_n^{(r)}$ for distinction. Asymptotic behaviour of V_n as $n \to \infty$ is the subject of this dissertation.

Suppose that for some real valued function ρ on $[0,T]$, $\sigma_G(s,t)$ is close to the values $\rho(|t-s|)$ as $t \to s$ uniformly in $s \in (0,T]$. The exact meaning of this assumption is formulated by Definition 2 below describing the class $\mathcal{LSI}(\rho)$ of Gaussian processes with a local variance ρ. Then one can consider a modified version of V_n, namely

$$\widetilde{V}_n := \widetilde{V}(G,H,\rho,m_n) := \sum_{i=1}^{m_n} H\left(\frac{G(t_i^n) - G(t_{i-1}^n)}{\rho(\Delta_n)}\right), \tag{2}$$

called the *weighted H-variation of G*. When $H(x) = |x|^r,\ x \in \mathbb{R}$, for some $r > 0$, $\widetilde{V}_n$ will be called the *weighted r-th power variation of G*. As in the case of V_n we denote this special case by $\widetilde{V}_n^{(r)}$.

If W is a standard Brownian motion on the time interval $[0,1]$, $m_n = 2^{-n}$ for all $n \in \mathbb{N}$ and $r = 2$, it has been known since Lévy [26] that, when properly normalized, the second power variation (or the quadratic variation)

$$2^{-n}V(W,H,2^n) = \sum_{i=1}^{2^n} \big(W(i2^{-n}) - W((i-1)2^{-n})\big)^2$$

converges to 1 almost surely. This result is viewed as one of the most celebrated theorems of stochastic processes and it has been extended to many directions. An extension to a wide class of Gaussian processes has been given by G. Baxter [8] and E. G. Gladyshev [20]. Gladyshev considered a normalized version of $\widetilde{V}(G,H,\rho,m_n)$ with a particular function ρ. Norvaiša [31] showed that Gladyshev's results are inapplicable to bifractional Brownian motion and subractional Brownian motion, which are recently discovered generalizations of a fractional Brownian motion (see Chapter 4). Norvaiša [31] extended Gladyshev's theorem to a class of Gaussian processes that includes bifractional and subfractional Brownian motion.

R.M. Dudley [16] proved a generalized version of the Lévy's result for noise processes. Moreover, he considered irregular partitions and proved that in order for the almost sure convergence to hold, the mesh of the partitions must go to 0 sufficiently fast, i.e.

$$\max_{i=1,\ldots,m_n} t_i^n - t_{i-1}^n = o(1/\log n), \quad \text{as } n\to\infty.$$

Klein and Giné [18] studied a modified version of $\widetilde{V}_n^{(2)}$ under hypotheses of [8] and [20] and studied irregular partitions as in [16] obtaining the same necessary condition for the mesh of the partitions. Following [18], Malukas [28] extended the results in [31] for general partitions and proved a central limit theorem in his setting. Using the ideas in Marcus and Rosen [30] and Shao [39], Norvaiša [32] further extended the almost sure convergence in [28]. In the case of regular partitions he proved that for a Gaussian process G from the class $\mathcal{LSI}(\rho)$ and under the hypotheses of Corollary 24 in [32] we have

$$\lim_{n\to\infty} \Delta_n \widetilde{V}_n^{(r)} = \mathbf{E}|Z|^r T, \quad \text{almost surely.}$$

Following the proof of Theorem 22 and Corollary 24 in [32] one could prove the same almost sure limit for $\Delta_n V_n^{(r)}$ as $n\to\infty$.

Throughout this dissertation all the random variables are defined on a complete probability space $(\Omega,\mathcal{F},\mathbf{P})$ if not stated otherwise.

For $n \in \mathbb{N}$ and $i = 1, \dots, n$ denote

$$X_{i,n} := \frac{G(t_i^n) - G(t_{i-1}^n)}{\sigma_G(t_i^n, t_{i-1}^n)}. \tag{3}$$

Then for all $n \in \mathbb{N}$ and $i = 1, \dots, n$, $\mathbf{E}X_{i,n} = 0$, $\mathbf{E}X_{i,n}^2 = 1$ and

$$V(G, H, n) = \sum_{i=1}^{n} H(X_{i,n}).$$

Let $H : \mathbb{R} \to \mathbb{R}$ be a function such that $\mathbf{E}H(Z) = 0$ and $\mathbf{E}H^2(Z) < \infty$. The minimal $m \geq 0$, such that $\mathbf{E}H(Z)H_m(Z) \neq 0$, where H_m is the m-th Hermite polynomial defined in (1.1), is called the *Hermite rank* of H (see [41], Definition 2.3). A classical result by Breuer and Major [13] states the following:

Theorem (Breuer-Major)**.** *Let* $(X_i)_{i\in\mathbb{N}}$ *be a centered stationary Gaussian family, with stationary meaning that there exists a function* $\rho : \mathbb{Z} \to \mathbb{R}$ *such that* $\mathbf{E}(X_i X_j) = \rho(i - j)$, $i, j \in \mathbb{N}$. *Assume further that* $\rho(0) = 1$ *and* $H : \mathbb{R} \to \mathbb{R}$ *has Hermite rank* $d \in \mathbb{N}$ *(see Section 1.2 for the definition of Hermite rank) and satisfies* $\mathbf{E}H(Z) = 0$. *Finally, assume that* $\sum_{i\in\mathbb{Z}} |\rho(i)|^d < \infty$. *Then* $\sigma^2 := \operatorname{var}(H(Z)) + 2\sum_{i=1}^{\infty} \operatorname{cov}(H(X_1), H(X_{1+i}))$ *is well-defined and finite. Moreover,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} H(X_i) \Rightarrow \sigma Z, \quad \text{as } n \to \infty.$$

Arcones [3] proved a corresponding theorem in the multivariate case and Giraitis and Surgailis [19] provided some continuous-time analogues of it. It is now usual for any central limit result involving Hermite ranks and series of covariance coefficients to be called a ”Breuer-Major Theorem”.

Guyon and León [21] proved a Breuer-Major theorem for $n^{-1/2}V(G, H, n)$ when G is a stationary process with a regularly varying covariance function, under a hypothesis relating the decay of the covariance function and the Hermite rank of H. Considering power variations of Gaussian and Gaussian related continuous-time processes, functional analogues of the Breuer-Major theorem in [21] were proved in [15], [6] and [7]. See, e.g., [34] and the references therein for some other variations and applications of the Breuer-Major Theorem.

For a Gaussian process G from the class $\mathcal{LSI}(\rho)$, under the hypotheses of Theorem 2.1 of this dissertation we prove a Breuer-Major theorem for $\widetilde{V}_n^{(r)}$. More specifically, we show that

$$\Delta_n^{1/2}\big(\widetilde{V}_n^{(r)} - \mathbf{E}\widetilde{V}_n^{(r)}\big) \Rightarrow \lambda_r Z, \quad \text{as } n \to \infty, \tag{4}$$

where the variance λ_r^2 depends on r and on the asymptotic behaviour as $n \to \infty$ of sums over $k \geq 1$ of powers of fractions

$$\eta_n(k) := \eta(k, \Delta_n) := \frac{[\rho((k+1)\Delta_n)]^2 + [\rho((k-1)\Delta_n)]^2 - 2[\rho(k\Delta_n)]^2}{2[\rho(\Delta_n)]^2} \tag{5}$$

(see (2.5) for the exact form of the variance λ_r^2 and Remark 2.3 following Theorem 2.1 for more specific comments). The same weak limit holds for $V_n^{(r)}$. Theorem 2.1 applies to Gaussian stochastic processes G with stationary increments having a regularly varying incremental variance $R(u) := E[G(s+u)-G(s)]^2$, $u \geq 0$ (subsection 4.2) as well as to subfractional Brownian motion (section 4.3) and bifractional Brownian motion (section 4.4).

For all $n \geq 1$ define functions $Y^n : [0,1] \to \mathbb{R}_+$ with values

$$Y_t^n(G, H) := \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left[H\left(X_{i,n}\right) - \mathbf{E}H(Z)\right] \tag{6}$$

$t \in [0,1]$. Under suitable hypotheses on G and H stated by Theorem 3.10 we also proved that there exists a constant $\lambda_{G,H}$ such that the weak convergence

$$Y^n(G, H) \Rightarrow \lambda_{G,H} W$$

holds in the space $\mathcal{D}[0,1]$ equipped with the Skorokhod topology as $n \to \infty$. Moreover, Theorem 3.10 applies to Gaussian processes G from the class $\mathcal{LSI}(\rho)$ as is shown in Theorem 4.11. This is a direct generalization of the results in [15] and [6].

In 1956 Esséen [17] proved the celebrated Berry-Esséen theorem:

Theorem (Berry-Esséen)**.** *Let* $(X_i)_{i\in\mathbb{N}}$ *be a sequence of independent and identically distributed random variables with* $\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 = 1$ *and* $\mathbf{E}|X_1|^3 < \infty$. *Define*

$$U_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i, \; n \in \mathbb{N}.$$

Then there exists a constant $c \in \mathbb{R}$ *such that for all* $n \in \mathbb{N}$

$$\sup_{x\in\mathbb{R}} |\mathbf{P}(U_n \leq x) - \mathbf{P}(Z \leq x)| \leq cn^{-1/2}.$$

Let $(X_i)_{i\in\mathbb{N}}$ be a centered Gaussian family, $H : \mathbb{R} \to \mathbb{R}$, $\gamma(n) \to 0$ as $n \to \infty$ and denote

$$S_n := \gamma(n) \sum_{i=1}^{n} H(X_i).$$

Suppose $S_n \Rightarrow S$, as $n \to \infty$, and S is a normal random variable. In view of the previous theorem, relations of the type

$$d_{\mathcal{G}}(S_n, S) := \sup_{g\in\mathcal{G}} |\mathbf{E}g(S_n) - \mathbf{E}g(S)| \leq \varphi(n), \quad n \in \mathbb{N},$$

for a suitable class of test functions $\mathcal{G}$ and $\varphi(n) \to 0$ as $n \to \infty$, are called "Berry-Esséen (type) bounds". By choosing particular classes $\mathcal{G}$ one obtains the Wasserstein, Kolmogorov and total variation distances between random variables respectively, denoted by d_W, d_{Kol}, d_{TV}. Specifically, one has for random variables X and Y

$$d_W(X,Y) := \sup\left\{ |\mathbf{E}g(X) - \mathbf{E}g(Y)| : \sup_{x\neq y} \frac{g(x) - g(y)}{x - y} \leq 1 \right\}, \tag{7}$$

$$d_{Kol}(X,Y) := \sup_{x\in\mathbb{R}} |\mathbf{P}(X \leq x) - \mathbf{P}(Y \leq x)|, \tag{8}$$

$$d_{TV}(X,Y) := \sup_{A\in\mathcal{B}(\mathbb{R})} |\mathbf{P}(X \in A) - \mathbf{P}(Y \in A)|. \tag{9}$$

Let $\mathbf{D} := \{d_W, d_{Kol}, d_{TV}\}$.

Berry-Esséen type bounds for the quadratic variation of fractional, bifractional, subfractional and weighted fractional Brownian motions have been obtained by Nourdin and Peccati [33] and Biermé et. al [10], Aazizi and Es-Sebaiy [1], Tudor [42] and Shen et al. [40] respectively. Nourdin et al. [34] considered H-variations of stationary Gaussian fields and proved a Berry-Esséen bound in their setting.

In this dissertation we prove a Berry-Esséen type bound for V_n. In particular, as a special case of Theorem 3.6 below and under its hypotheses we obtain for some $c \in \mathbb{R}$, all $n \geq 1$ and $d \in \mathbf{D}$

$$d^2\left(\frac{V_n - \mathbf{E}V_n}{\sqrt{\operatorname{var}(V_n)}}, Z\right) \leq \frac{c}{[\operatorname{var}(V_n))]^2} \max_{1\leq i\leq n} \sum_{j=1}^{n} r_n^2(i,j) \sum_{i,k,l=1}^{n} |r_n(k,l)||r_n(i,k)|,$$

where for all $n \geq 1$ and $(i,j) \in \{1,\ldots,n\}^2$, $r_n(i,j) := \mathbf{E}X_{i,n}X_{j,n}$.

In the cases of bifractional and subfractional Brownian motions we prove in Corollaries 4.6 and 4.9 that the Berry-Esséen bounds for more general H-variations is of the same order as for the quadratic variations or (more generally) the H_q-variations (for some values of the parameters; see [33]). Thus we extend the results in [42] and [1] and partially extend those in [33].

Apart from the independent interest in the problem of the limit behaviour of H-variations of Gaussian processes, some applications to parameter estimation of specific processes have been considered. Namely, a strongly consistent estimator of the covariance function of the process has been constructed in [20]. Similar estimators were also constructed in [24], [25], [28] among others. The estimator in [25] was given along with the convergence rate to the real value of the parameter. As shown in [21], the central limit theorem does not hold for fractional Brownian motion with the Hurst $H > 3/4$. To overcome this problem, generalized quadratic variations have been studied in [24], [14], [9] and others. Another type of applications is that of the estimation of the integrated volatility or the integrated variance in econometrics. For that the quadratic, power and multi-power variations have been used in [2], [4], [5], [7] among others. See [38] for an overview and some particular results on the topic.

The content presented in this thesis has been created and prepared by the author of the thesis together with his co-authors. The results obtained in the dissertation are original and all of them can be considered as new. The proofs of our theorems rely heavily on some recent results in the context of Malliavin calculus, namely in [23] and [33]. See Chapter 1 for these and related results. Chapter 2 contains the proof of our Breuer-Major theorem. It is based on the paper [29] written by the author of the thesis together with R. Norvaiša. In Chapter 3 we present the Berry-Esséen bound and a functional version of the Breuer-Major theorem. These result are from the paper [27] written by the author of the thesis and accepted for publication. In Chapter 4 we apply our theorems 2.1, 3.6 and 3.10 to particular Gaussian processes. Finally, in Chapter 5 we provide the conclusions.

Chapter 1

Background

1.1 Results from Malliavin calculus

Recent development of Malliavin analysis enabled to prove some convenient limit results on the Wiener space. Namely, the so called "Fourth Moment Theorem" discovered by Nualart and Peccati [36] which states that for a sequence of multiple stochastic integrals of a fixed order the convergence in distribution to a standard normal random variable is equivalent to convergence of the fourth moment. This result was extended for the multidimensional case by Peccati and Tudor [37]. Later, combining Stein's method and Malliavin calculus quantitative bounds for the Fourth Moment Theorem were given in [33]. Since then, these results have found many applications and this thesis is not an exception.

In this section we briefly recall the notions and results from Malliavin calculus we use to prove our theorems. A standard reference on Malliavin calculus is [35].

1.1.1 Wiener chaos decomposition and generalized multiple Wiener integrals

Let $\mathfrak{H}$ be a separable Hilbert space and a stochastic process $X := \{X(h) : h \in \mathfrak{H}\}$ be an isonormal Gaussian process over $\mathfrak{H}$ meaning that X is a centered Gaussian family indexed by the elements of $\mathfrak{H}$ and satisfying

$$\mathbf{E}X(h)X(g) = \langle h, g\rangle_{\mathfrak{H}},$$

for all $h, g \in \mathfrak{H}$.

Let $H_m,\ m \in \mathbb{N}$, denote the mth Hermite polynomial defined as

$$H_m(x) := (-1)^m \mathrm{e}^{\frac{x^2}{2}} \frac{d^m}{dx^m} \mathrm{e}^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}, \tag{1.1}$$

and $H_0 \equiv 1$.

For each $m \in \mathbb{N}$, let $\mathcal{H}_m$ be the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbf{P})$ generated by the random variables $\{H_m(X(h)) : h \in \mathfrak{H}, ||h||_{\mathfrak{H}} = 1\}$. The space $\mathcal{H}_m$ is called the mth Wiener chaos. It is well-known, that the space $L^2(\Omega, \sigma(X), \mathbf{P})$, where $\sigma(X)$ is the σ-algebra generated by X, can be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{H}_m,\ m = 0, 1, \ldots$ (e.g. Theorem 1.1.1 in [35]).

Given a separable Hilbert space $\mathfrak{H}$, for every $m \geq 2$, let $\mathfrak{H}^{\otimes m}$ and $\mathfrak{H}^{\odot m}$ be, respectively, the mth tensor product and the mth symmetric tensor product of $\mathfrak{H}$. The inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H} \otimes \mathfrak{H}}$ on the tensor product $\mathfrak{H} \otimes \mathfrak{H}$ is related to the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ by the equality

$$\langle g_1 \otimes h_1, g_2 \otimes h_2 \rangle_{\mathfrak{H} \otimes \mathfrak{H}} = \langle g_1, g_2 \rangle_{\mathfrak{H}} \langle h_1, h_2 \rangle_{\mathfrak{H}} \tag{1.2}$$

For each $m \geq 2$ one can define the mapping I_m from $\mathfrak{H}^{\odot m}$ to $\mathcal{H}_m$, called the abstract multiple Wiener integral, such that

$$I_m(h^{\otimes m}) = H_m(X(h)) \tag{1.3}$$

for each $h \in \mathfrak{H}$ with $||h||_{\mathfrak{H}} = 1$ (see [35], Chapter 1). For any $f \in \mathfrak{H}^{\odot m}$ (where $\mathfrak{H}^{\odot m}$ is equipped with the norm $\sqrt{m!}|| \cdot ||_{\mathfrak{H}^{\otimes m}}$) I_m satisfies

$$\mathbf{E}(I_m(f))^2 = m! ||f||^2_{\mathfrak{H}^{\otimes m}}. \tag{1.4}$$

Let $Y \in L^2(\sigma(X)) := L^2(\Omega, \sigma(X), \mathbf{P})$ satisfy

$$Y = \sum_{m=0}^{\infty} I_m(f_m), \tag{1.5}$$

where $I_0(f_0) := \mathbf{E}Y$ and $f_m \in \mathfrak{H}^{\odot m}$ for all $m \geq 2$. For every $m \geq 1$ let J_m be the orthogonal projection operator on the mth Wiener chaos $\mathcal{H}_m$ that is, $J_m Y = I_m(f_m)$.

1.1.2 Contractions

We need to recall a contraction of elements of tensor products of Hilbert spaces. Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. The collection $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : i_1, \ldots, i_n \geq 1\}$ is an orthonormal basis of $H^{\otimes n}$.

Let $1 \leq r \leq q$ be integers. Suppose that elements $f \in H^{\otimes r}$ and $g \in H^{\otimes q}$ have Fourier series expansions

$$\begin{aligned} f &= \sum_{j_1,\ldots,j_r=1}^{\infty} a(j_1, \ldots, j_r) e_{j_1} \otimes \cdots \otimes e_{j_r}, \\ g &= \sum_{k_1,\ldots,k_q=1}^{\infty} b(k_1, \ldots, k_q) e_{k_1} \otimes \cdots \otimes e_{k_q}. \end{aligned}$$

For every $p = 1, \ldots, r$ the *contraction* of order p of f and g is defined to be the element of $H^{\otimes r+q-2p}$ given by

$$\begin{aligned} f \otimes_p g := \sum_{z_1,\ldots,z_{r+q-2p}=1}^{\infty} \sum_{l_1,\ldots,l_p=1}^{\infty} & a(l_1, \ldots, l_p, z_1, \ldots, z_{r-p}) \\ & \times b(l_1, \ldots, l_p, z_{r-p+1}, \ldots, z_{r+q-2p}) e_{z_1} \otimes \cdots \otimes e_{z_{r+q-2p}}. \end{aligned}$$

We apply this formula to the case when $f = f_1 \otimes \ldots \otimes f_r \in H^{\otimes r}$ and $g = g_1 \otimes \ldots \otimes g_q \in H^{\otimes q}$. Using (1.2) the Fourier coefficients of f can be represented as

$$a(j_1, \ldots, j_r) = \langle f_1 \otimes \cdots \otimes f_r, e_{j_1} \otimes \cdots \otimes e_{j_r} \rangle_{H^{\otimes r}} = \prod_{i=1}^{r} \langle f_i, e_{j_i} \rangle_H,$$

and similarly the Fourier coefficients of g represented as $b(k_1, \ldots, k_q) = \prod_{i=1}^{q} \langle g_i, e_{k_i} \rangle_H$. Due to the linearity of the tensor product we have the contraction

$$\begin{aligned} f \otimes_p g = & \sum_{z_1,\ldots,z_{r+q-2p}=1}^{\infty} \sum_{l_1,\ldots,l_p=1}^{\infty} \prod_{i=1}^{p} \langle f_i, e_{l_i} \rangle_H \langle g_i, e_{l_i} \rangle_H \langle f_{p+1}, e_{z_1} \rangle_H \ldots \langle f_r, e_{z_{r-p}} \rangle_H \\ & \times \langle g_{p+1}, e_{z_{r-p+1}} \rangle_H \ldots \langle g_q, e_{z_{r+q-2p}} \rangle_H e_{z_1} \otimes \cdots \otimes e_{z_{r+q-2p}} \\ = & \prod_{i=1}^{p} \sum_{l_i=1}^{\infty} \langle f_i, e_{l_i} \langle g_i, e_{l_i} \rangle_H \rangle_H \sum_{z_1=1}^{\infty} \langle f_{p+1}, e_{z_1} \rangle_H e_{z_1} \otimes \cdots \otimes \sum_{z_{r-p}=1}^{\infty} \langle f_r, e_{z_{r-p}} \rangle_H e_{z_{r-p}} \\ & \otimes \sum_{z_{r-p+1}=1}^{\infty} \langle g_{p+1}, e_{z_{r-p+1}} \rangle_H e_{z_{r-p+1}} \otimes \cdots \otimes \sum_{z_{r+q-2p}=1}^{\infty} \langle g_q, e_{z_{r+q-2p}} \rangle_H e_{z_{r+q-2p}} \end{aligned}$$

$$= \prod_{i=1}^{p} \langle f_i, g_i \rangle_H f_{p+1} \otimes \cdots \otimes f_r \otimes g_{p+1} \otimes \cdots \otimes g_q.$$

Finally, let $m, M \in \mathbb{N}$ and $f_i^{\otimes m} \in H^{\otimes m}$, $i = 1, \dots, M$. For each $p \in \{1, \dots, m\}$ taking $r = q = m - p$ in the preceding formulas the following contraction of order p has the representation

$$\sum_{i=1}^{M} f_i^{\otimes m} \otimes_p \sum_{j=1}^{M} f_j^{\otimes m} = \sum_{i,j=1}^{M} f_i^{\otimes m} \otimes_p f_j^{\otimes m} = \sum_{i,j=1}^{M} \left(\langle f_i, f_j \rangle_H\right)^p f_i^{\otimes m-p} \otimes f_j^{\otimes m-p}. \tag{1.6}$$

1.1.3 Operators D and L^{-1}

Let $C_p^\infty(\mathbb{R}^n)$ denote the class of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and its derivatives have polynomial growth and let $\mathcal{S}(X) \subset L^2(\sigma(X))$ be a class of random variables of the type

$$Y = f(X(h_1), \dots, X(h_n)), \tag{1.7}$$

where $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in \mathfrak{H}$ for all $i = 1, \dots, n$. Let $Y \in \mathcal{S}(X)$ be as in (1.7). The (Malliavin) derivative DY of Y is defined to be the $\mathfrak{H}$-valued random element given by

$$DY := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(X(h_1), \dots, X(h_n)) h_i.$$

Definition 1. Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces, and $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ be a linear operator, defined on some subspace $\mathcal{D}(T) \subset \mathfrak{H}_1$. T is said to be *closable* if, given an arbitrary $x \in \mathfrak{H}_1$ a limit point of $\mathcal{D}(T)$, for all the approximating sequences $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ of x, such that Tx_n has a limit, such a limit is the same.

Proposition 1.1 (Proposition 1.2.1, [35])**.** *The operator $D : \mathcal{S}(X) \to L^2(\sigma(X), \mathfrak{H})$ is closable.*

The domain of the operator D in $L^2(\sigma(X))$, usually denoted by $\mathbb{D}^{1,2}$, is the closure of the class $\mathcal{S}(X)$ with respect to the norm

$$||Y||_{1,2} := \left[\mathbf{E}(Y^2) + \mathbf{E}\left(||DY||_{\mathfrak{H}}^2\right)\right]^{1/2}.$$

Proposition 1.2 (Proposition 1.2.2, [35]). *A random variable* $Y \in L^2(\sigma(X))$ *having representation* (1.5) *belongs to* $\mathbb{D}^{1,2}$ *if and only if*

$$\sum_{m=1}^{\infty} m m! ||f_m||^2_{\mathfrak{H}^{\otimes m}} < \infty.$$

Note that Propositions 1.1 and 1.2 imply that a random variable $Y \in \mathbb{D}^{1,2}$ having representation (1.5) has a Malliavin derivative

$$DY = \sum_{m=0}^{\infty} DI_m(f_m). \tag{1.8}$$

Let $Y \in L^2(\Omega)$. The operator L is defined as

$$LY := -\sum_{m=1} mJ_m Y,$$

provided this series converges in $L^2(\Omega)$, which is true, e.g., for all $Y \in \mathbb{D}^{1,2}$. Operator L is the *infinitesimal generator of the Ornstein-Uhlenbeck semigroup* (see [35], Proposition 1.4.2). For any $Y \in L^2(\Omega)$, we define the *pseudo-inverse* operator of L as

$$L^{-1}Y := -\sum_{m=1} \frac{1}{m} J_m Y.$$

One can check directly that for all $Y \in L^2(\Omega)$, $LL^{-1}Y = Y - \mathbf{E}Y$ (see [34], p.13). Assume that $Y \in L^2(\sigma(X))$ having representation (1.5) satisfies $\mathbf{E}Y = 0$. Then

$$L^{-1}Y = \sum_{m=1}^{\infty} -\frac{1}{m} I_m(f_m). \tag{1.9}$$

1.1.4 Central limit theorems

We present here a central limit theorem for a sequence of random variables which admit a Wiener chaos representation used in the proof of Theorem 2.1. Theorem 3 and Remark 1 in [23] give

Theorem 1.3 (Y. Hu and D. Nualart). *Let* $\mathfrak{H}$ *be a real separable infinite-dimensional Hilbert space and let* $I_m\colon \mathfrak{H}^{\odot m} \to \mathfrak{H}_m$, $m \geq 1$, *be the abstract multiple Wiener integrals. Let* $(F_n)_{n\geq 1}$ *be a sequence of square integrable and centered random variables* *with the Wiener chaos expansions*

$$F_n = \sum_{m=1}^{\infty} I_m(f_{m,n}).$$

for some $f_{m,n} \in \mathfrak{H}^{\odot m}$. *Suppose that*

(i) for all $m \geq 1$ *and* $n \geq 1$, $m!||f_{m,n}||^2_{\mathfrak{H}^{\otimes m}} \leq \delta_m$, *where* $\sum_{m=1}^{\infty} \delta_m < \infty$;

(ii) for every $m \geq 1$, $\lim_{n\to\infty} m!||f_{m,n}||^2_{\mathfrak{H}^{\otimes m}} = \sigma_m^2$;

(iii) for all $m \geq 2$, $p = 1, \ldots, m-1$, $\lim_{n\to\infty} ||f_{m,n} \otimes_p f_{m,n}||^2_{\mathfrak{H}^{\otimes 2(m-p)}} = 0$.

Then F_n *converges in distribution as* $n \to \infty$ *to a mean zero normal random variable with the variance* $\sigma^2 = \sum_{m=1}^{\infty} \sigma_m^2$.

Next result is Theorem 5 from [6] and is used in the proof of Theorem 3.10.

Theorem 1.4 (Barndorff-Nielsen et al.). *Let* $Y_n = (Y_n^1, \ldots, Y_n^d)$ *be a* d*-dimensional process which has a chaos representation*

$$Y_n^k = \sum_{m=1}^{\infty} I_m(f_{m,n}^k), \quad k = 1, \ldots, d,$$

with $f_{m,n}^k \in \mathfrak{H}^{\otimes m}$ *for all* k *and* n. *Suppose the following conditions hold:*

(i) *for any* $k = 1, \ldots, d$ *we have*

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{m=N+1}^{\infty} m!||f_{m,n}^k||^2_{\mathfrak{H}^{\otimes m}} = 0;$$

(ii) *for any* $m \geq 1$ *and* $k, l = 1, \ldots, d$ *we have constants* C_{kl}^m *such that*

$$\lim_{n\to\infty} m!||f_{m,n}^k||^2_{\mathfrak{H}^{\otimes m}} = C_{kk}^m,$$
$$\lim_{n\to\infty} \mathbf{E}[I_m(f_{m,n}^k) I_m(f_{m,n}^l)] = C_{kl}^m, \quad k \neq l,$$

and the matrix $C^m = (C_{kl}^m)_{1\leq k,l\leq d}$ *is positive definite for all* m;

(iii) $\sum_{m=1}^{\infty} C^m = C \in \mathbb{R}^{d\times d}$;

(iv) *for any* $m \geq 1$, $k = 1, \dots, d$ *and* $p = 1, \dots, m-1$

$$\lim_{n\to\infty} ||f^k_{m,n} \otimes_p f^k_{m,n}||^2_{\mathfrak{H}^{\otimes 2(m-p)}} = 0.$$

Then we have

$$Y_n \Rightarrow \xi, \quad \xi \sim N_d(0, C).$$

1.1.5 Nourdin-Peccati bound

Using Stein's method and Malliavin calculus Nourdin and Peccati [33] proved the following theorem which plays a central role in the proof of Theorem 3.6.

Theorem 1.5 (Theorem 3.1, [33])**.** *Let* $Y \in \mathbb{D}^{1,2}$ *be such that* $\mathbf{E}Y = 0$ *and* $d \in \mathbf{D}$. *If* $d \neq d_W$*, assume also that the law of* Y *is absolutely continuous. Then*

$$d(Y, Z) \leq 2\mathbf{E}\left[(1 - \langle DY, DL^{-1}Y\rangle_{\mathfrak{H}})^2\right]^{1/2}.$$

1.2 Technical lemmas

A Gaussian vector $(Z_1, \dots, Z_p)$, $p \geq 2$, is said to be *standard Gaussian* if $\mathbf{E}Z_i = 0$ and $\mathbf{E}Z_i^2 = 1$ for all $i = 1, \dots, p$.

Lemma 1.6 (Lemma 3.1, [41])**.** *Let* $p \geq 2$ *and suppose* $(Z_1, \dots, Z_p)$ *is standard Gaussian. Then*

$$\mathbf{E}|H_{m_1}(Z_1)\dots H_{m_p}(Z_p)| \leq \prod_{j=1}^{p}(p-1)^{\frac{m_j}{2}}\sqrt{m_j!}.$$

Next lemma is similar to the part (i) of Proposition 3.1 in [41]. See Definition 3 for the definition of the classes $\mathcal{F}_q$.

Lemma 1.7. *Let* $q \in \mathbb{N}_0$, $F \in \mathcal{F}_q$ *and denote* $a_m := \mathbf{E}F(Z)H_m(Z)$, $m \geq 2$. *Let* $(Z_1, \dots, Z_4)$ *be standard Gaussian,* $(q_1, \dots, q_4) \in \mathbb{R}^4$ *be such that* $q_i \leq q$ *for all* $i = 1, \dots, 4$, *and* $(u_1, \dots, u_4) \in \{0, 1\}^4$ *for all* $i = 1, \dots, 4$. *Then*

$$\sum_{m_1,\dots,m_4=2}^{\infty} \left| \frac{a_{m_1}\dots a_{m_4} m_1^{q_1}\dots m_4^{q_4}}{m_1!\dots m_4!}\mathbf{E}H_{m_1-u_1}(Z_1)\dots H_{m_4-u_4}(Z_4)\right| < \infty.$$

Proof. By Lemma 1.6 and the fact that $F \in \mathcal{F}_q$ we have

$$\sum_{m_1,\ldots,m_4=2}^{\infty} \left| \frac{a_{m_1} \ldots a_{m_4} m_1^{q_1} \ldots m_4^{q_4}}{m_1! \ldots m_4!} \mathbf{E} H_{m_1-u_1}(Z_1) \ldots H_{m_4-u_4}(Z_4) \right|$$
$$\leq \prod_{j=1}^{4} \sum_{m=2}^{\infty} \frac{|a_m| m^{q_j}}{m!} 3^{\frac{m-u_j}{2}} \sqrt{(m-u_j)!} \leq \left(\sum_{m=2}^{\infty} \frac{|a_m| m^q}{\sqrt{m!}} 3^{\frac{m}{2}} \right)^4 < \infty,$$

which ends the proof. □

Let $n \geq 2$ and $X := (X_1, \ldots, X_n)$ be standard Gaussian with a covariance matrix $(r(i,j))_{1 \leq i,j \leq n}$. Let $2 \leq p \leq n$. To prove our theorems we will need a lemma for the expression of the mixed moments $\mathbf{E} H_{m_1}(X_{i_1}) \ldots H_{m_p}(X_{i_p})$, $(m_1, \ldots, m_p) \in I \subset \mathbb{N}^p$, $(i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$. We formulate the lemma using notation presented below which will also be used in our proofs of the main results in Chapter 3.

Define a function $q : I \to \mathbb{R}_+$ with values

$$q(\mathbf{m}) := \frac{1}{2} \sum_{j=1}^{p} m_j \tag{1.10}$$

for $\mathbf{m} = (m_1, \ldots, m_p) \in I$. For $\mathbf{m} \in I$ and $\mathbf{i} = (i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$ denote

$$\mathcal{E}(\mathbf{m}, \mathbf{i}) := \mathbf{E} H_{m_1}(X_{i_1}) \ldots H_{m_p}(X_{i_p}). \tag{1.11}$$

Denote $\mathbf{1} := (1, \ldots, 1) \in \mathbb{N}^p$. For an $x = (x_1, \ldots, x_p) \in \mathbb{N}^p$ let $x! := x_1! \ldots x_p!$. Let Γ be the gamma function. For $\mathbf{m} \in I$ denote

$$\mathcal{C}(\mathbf{m}) := \frac{\mathbf{m}!}{2^{q(\mathbf{m})} \Gamma(q(\mathbf{m}))} \chi \left\{ \max_{i=1,\ldots,p} m_i \leq q(\mathbf{m}) \text{ and } q(\mathbf{m}) \in \mathbb{N} \right\}. \tag{1.12}$$

Given an $\mathbf{m} \in I$, a function $\tau : \{1, \ldots, 2q(\mathbf{m})\} \to \{1, \ldots, p\}$ and any $\mathbf{i} = (i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$ define

$$R(\tau, \mathbf{i}) := r\big(i_{\tau(1)}, i_{\tau(2)}\big) \ldots r\big(i_{\tau(2q(\mathbf{m})-1)}, i_{\tau(2q(\mathbf{m}))}\big). \tag{1.13}$$

Finally, for a finite set A, let $|A|$ denote the number of its elements.

Lemma 1.8 (Lemma 3.2, [41])**.** *Let $n \geq 2$ and $X := (X_1, \ldots, X_n)$ be standard Gaussian with a covariance matrix $(r(i,j))_{1 \leq i,j \leq n}$. Let $2 \leq p \leq n$, $I \subset \mathbb{N}^p$, $\mathbf{m} =$*

$(m_1,\dots,m_p) \in I$ *and* $\mathbf{i} = (i_1,\dots,i_p) \in \{1,\dots,n\}^p$. *Then*

$$\mathcal{E}(\mathbf{m},\mathbf{i}) = \mathcal{C}(\mathbf{m}) \sum_{\tau \in \mathcal{T}(\mathbf{m})} R(\tau,\mathbf{i})$$

where $\mathcal{T}(\mathbf{m})$ *is the collection of functions* $\tau : \{1,\dots,2q(\mathbf{m})\} \to \{1,\dots,p\}$ *such that*

(*i*) $\tau(2v-1) \neq \tau(2v)$, $v = 1,\dots,q(\mathbf{m})$;

(*ii*) $|\tau^{-1}(\{j\})| = m_j, \ j = 1,\dots,p$.

As a special case of Lemma 3 one obtains the well-known fact that for any standard Gaussian vector (U,Y) Hermite polynomials satisfy

$$\mathbf{E} H_n(U) H_m(Y) = m!\delta_{nm} (\mathbf{E}UY)^m, \tag{1.14}$$

where δ is the Kronecker delta function.

1.3 Notation

Let $G = \{G(t)\colon\ t \in [0,T]\}$ be a mean zero Gaussian process. The covariance function of G is a function $\Gamma_G\colon\ [0,T]^2 \to \mathbb{R}$ with values

$$\Gamma_G(s,t) = \mathbf{E}G(t)G(s), \quad (s,t) \in [0,T]^2. \tag{1.15}$$

Let $\pi = \{t_i\colon\ i = 0,\dots,m\}$ be a partition of $[0,T]$, i.e. a set of numbers t_i such that $0 = t_0 < t_1 < \cdots < t_m = T$. For a function $F : [0,T] \to \mathbb{R}$, its increment over the interval $[t_{i-1},t_i]$ is $\Delta^{\pi}_{t_i} F := F(t_i) - F(t_{i-1})$. For a two variable function $f : [0,T]^2 \to \mathbb{R}$, its double increment over the rectangle $[t_{i-1},t_i] \times [t_{j-1},t_j]$ is

$$\square^{\pi}_{t_i,t_j} f := f(t_i,t_j) - f(t_i,t_{j-1}) - f(t_{i-1},t_j) + f(t_{i-1},t_{j-1}). \tag{1.16}$$

In this dissertation only regular partitions of $[0,T]$ are considered. Namely, given an increasing sequence of positive integers $(m_n)_{n\in\mathbb{N}}$, for each $n \in \mathbb{N}$, the regular partition π_n is defined by equally spaced points $t^n_i = iT/m_n$, $i = 0,\dots,m_n$, and so its mesh is $\Delta_n := \max\{t^n_i - t^n_{i-1}\colon\ i = 1,\dots,m_n\} = T/m_n$. In this case we write $\Delta^n_i F := \Delta^{\pi_n}_{t_i} F$ and $\square^n_{i,j} f := \square^{\pi_n}_{t^n_i,t^n_j} f$.

For sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we will say that $a_n \trianglelefteq b_n$ if $\sup_{n\in\mathbb{N}} |a_n|/|b_n| < \infty$. We will also say that $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. The following relation will be useful

$$\sum_{k=1}^{n} k^{\alpha} \sim \begin{cases} \frac{n^{\alpha+1}}{\alpha+1} \trianglelefteq n^{\alpha+1} & \text{if } \alpha > -1, \\ \log n & \text{if } \alpha = -1, \end{cases} \tag{1.17}$$

as $n \to \infty$.

Chapter 2

A central limit theorem

In this chapter we prove a central limit theorem for a weighted r-th power variation of a Gaussian process having a local variance.

2.1 Formulation of the main result

The class of Gaussian processes considered in this chapter is defined as follows.

Definition 2. Let $T > 0$ and let $R[0,T]$ be a set of functions $\rho\colon\ [0,T] \to \mathbb{R}_+$ such that $\rho(0) = 0$, ρ is continuous at zero, and for each $\delta \in (0,T)$,

$$0 < \inf\{\rho(u)\colon\ u \in [\delta,T]\} \leq \sup\{\rho(u)\colon\ u \in [\delta,T]\} < \infty. \tag{2.1}$$

Let $G = \{G(t)\colon\ t \in [0,T]\}$ be a mean zero Gaussian stochastic process with the incremental variance function σ_G^2 defined on $[0,T]^2 := [0,T] \times [0,T]$ with values

$$\sigma_G^2(s,t) := E[G(t) - G(s)]^2, \qquad (s,t) \in [0,T]^2.$$

We say that G has a *local variance* if there is a function $\rho \in R[0,T]$ such that $(A1)$ and $(A2)$ hold, where

$(A1)$ there is a finite constant L such that for all $(s,t) \in [0,T]^2$

$$\sigma_G(s,t) \leq L\rho(|t-s|);$$

$(A2)$ for each $\epsilon \in (0,T)$

$$\lim_{\delta\downarrow 0}\sup\Big\{\Big|\frac{\sigma_G(s,s+h)}{\rho(h)}-1\Big|\colon\ s\in[\epsilon,T),\ h\in(0,\delta\wedge(T-s)]\Big\}=0. \tag{2.2}$$

We say that G has locally stationary increments if G has a local variance with some $\rho \in R[0,T]$ and we write $G \in \mathcal{LSI}(\rho)$.

Recall the notation in Section 1.3. We are now ready to formulate the main result of this chapter.

Theorem 2.1. *Let $r > 0$, let $T > 0$ and let $G = \{G(t)\colon\ t \in [0,T]\} \in \mathcal{LSI}(\rho)$ with some $\rho \in R[0,T]$. Let $(m_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integers such that $\Delta_n = T/m_n \to 0$ as $n \to \infty$. Suppose that*

(a) *there is a constant $C_1 > 0$ such that $\sigma_G(s,t) \geq C_1\rho(|t-s|)$ for each $(s,t) \in [0,T]^2$;*

(b) *for every integer $m \geq 2$ there is a real number Ψ_m such that the following limit exists and the equality*

$$\lim_{n\to\infty}\sum_{k=1}^{y_n}\big[\eta(k,\Delta_n)\big]^m=\Psi_m \tag{2.3}$$

holds for every increasing and unbounded sequence of positive integers $(y_n)_{n\in\mathbb{N}}$ with values $y_n \leq m_n - 2$ for each $n \in \mathbb{N}$ (the function η is defined by (5));

(c) *for every integer $m \geq 2$ the following limit exists and the equality*

$$\lim_{n\to\infty}\frac{\Delta_n}{[\rho(\Delta_n)]^{2m}}\sum_{i,j=2}^{m_n}\big|\Box^n_{i,j}[\Gamma_G-2^{-1}\tilde{\rho}]\big|^m=0$$

holds, where $\tilde{\rho}(s,t) := -[\rho(|t-s|)]^2$ for $(s,t) \in [0,T]^2$.

Then the central limit theorem

$$\Delta_n^{1/2}\big(\widetilde{V}_n^{(r)}-\mathbf{E}\widetilde{V}_n^{(r)}\big)=\sqrt{\Delta_n}\sum_{i=1}^{m_n}\left[\left(\frac{|\Delta_i^nG|}{\rho(\Delta_n)}\right)^r-\mathbf{E}\left(\frac{|\Delta_i^nG|}{\rho(\Delta_n)}\right)^r\right]\Rightarrow\lambda_rZ, \tag{2.4}$$

as $n \to \infty$, holds, where the variance

$$\lambda_r^2 := T \sum_{m=2}^{\infty} a_{rm}^2 m!(1 + 2\Psi_m), \tag{2.5}$$

here Ψ_m are defined by (2.3) and the coefficients a_{rm} are given by

$$a_m := a_{rm} := (m!)^{-1}\mathbf{E}[(|Z|^r - \mathbf{E}|Z|^r)H_m(Z)], \tag{2.6}$$

with H_m, $m \geq 2$, being the Hermite polynomials (see (1.1)).

Remark 2.2. In the case $r = 2$ the variance (2.5) can be given a simpler form. Let $c_r := \mathbf{E}|Z|^r$ for any $r > 0$. Note that $Z^2 - \mathbf{E}Z^2 = H_2(Z)$ and $c_4 = 3$. By (1.14), for $m \geq 2$

$$a_{2m} = \frac{1}{2}\mathbf{E}[H_2(Z)H_m(Z)] = \frac{1}{2}(c_4 - 1)\delta_{2m} = \delta_{2m},$$

where δ is the Kronecker delta function. Therefore $\lambda_2 = 2T(1 + 2\Psi_2)$.

Remark 2.3. We explain some notation to make sense of hypotheses (b) and (c) of the preceding theorem. Using (1.16) one can check that

$$\square_{i,j}^n \Gamma_G = \mathbf{E}[\Delta_i^n G \Delta_j^n G] \quad \text{and} \quad \frac{\square_{i,j}^n \tilde{\rho}}{2[\rho(\Delta_n)]^2} = \eta(|i-j|, \Delta_n) \tag{2.7}$$

for each $i, j \in \{1, \dots, m_n\}$ and $n \in \mathbb{N}$; here and elswhere $\eta(0, \Delta_n) := 1$. For each such i, j, n, letting

$$z_n(i,j) := \frac{\mathbf{E}[\Delta_i^n G \Delta_j^n G]}{[\rho(\Delta_n)]^2} - \eta(|i-j|, \Delta_n) = \frac{\square_{i,j}^n [\Gamma_G - 2^{-1}\tilde{\rho}]}{[\rho(\Delta_n)]^2} \tag{2.8}$$

it follows that

$$\frac{\mathbf{E}[\Delta_i^n G \Delta_j^n G]}{[\rho(\Delta_n)]^2} = \eta(|i-j|, \Delta_n) + z_n(i,j). \tag{2.9}$$

Hypothesis (b) involves the first term of the right side of the preceding decomposition and describes the variance (2.5) via the limits Ψ_m. Hypothesis (c) involves the second term of the right side of the same decomposition and requires its negligibility. Hypothesis (c) is trivially satisfied with ρ being the incremental variance of the Gaussian process G when its increments are stationary.

Remark 2.4. If G is a Gaussian process with possibly non-stationary increments then the sum $\widetilde{V}_n^{(r)}$ may be different from the r-th power variation $V_n^{(r)}$. Under the hypotheses of Corollary 24 in [32], we have

$$V_n^{(r)} \to c_r T, \quad \text{almost surely.}$$

Also, under the hypotheses of Theorem 2.1, the asymptotic relation

$$\Delta_n^{-1/2}\big(\Delta_n V_n^{(r)} - c_r T\big) \Rightarrow \lambda_r Z, \quad \text{as } n \to \infty, \tag{2.10}$$

holds, where the variance λ_r^2 is defined by (2.5). The proofs in both cases are the same as the proofs for the weighted r-th power variation sums $\widetilde{V}_n^{(r)}$.

Remark 2.5. The proof of Theorem 2.1 given in the next section can be used to prove the asymptotic relation (2.10) for the general case of V_n. Let μ be a standard Gaussian measure on $\mathbb{R}$, and let $H : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\mathbf{E}H^2(Z) < \infty$ and H has Hermite rank $k \geq 2$. Then H has the unique expansion

$$H = \sum_{m \geq k} a_{H,m} H_m,$$

with respect to Hermite polynomials H_m, $m \in \mathbb{N}$. Here $a_{H,m} = (m!)^{-1}\mathbf{E}[(H(Z) - \mathbf{E}H(Z))H_m(Z)]$. Thus, under the hypotheses of Theorem 2.1, we have

$$\Delta_n^{-1/2}\left(\Delta_n V_n - \mathbf{E}H(Z)T\right) \Rightarrow \lambda_H Z, \quad \text{as } n \to \infty,$$

where the variance

$$\lambda_H^2 = T \sum_{m=k}^{\infty} a_{H,m}^2 m!(1 + 2\Psi_m),$$

and Ψ_m are defined by (2.3). We do not know at this writing whether the central limit theorem (2.4) extends similarly under the same hypotheses of Theorem 2.1 and H in the general case of $\widetilde{V}_n$.

2.2 Proof of the main result

For each $n \in \mathbb{N}$ and $i \in \{1, \ldots, m_n\}$ let $w_{i,n} := (\mathbf{E}(\Delta_i^n G)^2)^{1/2}$ and $v_{i,n} := w_{i,n}/\rho(\Delta_n)$. By $(A1)$ from Definition 2 and by the hypothesis (a), for some $C_2 > 0$, all $n \in \mathbb{N}$ and each $i \in \{1, \ldots, m_n\}$, we have

$$C_1 \leq v_{i,n} \leq C_2. \tag{2.11}$$

By the definition of $c_r = \mathbf{E}|Z|^r$, we have for each $n \geq 1$

$$\mathbf{E}V_n = c_r \sum_{i=1}^{m_n} \left(\frac{w_{i,n}}{\rho(\Delta_n)} \right)^r = c_r \sum_{i=1}^{m_n} v_{i,n}^r. \tag{2.12}$$

For $n \in \mathbb{N}$ and $i = 1, \ldots, m_n$ denote $h_{i,n} := \Delta_i^n G / w_{i,n}$. We shall separate the first term in the sum

$$\begin{aligned}\Delta_n^{1/2} (V_n - \mathbf{E}V_n) &= \Delta_n^{1/2} \sum_{i=1}^{m_n} v_{i,n}^r \left(|h_{i,n}|^r - c_r\right) \\ &= \Delta_n^{1/2} v_{1,n}^r \left(|h_{1,n}|^r - c_r\right) + \Delta_n^{1/2} \sum_{i=2}^{m_n} v_{i,n}^r \left(|h_{i,n}|^r - c_r\right) =: R_n + Y_n.\end{aligned}$$

By Chebyshev's inequality, for any $\delta > 0$, we have

$$\begin{aligned}\mathbf{P}(|R_n| > \delta) &\leq \delta^{-2} \Delta_n v_{1,n}^{2r} \left(\mathbf{E}h_{1,n}^{2r} - 2c_r \mathbf{E}|h_{1,n}|^r + c_r^2 \right) \\ &= \delta^{-2} \Delta_n v_{1,n}^{2r} (c_{2r} - c_r^2).\end{aligned}$$

Using (2.11), since $\Delta_n \to 0$ as $n \to \infty$, it follows that $R_n = o_P(1)$ as $n \to \infty$. By Slutsky's lemma, it is enough to prove that

$$Y_n \Rightarrow \lambda_r Z, \quad \text{as } n \to \infty, \tag{2.13}$$

where λ_r^2 is given by (2.5).

Let $H(x) := |x|^r - c_r$ for $x \in \mathbb{R}$, and let μ be a standard Gaussian measure on $\mathbb{R}$. Since Hermite polynomials H_m form an orthogonal basis for the Hilbert space $L^2(\mathbb{R}, \mu)$ and $H \in L^2(\mathbb{R}, \mu)$, we have the expansion

$$H = \sum_{m=0}^{\infty} a_m H_m = \sum_{m=2}^{\infty} a_m H_m, \tag{2.14}$$

where the second equality holds due to the fact

$$a_0 = \mathbf{E}H_0(Z)H(Z) = \mathbf{E}(|Z|^r - c_r) = 0,$$
$$a_1 = \mathbf{E}H_1(Z)H(Z) = \mathbf{E}Z(|Z|^r - c_r) = \mathbf{E}Z|Z|^r = 0,$$

i.e. H has Hermite rank greater than 1. Then each Y_n has the corresponding expansion

$$Y_n = \Delta_n^{1/2} \sum_{i=2}^{m_n} v_{i,n}^r H(h_{i,n}) = \sum_{m=2}^{\infty} \left(a_m \Delta_n^{1/2} \sum_{i=2}^{m_n} v_{i,n}^r H_m(h_{i,n}) \right). \tag{2.15}$$

To deal with this expansion we use notation and results concerning the Wiener chaos decomposition and the abstract multiple Wiener integral I_m described in Section 1.1. Let $\mathfrak{H}$ be the closure in $L^2 = L^2(\Omega, \mathcal{F}, \mathbf{P})$ of all finite linear combinations of elements of $\{\Delta_i^n G / w_{i,n},\ i = 1, \dots, m_n,\ n \in \mathbb{N}\}$. Then $\mathfrak{H}$ is a separable Hilbert space with the inner product being the covariance of its elements, and so the identity map on $\mathfrak{H}$ is an isonormal Gaussian process. Let I_m be the multiple Wiener integral on $\mathfrak{H}^{\odot m}$ and J_m be the orthogonal projection operator on $\mathcal{H}_m$.

Since the L^2-norm of $h_{i,n}$ is one, by (1.3) we have

$$H_m(h_{i,n}) = I_m(h_{i,n}^{\otimes m}). \tag{2.16}$$

For each $m \geq 2$ and $n \geq 1$ let

$$f_{m,n} := a_m \Delta_n^{1/2} \sum_{i=2}^{m_n} v_{i,n}^r h_{i,n}^{\otimes m}. \tag{2.17}$$

Since I_m is linear, by (2.15) and (2.16) we have the following Wiener chaos representation for Y_n

$$Y_n = \sum_{m=2}^{\infty} I_m(f_{m,n}). \tag{2.18}$$

By the fact that $I_m(f) \in \mathfrak{H}_m$ for any $f \in \mathcal{H}^{\otimes m}$, (2.18) and (1.4) we have for all $n \geq 1$ and $m \geq 2$

$$m! \, \|f_{m,n}\|^2_{\mathfrak{H}^{\otimes m}} = \mathbf{E}(J_m Y_n)^2. \tag{2.19}$$

According to Theorem 1.3, by (2.19), in order to prove (2.13) one needs to check the following conditions:

(i) for every $n \geq 1$ and $m \geq 1$, $\mathbf{E}(J_m Y_n)^2 \leq \delta_m$, where $\sum_{m=1}^{\infty} \delta_m < \infty$;

(ii) for every $m \geq 1$, there exists $\lim_{n\to\infty} \mathbf{E}(J_m Y_n)^2 =: \sigma_m^2$;

(iii) for every $m \geq 2$, $p = 1, \dots, m-1$,

$$\lim_{n\to\infty} ||f_{m,n} \otimes_p f_{m,n}||^2_{\mathfrak{H}^{\otimes 2(m-p)}} = 0.$$

Orthogonality of the Hermite polynomials implies orthogonality of the Wiener chaoses $\mathcal{H}_n$ and $\mathcal{H}_m$ for $n \neq m$. Therefore $J_1 Y_n = 0$ for each $n \geq 1$, and we need to prove conditions 1. and 2. only for $m \geq 2$. Provided these three conditions are satisfied, Theorem 1.3 assures that (2.13) holds with

$$\lambda_r^2 = \sum_{m=2}^{\infty} \sigma_m^2.$$

For $n \geq 1$ and $i, j \in \{2, \dots, m_n\}$, let

$$r_n(i,j) := \mathbf{E}(h_{i,n} h_{j,n}). \tag{2.20}$$

By Cauchy-Schwarz inequality, $|r_n(i,j)| \leq 1$ for all $n \geq 1$ and $i, j \in \{2, \dots, m_n\}$. Then for any $m \geq 2$

$$\left| \sum_{2\leq j<i\leq m_n} [r_n(i,j)]^m \right| \leq \sum_{2\leq j<i\leq m_n} [r_n(i,j)]^2. \tag{2.21}$$

To prove (i), let $m \geq 2$. By (2.15) we have

$$J_m Y_n = a_m \Delta_n^{1/2} \sum_{i=2}^{m_n} v_{i,n}^r H_m(h_{i,n}),$$

hence by (1.14), (2.11) and (2.21)

$$\begin{aligned} \mathbf{E}(J_m Y_n)^2 &= a_m^2 m! \Delta_n \sum_{i,j=2}^{m_n} v_{i,n}^r v_{j,n}^r [r_n(i,j)]^m \\ &= a_m^2 m! \left(\Delta_n \sum_{i=2}^{m_n} v_{i,n}^{2r} + 2\Delta_n \sum_{2\leq j<i\leq m_n} v_{i,n}^r v_{j,n}^r [r_n(i,j)]^m \right) \end{aligned} \tag{2.22}$$

$$\leq a_m^2 m! \left(C_2^{2r} T + 2\Delta_n \sum_{2\leq j<i\leq m_n} v_{i,n}^r v_{j,n}^r [r_n(i,j)]^2 \right). \tag{2.23}$$

By (2.20) and (2.9), for each $n \geq 1$ and $i, j \in \{2, \ldots, m_n\}$ we have the decomposition

$$r_n(i,j) = \frac{\mathbf{E}(\Delta_i^n G \Delta_j^n G)}{w_{i,n} w_{j,n}} = \frac{1}{v_{i,n} v_{j,n}} \big[\eta_n(|i-j|) + z_n(i,j)\big], \tag{2.24}$$

where η_n is defined by (5) and z_n is defined by (2.8). We will prove that for each $p \geq 2$,

$$\lim_{n\to\infty} \Delta_n \sum_{2\leq j<i\leq m_n} v_{i,n}^r v_{j,n}^r [r_n(i,j)]^p = \lim_{n\to\infty} \Delta_n \sum_{2\leq j<i\leq m_n} \frac{\eta_n(i-j)^p}{(v_{i,n} v_{j,n})^{p-r}} = T\Psi_p. \tag{2.25}$$

By hypothesis (b), we have

$$\lim_{n\to\infty} \sum_{k=1}^{m_n-2} \eta_n^2(k) < \infty. \tag{2.26}$$

Then there exists an $M > 0$ such that

$$\sup\{|\eta_n(k)| : 1 \leq k \leq m_n - 2,\, n \geq 1\} \leq M. \tag{2.27}$$

Let $Q_n := \{k \in \{1, \ldots, m_n - 2\} : |\eta_n(k)| > 1\}$. We will show that there exists a $K > 0$ such that

$$\mathrm{card}(Q_n) \leq K \quad \text{for all } n \geq 1. \tag{2.28}$$

Suppose it is not true. Then there exists a subsequence (Q_{n_i}) such that $\mathrm{card}(Q_{n_i}) > i$ for each i, and so

$$\sum_{k=1}^{m_{n_i}-2} \eta_{n_i}^2(k) \geq \sum_{k\in Q_{n_i}} \eta_{n_i}^2(k) > i$$

for each i. This contradiction proves (2.28).

Let $p \geq 2$, and let $Q_n^c := \{1, \ldots, m_n - 2\} \backslash Q_n$. By (2.27) and (2.28) we have

$$\sum_{k=1}^{m_n-2} |\eta_n(k)|^p = \sum_{k\in Q_n} |\eta_n(k)|^p + \sum_{k\in Q_n^c} |\eta_n(k)|^p \leq KM^p + \sum_{k\in Q_n^c} \eta_n^2(k),$$

and so by (2.26), it follows that

$$A_p := \limsup_{n\to\infty} \sum_{k=1}^{m_n-2} |\eta_n(k)|^p < \infty. \tag{2.29}$$

Let $0 < \varepsilon < 1/2$ and let $p \geq 2$. Let $Q_{1n} := [\lceil (m_n-2)\varepsilon \rceil, m_n - 2] \cap Q_n$ and $Q_{1n}^c := [\lceil (m_n-2)\varepsilon \rceil, m_n - 2] \cap Q_n^c$. If p is odd, by hypothesis (b) with $y_n = m_n - 2$ and with $y_n = \lceil (m_n-2)\varepsilon \rceil$, we have

$$\begin{aligned}\sum_{k=\lceil (m_n-2)\varepsilon\rceil}^{m_n-2} |\eta_n(k)|^p &= \sum_{k\in Q_{1n}} |\eta_n(k)|^p + \sum_{k\in Q_{1n}^c} |\eta_n(k)|^p \\ &\leq \sum_{k=\lceil (m_n-2)\varepsilon\rceil}^{m_n-2} \eta_n^{p+1}(k) + \sum_{k=\lceil (m_n-2)\varepsilon\rceil}^{m_n-2} \eta_n^2(k) \to 0\end{aligned}$$

as $n \to \infty$. If p is even the same consequence follows from hypothesis (b) immediately. Since $\Delta_n m_n = T$, we have

$$\begin{aligned}\Delta_n \sum_{k=1}^{m_n-2} k|\eta_n(k)|^p &= \Delta_n \sum_{k=1}^{\lfloor (m_n-2)\varepsilon\rfloor} k|\eta_n(k)|^p + \Delta_n \sum_{k=\lceil (m_n-2)\varepsilon\rceil}^{m_n-2} k|\eta_n(k)|^p \\ &\leq \varepsilon T \sum_{k=1}^{m_n} |\eta_n(k)|^p + T \sum_{k=\lceil (m_n-2)\varepsilon\rceil}^{m_n-2} |\eta_n(k)|^p.\end{aligned}$$

for each $n \geq 1$. Since ε is arbitrary, by (2.29) and hypothesis (b), we conclude that

$$\lim_{n\to\infty} \Delta_n \sum_{k=1}^{m_n-2} k|\eta_n(k)|^p = 0. \tag{2.30}$$

By hypothesis (b), (2.29) and (2.30) it follows that

$$\begin{aligned}\limsup_{n\to\infty} \Delta_n \sum_{i,j=2}^{m_n} |\eta_n(|i-j|)|^p &= T + 2\limsup_{n\to\infty} \Delta_n \sum_{2\leq j<i\leq m_n} |\eta_n(i-j)|^p \\ &= T + 2\limsup_{n\to\infty} \Delta_n \sum_{k=1}^{m_n-2} \left(\frac{T}{\Delta_n} - 1 - k\right) |\eta_n(k)|^p \\ &= T(1+2A_p) < \infty \end{aligned} \tag{2.31}$$

for each $p \geq 2$.

We prove next that

$$S_n := \Delta_n \sum_{2\le j<i\le m_n}^{m_n} \frac{\eta_n(i-j)^p}{(v_{i,n}v_{j,n})^{p-r}} \to T\Psi_p \quad \text{as } n\to\infty \tag{2.32}$$

for each $p \geq 2$, which is the second equality in (2.25) above. Let $p \geq 2$ and let $\varepsilon > 0$. For each $n \geq 1$, let $N_n(\varepsilon) := \lceil \varepsilon/\Delta_n \rceil$. By condition $(A2)$ of Definition 2, we have

$$\lim_{n\to\infty} \sup \{|1-v_{i,n}|\colon\ N_n(\varepsilon) \le i \le m_n\} = 0. \tag{2.33}$$

For each $n \geq 1$, we split the sum S_n into two parts as follows

$$S_{1n} + S_{2n} := \Delta_n \left(\sum_{j=2}^{N_n(\varepsilon)-1} \sum_{i=j+1}^{m_n} + \sum_{N_n(\varepsilon)\le j<i\le m_n} \right) \frac{\eta_n(i-j)^p}{(v_{i,n}v_{j,n})^{p-r}} = S_n. \tag{2.34}$$

Due to (2.11), for each $n \geq 1$ we have

$$\begin{aligned} |S_{1n}| &\le \Delta_n \sum_{j=2}^{N_n(\varepsilon)-1} \sum_{i=j+1}^{m_n} \frac{|\eta_n(i-j)|^p}{(v_{i,n}v_{j,n})^{p-r}} \\ &\le \frac{\Delta_n C_2^{2r}}{C_1^{2p}} \frac{\varepsilon}{\Delta_n} \sum_{k=1}^{m_n-2} |\eta_n(k)|^p \le \frac{\varepsilon C_2^{2r}}{C_1^{2p}} \sum_{k=1}^{m_n} |\eta_n(k)|^p. \end{aligned}$$

Also for each $n \geq 1$, we have

$$\Big|S_{2n} - \Delta_n \sum_{N_n(\varepsilon)\le j<i\le m_n} \eta_n(i-j)^p\Big| \le \max_{N_n(\varepsilon)\le j<i\le m_n} \big|(v_{i,n}v_{j,n})^{r-p} - 1\big| \Delta_n \sum_{2\le j<i\le m_n} |\eta_n(i-j)|^p$$

The right side of the preceding inequality approaches zero as $n \to \infty$ by (2.31) and (2.33). By (2.30) and hypothesis (b)

$$\begin{aligned} \lim_{n\to\infty} S_{2n} &= \lim_{n\to\infty} \Delta_n \sum_{N_n(\varepsilon)\le j<i\le m_n} \eta_n(i-j)^p \\ &= \lim_{n\to\infty} \Delta_n \sum_{k=1}^{m_n-N_n(\varepsilon)-1} (m_n - N_n(\varepsilon) - k)\eta_n(k)^p = (T-\varepsilon)\Psi_p. \end{aligned}$$

Let $C := C_1^{-2p} C_2^{2r} A_p$. Then $C < \infty$ by (2.29), and

$$(T-\varepsilon)\Psi_p - \varepsilon C \le \liminf_{n\to\infty} S_n \le \limsup_{n\to\infty} S_n \le (T-\varepsilon)\Psi_p + \varepsilon C.$$

Since ε is arbitrary, we conclude that (2.32) holds.

Now we turn to the proof of the first equality in (2.25) above. Let $p \geq 2$. By (2.24) and using the binomial theorem we have

$$\begin{aligned} A_n := \Delta_n \sum_{2\leq j<i\leq m_n} (v_{i,n}v_{j,n})^r \left[r_n(i,j)\right]^p \\ = \Delta_n \sum_{2\leq j<i\leq m_n} (v_{i,n}v_{j,n})^r \left[\frac{\eta_n(i-j)+z_n(i,j)}{v_{i,n}v_{j,n}}\right]^p \\ = \Delta_n \sum_{2\leq j<i\leq m_n} \sum_{l=0}^{p} \binom{p}{l} \frac{\eta_n(i-j)^l z_n(i,j)^{p-l}}{(v_{i,n}v_{j,n})^{p-r}} \end{aligned} \tag{2.35}$$

for each $n \in \mathbb{N}$. Due to hypothesis (c), (2.31) and Cauchy-Schwartz inequality for double sums, for $1 \leq p \leq m$ and $l = m - p$ we have

$$\begin{aligned} &\Delta_n \sum_{2\leq j<i\leq m_n} |\eta_n(i-j)|^l |z_n(i,j)|^p \\ &\leq \Delta_n \left(\sum_{2\leq j<i\leq m_n} \eta_n(i-j)^{2l}\right)^{\frac{1}{2}} \left(\sum_{i,j=2}^{m_n} z_n(i,j)^{2p}\right)^{\frac{1}{2}} \\ &= \left(\Delta_n \sum_{2\leq j<i\leq m_n} \eta_n(i-j)^{2l}\right)^{\frac{1}{2}} \left(\Delta_n \sum_{i,j=2}^{m_n} z_n(i,j)^{2p}\right)^{\frac{1}{2}} \to 0, \end{aligned} \tag{2.36}$$

as $n \to \infty$. Therefore by (2.36) and (2.11) we bound

$$\begin{aligned} &\Delta_n \sum_{2\leq j<i\leq m_n} \frac{|\eta_n(i-j)|^l |z_n(i,j)|^p}{(v_{i,n}v_{j,n})^{m-r}} \\ &\qquad \leq \frac{C_2^{2r}\Delta_n}{C_1^{2m}} \sum_{2\leq j<i\leq m_n} |\eta_n(i-j)|^l |z_n(i,j)|^p \to 0, \end{aligned} \tag{2.37}$$

as $n \to \infty$. Then, by (2.35), (2.32) and (2.37), it follows that

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \Delta_n \sum_{2\leq j<i\leq m_n} \frac{\eta_n(i-j)^p}{(v_{i,n}v_{j,n})^{p-r}} = T\Psi_p,$$

and so (2.25) holds true for each $p \geq 2$.

By (2.23) and (2.25) with $p = 2$ we can bound

$$\mathbf{E}(J_m Y_n)^2 \leq a_m^2 m!(C_2^r T + 2C) =: \delta_m,$$

where

$$C := \sup_n \left\{ \Delta_n \sum_{2\leq j<i\leq m_n} (v_{i,n}v_{j,n})^r \left[r_n(i,j)\right]^2 \right\} < \infty.$$

Since $\sum_{m=2}^{\infty} a_m^2 m! = \mathbf{E}H(Z)^2 < \infty$, we have that $\sum_{m=2}^{\infty} \delta_m < \infty$.

For (ii), consider a function $f : [0,\infty) \to \mathbb{R}$ with values $f(x) := x^r - 1$. By the mean value theorem for any $x \in \mathbb{R}$, $x \neq 1$, there exists a ξ_x between 1 and x such that

$$f(x) = r\xi_x^{r-1}(x-1).$$

Let $\varepsilon > 0$ and $N_n(\varepsilon) := \lceil \varepsilon/\Delta_n \rceil$. By (2.33) we have that $1/2 < v_{i,n} < 3/2$ for all n large enough and $i = N_n(\varepsilon), \ldots, m_n$. Then for all n large enough and $i = N_n(\varepsilon), \ldots, m_n$

$$|v_{in}^r - 1| = |f(v_{i,n})| \leq R|v_{i,n} - 1|,$$

with $R := r\max\{(1/2)^{r-1}, (3/2)^{r-1}\}$.

By (2.11) and (2.33) we have

$$\begin{aligned}\left| \Delta_n \sum_{i=2}^{m_n} v_{i,n}^r - T + \Delta_n \right| &\leq \Delta_n \sum_{i=2}^{N_n(\varepsilon)} (v_{i,n}^r + 1) + \Delta_n \sum_{i=N_n(\varepsilon)+1}^{m_n} |v_{i,n}^r - 1| \\ &\leq (1 + C_2^r)\Delta_n N_n(\varepsilon) + o(1)R\Delta_n(m_n - N_n(\varepsilon)) \\ &\to (1 + C_2^r)\varepsilon,\end{aligned}$$

as $n \to \infty$. Since ε is arbitrary, we conclude that

$$\lim_{n\to\infty} \Delta_n \sum_{i=2}^{m_n} v_{i,n}^r = T.$$

Combine this fact with (2.22) and (2.35) to get for all $m \geq 2$

$$\lim_{n\to\infty} \mathbf{E}(J_m Y_n)^2 = a_m^2 m! T\left(1 + 2\Psi_m\right).$$

Finally, to check (iii), let $1 \leq p \leq m-1$. Recall notation (2.17) of $f_{m,n}$. Using the representation (1.6) of the contraction of order p we have

$$A_n := f_{m,n} \otimes_p f_{m,n} = \Delta_n a_m^2 \left(\sum_{i=2}^{m_n} v_{i,n}^r h_{i,n}^{\otimes m} \right) \otimes_p \left(\sum_{j=2}^{m_n} v_{j,n}^r h_{j,n}^{\otimes m} \right)$$
$$= \Delta_n a_m^2 \sum_{i,j=2}^{m_n} v_{i,n}^r v_{j,n}^r r_n^p(i,j) \left(h_{i,n}^{\otimes(m-p)} \otimes h_{j,n}^{\otimes(m-p)} \right).$$

We have to prove that $B_n \to 0$ as $n \to \infty$, where

$$B_n := ||A_n||^2_{\mathcal{H}^{\otimes 2(m-p)}} = \langle A_n, A_n \rangle_{\mathcal{H}^{\otimes 2(m-p)}} \tag{2.38}$$
$$= \Delta_n^2 a_m^4 \sum_{i,j,k,l=2}^{m_n} r_n^p(i,j) r_n^p(k,l) v_{i,n}^r v_{j,n}^r v_{k,n}^r v_{l,n}^r$$
$$\times \left\langle h_{i,n}^{\otimes(m-p)} \otimes h_{j,n}^{\otimes(m-p)}, h_{k,n}^{\otimes(m-p)} \otimes h_{l,n}^{\otimes(m-p)} \right\rangle_{\mathcal{H}^{\otimes 2(m-p)}}.$$

Using the property (1.2) of the inner product on the tensor product of Hilbert spaces we have

$$B_n = \Delta_n^2 a_m^4 \sum_{i,j,k,l=2}^{m_n} v_{i,n}^r v_{j,n}^r v_{k,n}^r v_{l,n}^r r_n^p(i,j) r_n^p(k,l) r_n^{m-p}(i,k) r_n^{m-p}(j,l).$$

By (2.11) and (2.24), and by the binomial formula, we bound

$$\frac{C_1^{4m}}{C_2^{4r} a_m^4} B_n \leq \Delta_n^2 \sum_{i,j,k,l=2}^{m_n} \left(|\eta_n(|i-j|)| + |z_n(i,j)| \right)^p \left(|\eta_n(|k-l|)| + |z_n(k,l)| \right)^p$$
$$\times \left(|\eta_n(|i-k|)| + |z_n(i,k)| \right)^{m-p} \left(|\eta_n(|j-l|)| + |z_n(j,l)| \right)^{m-p}$$
$$= \Delta_n^2 \sum_{s,t=0}^{p} \sum_{u,v=0}^{m-p} \binom{p}{s} \binom{p}{t} \binom{m-p}{u} \binom{m-p}{v} D_n(s,t,u,v),$$

where

$$D_n(s,t,u,v) := \sum_{i,j,k,l=2}^{m_n} |\eta_n(|i-j|)|^{p-s} |z_n(i,j)|^s |\eta_n(|k-l|)|^{p-t} |z_n(k,l)|^t$$
$$\times |\eta_n(|i-k|)|^{m-p-u} |z_n(i,k)|^u |\eta_n(|j-l|)|^{m-p-v} |z_n(j,l)|^v.$$

Let s,t,u,v be such that $s+t+u+v \neq 0$ and let $D_n := D_n(s,t,u,v)$. Then, by the

Cauchy-Schwartz inequality, we can bound

$$
\begin{aligned}
D_n \le & \left(\sum_{i,j,k,l=2}^{m_n} \left[\eta_n(|i-j|)^{p-s} z_n(i,j)^s \eta_n(|k-l|)^{p-t} z_n(k,l)^t \right]^2 \right)^{1/2} \\
& \times \left(\sum_{i,j,k,l=2}^{m_n} \left[\eta_n(|i-k|)^{m-p-u} z_n(i,k)^u \eta_n(|j-l|)^{m-p-v} z_n(j,l)^v \right]^2 \right)^{1/2} \\
= & \left(\sum_{i,j=2}^{m_n} \left[\eta_n(|i-j|)^{p-s} z_n(i,j)^s \right]^2 \sum_{k,l=2}^{m_n} \left[\eta_n(|k-l|)^{p-t} z_n(k,l)^t \right]^2 \right)^{1/2} \\
& \times \left(\sum_{i,k=2}^{m_n} \left[\eta_n(|i-k|)^{m-p-u} z_n(i,k)^u \right]^2 \sum_{j,l=2}^{m_n} \left[\eta_n(|j-l|)^{m-p-v} z_n(j,l)^v \right]^2 \right)^{1/2} .
\end{aligned}
$$

By the choice of s, t, u, v, (2.36) and (2.31) we conclude that $\Delta_n^2 D_n \to 0$, as $n \to \infty$, and

$$
\begin{aligned}
\frac{C_1^{4m}}{C_2^{4r} a_m^4} B_n \le \Delta_n^2 & \sum_{i,j,k,l=2}^{m_n} |\eta_n(|i-j|)|^p |\eta_n(|k-l|)|^p \\
& \times |\eta_n(|i-k|)|^{m-p} |\eta_n(|j-l|)|^{m-p} + o(1) =: E_n + o(1),
\end{aligned}
$$

as $n \to \infty$. Change the summation variables as follows

$$\tilde{k} := k - l, \quad \tilde{j} := j - l \quad \text{and} \quad \tilde{i} := i - l.$$

Then

$$i - j = \tilde{i} - \tilde{j} \quad \text{and} \quad i - k = \tilde{i} - \tilde{k}.$$

By Hölder's inequality, we see that

$$
\begin{aligned}
E_n & \le T \Delta_n \sum_{i,j,k=0}^{m_n - 2} |\eta_n(|i-j|)|^p |\eta_n(k)|^p |\eta_n(|i-k|)|^{m-p} |\eta_n(j)|^{m-p} \\
& \le T \left(\Delta_n \sum_{i=0}^{m_n} \left(\sum_{k=0}^{m_n} |\eta_n(|i-k|)|^{m-p} |\eta_n(k)|^p \right)^2 \right)^{1/2} \\
& \quad \times \left(\Delta_n \sum_{i=0}^{m_n} \left(\sum_{j=0}^{m_n} |\eta_n(|i-j|)|^p |\eta_n(j)|^{m-p} \right)^2 \right)^{1/2} =: T U_n W_n.
\end{aligned}
$$

Let $\varepsilon > 0$ and let $a, b \geq 1$. By the Cauchy-Schwartz inequality, hypothesis (b) and by a change of summation variables $k := |i-j|$ we have the following three bounds

$$\begin{aligned} U_{1n}(a,b) &:= \Delta_n \sum_{i=0}^{[m_n\varepsilon]} \left(\sum_{j=0}^{m_n} |\eta_n(|i-j|)|^a |\eta_n(j)|^b \right)^2 \\ &\leq \Delta_n \sum_{i=0}^{[m_n\varepsilon]} \sum_{j=0}^{m_n} \eta_n(|i-j|)^{2a} \sum_{j=0}^{m_n} \eta_n(j)^{2b} \\ &\leq 2\Delta_n([m_n\varepsilon]+1) \sum_{j=0}^{m_n} \eta_n(j)^{2a} \sum_{j=0}^{m_n} \eta_n(j)^{2b} \to 2T\varepsilon(1+\Psi_{2a})(1+\Psi_{2b}), \end{aligned}$$

as $n \to \infty$,

$$\begin{aligned} U_{2n}(a,b) &:= \Delta_n \sum_{i=[m_n\varepsilon]+1}^{m_n} \left(\sum_{j=0}^{[m_n\varepsilon/2]} |\eta_n(|i-j|)|^a |\eta_n(j)|^b \right)^2 \\ &\leq \Delta_n \sum_{i=[m_n\varepsilon]+1}^{m_n} \sum_{j=0}^{[m_n\varepsilon/2]} \eta_n(|i-j|)^{2a} \sum_{j=0}^{[m_n\varepsilon/2]} \eta_n(j)^{2b} \\ &\leq \Delta_n(m_n - [m_n\varepsilon]) \sum_{k=[m_n\varepsilon/2]}^{m_n} \eta_n(k)^{2a} \sum_{j=0}^{[m_n\varepsilon/2]} \eta_n(j)^{2b} \to 0, \end{aligned}$$

as $n \to \infty$, and

$$\begin{aligned} U_{3n}(a,b) &:= \Delta_n \sum_{i=[m_n\varepsilon]+1}^{m_n} \left(\sum_{j=[m_n\varepsilon/2]+1}^{m_n} |\eta_n(|i-j|)|^a |\eta_n(j)|^b \right)^2 \\ &\leq \Delta_n \sum_{i=[m_n\varepsilon]+1}^{m_n} \sum_{j=[m_n\varepsilon/2]+1}^{m_n} \eta_n(|i-j|)^{2a} \sum_{j=[m_n\varepsilon/2]+1}^{m_n} \eta_n(j)^{2b} \\ &\leq 2\Delta_n(m_n - [m_n\varepsilon]) \sum_{k=0}^{m_n-[m_n\varepsilon]-1} \eta_n(k)^{2a} \sum_{j=[m_n\varepsilon/2]+1}^{m_n} \eta_n(j)^{2b} \to 0, \end{aligned}$$

as $n \to \infty$. Therefore

$$\begin{aligned} \limsup_{n\to\infty} U_n^2 &\leq \limsup_{n\to\infty} (U_{1n}(m-p,p) + 2U_{2n}(m-p,p) + 2U_{3n}(m-p,p)) \\ &\leq 2T\varepsilon(1+\Psi_{2a})(1+\Psi_{2b}), \end{aligned}$$

and, since ε is arbitrary, $U_n \to 0$ as $n \to \infty$. Analogously, one can show that $W_n \to 0$ as $n \to \infty$. We have proved that $B_n \to 0$ as $n \to \infty$, where B_n is defined by (2.38), and so the third hypothesis of Theorem 1.3 holds. Theorem 1.3 yields (2.13) and the proof of Theorem 2.1 is complete.

Chapter 3

Berry-Esséen bound and a FCLT

Recall notation (3). For $n \in \mathbb{N}$ denote $X_n := (X_{i,n},\ i = 1, \ldots, n)$. Then $(X_n)_{n\in\mathbb{N}}$ is a sequence of standard Gaussian vectors. In this chapter we prove two theorems for general sequences of standard Gaussian vectors.

3.1 Class of functions H

We define the class of functions H we study. As in (1.1), H_m denotes the m-th Hermite polynomial.

Definition 3. Let $q \in \mathbb{N}_0$. A function $H : \mathbb{R} \to \mathbb{R}$ is said to belong to the class $\mathcal{F}_q$ if

(a) H is continuous and such that there exists a countable partition of $\mathbb{R}$, $\{[a_j, b_j]\}_{j\geq 1}$, such that the restriction of H onto each $[a_j, b_j]$ has a continuous inverse;

(b) $\mathbf{E}H^2(Z) < \infty$, $\mathbf{E}H(Z)Z = 0$ and

$$\sum_{m=0}^{\infty} \frac{|\mathbf{E}H(Z)H_m(Z)|m^q}{\sqrt{m!}} 3^{\frac{m}{2}} < \infty.$$

Remark 3.1. Clearly, the inclusion $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \ldots$ holds. By (b) of Definition 3 and the fact that $H_0(x) = 1$ and $H_1(x) = x$ for all $x \in \mathbb{R}$, we have that all $F \in \mathcal{F}_0$, such that $\mathbf{E}F(Z) = 0$, have the Hermite rank greater than or equal to 2.

Example 3.2. Let H be any polynomial of degree $k \in \mathbb{N}$ satisfying $\mathbf{E}H(Z)Z = 0$ and

$p, m \in \mathbb{N}$ be such that $p \le k < m$. Then repeated integration by parts leads to

$$\begin{aligned}\sqrt{2\pi}\mathbf{E}Z^p H_m(Z) &= \int_{\mathbb{R}} x^p H_m(x) \mathrm{e}^{-\frac{x^2}{2}} dx = (-1)^m \int_{\mathbb{R}} x^p \frac{d^m}{dx^m} \mathrm{e}^{-\frac{x^2}{2}} dx \\ &= -x^p H_{m-1}(x) \mathrm{e}^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + (-1)^{m+1} 2p \int_{\mathbb{R}} x^{p-1} \frac{d^{m-1}}{dx^{m-1}} \mathrm{e}^{-\frac{x^2}{2}} dx = \\ &= \ldots = (-1)^{m+p} p! \int_{\mathbb{R}} \frac{d^{m-p}}{dx^{m-p}} \mathrm{e}^{-\frac{x^2}{2}} dx \\ &= (-1)^{m+p} p! \sqrt{2\pi} \mathbf{E} H_{m-p}(Z) = 0.\end{aligned}$$

It then follows that H belongs to $\mathcal{F}_q$ for all $q \in \mathbb{N}_0$. Thus all the Hermite polynomials $H_m,\ m \ge 2$, and their finite linear combinations belong to $\mathcal{F}_q$ for all $q \in \mathbb{N}_0$ as well. Usually, a random variable $H(Z)$ is assumed to be in a fixed Wiener chaos, but in this chapter the random variable $H(Z)$ can have an infinite Wiener chaos expansion.

Example 3.3. A function $H^{(p)}$ with values $H^{(p)}(x) := |x|^p,\ p > 0$, is a polynomial if and only if p is an even integer. For a general p and any $m \in \mathbb{N}$ such that $m > p$ one has

$$\begin{aligned}\sqrt{2\pi}\mathbf{E}|Z|^p H_m(Z) &= 2\int_0^{\infty} x^p H_m(x) \mathrm{e}^{-\frac{x^2}{2}} dx = (-1)^m 2 \int_0^{\infty} x^p \frac{d^m}{dx^m} \mathrm{e}^{-\frac{x^2}{2}} dx \\ &= -2x^p H_{m-1}(x) \mathrm{e}^{-\frac{x^2}{2}} \Big|_0^{\infty} + (-1)^{m+1} 2p \int_0^{\infty} x^{p-1} \frac{d^{m-1}}{dx^{m-1}} \mathrm{e}^{-\frac{x^2}{2}} dx \\ &= \ldots = \\ &= (-1)^{m+[p]} 2p(p-1)\cdots(p-[p]+1) \int_0^{\infty} x^{p-[p]} \frac{d^{m-[p]}}{dx^{m-[p]}} \mathrm{e}^{-\frac{x^2}{2}} dx.\end{aligned}$$

However, if m is odd, $\mathbf{E}|Z|^p H_m(Z) = 0$, since H_m are odd functions for odd m. By the previous formula, in the case of odd p we have

$$\begin{aligned}\mathbf{E}H^{(p)}(Z) H_m(Z) &= -\frac{1}{\sqrt{2\pi}} 2p! \int_0^{\infty} \frac{d^{m-p}}{dx^{m-p}} \mathrm{e}^{-\frac{x^2}{2}} dx = -\frac{1}{\sqrt{2\pi}} 2p! H_{m-p-1}(x) \mathrm{e}^{-\frac{x^2}{2}} \Big|_0^{\infty} \\ &= (-1)^{\frac{m}{2}} \sqrt{\frac{2}{\pi}} p! (m-p-2)!!.\end{aligned}$$

for all $m > p$. For $n = 2k-1$ with some $k \in \mathbb{N}$ one has by the Stirling's approximation

$$n!! = \frac{(2k)!}{2^k k!} \sim \sqrt{2} \left(\frac{2k}{e}\right)^k, \quad \text{as } k \to \infty. \tag{3.1}$$

Suppose $m = 2k$ and $p = 2l+1$ for some $k, l \in \mathbb{N}$. Then by the Stirling's approximation and (3.1) we have

$$\begin{aligned}\frac{(m-p-2)!!}{\sqrt{m!}}3^{\frac{m}{2}} &\sim \frac{\sqrt{2}(2(k-l-1))^{k-l-1}\mathrm{e}^k}{\mathrm{e}^{k-l-1}(4\pi k)^{1/4}(2k)^k}3^k \\ &= \frac{\sqrt{2}}{(4\pi)^{1/4}}\left(\frac{e}{2}\right)^{l+1}\frac{(k-l-1)^{k-l-1}3^k}{k^k}\end{aligned}$$

as $k\to\infty$, thus in this case $H^{(p)}$ does not belong to $\mathcal{F}_0$. The general case when p is a non-integer positive real number remains an unsolved problem.

Example 3.4. A less trivial example is a function H with values $H(x) := \mathrm{e}^{ax^2/2}$ with $0 < a < 1/3$. It is argued in [41] that in this case

$$\frac{\mathbf{E}H(Z)H_m(Z)}{\sqrt{m!}}3^{\frac{m}{2}} = O(m^{-\frac{1}{4}}\lambda^{\frac{m}{2}})$$

with some $\lambda < 1$ as $m\to\infty$, thus $H \in \mathcal{F}_q$ for all $q \in \mathbb{N}_0$.

Example 3.5. By the proof of Lemma 5.1 in [10] we have

$$\cos x = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!\sqrt{\mathrm{e}}}H_{2m}(x) \quad \text{and} \quad \sin x = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!\sqrt{\mathrm{e}}}H_{2m+1}(x)$$

for almost all $x \in \mathbb{R}$. By (1.14) we thus conclude that for all $m \in \mathbb{N}$

$$\begin{aligned}&|\mathbf{E}\cos(Z)H_{2m}(Z)| = 1/\sqrt{\mathrm{e}} = |\mathbf{E}\sin(Z)H_{2m+1}(Z)| \quad \text{and} \\ &\mathbf{E}\cos(Z)H_{2m+1}(Z) = 0 = \mathbf{E}\sin(Z)H_{2m}(Z).\end{aligned}$$

Then for all $q \in \mathbb{N}_0$

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{|\mathbf{E}\cos(Z)H_m(Z)|m^q}{\sqrt{m!}}3^{m/2} &= \sum_{m=0}^{\infty}\frac{|\mathbf{E}\cos(Z)H_{2m}(Z)|(2m)^q}{\sqrt{(2m)!}}3^m \\ &= \sum_{m=0}^{\infty}\frac{(2m)^q}{\sqrt{(2m)!\mathrm{e}}}3^m < \infty\end{aligned}$$

and likewise for the sine, therefore both sine and cosine belong to $\mathcal{F}_q$ for all $q \in \mathbb{N}_0$.

3.2 Berry-Esséen bound

Let $n \in \mathbb{N}$ and $X := (X_i,\ i = 1, \dots, n)$ be a standard Gaussian vector. Given a function $H : \mathbb{R} \to \mathbb{R}$ and a vector $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$ we define

$$S(X, H, \alpha) := \sum_{i=1}^{n} \alpha_i H(X_i), \tag{3.2}$$

called the *weighted H-sum of X with weights* α and

$$W(X, H, \alpha) := \frac{S(X, H, \alpha) - \mathbf{E}S(X, H, \alpha)}{\sqrt{\mathrm{var}(S(X, H, \alpha))}}.$$

Theorem 3.6. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of standard Gaussian vectors, $X_n := (X_{i,n},\ i = 1, \dots, n),\ n \in \mathbb{N}$, with covariance matrices $(r_n(i,j))_{1\le i,j\le n}$. Let $H \in \mathcal{F}_1$, $d \in \mathbf{D}$ and $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of vectors $\alpha_n := (\alpha_{1,n}, \dots, \alpha_{n,n}) \in [\beta_1, \beta_2]^n,\ n \in \mathbb{N}$, where $0 < \beta_1 \le \beta_2 < \infty$. Then there exists a $c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$*

$$\begin{aligned} d^2\left(W(X_n, H, \alpha_n), Z\right) \le \frac{c}{[\mathrm{var}(S(X_n, H, \alpha_n))]^2} \max_{1\le i\le n} \sum_{j=1}^{n} r_n^2(i,j) \\ \times \sum_{i,k,l=1}^{n} |r_n(k,l)||r_n(i,k)|. \end{aligned}$$

Remark 3.7. It might be surprising, that the bound is of the same order for all the metrics in $\mathbf{D}$. However, it is a consequence of the fact, that the random variables considered have densities as shown in Lemma 3.13.

Remark 3.8. The more general case when $X_n = (X_{i,n},\ i = 1, \dots, m_n)$ for some unbounded sequence $(m_n)_{n\in\mathbb{N}}$ can be treated identically.

Proof. Fix an $n \in \mathbb{N}$ and denote $h_i := X_{i,n}$, $w_i := \alpha_{i,n}$ for all $i = 1, \dots, n$ and $r(i,j) := r_n(i,j)$ for all $1 \le i,j \le n$. Let $h := (h_1, \dots, h_n)$, $w := (w_1, \dots, w_n)$, $\sigma^2 := \mathrm{var}(S(h, H, w))$ and $F(x) := H(x) - \mathbf{E}H(Z),\ x \in \mathbb{R}$. Then

$$Y := W(h, H, w) = \frac{1}{\sigma} \sum_{i=1}^{n} w_i F(h_i). \tag{3.3}$$

Since $H \in \mathcal{F}_1$, we have $F \in L^2(\mathbb{R}, \mu)$, where μ is a standard Gaussian measure, and F has Hermite rank greater or equal to 2 (see Remark 3.1), thus F can be expanded in the base of Hermite polynomials as follows

$$F = \sum_{m=2}^{\infty} \frac{a_m}{m!} H_m, \tag{3.4}$$

where

$$a_m := \mathbf{E}F(Z)H_m(Z), \quad m \geq 2.$$

Let $\mathfrak{H}$ be the closure in $L^2 = L^2(\Omega, \mathcal{F}, \mathbf{P})$ of all finite linear combinations of elements of $(X_n)_{n\in\mathbb{N}}$. Then $\mathfrak{H}$ is a separable Hilbert space with the inner product being the covariance of its elements, and so the identity map on $\mathfrak{H}$ is an isonormal Gaussian process. Let I_m be the multiple Wiener integral on $\mathfrak{H}^{\odot m}$ and J_m be the orthogonal projection operator on $\mathcal{H}_m$.

For all $m \geq 2$ let

$$f_m := \frac{1}{\sigma} \sum_{i=1}^{n} w_i \frac{a_m}{m!} h_i^{\otimes m}.$$

Then by (3.3), (3.4) and (1.3) we have the following Wiener chaos expansion for Y

$$Y = \frac{1}{\sigma} \sum_{i=1}^{n} \sum_{m=2}^{\infty} w_i \frac{a_m}{m!} H_m(h_i) = \sum_{m=2}^{\infty} I_m(f_m). \tag{3.5}$$

By (1.4), (3.5) and (1.14)

$$m!||f_m||^2_{\mathfrak{H}^{\otimes m}} = \mathbf{E}\left(I_m(f_m)\right)^2 = \mathbf{E}(J_m Y)^2 = \frac{1}{\sigma^2} \sum_{i,j=1}^{n} w_i w_j \frac{a_m^2}{m!} r^m(i,j).$$

By the fact that $F \in \mathcal{F}_1$ we have

$$\sum_{m=2}^{\infty} \frac{a_m^2}{m!} < \sum_{m=2}^{\infty} \frac{m a_m^2}{m!} < \sum_{m=2}^{\infty} \frac{m a_m^2}{m!} 3^{\frac{m}{2}} < \infty. \tag{3.6}$$

Since $|r(i,j)| \leq 1$ for all $i, j \in \{1, \ldots, n\}$ and by (3.6) we get

$$\sum_{m=2}^{\infty} m m! ||f_m||^2_{\mathfrak{H}^{\otimes m}} = \sum_{m=2}^{\infty} \frac{m a_m^2}{m!} \sum_{i,j=1}^{n} w_i w_j r^m(i,j)$$

$$\leq \sum_{i,j=1}^{n} w_i w_j r^2(i,j) \sum_{m=2}^{\infty} \frac{m a_m^2}{m!} < \infty,$$

and by Proposition 1.2, $Y \in \mathbb{D}^{1,2}$.

By the definition of the derivative operator and (1.8) we have

$$DY = \frac{1}{\sigma} \sum_{i=1}^{n} \sum_{m=2}^{\infty} w_i \frac{a_m}{(m-1)!} H_{m-1}(h_i) h_i. \tag{3.7}$$

By the definition of the operator L^{-1} we have

$$L^{-1}Y = -\frac{1}{\sigma} \sum_{j=1}^{n} \sum_{m=2}^{\infty} w_i \frac{a_m}{m m!} H_m(h_j).$$

Then by (1.8)

$$DL^{-1}Y = -\frac{1}{\sigma} \sum_{j=1}^{n} \sum_{m=2}^{\infty} w_i \frac{a_m}{m!} H_{m-1}(h_j) h_j. \tag{3.8}$$

By Lemma 3.13, Y is absolutely continuous. Hence, by Theorem 1.5

$$d^2(Y,Z) \leq 2\mathbf{E}\left[1 - \left\langle DY, -DL^{-1}Y \right\rangle_{\mathfrak{H}}\right]^2. \tag{3.9}$$

By (3.7) and (3.8) and since $\langle h_i, h_j \rangle_{\mathfrak{H}} = \mathbf{E} h_i h_j = r(i,j)$ for all $1 \leq i, j \leq n$, we have

$$\begin{aligned} e := \left\langle DY, -DL^{-1}Y \right\rangle_{\mathfrak{H}} &= \frac{1}{\sigma^2} \sum_{i,j=1}^{n} w_i w_j b_i c_j \langle h_i, h_j \rangle_{\mathfrak{H}} \\ &= \frac{1}{\sigma^2} \sum_{i,j=1}^{n} w_i w_j b_i c_j r(i,j), \end{aligned} \tag{3.10}$$

where

$$b_i := \sum_{m=2}^{\infty} \frac{a_m}{(m-1)!} H_{m-1}(h_i) \quad \text{and} \quad c_j := \sum_{m=2}^{\infty} \frac{a_m}{m!} H_{m-1}(h_j).$$

By (3.5), (3.6), (1.14) and the Fubini-Tonelli theorem we have

$$\sigma^2 = \sum_{i,j=1}^{n} \sum_{m=2}^{\infty} w_i w_j \frac{a_m^2}{m!} r^m(i,j) \tag{3.11}$$

and

$$\mathbf{E}b_ic_j = \sum_{m=2}^{\infty} m\frac{a_m^2}{(m!)^2}(m-1)!r^{m-1}(i,j) = \sum_{m=2}^{\infty}\frac{a_m^2}{m!}r^{m-1}(i,j).$$

Therefore, by (3.10) we have

$$\mathbf{E}e = \frac{1}{\sigma^2}\sum_{i,j=1}^{n}\sum_{m=2}^{\infty} w_iw_j\frac{a_m^2}{m!}r^m(i,j) = 1$$

and

$$\mathbf{E}(1-e)^2 = 1 - 2\mathbf{E}e + \mathbf{E}e^2 = \mathbf{E}e^2 - 1. \tag{3.12}$$

Recall notation (1.10)-(1.13) applied to the vector $(h_1,\ldots,h_n)$, $p=4$ and $I := \{2,3,\ldots\}^4$. Then by Lemma 1.7 we have

$$\mathbf{E}b_ic_jb_kc_l = \sum_{\mathbf{m}\in I}\frac{a_{m_1}a_{m_2}a_{m_3}a_{m_4}}{(m_1-1)!(m_2)!(m_3-1)!(m_4)!}\mathcal{E}(\mathbf{m}-\mathbf{1},(i,j,k,l)) < \infty, \tag{3.13}$$

and term by term integration in (3.13) is legitimate.

By Lemma 1.8 we have

$$\mathcal{E}(\mathbf{m}-\mathbf{1},(i,j,k,l)) = \mathcal{C}(\mathbf{m}-\mathbf{1})\sum_{\tau\in\mathcal{T}(\mathbf{m}-\mathbf{1})} R(\tau,(i,j,k,l)). \tag{3.14}$$

Let $\mathbf{m}\in I$ and $\tau\in\mathcal{T}(\mathbf{m}-\mathbf{1})$. Note that $R(\tau,(i,j,k,l)) = r^s(i,j)r^t(k,l)$ for some $s,t\in\mathbb{N}$ if and only if $m_1=m_2$ and $m_3=m_4$. Let

$$\mathcal{T}_1(\mathbf{m}-\mathbf{1}) := \{\tau\in\mathcal{T}(\mathbf{m}-\mathbf{1}) : |\{v : \{\tau(2v-1),\tau(2v)\} = \{1,2\}\}| = m_1-1 \\ \text{and } |\{v : \{\tau(2v-1),\tau(2v)\} = \{3,4\}\}| = m_3-1\}.$$

Then

$$|\mathcal{T}_1(\mathbf{m}-\mathbf{1})| = \frac{2^{q(\mathbf{m}-\mathbf{1})}q(\mathbf{m}-\mathbf{1})!}{(m_1-1)!(m_3-1)!}.$$

Also, denote $\mathcal{T}_2(\mathbf{m}) := \mathcal{T}(\mathbf{m})\backslash\mathcal{T}_1(\mathbf{m})$. Then

$$\sum_{\tau\in\mathcal{T}_1(\mathbf{m}-\mathbf{1})} R(\tau,(i,j,k,l)) = \frac{2^{q(\mathbf{m}-\mathbf{1})}q(\mathbf{m}-\mathbf{1})!}{(m_1-1)!(m_3-1)!} \times r^{m_1-1}(i,j)r^{m_3-1}(k,l)\delta_{m_1m_2}\delta_{m_3m_4}, \tag{3.15}$$

where δ is the Kronecker delta function. For all $\mathbf{m}\in I$ and $(i_1,i_2,i_3,i_4)\in\{1,\ldots,n\}^4$ denote

$$a(\mathbf{m}) := \frac{a_{m_1}a_{m_2}a_{m_3}a_{m_4}}{(m_1-1)!(m_2)!(m_3-1)!(m_4)!} \quad \text{and} \quad w_{i_1i_2i_3i_4} := w_{i_1}w_{i_2}w_{i_3}w_{i_4}.$$

By (3.11) we have

$$\sigma^4 = \sum_{i,j,k,l=1}^{n}\sum_{m_1,m_2=2}^{\infty} w_{ijkl}\frac{a_{m_1}^2}{(m_1)!}\frac{a_{m_2}^2}{(m_2)!}r^{m_1}(i,j)r^{m_2}(k,l). \tag{3.16}$$

Then by (3.10), (3.11), (3.13), (3.14), (3.15) and (3.16)

$$\begin{aligned}
\mathbf{E}e^2-1 &= \frac{1}{\sigma^4}\sum_{i,j,k,l=1}^{n} r(i,j)r(k,l)w_{ijkl}\sum_{\mathbf{m}\in I}a(\mathbf{m})\mathcal{E}(\mathbf{m}-\mathbf{1},(i,j,k,l))-1\\
&= \frac{1}{\sigma^4}\sum_{i,j,k,l=1}^{n} w_{ijkl}r(i,j)r(k,l)\Bigg[\sum_{\mathbf{m}\in I}a(\mathbf{m})\mathcal{C}(\mathbf{m}-\mathbf{1})\sum_{\tau\in\mathcal{T}_2(\mathbf{m}-\mathbf{1})}R(\tau,(i,j,k,l))\\
&\quad+\sum_{m_1,m_2=2}^{\infty}\frac{a_{m_1}^2}{(m_1)!}\frac{a_{m_2}^2}{(m_2)!}r^{m_1-1}(i,j)r^{m_2-1}(k,l)\Bigg]-1\\
&= \frac{1}{\sigma^4}\sum_{i,j,k,l=1}^{n} w_{ijkl}r(i,j)r(k,l)\sum_{\mathbf{m}\in I}a(\mathbf{m})\mathcal{C}(\mathbf{m}-\mathbf{1})\sum_{\tau\in\mathcal{T}_2(\mathbf{m}-\mathbf{1})}R(\tau,(i,j,k,l))\\
&\le \frac{\beta_2^4}{\sigma^4}\sum_{i,j,k,l=1}^{n}|r(i,j)||r(k,l)|\sum_{\mathbf{m}\in I}|a(\mathbf{m})|\mathcal{C}(\mathbf{m}-\mathbf{1})\sum_{\tau\in\mathcal{T}_2(\mathbf{m}-\mathbf{1})}|R(\tau,(i,j,k,l))|.
\end{aligned} \tag{3.17}$$

Note, that by Lemma 1.8

$$\mathcal{C}(\mathbf{m}-\mathbf{1})\sum_{\tau\in\mathcal{T}(\mathbf{m}-\mathbf{1})}1 = \mathbf{E}H_{m_1-1}(Z)H_{m_2-1}(Z)H_{m_3-1}(Z)H_{m_4-1}(Z).$$

By Lemma 1.7 with $d_1=d_3=1,\ d_2=d_4=0$, $u_1=\ldots=u_4=1$ and $Z_1=\ldots=$

$Z_4 = Z$ we can bound

$$\sum_{\mathbf{m}\in I} |a(\mathbf{m})| \mathcal{C}(\mathbf{m}-\mathbf{1}) \sum_{\tau\in\mathcal{T}_2(\mathbf{m}-\mathbf{1})} 1 \leq \sum_{\mathbf{m}\in I} |a(\mathbf{m})| \mathcal{C}(\mathbf{m}-\mathbf{1}) \sum_{\tau\in\mathcal{T}(\mathbf{m}-\mathbf{1})} 1 < \infty. \quad (3.18)$$

For all $\mathbf{m} \in I$ and $\tau \in \mathcal{T}(\mathbf{m}-\mathbf{1})$ denote

$$D(\tau) := \sum_{i,j,k,l=1}^{n} |r(i,j)||r(k,l)||R(\tau,(i,j,k,l))|.$$

By the fact that for all $\mathbf{m} \in I$, $\tau \in \mathcal{T}(\mathbf{m}-\mathbf{1})$ and $(i,j,k,l) \in \{1,\ldots,n\}^4$ we have $|R(\tau,(i,j,k,l))| \leq 1$ and (3.18) we can get the sum in (3.17) inside the series to get

$$\mathbf{E}e^2 - 1 \leq \frac{\beta_2^4}{\sigma^4} \sum_{\mathbf{m}\in I} |a(\mathbf{m})| \mathcal{C}(\mathbf{m}-\mathbf{1}) \sum_{\tau\in\mathcal{T}_2(\mathbf{m}-\mathbf{1})} D(\tau).$$

Let $\mathbf{m} \in I$ and $\tau \in \mathcal{T}_2(\mathbf{m}-\mathbf{1})$. Then by the definition of $\mathcal{T}_2(\mathbf{m}-\mathbf{1})$ for some $v \in \{1,\ldots,q(\mathbf{m}-\mathbf{1})\}$, $\{\tau(2v-1),\tau(2v)\} \neq \{1,2\}$ and $\{\tau(2v-1),\tau(2v)\} \neq \{3,4\}$. Due to symmetry we can assume that $\{\tau(2v-1),\tau(2v)\} = \{1,3\}$. Since $\mathbf{m} \in I$, $m_2 \geq 2$. By Lemma 1.8 there are $m_2 - 1 \geq 1$ coordinates of τ equal to 2. Therefore there exists a $w \in \{1,\ldots,q(\mathbf{m}-\mathbf{1})\}$ such that $\{\tau(2w-1),\tau(2w)\} = \{2,s\}$ for some $s \in \{1,3,4\}$. Assume that $s = 3$, which is the most general case. Since for all $(i,j) \in \{1,\ldots,n\}^2$, $|r(i,j)| \leq 1$, we can estimate

$$|R(\tau,(i,j,k,l))| \leq |r(i,k)||r(j,k)|. \quad (3.19)$$

Applying (3.19) and the inequality $|ab| \leq \frac{1}{2}(a^2+b^2)$ we can bound

$$\begin{aligned} D(\tau) &\leq \sum_{i,j,k,l=1}^{n} |r(i,j)||r(k,l)||r(i,k)||r(j,k)| \\ &\leq \frac{1}{2} \sum_{i,k,l=1}^{n} |r(k,l)||r(i,k)| \sum_{j=1}^{n} \left(r^2(i,j) + r^2(j,k)\right) \\ &\leq \max_{1\leq i\leq n} \sum_{j=1}^{n} r^2(i,j) \sum_{i,k,l=1}^{n} |r(k,l)||r(i,k)|. \quad (3.20) \end{aligned}$$

Hence, by (3.9), (3.12), (3.17), (3.18) and (3.20) we have

$$d^2(Y,Z) \le \frac{c}{\sigma^4} \max_{1\le i\le n} \sum_{j=1}^{n} r^2(i,j) \sum_{i,k,l=1}^{n} |r(k,l)||r(i,k)|,$$

where

$$c := 2\beta_2^4 \sum_{\mathbf{m}\in I} |a(\mathbf{m})|\mathcal{C}(\mathbf{m}-\mathbf{1}) \sum_{\tau\in\mathcal{T}_2(\mathbf{m}-\mathbf{1})} 1,$$

and the theorem is proved. □

3.3 Functional CLT

In this section we prove a functional CLT for the partial sum process related to the sequence of H-variations of a Gaussian process. We prove an auxiliary lemma first.

Lemma 3.9. *Let I be any set with four elements, and $D_I := \{(i,i):\ i\in I\}$. For every symmetric function $f: I^2\backslash D_I \to \mathbb{N}_0$ such that for all $i\in I$*

$$\sum_{j\in I\backslash\{i\}} f(i,j) \ge 2, \tag{3.21}$$

at least one of the following is true:

(a) *there exist some distinct i,j,k,l in I such that $f(i,j)\ge 1$ and $f(k,l)\ge 1$;*

(b) *for some $i\in I$, $f(i,j)\ge 2$ for all $j\in I\backslash\{i\}$.*

Proof. Without loss of generality we can assume that $I=\{1,2,3,4\}$. By (3.21), without loss of generality, we can assume $f(1,2)\ge 1$. If $f(3,4)\ge 1$, the statement holds. Suppose $f(3,4)=0$. By (3.21), $f(3,1)\ge 1$ or $f(3,2)\ge 1$. Without loss of generality, we can assume $f(3,1)\ge 1$. By (3.21) again, $f(4,1)\ge 1$ or $f(4,2)\ge 1$. If $f(4,2)\ge 1$, the statement holds. Suppose $f(4,2)=0$ and $f(4,1)\ge 1$. Then, if $f(3,2)\ge 1$, the statement holds. Suppose $f(3,2)=0$. Since $f(2,3)=0$ and $f(2,4)=0$, by (3.21), $f(2,1)\ge 2$. Similarly, $f(3,1)\ge 2$ and $f(4,1)\ge 2$. The proof is finished. □

Theorem 3.10. *Let $X_n := (X_{i,n},\ i=1,\dots,n),\ n\in\mathbb{N}$, be a sequence of standard Gaussian vectors with covariance matrices $(r_n(i,j))_{1\le i,j\le n},\ n\in\mathbb{N}$, a function $H\in\mathcal{F}_0$*

and sequence of vectors $\alpha_n := (\alpha_{1,n}, \ldots, \alpha_{n,n}) \in [\beta_1, \beta_2]^n$*, where* $0 < \beta_1 \le \beta_2 < \infty$*. For all* $n \in \mathbb{N}$ *define functions* $Y^n : [0,1] \to \mathbb{R}_+$ *with values*

$$Y_t^n := \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \alpha_{i,n}[H(X_{i,n}) - \mathbf{E}H(Z)], \tag{3.22}$$

$t \in [0,1]$*. Suppose that*

(a) *there exist an* $M \in \mathbb{R}$ *such that*

$$\sup_n \left\{ \max_{1 \le i \le n} \sum_{j=1}^{n} r_n^2(i,j) \right\} = M; \tag{3.23}$$

(b) *for every* $m \ge 2$ *there exists a real number* Φ_m *such that for all* $s,t \in [0,1]$*, satisfying* $s < t$*, the following limit exists*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=[ns]+1}^{[nt]} \alpha_{i,n}\alpha_{j,n} r_n^m(i,j) = (t-s)\Phi_m; \tag{3.24}$$

(c) *for all* $s,t,u,v \in [0,1]$*, such that* $s < t \le u < v$*, we have that the following limit exists*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=[nu]+1}^{[nv]} \sum_{j=[ns]+1}^{[nt]} r_n^2(i,j) = 0; \tag{3.25}$$

(d) *for every* $m \ge 2$*,* $1 \le p \le m-1$ *and all* $s,t \in [0,1]$*, such that* $s < t$*, we have that the following limit exists*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j,k,l=[ns]+1}^{[nt]} |r_n^p(i,j) r_n^p(k,l) r_n^{m-p}(i,k) r_n^{m-p}(j,l)| = 0. \tag{3.26}$$

Then in the space $\mathcal{D}[0,1]$ *equipped with the Skorokhod topology it holds that*

$$Y^n \Rightarrow \lambda W,$$

as $n \to \infty$, Y^n is defined in (3.22) and

$$\lambda^2 := \sum_{m=2}^{\infty} \frac{a_m^2}{m!} \Phi_m \quad \textit{with} \quad a_m := \mathbf{E} H(Z) H_m(Z),\ m \geq 2. \tag{3.27}$$

Remark 3.11. Hypotheses $(b), (c), (d)$ correspond to the hypotheses $(ii), (iii), (iv)$ of Theorem 1.4. Hypothesis (b) determines the limiting variance λ^2, through the limiting variances of the increments of Y^n. By hypothesis (c) the correlation of disjoint increments of Y^n should vanish in the limit. The form of the limiting variance λ^2 is standard in the study of H-variations and has been obtained in [13], [21], [15], [6] among others.

Proof. By Theorem 15.6 in [11] it is sufficient to show that for any $s, t, u \in [0, 1]$ such that $s \leq u \leq t$, there exists a constant C such that for all $n \in \mathbb{N}$

$$\mathbf{E}(Y_u^n - Y_s^n)^2 (Y_t^n - Y_u^n)^2 \leq C(t - s)^2 \tag{3.28}$$

and for any $t_1 < \ldots < t_k \in [0, 1]$ we have as $n \to \infty$

$$\left(Y_{t_1}^n, Y_{t_2}^n - Y_{t_1}^n, \ldots, Y_{t_k}^n - Y_{t_{k-1}}^n\right) \Rightarrow \lambda \left(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}}\right). \tag{3.29}$$

We start with (3.28). Fix an $n \in \mathbb{N}$. Since $H \in \mathcal{F}_0$, its Hermite rank is greater or equal to 2, and with a_m defined in (3.27) we have

$$H - \mathbf{E} H(Z) = \sum_{m=2}^{\infty} \frac{a_m}{m!} H_m,$$

thus for all $t \in [0, 1]$

$$Y_t^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \alpha_{i,n} \sum_{m=2}^{\infty} \frac{a_m}{m!} H_m(X_{i,n}). \tag{3.30}$$

Let $I := \{2, 3, \ldots\}^4$ and define a function $a : I \to \mathbb{N}$ with values

$$a(\mathbf{m}) := \frac{a_{m_1} a_{m_2} a_{m_3} a_{m_4}}{m_1! m_2! m_3! m_4!}$$

for $\mathbf{m} \in I$. Let $s, t \in [0, 1]$ be such that $s < t$. Recall notation (1.10)-(1.13) applied to the vector $(X_{1,n}, \ldots, X_{n,n})$, $p = 4$ and $I = \{2, 3, \ldots\}^4$. Then by Lemma 1.8, (1.11) and (1.12) we have

$$\mathcal{E}(\mathbf{m},(i,j,k,l)) = \mathcal{C}(\mathbf{m}) \sum_{\tau\in\mathcal{T}(\mathbf{m})} R(\tau,(i,j,k,l)) \tag{3.31}$$

for all $\mathbf{m} \in I$ and $(i,j,k,l) \in \{1,\dots,n\}^4$. By Lemma 1.8

$$\mathcal{C}(\mathbf{m}) \sum_{\tau\in\mathcal{T}(\mathbf{m})} 1 = \mathbf{E}H_{m_1}(Z)H_{m_2}(Z)H_{m_3}(Z)H_{m_4}(Z). \tag{3.32}$$

Since for all $\mathbf{m} \in I, \tau \in \mathcal{T}(\mathbf{m})$ and $(i,j,k,l) \in \{1,\dots,n\}^4$ we have $|R(\tau,(i,j,k,l))| \le 1$, by (3.32) and Lemma 1.7 with $d_1 = \dots = d_4 = u_1 = \dots = u_4 = 0$ and $Z_1 = \dots = Z_4 = Z$ we can bound

$$\sum_{\mathbf{m}\in I} |a(\mathbf{m})|\mathcal{C}(\mathbf{m}) \sum_{\tau\in\mathcal{T}(\mathbf{m})} |R(\tau,(i,j,k,l))| \le \sum_{\mathbf{m}\in I} |a(\mathbf{m})|\mathcal{C}(\mathbf{m}) \sum_{\tau\in\mathcal{T}(\mathbf{m})} 1 < \infty. \tag{3.33}$$

For all $\mathbf{m} \in I$ and $\tau \in \mathcal{T}(\mathbf{m})$ denote

$$D(\tau) := \frac{1}{n^2} \sum_{i,j,k,l=[ns]+1}^{[nt]} |R(\tau,(i,j,k,l))|.$$

Then by (3.22), (3.31) and (3.33) and the Tonelli-Fubini theorem we can change the order of integration and then get a sum inside a series to obtain

$$\begin{aligned}\mathbf{E}(Y_t^n - Y_s^n)^4 &= \frac{1}{n^2} \sum_{i,j,k,l=[ns]+1}^{[nt]} \alpha_{i,n}\alpha_{j,n}\alpha_{k,n}\alpha_{l,n} \sum_{\mathbf{m}\in I} a(\mathbf{m})\mathcal{C}(\mathbf{m}) \sum_{\tau\in\mathcal{T}(\mathbf{m})} R(\tau,(i,j,k,l)) \\ &\le \beta_2^4 \sum_{\mathbf{m}\in I} |a(\mathbf{m})|\mathcal{C}(\mathbf{m}) \sum_{\tau\in\mathcal{T}(\mathbf{m})} D(\tau). \end{aligned} \tag{3.34}$$

Let $\mathbf{m} \in I$ and $\tau \in \mathcal{T}(\mathbf{m})$. For $x, y \in \{1,2,3,4\},\ x \ne y$, define

$$f_\tau(x,y) := \sum_{v=1}^{q(\mathbf{m})} \chi\left\{\{x,y\} = \{\tau(2v-1),\tau(2v)\}\right\},$$

that is, if, for example, $x = 1$ and $y = 2$, then $f_\tau(x,y)$ is the power of $r_n(i,j)$ in

$R(\tau,(i,j,k,l))$. By (ii) of Lemma 1.8 we have for all $i \in \{1,2,3,4\}$

$$\sum_{j\in\{1,2,3,4\}\setminus\{i\}} f_\tau(i,j) \geq 2, \tag{3.35}$$

therefore by Lemma 3.9 at least one of the following is true:

1. ($f_\tau(1,2) \geq 1$ and $f_\tau(3,4) \geq 1$) or ($f_\tau(1,3) \geq 1$ and $f_\tau(2,4) \geq 1$) or ($f_\tau(1,4) \geq 1$ and $f_\tau(2,3) \geq 1$);
2. for some $i \in \{1,2,3,4\}$, $f_\tau(i,j) \geq 2$ for all $j \in \{1,2,3,4\}\setminus\{i\}$.

Consider case 1. Suppose $f_\tau(1,2) \geq 1$ and $f_\tau(3,4) \geq 1$ (the other two cases are symmetric). By (3.35) we have the following possible cases:

(i) $f_\tau(1,2) \geq 2$ and $f_\tau(3,4) \geq 2$;

(ii) $f_\tau(1,2) \geq 2$ and $f_\tau(3,u) \geq 1$ for some $u \in \{1,2\}$;

(iii) $f_\tau(1,2) = 1$ and $f_\tau(1,3) \geq 1$ and $f_\tau(4,u) \geq 1$ for some $u \in \{1,2,3\}$;

(iv) $f_\tau(1,2) = 1$ and $f_\tau(1,4) \geq 1$ and $f_\tau(3,u) \geq 1$ for some $u \in \{1,2,4\}$.

Note that the cases (iii) and (iv) are symmetric. In case (i) we have that the power of $r_n(i,j)$ and $r_n(k,l)$ in $R(\tau,(i,j,k,l))$ is at least 2. Since $|r_n(i,j)| \leq 1$ for all $i,j \in \{[ns]+1,\ldots,[nt]\}$ we can bound

$$|R(\tau,(i,j,k,l))| \leq r_n^2(i,j) r_n^2(k,l).$$

Therefore, by (3.23)

$$\begin{aligned} D(\tau) &\leq \frac{1}{n^2} \sum_{i,j,k,l=[ns]+1}^{[nt]} r_n^2(i,j) r_n^2(k,l) \\ &= \frac{1}{n^2} \left(\sum_{i,j=[ns]+1}^{[nt]} r_n^2(i,j) \right)^2 \leq M^2 \left(\frac{[nt]-[ns]}{n} \right)^2. \end{aligned} \tag{3.36}$$

In case (ii) we bound (if $f_\tau(3,1) \geq 1$; the other case is symmetric)

$$|R(\tau,(i,j,k,l))| \leq r_n^2(i,j) |r_n(k,l) r_n(k,i)|.$$

Then by the inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$ and (3.23) we have

$$\begin{aligned} D(\tau) &\leq \frac{1}{n^2} \sum_{i,j,k,l=[ns]+1}^{[nt]} r_n^2(i,j)|r_n(k,l)r_n(k,i)| \\ &\leq \frac{1}{2n^2} \sum_{i,j,l=[ns]+1}^{[nt]} r_n^2(i,j) \sum_{k=[ns]+1}^{[nt]} \big(r_n^2(k,l) + r_n^2(k,i)\big) \\ &\leq M^2 \left(\frac{[nt]-[ns]}{n}\right)^2. \end{aligned} \tag{3.37}$$

Finally, in case (iii) we bound for some $s \in \{i,j,k\}$

$$|R(\tau,(i,j,k,l))| \leq |r_n(i,j)r_n(k,l)r_n(i,k)r_n(l,s)|$$

Again, by the inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$ and (3.23) we have

$$\begin{aligned} D(\tau) &\leq \frac{1}{n^2} \sum_{i,j,k,l=[ns]+1}^{[nt]} |r_n(i,j)r_n(k,l)r_n(i,k)r_n(l,s)| \\ &\leq \frac{1}{2n^2} \sum_{i,j,k=[ns]+1}^{[nt]} |r_n(i,j)r_n(i,k)| \sum_{l=[ns]+1}^{[nt]} \big(r_n^2(k,l) + r_n^2(l,s)\big) \\ &\leq \frac{M}{2n^2} \sum_{j,k=[ns]+1}^{[nt]} \sum_{i=[ns]+1}^{[nt]} \big(r_n^2(i,j) + r_n^2(i,k)\big) \\ &\leq M^2 \left(\frac{[nt]-[ns]}{n}\right)^2. \end{aligned} \tag{3.38}$$

If, on the other hand, case 2. happens, then without loss of generality we can assume, that $f_\tau(1,j) \geq 2$ for all $j \in \{2,3,4\}$. Then

$$|R(\tau,(i,j,k,l))| \leq r_n^2(i,j)r_n^2(i,k)$$

and by (3.23)

$$\begin{aligned} D(\tau) &\leq \frac{1}{n^2} \sum_{i,j,k,l=[ns]+1}^{[nt]} r_n^2(i,j)r_n^2(i,k) \leq \frac{1}{n^2} \sum_{i,j,l=[ns]+1}^{[nt]} r_n^2(i,j) \sum_{k=[ns]+1}^{[nt]} r_n^2(i,k) \\ &\leq M^2 \left(\frac{[nt]-[ns]}{n}\right)^2. \end{aligned} \tag{3.39}$$

By (3.34), (3.36), (3.37), (3.38), (3.39) and (3.33) we bound

$$\mathbf{E}(Y_t^n - Y_s^n)^4 \leq C \left(\frac{[nt] - [ns]}{n} \right)^2, \tag{3.40}$$

where

$$C := \beta_2^4 M^2 \sum_{\mathbf{m} \in I} |a(\mathbf{m})| \mathcal{C}(\mathbf{m}) \sum_{\tau \in \mathcal{T}(\mathbf{m})} 1.$$

By the Cauchy-Schwartz inequality, for any $t_1 \leq t \leq t_2$, (3.40) gives

$$\mathbf{E}(Y_{t_2}^n - Y_t^n)^2 (Y_t^n - Y_{t_1}^n)^2 \leq C \left(\frac{[nt_2] - [nt]}{n} \right) \left(\frac{[nt] - [nt_1]}{n} \right) \leq C(t_2 - t_1)^2.$$

Let us show (3.29). Let $0 = t_0 < t_1 < \ldots < t_u \in [0, 1]$ and denote $Z_v^n := Y_{t_v}^n - Y_{t_{v-1}}^n,\ n \in \mathbb{N},\ v = 1, \ldots, u$. Let $\mathfrak{H}$ be the closure in $L^2 = L^2(\Omega, \mathcal{F}, \mathbf{P})$ of all finite linear combinations of $\{Z_i^n, i = 1 \ldots, u,\ n \in \mathbb{N}\}$. Then $\mathfrak{H}$ is a separable Hilbert space with the inner product being the covariance of its elements, and so the identity map on $\mathfrak{H}$ is an isonormal Gaussian process. Let I_m be the multiple Wiener integral on $\mathfrak{H}^{\odot m}$ and J_m be the orthogonal projection operator on $\mathcal{H}_m$.

For all $m \geq 2,\ n \in \mathbb{N},\ v = 1, \ldots, u$ let

$$f_{m,n}^v := \frac{1}{\sqrt{n}} \sum_{i=[nt_{v-1}]+1}^{[nt_v]} \alpha_{i,n} \frac{a_m}{m!} X_{i,n}^{\otimes m}$$

By (3.30) and (1.3) we have for all $n \in \mathbb{N},\ v = 1, \ldots, u$

$$Z_v^n = \frac{1}{\sqrt{n}} \sum_{i=[nt_{v-1}]+1}^{[nt_v]} \alpha_{i,n} \sum_{m=2}^{\infty} \frac{a_m}{m!} H_m(X_{i,n}) = \sum_{m=2}^{\infty} I_m(f_{m,n}^v),$$

and we want to apply Theorem 1.4. For that we need to check its hypotheses $(i) - (iv)$. By (1.4) for $(i) - (iii)$ it is sufficient that for all $m \geq 2$, $v = 1, \ldots, u$ and $n \in \mathbb{N}$ there exist sequences $(\delta_m)_{m \geq 2}$ and $(\sigma_m^2)_{m \geq 2}$ such that

$$\mathbf{E}(J_m Z_v^n)^2 \leq \delta_m \quad \text{and} \quad \lim_{n \to \infty} \mathbf{E}(J_m Z_v^n)^2 = \sigma_m^2$$

with $\sum_{m\geq 2}\delta_m<\infty$, and for all $q,v=1,\dots,u$ such that $q\neq v$

$$\lim_{n\to\infty}\mathbf{E}J_mZ_v^nJ_mZ_q^n=0.$$

Let $v\in\{1,\dots,u\}$ and $m\geq 2$. Then

$$\mathbf{E}(J_mZ_v^n)^2=\frac{1}{n}\sum_{i,j=[nt_{v-1}]+1}^{[nt_v]}\alpha_{i,n}\alpha_{j,n}\frac{a_m^2}{m!}r_n^m(i,j).$$

Since $|r_n(i,j)|\leq 1$, by (3.24) for all n large enough and $i,j\in\{1,\dots,n\}$ we can bound

$$\mathbf{E}(J_mZ_v^n)^2\leq\frac{a_m^2}{m!}\frac{1}{n}\sum_{i,j=[nt_{v-1}]+1}^{[nt_v]}\alpha_{i,n}\alpha_{j,n}r_n^2(i,j)\leq\frac{a_m^2}{m!}(t_v-t_{v-1})(\Phi_2+1)=:\delta_m.$$

Since $H\in\mathcal{F}_0$ we have $\sum_{m\geq 2}\delta_m<\infty$. By (3.24) we also have

$$\lim_{n\to\infty}\mathbf{E}(J_mZ_v^n)^2=(t_v-t_{v-1})\frac{a_m^2}{m!}\Phi_m.$$

Let $m\geq 2$ and $q,v=1,\dots,u$ be such that $q\neq v$. Then by (3.25)

$$\begin{aligned}\mathbf{E}J_mZ_v^nJ_mZ_q^n&=\frac{a_m^2}{m!}\frac{1}{n}\sum_{i=[nt_{v-1}]+1}^{[nt_v]}\sum_{j=[nt_{q-1}]+1}^{[nt_q]}\alpha_{i,n}\alpha_{j,n}r_n^m(i,j)\\&\leq\frac{a_m^2\beta_2^2}{m!}\frac{1}{n}\sum_{i=[nt_{v-1}]+1}^{[nt_v]}\sum_{j=[nt_{q-1}]+1}^{[nt_q]}r_n^2(i,j)\to 0,\end{aligned}$$

as $n\to\infty$. Finally, for the hypothesis (iv) by (1.6) we have

$$f_{m,n}^v\otimes_p f_{m,n}^v=\frac{a_m}{m!}\frac{1}{n}\sum_{i,j=[nt_{v-1}]+1}^{[nt_v]}\alpha_{i,n}\alpha_{j,n}r_n^p(i,j)X_{i,n}^{\otimes(m-p)}\otimes X_{j,n}^{\otimes(m-p)}$$

and by (1.2) and (3.26)

$$\begin{aligned}||f_{m,n}^v\otimes_p f_{m,n}^v||^2_{\mathfrak{H}^{\otimes m}}&=\langle f_{m,n}^v\otimes_p f_{m,n}^v,f_{m,n}^v\otimes_p f_{m,n}^v\rangle_{\mathfrak{H}^{\otimes 2(2m-p)}}\\&=\frac{a_m^2}{(m!)^2}\frac{1}{n^2}\sum_{i,j,k,l=[nt_{v-1}]+1}^{[nt_v]}\alpha_{i,n}\alpha_{j,n}\alpha_{k,n}\alpha_{l,n}\\&\quad\times r_n^p(i,j)r_n^p(k,l)r_n^{m-p}(i,k)r_n^{m-p}(j,l)\end{aligned}$$

$$\leq \frac{a_m^2 \beta_2^4}{(m!)^2} \frac{1}{n^2} \sum_{i,j,k,l=[nt_{v-1}]+1}^{[nt_v]} |r_n^p(i,j) r_n^p(k,l) r_n^{m-p}(i,k) r_n^{m-p}(j,l)|,$$

which converges to 0 as $n \to \infty$. By Theorem 1.4 we conclude that

$$(Z_1^n, \ldots, Z_u^n) \Rightarrow \xi, \quad \xi \sim N_u(0, \lambda^2 \operatorname{diag}(t_1, t_2 - t_1, \ldots, t_u - t_{u-1}))$$

where λ is defined in (3.27), thus we have proved (3.29) and our theorem. □

3.4 Appendix

We present here a proof that the random variable Y in the proof of Theorem 3.6 is absolutely continuous. We prove an auxiliary lemma first.

Lemma 3.12. *Let $A \subset \mathbb{R}$ have Lebesgue measure zero, $n \in \mathbb{N}$, $D \subset \mathbb{R}^n \times A$ be closed and $g : D \to \mathbb{R}$ be a continuous function. Then*

$$S := \{(z, g(z,c)) : (z,c) \in D\}$$

also has Lebesgue measure zero.

Proof. Let $N \in \mathbb{N}$ and denote $I_N := [-N, N]$. Let $\varepsilon > 0$. Then for some $k \in \mathbb{N}$ there exist $\{a_i, b_i\}_{i=1}^k \subset \mathbb{R}$ such that

$$A \cap I_N \subset \bigcup_{i=1}^k [a_i, b_i], \quad \text{and} \quad 0 < b_i - a_i < \varepsilon/k, \ i = 1, \ldots, k, \tag{3.41}$$

where the intervals (a_i, b_i) are pairwise disjoint.

Since g is continuous, it is uniformly continuous on $\left(I_N^n \times \bigcup_{i=1}^k [a_i, b_i]\right) \cap D =: D_N$. Let $\eta > 0$ and $\delta > 0$ be such that $|g(x) - g(y)| < \eta$ holds whenever $x, y \in D_N$ satisfy $||x - y||_2 < \delta$.

Let $(B_j)_{j=1}^J$ be a regular partition of I_N^n into hypercubes such that their interiors are pairwise disjoint and let $(C_l)_{l=1}^L$ be a regular partition of $\bigcup_{i=1}^k [a_i, b_i]$ such that their interiors are pairwise disjoint and for all $j = 1, \ldots, J$ and $l = 1, \ldots, L$

$$||x - y||_2 < \delta \text{ for all } x, y \in B_j \times C_l.$$

By (3.41) we have that for some constant $R \in \mathbb{R}$

$$L = (\varepsilon/\delta)R. \tag{3.42}$$

For all $j = 1, \dots, J$ and $l = 1, \dots, L$ let b_{jl} be any element of $(B_j \times C_l) \cap D_N$. Then for all $j = 1, \dots, J$ and $l = 1, \dots, L$

$$g(z,c) \in [g(b_{jl}) - \eta, g(b_{jl}) + \eta], \tag{3.43}$$

for all $(z,c) \in (B_j \times C_l) \cap D_N$.

We will show that the set

$$S_N := \{(z, g(z,c)) : z \in I_N^n, c \in A \cap I_N, (z,c) \in D\}$$

has Lebesgue measure zero. By (3.41) and (3.43) we have

$$\begin{aligned} S_N &\subset \{(z, g(z,c)) : (z,c) \in D_N\} \\ &= \bigcup_{j=1}^{J} \bigcup_{l=1}^{L} \{(z, g(z,c)) : (z,c) \in D_N \cap (B_j \times C_l)\} \\ &\subset \bigcup_{j=1}^{J} \bigcup_{l=1}^{L} B_j \times [g(b_{jl}) - \eta, g(b_{jl}) + \eta]. \end{aligned}$$

Therefore, due to the fact that (B_j) are pairwise disjoint and by (3.42) we can bound

$$\begin{aligned} \lambda^{n+1}(S_N) &\le \sum_{j=1}^{J} \sum_{l=1}^{L} \lambda^{n+1}(B_j \times [g(b_{jl}) - \eta, g(b_{jl}) + \eta]) \\ &= \sum_{j=1}^{J} \lambda^n(B_j) \sum_{l=1}^{L} \lambda([g(b_{jl}) - \eta, g(b_{jl}) + \eta]) \\ &= 2\eta L (2N)^n = 2\eta\delta^{-1} R (2N)^n \varepsilon, \end{aligned}$$

where λ^d is the Lebesgue measure on $\mathbb{R}^d$. Since ε is arbitrary, we conclude that $\lambda^{n+1}(S_N) = 0$ for all $N \in \mathbb{N}$. Since $S = \bigcup_{n \in \mathbb{N}} S_N$, $\lambda^{n+1}(S) = 0$. □

Lemma 3.13. *Let $G := (X_i,\ i = 1, \dots, n)$ be a Gaussian vector, a function $H \in \mathcal{F}_0$ and $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$. Then $S(G, H, \alpha)$ defined in (3.2) has a density.*

Proof. Let $A \subset \mathbb{R}_+$ have Lebesgue measure zero and $f : \mathbb{R}^n \to \mathbb{R}_+$ be a function with values

$$f(x_1, \dots, x_n) = \sum_{i=1}^{n} \alpha_i H(x_i),\ (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Then $S(G, H, \alpha) = f(G)$. Let $\{[a_j, b_j]\}_{j \geq 1}$ be the partition from Definition 2 for the function H. For $j \geq 1$ let $H^{(j)}$ be the restriction of H onto $[a_j, b_j]$ and denote

$$D_j := \left\{ (z, c) \in \mathbb{R}^{n-1} \times A :\ \exists x \in [a_j, b_j] :\ \alpha_n H^{(j)}(x) = c - \sum_{i=1}^{n-1} \alpha_i H(z_i) \right\}.$$

Then D_j is closed for all $j \geq 1$. Let $J := \{j \geq 1 :\ D_j \neq \emptyset\}$ and for all $j \in J$ define functions $g_j : D_j \to \mathbb{R}$ with values

$$g_j(z, c) := (H^{(j)})^{-1} \left(\alpha_n^{-1} \left(c - \sum_{i=1}^{n-1} \alpha_i H(z_i) \right) \right),$$

$(z, c) \in D_j$. Then

$$f^{-1}(A) = \{x \in \mathbb{R}^n :\ f(x) = c \text{ for some } c \in A\} = \bigcup_{j \in J} \{(z, g_j(z, c)) :\ (z, c) \in D_j\}.$$

Since for all $j \in J$, g_j is continuous on D_j, by Lemma 3.12, $f^{-1}(A)$ has Lebesgue measure zero and therefore

$$\mathbf{P}(S(G, H, \alpha) \in A) = \mathbf{P}(f(G) \in A) = \mathbf{P}(G \in f^{-1}(A)) = 0.$$

Then by the Radon-Nikodym theorem $S(G, H, \alpha)$ has a density. □

Chapter 4

Applications

In this section we apply Theorems 2.1, 3.6 and 3.10 to several Gaussian processes.

4.1 Fractional Brownian motion

A fractional Brownian motion $B_H := \{B_H(t),\ t \in [0,T]\}$ with the Hurst index $H \in (0,1)$ is a mean zero Gaussian stochastic process with the covariance function

$$F_H(s,t) := \Gamma_{B_H}(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

for $(s,t) \in [0,T]^2$. Its incremental variance function is given by

$$\sigma^2_{B_H}(s,t) = |s-t|^{2H}, \quad (s,t) \in [0,T]^2. \tag{4.1}$$

By Proposition 15 in [31], $B_H \in \mathcal{LSI}(\rho_H)$ with $\rho_H(u) = u^H = \sigma_{B_H}(s+u,s)$, $u \geq 0$. In the case $\rho = \rho_H$, the function η with values (5) is equal to the function η_H given by

$$\eta_H(k) := 2^{-1}\left[(k+1)^{2H} + (k-1)^{2H} - 2k^{2H}\right] = \eta(k,\Delta) \tag{4.2}$$

for each $k \geq 1$ and $\Delta > 0$. Let $\eta_H(0) := 1$. Then $\eta_H(k) = \eta_H(-k)$ for all $k \in \mathbb{Z}$ and η_H behaves asymptotically as

$$\eta_H(k) \sim H(2H-1)|k|^{2H-2},\ |k| \to \infty. \tag{4.3}$$

Let $r > 0$ and $H(x) := |x|^r,\ x \in \mathbb{R}$. For each $n \in \mathbb{N}$ we have

$$V(B_H, H, m_n) = \widetilde{V}(B_H, H, \rho_H, m_n)$$

for any increasing sequence (m_n) of positive integers (cf. (1) and (2)).

The following statement is known (see e.g. [13]). We prove it as a consequence of Theorem 2.1.

Corollary 4.1. *Let $r > 0$, $H \in (0, 3/4)$ and $(m_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integers such that $\Delta_n = T/m_n \to 0$ as $n \to \infty$. Then*

$$\sqrt{\Delta_n} \sum_{i=1}^{m_n} \left[\left(\frac{|\Delta_i^n B_H|}{\Delta_n^H} \right)^r - c_r \right] \Rightarrow \lambda_r Z, \quad \text{as } n \to \infty,$$

where the variance

$$\lambda_r^2 = T \sum_{m=2}^{\infty} a_{rm}^2 m! \left(1 + 2 \sum_{k=1}^{\infty} \eta_H(k)^m \right)$$

and coefficients a_{rm} are defined by (2.6).

Proof. It is enough to check the hypotheses of Theorem 2.1 for the Gaussian process $G = B_H$. It is clear that B_H satisfies the hypothesis (a) of Theorem 2.1 with $C_1 = 1$. Since $H < 3/4$, by (4.3) and (4.2), the hypothesis (b) of Theorem 2.1 holds for B_H.

By (2.7), for each $n \in \mathbb{N}$ and $(i, j) \in \{1, 2, \ldots, m_n\}$, we have $\square_{i,j}^n F_H = 2^{-1} \square_{i,j}^n \tilde{\rho}$, where $\tilde{\rho}(s, t) = -[\rho_H(|s - t|)]^2 = -|s - t|^{2H}$ for $(s, t) \in [0, T]^2$. Thus the hypothesis (c) of Theorem 2.1 holds for B_H. The conclusion of Corollary 4.1 now follows from Theorem 2.1. $\square$

Corollary 4.2. *Let $H \in (0, 3/4]$, $F \in \mathcal{F}_1$, $d \in \mathbf{D}$ and for all $n \in \mathbb{N}$ denote*

$$V_n := \sum_{i=1}^{n} F\left(n^H \Delta_i^n B_H\right) \quad \text{and} \quad Z_n := \frac{V_n - \mathbf{E} V_n}{\sqrt{\operatorname{var}(V_n)}}.$$

Then for some $c \in \mathbb{R}$ and for all $n \in \mathbb{N}$

$$d(Z_n, Z) \le c \begin{cases} n^{-1/2} & \text{if } H \in (0, 1/2), \\ n^{2H-3/2} & \text{if } H \in [1/2, 3/4), \\ (\log n)^{-1/2} & \text{if } H = 3/4. \end{cases}$$

Remark 4.3. If $F = H_q$, when $0 < H \le 1/2$ the same rates of convergence of V_n for fractional Brownian motion with the Hurst index H have been obtained in [33] (Theorem 4.1), [34] (Example 2.7) and [10] (Propositions 6.6 and 6.7). In all of the aforementioned papers the bound has been improved when $1/2 < H < 3/4$ and a bound was also proved in the case $H \in [3/4, 1-1/(2q))$. The rates in [10] are shown to be optimal. One cannot expect to prove convergence to a normal random variable when $H > 3/4$ in Corollary 4.2. For general F-variations it was shown in [21] (Theorem 2) that in case F has Hermite rank 2, V_n converges in distribution to a random variable in the second Wiener chaos.

Proof. Let $B_n := (n^H \Delta_i^n B_H,\ i = 1, \dots, n)$ and $\alpha_n := (1, \dots, 1) \in \mathbb{N}^n$. By (4.1) for all $n \in \mathbb{N}$ B_n is a standard Gaussian vector and by (3.2) we get $Z_n = W(B_n, F, \alpha_n)$. Therefore our aim is to apply Theorem 3.6 to Z_n.

By (2.7) and (4.2) for $i, j \in \{1, \dots, n\}$

$$r_n(i,j) = T^{-2H} n^{2H} \mathbf{E} \Delta_i^n B_H \Delta_j^n B_H = T^{-2H} n^{2H} \Box_{i,j}^n F_H = \eta_H(i-j).$$

Then by (4.3)

$$\sum_{j=1}^n r_n^2(i,j) = \sum_{j=1}^n \eta_H^2(i-j) = \sum_{r=-i+1}^{n-i} \eta_H^2(r) \le 2 \sum_{r=0}^n \eta_H^2(r) \trianglelefteq \sum_{r=1}^n r^{4(H-1)} \tag{4.4}$$

and for all $m \ge 2$

$$\begin{aligned} \sum_{i,j=1}^n r_n^m(i,j) &= n + 2 \sum_{1 \le i < j \le n} \eta_H^m(i-j) = n + 2 \sum_{k=1}^{n-1} (n-k) \eta_H^{2m}(k) \\ &\sim n + 2H(2H-1) \sum_{k=1}^{n-1} (n-k) k^{m(2H-2)}, \end{aligned}$$

as $n \to \infty$. By (3.11) we thus have

$$\operatorname{var}(V_n) = \sum_{m=2}^\infty \frac{a_m^2}{m!} \sum_{i,j=1}^n r_n^m(i,j) \sim c_{F,H} \begin{cases} n & \text{if } H \in (0, 3/4), \\ n \log n & \text{if } H = 3/4, \end{cases} \tag{4.5}$$

as $n \to \infty$, where $a_m := \mathbf{E} F(Z) H_m(Z),\ Z \sim N(0,1),\ m \ge 2$ and the constant $c_{F,H}$

does not depend on n. Also, by (4.3)

$$\sum_{i,k,l=1}^{n} |r_n(i,k)r_n(k,l)| = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} |r_n(i,k)| \right)^2 = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} |\eta_H(i-k)| \right)^2$$
$$\leq \sum_{k=1}^{n} \left(2 \sum_{r=0}^{n} |\eta_H(r)| \right)^2 \trianglelefteq n \left(\sum_{r=1}^{n} r^{2(H-1)} \right)^2 . \tag{4.6}$$

When $0 < H < 1/2$, we have $\sum_{r=1}^{\infty} r^{2(H-1)} < \infty$. Assume $1/2 < H \leq 3/4$. By (1.17) we have

$$n \left(\sum_{r=1}^{n} r^{2(H-1)} \right)^2 \trianglelefteq n^{4H-1}. \tag{4.7}$$

Finally, set $H = 1/2$. It is well-known that $B_{1/2}$ is a standard Brownian motion and in this case $r_n(i,k) = 0$ whenever $i \neq k$. Therefore

$$\sum_{i,k,l=1}^{n} |r_n(i,k)r_n(k,l)| = \sum_{k=1}^{n} r_n^2(k,k) = n. \tag{4.8}$$

By Theorem 3.6, (4.4), (4.5), (4.6), (4.8) and (4.7) there exists a constant $c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$

$$d^2(Z_n, Z) \leq \frac{c}{[\mathrm{var}(V_n)]^2} \max_{1 \leq i \leq n} \sum_{j=1}^{n} r_n^2(i,j) \sum_{i,k,l=1}^{n} |r_n(i,k)r_n(k,l)|$$
$$\trianglelefteq \begin{cases} n^{-1} & \text{if } H \in (0, 1/2), \\ n^{4H-3} & \text{if } H \in [1/2, 3/4), \\ (\log n)^{-1} & \text{if } H = 3/4, \end{cases}$$

which completes the proof. □

4.2 A class of Gaussian processes with stationary increments

Following Guyon and León [21], Barndorff-Nielsen et al. [6] considered a class of Gaussian processes defined as follows. Let $G = \{G(t) : t \in [0,T]\}$ be a mean zero Gaussian process with stationary increments. Let $R\colon [0,\infty) \to \mathbb{R}$ be a function with values

$$R(t) := \sigma_G^2(s, s+t) = \mathbf{E}(G(s+t) - G(s))^2, \quad t \geq 0. \tag{4.9}$$

In [6], the Gaussian process G is assumed to satisfy the conditions $(i) - (iii)$, where

(i) $R(t) = t^\beta L_0(t)$ for some $\beta \in (0, 2)$ and some positive slowly varying at 0 function L_0, which is continuous on $(0, \infty)$;

(ii) $R''(t) = t^{\beta-2} L_2(t)$ for some slowly varying at 0 function L_2, which is continuous on $(0, \infty)$;

(iii) there exists a $b \in (0, 1)$ such that

$$\limsup_{x \to 0} \sup_{y \in [x, x^b]} \frac{|L_2(y)|}{|L_0(x)|} < \infty.$$

Under $(i) - (iii)$ with $\beta \in (0, 3/2)$ a functional central limit theorem in the Skorokhod space $D([0, T]^2)$ is proved in [6, Theorem 6]. This yields a central limit theorem for $V(G, r, R^{1/2}, n)$ under the same hypotheses.

We will show that the hypotheses $(i) - (iii)$ with $\beta \in (0, 3/2)$ imply the hypotheses of Theorem 2.1 for any Gaussian process with stationary increments. It is clear that $G \in \mathcal{LSI}(\rho)$ with $\rho = R^{1/2}$, and the hypothesis (a) of Theorem 2.1 holds with $C_1 = 1$. As for hypothesis (b) we have

$$\eta_R(k, \Delta) := \frac{R((k+1)\Delta) + R((k-1)\Delta) - 2R(k\Delta)}{2R(\Delta)} = \eta(k, \Delta)$$

for each $k \geq 1$ and $\Delta > 0$. By Lemma 1 in [6] we have that for any $m \geq 2$

$$\lim_{n \to \infty} \sum_{k=1}^{y_n} \eta_R^m(k, 1/n) = \sum_{k=1}^{\infty} \eta_\beta^m(k) =: \Psi_m < \infty,$$

where η_β is defined by (4.2) and (y_n) is any increasing and unbounded sequence of positive integers with values $y_n \leq n - 1$ for each $n \geq 2$. Thus the hypothesis (b) of Theorem 2.1 holds for G. Due to stationarity of increments, by (2.7) we have $\Box_{i,j}^n \Gamma_G = 2^{-1} \Box_{i,j}^n \tilde{\rho}$ for all $n \in \mathbb{N}$ and $i, j \in \{1, \ldots, n\}$, where $\tilde{\rho}(s, t) := -R(|t - s|)$, $(s, t) \in [0, T]^2$. Thus the hypothesis (c) of Theorem 2.1 holds for G, and so the conclusion of Theorem 2.1 must hold. We have proved the following corollary.

Corollary 4.4. *Let* $G = \{G(t)\colon\ t \in [0,T]\}$ *be a mean zero Gaussian process with stationary increments and let* R *be a function with values* (4.9) *satisfying the hypotheses* $(i)-(iii)$ *with* $\beta \in (0,3/2)$. *Let* $r > 0$ *and* $\Delta_n := T/n$ *for each* $n \in \mathbb{N}$. *Then*

$$\sqrt{n}\sum_{i=1}^{n}\left[\left(\frac{|\Delta_i^n G|}{\sqrt{R(1/n)}}\right)^r - c_r\right] \Rightarrow \lambda_r Z, \quad \text{as } n \to \infty,$$

where the variance

$$\lambda_r^2 = T\sum_{m=2}^{\infty} a_{rm}^2 m!\left(1 + 2\sum_{k=1}^{\infty}\eta_\beta(k)^m\right)$$

and coefficients a_{rm} *are defined by* (2.6).

4.3 Subfractional Brownian motion

A *sub-fractional Brownian motion* with index H is a mean zero Gaussian stochastic process $G_H = \{G_H(t) : t \in [0,T]\}$ having the covariance function R_H with values

$$R_H(s,t) := s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right],$$

$(s,t) \in [0,T]^2$ (see [12]). Its incremental variance function is given by

$$\sigma_{G_H}^2(s,t) = |s-t|^{2H} + (s+t)^{2H} - 2^{2H-1}\left[t^{2H} + s^{2H}\right],$$

$(s,t) \in [0,T]^2$. The Gaussian process G_H has no stationary increments. By Proposition 17 in [31], $G_H \in \mathcal{LSI}(\rho_H)$ with $\rho_H(u) = u^H$, $u > 0$. The following bound is from [12]:

$$\beta_1 \le \frac{\sigma_{G_H}(s,t)}{\rho_H(|s-t|)} \le \beta_2, \tag{4.10}$$

where $\beta_1 := \sqrt{2 - 2^{2H-1}} \wedge 1$ and $\beta_2 := \sqrt{2 - 2^{2H-1}} \vee 1$ for all $t, s \in \mathbb{R}$.

Corollary 4.5. *Let* $H \in (0,3/4)$, *let* $r > 0$ *and let* $(m_n)_{n\in\mathbb{N}}$ *be an increasing sequence of positive integers such that* $\Delta_n = T/m_n \to 0$ *as* $n \to \infty$. *Then*

$$\sqrt{\Delta_n}\sum_{i=1}^{m_n}\left[\left(\frac{|\Delta_i^n G_H|}{\Delta_n^H}\right)^r - \mathbf{E}\left(\frac{|\Delta_i^n G_H|}{\Delta_n^H}\right)^r\right] \Rightarrow \lambda_r Z, \quad \text{as } n \to \infty,$$

where the variance

$$\lambda_r^2 = T \sum_{m=2}^{\infty} a_{rm}^2 m! \left(1 + 2\sum_{k=1}^{\infty} \eta_H(k)^m\right),$$

η_H is defined by (4.2) and coefficients a_{rm} are defined by (2.6).

Proof. It is enough to check the hypotheses of Theorem 2.1 for the Gaussian process $G = G_H$. We can assume that $H \neq 1/2$. By (4.10), G_H satisfies hypothesis (a). Since the local variance function ρ_H for G_H is the same as in the case of fractional Brownian motion, we conclude by the same arguments as in the proof of Corollary 4.1 that the hypothesis (b) of Theorem 2.1 holds for G_H.

To check the hypothesis (c) let $f(x,y) := (x+y)^{2H}$ for $(x,y) \in R_+^2$. By (2.7), for each $i,j \in \{2,\ldots,m_n\}$ we have

$$\begin{aligned} z(i,j) &:= \frac{\square_{i,j}^n [R_H - 2^{-1}\tilde{\rho}_H]}{\rho_H(\Delta_n)]^2} = \frac{\square_{i,j}^n R_H}{\Delta_n^{2H}} - \eta_H(|i-j|) \\ &= -\frac{1}{2}\left[(i+j)^{2H} + (i+j-2)^{2H} - 2(i+j-1)^{2H}\right] \\ &= -\frac{1}{2}\int_{i-1}^{i}\int_{j-1}^{j} \frac{\partial^2 f(x,y)}{\partial x \partial y} dxdy = -C\int_{i-1}^{i}\int_{j-1}^{j} \frac{dxdy}{(x+y)^{2(1-H)}}, \end{aligned}$$

where $C := H|2H-1| > 0$. Let $m \geq 2$ and $N \geq 2$ be integers, and let $\alpha := 2^{m(H-1)}$. Using the inequality $(x+y)^2 \geq 2xy$ and the fact that $H < 1$, it follows that

$$\begin{aligned} A_N :=& C^{-m}\sum_{i,j=2}^{N} |z(i,j)|^m = \sum_{i,j=2}^{N}\left(\int_{i-1}^{i}\int_{j-1}^{j}\frac{dxdy}{(x+y)^{2(1-H)}}\right)^m \\ \leq& \alpha \sum_{i,j=2}^{N}\left(\int_{i-1}^{i}\int_{j-1}^{j}\frac{dxdy}{(xy)^{1-H}}\right)^m = \alpha\left(\sum_{i=2}^{N}\left(\int_{i-1}^{i}\frac{dx}{x^{1-H}}\right)^m\right)^2 \\ \leq& \alpha\left(\sum_{i=1}^{N} i^{m(H-1)}\right)^2. \end{aligned} \tag{4.11}$$

Since $H < 3/4$ and $m \geq 2$, we have $m(H-1) < -1/2$, and so there exists a $\delta > 0$ such that $m(H-1) - 1/2 + \delta < -1$. Then

$$(N^{-1}A_N)^{1/2} \leq \frac{\sqrt{\alpha}}{N^{1/2}}\sum_{i=1}^{N} i^{m(H-1)} \leq \frac{\sqrt{\alpha}}{N^{\delta}}\sum_{i=1}^{N} i^{m(H-1)-1/2+\delta}.$$

Taking $N = m_n$ and letting $n \to \infty$, it follows that the hypothesis (c) holds for G_H. The proof of Corollary 4.5 is complete. □

Corollary 4.6. *Let* $H \in (0, 3/4]$, $F \in \mathcal{F}_1$, $d \in \mathbf{D}$ *and for all* $n \in \mathbb{N}$ *denote*

$$V_n := \sum_{i=1}^{n} F\left(\frac{\Delta_i^n G_H}{\sigma_{G_H}(t_i^n, t_{i-1}^n)}\right) \quad \textit{and} \quad Z_n := \frac{V_n - \mathbf{E}V_n}{\sqrt{\operatorname{var}(V_n)}}.$$

Then for some $c \in \mathbb{R}$ *and all* $n \in \mathbb{N}$

$$d(Z_n, Z) \le c \begin{cases} n^{-1/2} & \textit{if } H \in (0, 1/2), \\ n^{2H-3/2} & \textit{if } H \in [1/2, 3/4), \\ (\log n)^{-1/2} & \textit{if } H = 3/4. \end{cases}$$

Remark 4.7. The same rates of convergence for the H_q-variations of subfractional Brownian motion have been obtained in [42] (Theorem 3.1). When $H = 1/2$ we have the case of a Brownian motion, which in turn is a special case of fractional Brownian motion. It then follows from Remark 4.3 that the rate for $H = 1/2$ is optimal. Optimality of the rates when $H \in [0, 3/4] \backslash \{1/2\}$ is still an open problem.

Proof. Let $B_n := (\Delta_i^n G_H / \sigma_{G_H}(t_i^n, t_{i-1}^n),\ i = 1, \dots, n)$ and $\alpha_n := (1, \dots, 1) \in \mathbb{R}^n$. Then B_n is a standard Gaussian vector and by (3.2) we get $Z_n = W(B_n, F, \alpha_n)$. Therefore our aim is to apply Theorem 3.6 to Z_n.

For all $i, j \in \{1, \dots, n\}$ we have

$$r_n(i, j) = \mathbf{E} \frac{\Delta_i^n G_H \Delta_j^n G_H}{\sigma_{G_H}(t_i^n, t_{i-1}^n) \sigma_{G_H}(t_j^n, t_{j-1}^n)}.$$

Since $H \in (0, 3/4]$, by (4.10) we have

$$(2/3) T^{-2H} n^{2H} \left| \mathbf{E} \Delta_i^n G_H \Delta_j^n G_H \right| \le |r_n(i, j)| \le 2 T^{-2H} n^{2H} \left| \mathbf{E} \Delta_i^n G_H \Delta_j^n G_H \right|. \quad (4.12)$$

Let η_H be as in (4.2). It is easy to check that for all $i, j \in \{1, \dots, n\}$

$$T^{-2H} n^{2H} \mathbf{E} \Delta_i^n G_H \Delta_j^n G_H = \eta_H(|i - j|) + z(i, j), \quad (4.13)$$

where

$$z(i, j) := 2^{-1} \left[2(i + j - 1)^{2H} - (i + j)^{2H} - (i + j - 2)^{2H} \right].$$

In the proof of Theorem 2.1 it was shown that if $H < 3/4$, then for all $m \geq 2$ we have

$$\sum_{i,j=1}^{n} r_n^m(i,j) \sim nT\left(1 + 2\sum_{k=1}^{\infty} \eta_H^m(k)\right), \quad \text{as } n \to \infty.$$

Consider the case when $H = 3/4$. By (4.11) for all $p \geq 2$

$$\sum_{i,j=1}^{n} |z(i,j)|^p \trianglelefteq \left(\sum_{k=1}^{n} k^{-p/4}\right)^2 \trianglelefteq n, \quad \text{as } n \to \infty, \tag{4.14}$$

by (1.17). Also, by (1.17) and (4.3) for all $p \geq 2$

$$\begin{aligned}\sum_{i,j=1}^{n} \eta_{3/4}^p(|i-j|) &= n + 2\sum_{k=1}^{n-1}(n-k)\eta_{3/4}^p(k) \sim n + \frac{3}{4}\left[n\sum_{k=1}^{n-1} k^{-p/2} - \sum_{k=1}^{n-1} k^{-p/2+1}\right] \\ &\trianglelefteq \begin{cases} n\log n & \text{if } p = 2, \\ n & \text{if } p > 2, \end{cases} \quad \text{as } n \to \infty. \end{aligned} \tag{4.15}$$

By the Cauchy-Schwarz inequality, (4.14) and (4.15) for all $m \geq 2$ and $1 \leq p \leq m$ we have

$$\begin{aligned}&\frac{1}{n\log n}\sum_{i,j=1}^{n} |\eta_{3/4}^{m-p}(|i-j|)z^p(i,j)| \\ &\qquad \leq \left(\frac{1}{n\log n}\sum_{i,j=1}^{n} \eta_{3/4}^{2m-2p}(|i-j|)\right)^{1/2}\left(\frac{1}{n\log n}\sum_{i,j=1}^{n} z^{2p}(i,j)\right)^{1/2}, \end{aligned} \tag{4.16}$$

which converges to 0 as $n \to \infty$. By (4.13), the binomial theorem, (4.15) and (4.16) we have for all $m \geq 2$

$$\begin{aligned}\sum_{i,j=1}^{n} T^{-3m/2} n^{3m/2}\left(\mathbf{E}\Delta_i^n G_H \Delta_j^n G_H\right)^m &= \sum_{i,j=1}^{n} [\eta_{3/4}(|i-j|) + z(i,j)]^m \\ &= \sum_{i,j=1}^{n}\sum_{p=0}^{m}\binom{m}{p}\eta_{3/4}^{m-p}(|i-j|)z^p(i,j) \\ &= \sum_{i,j=1}^{n} \eta_{3/4}^m(|i-j|) + o(n\log n), \end{aligned} \tag{4.17}$$

as $n \to \infty$. By (3.11), (4.12), (4.17) and (4.15) we thus have

$$\operatorname{var}(V_n) = \sum_{m=2}^{\infty} \frac{a_m^2}{m!} \sum_{i,j=1}^{n} r_n(i,j)^m \sim Cn \log n, \tag{4.18}$$

as $n \to \infty$, where $a_m := \mathbf{E}F(Z)H_m(Z),\ m \geq 2$ and the constant C does not depend on n.

The proof of Theorem 3.1 in [42] shows that for all $H \in (0, 3/4]$ and all $i = 1, \dots, n$

$$\sum_{j=1}^{n} r_n^2(i,j) \leq 1 + \sum_{j=1}^{n} j^{4(H-1)} \tag{4.19}$$

and

$$\sum_{i,k,l=1}^{n} |r_n(i,k) r_n(k,l)| \trianglelefteq n \left(\sum_{k=1}^{n-1} k^{2(H-1)} \right)^2 \tag{4.20}$$

as $n \to \infty$. By (4.19), (4.20), (1.17), (4.18) and Theorem 3.6 there exists a constant $c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$

$$\begin{aligned} d^2(Z_n, Z) &\leq \frac{c}{[\operatorname{var}(V_n)]^2} \max_{1 \leq i \leq n} \sum_{j=1}^{n} r_n^2(i,j) \sum_{i,k,l=1}^{n} |r_n(k,l)||r_n(i,k)| \\ &\trianglelefteq \begin{cases} n^{-1} & \text{if } H \in (0, 1/2), \\ n^{4H-3} & \text{if } H \in [1/2, 3/4), \\ (\log n)^{-1} & \text{if } H = 3/4, \end{cases} \end{aligned}$$

finishing the proof. □

4.4 Bifractional Brownian motion

Let $0 < T < \infty$, $0 < H < 1$ and $0 < K \leq 1$. A *bifractional Brownian motion* with parameters (H, K) is a mean zero Gaussian stochastic process $B_{H,K} = \{B_{H,K}(t) : t \in [0, T]\}$ having the covariance function $C_{H,K}$ with values

$$C_{H,K}(s,t) := 2^{-K} \left\{ (t^{2H} + s^{2H})^K - |t - s|^{2HK} \right\},$$

$(s,t) \in [0,T]^2$ (see [22]). The Gaussian process $B_{H,K}$ is a selfsimilar stochastic process of order $HK \in (0,1)$ and its incremental variance function is given by

$$\sigma^2_{B_{H,K}}(s,t) = 2^{1-K}[|t-s|^{2HK} - (t^{2H} + s^{2H})^K] + t^{2HK} + s^{2HK}$$

for each $s,t \geq 0$. The Gaussian process $B_{H,K}$ has no stationary increments. By Proposition 18 in [31], $B_{H,K} \in \mathcal{LSI}(\rho_{HK})$ with $\rho_{HK}(u) = 2^{(1-K)/2}u^{HK}$, $u > 0$. By Proposition 3.1 in [22] we have

$$\frac{1}{\sqrt{2}} \leq \frac{\sigma_{B_{H,K}}(s,t)}{\rho_{H,K}(|s-t|)} \leq \sqrt{2} \tag{4.21}$$

for all $t,s \in \mathbb{R}$.

Corollary 4.8. *Let $HK \in (0,3/4)$, let $r > 0$ and let $(m_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integers such that $\Delta_n = T/m_n \to 0$ as $n \to \infty$. Then*

$$\sqrt{\Delta_n}\sum_{i=1}^{m_n}\left[\left(\frac{|\Delta_i^n B_{HK}|}{\Delta_n^{HK}}\right)^r - \mathbf{E}\left(\frac{|\Delta_i^n B_{HK}|}{\Delta_n^{HK}}\right)^r\right] \Rightarrow \lambda_r Z, \quad \text{as } n \to \infty,$$

where the variance

$$\lambda_r^2 = T\sum_{m=2}^{\infty} a_{rm}^2 m!\left(1 + 2^{1-m}\sum_{k=1}^{\infty}\left((k+1)^{2HK} + (k-1)^{2HK} - 2k^{2HK}\right)^m\right).$$

and coefficients a_{rm} are defined in (2.6).

Proof. It is enough to check the hypotheses of Theorem 2.1 for the Gaussian process $G = B_{H,K}$. By (4.21) $B_{H,K}$ satisfies the hypothesis (a) with $C_1 = 2^{-1/2}$.

Recalling notation (5) of $\eta(k,\Delta_n)$ we have for $k \geq 1$

$$\begin{aligned}\eta_n(k) &= 2^{-1}\Delta_n^{-2HK}\left(\Delta_n^{2HK}(k+1)^{2HK} + \Delta_n^{2HK}(k-1)^{2HK} - 2\Delta_n^{2HK}k^{2HK}\right)\\ &= 2^{-1}\left((k+1)^{2HK} + (k-1)^{2HK} - 2k^{2HK}\right) =: \eta_{HK}(k).\end{aligned}$$

Since $HK < 3/4$, by (4.3) the hypothesis (b) of Theorem 2.1 holds for $B_{H,K}$.

To check the hypothesis (c) let $f(x,y) := (x^{2H} + y^{2H})^K$, $(x,y) \in R_+^2$. By (2.7) with $\Gamma_G = C_{H,K}$ and $\rho = \rho_{HK}$, for each $i,j \in \{2,\ldots,m_n\}$ we have

$$z(i,j) := \frac{\square_{i,j}^n[C_{H,K} - 2^{-1}\tilde{\rho}_{HK}]}{[\rho_{HK}(\Delta_n)]^2} = \frac{\square_{i,j}^n C_{HK}}{\Delta_n^{2HK}} - \eta_{HK}(|i-j|)$$

$$
\begin{aligned}
&= \frac{1}{2}\big[(i^{2H}+j^{2H})^K + ((i-1)^{2H}+(j-1)^{2H})^K \\
&\quad -(i^{2H}+(j-1)^{2H})^K - ((i-1)^{2H}+j^{2H})^K\big] \\
&= \frac{1}{2}\int_{i-1}^{i}\int_{j-1}^{j}\frac{\partial^2 f(x,y)}{\partial x\partial y}dxdy = C\int_{i-1}^{i}\int_{j-1}^{j}\frac{dxdy}{(x^{2H}+y^{2H})^{2-K}},
\end{aligned}
$$

where $C := 2K(1-K)H^2 > 0$.

Let $m \geq 2$ and let $N \geq 2$ be integers, and let $\alpha := 2^{m(K-2)}$. Using the inequality $(x+y)^2 \geq 2xy$ and the fact that $HK < 3/4$, we have

$$
\begin{aligned}
A_N :=& C^{-m}\sum_{i,j=2}^{N}|z(i,j)|^m = \sum_{i,j=2}^{N}\left(\int_{i-1}^{i}\int_{j-1}^{j}\frac{(xy)^{2H-1}}{(x^{2H}+y^{2H})^{2-K}}dxdy\right)^m \\
&\leq \alpha\sum_{i,j=2}^{N}\left(\int_{i-1}^{i}\int_{j-1}^{j}\frac{(xy)^{2H-1}}{(xy)^{H(2-K)}}dxdy\right)^m = \alpha\left(\sum_{i=2}^{N}\left(\int_{i-1}^{i}x^{HK-1}dx\right)^m\right)^2 \\
&\leq \alpha\left(\sum_{i=1}^{N}i^{m(HK-1)}\right)^2. \qquad (4.22)
\end{aligned}
$$

Since $HK < 3/4$ and $m \geq 2$, $m(HK-1) < -1/2$, and so there exists a $\delta > 0$ such that $m(HK-1) - 1/2 + \delta < -1$. Then

$$
(N^{-1}A_N)^{1/2} \leq \frac{\sqrt{\alpha}}{N^{1/2}}\sum_{i=1}^{N}i^{m(HK-1)} \leq \frac{\sqrt{\alpha}}{N^{\delta}}\sum_{i=1}^{N}i^{m(HK-1)-1/2+\delta}.
$$

Taking $N = m_n$ and letting $n \to \infty$, it follows that the hypothesis (c) holds for $B_{H,K}$. The proof of Corollary 4.8 is complete. □

Corollary 4.9. *Let $HK \in (0, 3/4]$, $F \in \mathcal{F}_1$, $d \in \mathbf{D}$ and for all $n \in \mathbb{N}$ denote*

$$
V_n := \sum_{i=1}^{n}F\left(\frac{\Delta_i^n B_{H,K}}{\sigma_{B_{H,K}}(t_i^n, t_{i-1}^n)}\right) \quad \textit{and} \quad Z_n := \frac{V_n - \mathbf{E}V_n}{\sqrt{\mathrm{var}(V_n)}}.
$$

Then for some $c \in \mathbb{R}$ and all $n \in \mathbb{N}$

$$
d(Z_n, Z) \leq c\begin{cases} n^{-1/2} & \textit{if } HK \in (0,1/2), \\ n^{2HK-3/2} & \textit{if } HK \in [1/2,3/4), \\ (\log n)^{-1/2} & \textit{if } HK = 3/4. \end{cases}
$$

Remark 4.10. The same rates of convergence for the quadratic variation of bifractional

Brownian motion have been obtained in [1] (Theorem 3.2). Since fractional Brownian motion is a special case of a bifractional Brownian motion (when $K = 1$), it follows from Remark 4.3 that the rate for $HK \in (0, 1/2]$ is optimal. Optimality of the rates when $HK \in (1/2, 3/4]$ is still an open problem.

Proof. Let $B_n := (\Delta_i^n B_{H,K}/\sigma_{B_{H,K}}(t_i^n, t_{i-1}^n),\ i = 1, \dots, n)$ and $\alpha_n := (1, \dots, 1) \in \mathbb{R}^n$. Then B_n is a standard Gaussian vector and by (3.2) we get $Z_n = W(B_n, F, \alpha_n)$. Therefore our aim is to apply Theorem 3.6 to Z_n.

For all $i, j \in \{1, \dots, n\}$ we have

$$r_n(i,j) = \mathbf{E}\frac{\Delta_i^n B_{H,K}\Delta_j^n B_{H,K}}{\sigma_{B_{H,K}}(t_i^n, t_{i-1}^n)\sigma_{B_{H,K}}(t_j^n, t_{j-1}^n)}.$$

By (4.21) we have

$$\begin{aligned} T^{-2HK}n^{2HK}\left|\mathbf{E}\Delta_i^n B_{H,K}\Delta_j^n B_{H,K}\right| &\le |r_n(i,j)| \\ &\le 2T^{-2HK}n^{2HK}\left|\mathbf{E}\Delta_i^n B_{H,K}\Delta_j^n B_{H,K}\right|. \end{aligned} \tag{4.23}$$

Let η_{HK} be as in (4.2) with H replaced by HK. It is easy to check that for all $i, j \in \{1, \dots, n\}$

$$T^{-2HK}n^{2HK}\mathbf{E}\Delta_i^n B_{H,K}\Delta_j^n B_{H,K} = \eta_{HK}(|i-j|) + z(i,j), \tag{4.24}$$

where

$$\begin{aligned} z(i,j) :=& 2^{-1}\left[(i^{2H}+j^{2H})^K + ((i-1)^{2H}+(j-1)^{2H})^K \right. \\ & \left. -(i^{2H}+(j-1)^{2H})^K - ((i-1)^{2H}+j^{2H})^K\right]. \end{aligned}$$

In the proof of Theorem 2.1 it was shown that if $HK < 3/4$, then for all $m \ge 1$ we have

$$\sum_{i,j=1}^{n} r_n^m(i,j) \sim nT\left(1 + 2\sum_{k=1}^{\infty}\eta_{HK}^m(k)\right), \quad \text{as } n \to \infty.$$

Consider the case when $HK = 3/4$. By (4.22) for all $p \ge 2$

$$\sum_{i,j=1}^{n} |z(i,j)|^p \trianglelefteq \left(\sum_{k=1}^{n} k^{-p/4}\right)^2 \trianglelefteq n, \quad \text{as } n \to \infty, \tag{4.25}$$

by (1.17).

By (4.25) and (4.15) we have that (4.16) holds for the function z of this corollary, thus by the same arguments as in (4.17) we have for all $m \geq 2$

$$\sum_{i,j=1}^{n} T^{-3m/2} n^{3m/2} \left(\mathbf{E} \Delta_i^n B_{H,K} \Delta_j^n B_{H,K} \right)^m = \sum_{i,j=1}^{n} \eta_{3/4}^m(|i-j|) + o(n \log n) \quad (4.26)$$

as $n \to \infty$. By (3.11), (4.23), (4.26) and (4.15) we thus have

$$\operatorname{var}(V_n) = \sum_{m=2}^{\infty} \frac{a_m^2}{m!} \sum_{i,j=1}^{n} r_n^m(i,j) \sim C n \log n, \quad (4.27)$$

as $n \to \infty$, where $a_m := \mathbf{E} F(Z) H_m(Z),\ m \geq 2$, and the constant C does not depend on n.

In the proof of Theorem 3.1 in [1] it is shown that

$$\sum_{j=1}^{n} r_n^2(i,j) \leq 1 + \sum_{j=1}^{n} j^{4(HK-1)} \quad (4.28)$$

and

$$\sum_{i,k,l=1}^{n} |r_n(i,k) r_n(k,l)| \trianglelefteq \sum_{k=1}^{n-1} k^{4HK-2} + n \left(\sum_{k=1}^{n-1} k^{2HK-2} \right)^2 \quad (4.29)$$

as $n \to \infty$. By (4.28), (4.29), (1.17), (4.27) and Theorem 3.6 there exists a constant $c \in \mathbb{R}$ such that

$$\begin{aligned} d^2(Z_n, Z) &\leq \frac{c}{[\operatorname{var}(V_n)]^2} \max_{1 \leq i \leq n} \sum_{j=1}^{n} r_n^2(i,j) \sum_{i,k,l=1}^{n} |r_n(k,l)||r_n(i,k)| \\ &\trianglelefteq \begin{cases} n^{-1} & \text{if } HK \in (0, 1/2), \\ n^{4HK-3} & \text{if } HK \in [1/2, 3/4), \\ (\log n)^{-1} & \text{if } HK = 3/4, \end{cases} \end{aligned}$$

as $n \to \infty$, finishing the proof. □

4.5 Processes with a local variance

Our next step is to apply Theorem 3.10 to processes having a local variance. Some of the hypotheses of our Theorem 4.11 come from Theorem 2.1. See Remark 2.3 for an explication of these hypotheses.

Theorem 4.11. *Let $T > 0$ and $G := \{G_t,\ t \in [0,T]\}$ be a mean zero Gaussian process from the class $\mathcal{LSI}(\rho)$ with some $\rho \in R[0,T]$, $H \in \mathcal{F}_0$ and*

(a) *there is a finite constant $C_1 > 0$ such that for all $(s,t) \in [0,T]^2$*

$$C_1\rho(|t-s|) \le \sigma_G(s,t); \tag{4.30}$$

(b) *it holds that*

$$\sup_n \left\{ \max_{1\le i\le n} \sum_{j=1}^{n} \left[\mathbf{E}\left(\frac{\Delta_i^n G \Delta_j^n G}{\sigma_G(t_i^n, t_{i-1}^n)\sigma_G(t_j^n, t_{j-1}^n)} \right) \right]^2 \right\} < \infty; \tag{4.31}$$

(c) *for every integer $m \ge 2$ there is a real number Ψ_m such that the following limit exists and the equality*

$$\lim_{n\to\infty} \sum_{k=1}^{y_n} [\eta(k,n)]^m = \Psi_m \tag{4.32}$$

holds for every increasing and unbounded sequence of positive integers $(y_n)_{n\in\mathbb{N}}$ with values $y_n \le n-2$ for each $n \in \mathbb{N}$ (the function η is defined by (4.2));

(d) *for every $p \ge 2$*

$$\lim_{n\to\infty} \frac{1}{n[\rho(1/n)]^{2p}} \sum_{i,j=2}^{n} \left| \Box_{i,j}^n \left[\Gamma_G - 2^{-1}\tilde{\rho} \right] \right|^p = 0,$$

where $\tilde{\rho}(s,t) := -\rho^2(|t-s|)$, $(s,t) \in [0,T]^2$.

Then in the space $\mathcal{D}[0,1]$ equipped with the Skorokhod topology

$$Y^n(G,H) \Rightarrow \lambda_{G,H} W \text{ as } n \to \infty,$$

where Y^n is defined by (6) *and*

$$\lambda_{G,H}^2 := T \sum_{m=2}^{\infty} \frac{a_{H,m}^2}{m!} (1 + 2\Psi_m)$$

and with $H_m,\ m \geq 2$, defined in (1.1) *we denote*

$$a_{H,m} := \mathbf{E} H(Z) H_m(Z). \tag{4.33}$$

Proof. We will apply Theorem 3.10. For that we need to check its hypotheses $(b) - (d)$ since (a) coincides with (4.31). For $n \in \mathbb{N}$ and $(i,j) \in \{1, \ldots, n\}^2$ let

$$r_n(i,j) := \mathbf{E} \left(\frac{\Delta_i^n G \Delta_j^n G}{\sigma_G(t_i^n, t_{i-1}^n) \sigma_G(t_j^n, t_{j-1}^n)}. \right)$$

Let $I = J = \{1, \ldots, n\}$. In the proof of Theorem 2.1 it was shown that under the hypotheses given we have for all $m \geq 2$

$$\frac{1}{n} \sum_{i \in I} \sum_{j \in J} r_n^m(i,j) = \frac{1}{n} \sum_{i \in I} \sum_{j \in J} [\eta_n(|i-j|)]^m + o(1), \tag{4.34}$$

as $n \to \infty$, and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i \in I} \sum_{j \in J} [\eta_n(|i-j|)]^m = T(1 + 2\Psi_m).$$

By similar arguments one would show the equality in (4.34) for any $I, J \subset \{1, \ldots, n\}$, and conclude that for any $0 \leq s < t \leq 1$ and all $m \geq 2$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i,j=[ns]+1}^{[nt]} r_n^m(i,j) = T(t-s)(1 + 2\Psi_m).$$

It was also shown that for every $m \geq 2$, $1 \leq p \leq m-1$ we have

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{i,j,k,l=1}^{n} |r_n^p(i,j) r_n^p(k,l) r_n^{m-p}(i,k) r_n^{m-p}(j,l)| = 0,$$

thus we are left to check the hypothesis (c). Let $s, t, u, v \in [0,1]$ be such that $s < t \leq$

$u < v$. Assume $t = u$, with the case $t < u$ being easier. By (4.34) we can bound

$$\frac{1}{n}\sum_{i=[nu]+1}^{[nv]}\sum_{j=[ns]+1}^{[nt]} r_n^2(i,j) = \frac{1}{n}\sum_{i=[nu]+1}^{[nv]}\sum_{j=[ns]+1}^{[nt]} [\eta_n(|i-j|)]^2 + o(1)$$
$$\leq \frac{1}{n}\sum_{i=1}^{[nv]-[nt]} ([nt]-[ns])\eta_n^2([nt]-[ns]+i-1)$$
$$+ \frac{1}{n}\sum_{i=1}^{[nt]-[ns]} i\eta_n^2(i) + o(1).$$

By (2.30) the second term on the right hand side converges to zero. As for the first one, by (4.32) we can write

$$\sum_{i=1}^{[nv]-[nt]} \eta_n^2([nt]-[ns]+i-1) = \sum_{k=[nt]-[ns]}^{[nv]-[ns]-1} \eta_n^2(k)$$
$$= \sum_{k=1}^{[nv]-[ns]-1} \eta_n^2(k) + \sum_{k=1}^{[nt]-[ns]-1} \eta_n^2(k) \to 0,$$

as $n \to \infty$, which completes the proof. □

Corollary 4.12. *Theorem 4.11 applies to subfractional Brownian motion process with index $H < 3/4$ and bifractional Brownian motion process with parameters (H, K) satisfying $HK < 3/4$.*

Proof. Bifractional Brownian motion with parameters (H, K) belongs to $\mathcal{LSI}(\rho_{H,K})$ with $\rho_{H,K}(u) = 2^{(1-K)/2}u^{HK},\ u > 0$, (see Section 4.4) and subfractional Brownian motion with parameter H belongs to $\mathcal{LSI}(\rho_H)$ with $\rho_H(u) = u^H,\ u > 0$, (see Section 4.3). Hypotheses $(a), (c), (d)$ were checked for subfractional and bifractional Brownian motions in Corollaries 4.5 and 4.8 respectively. Hypothesis (b) holds for subfractional Brownian motion by (4.19) and for bifractional Brownian motion by (4.28). This completes the proof. □

Remark 4.13. Given a process $G \in \mathcal{LSI}(\rho)$ with some $\rho \in R[0, T]$, and a function $H : \mathbb{R} \to \mathbb{R}$ one could consider the modified H-variations of G with $\sigma_G(t_i^n, t_{i-1}^n)$ replaced by $\rho(t_i^n - t_{i-1}^n)$ in (1). If there exists a function $g : \mathbb{R}_+ \to \mathbb{R}_+$, which is bounded on compact intervals, such that $H(xy) = g(x)H(y)$ for all $x, y \in \mathbb{R}$ (which is the case with the power function, for example), and there exists a constant K such that

$\sigma_G(s,t) \geq K\rho(|t-s|)$ for all $(s,t) \in [0,T]^2$, then

$$\sum_{i=1}^{n} H\left(\frac{G(t_i^n) - G(t_{i-1}^n)}{\rho(t_i^n - t_{i-1}^n)}\right) = \sum_{i=1}^{n} g\left(\frac{\sigma_G(t_i^n, t_{i-1}^n)}{\rho(t_i^n - t_{i-1}^n)}\right) H\left(\frac{G(t_i^n) - G(t_{i-1}^n)}{\sigma_G(t_i^n, t_{i-1}^n)}\right), \quad (4.35)$$

and by (4.10) and (4.21) we can apply Theorem 3.6 to bifractional and subfractional Brownian motions to get the same order of the Berry-Esséen bound as in Corollaries 4.6 and 4.9.

If g is also continuous at 1 and $g(1) = 1$ then one can prove an analogue of Theorem 4.11 to bifractional and subfractional Brownian motions with the partial sum process corresponding to (4.35).

Chapter 5

Conclusions

During the doctoral studies at Vilnius University, we have studied the H-variations of Gaussian processes with possibly non-stationary increments. In this last Chapter, a brief summary of the results obtained is given.

- In the second Chapter we investigated the special case of power variations of Gaussian processes having a local variance. We obtained a central limit theorem for these variations, which is a generalization of some previous results.
- In the third Chapter we considered general sequences of standard Gaussian vectors and the weighted H-sums of these vectors. We proved a Berry-Esséen bound for three distances between these H-sums and a standard Gaussian random variable. In this chapter we also considered partial sum processes corresponding to these H-sums and proved a functional central limit theorem for them.
- In the fourth Chapter we applied the results obtained in the previous chapters to fractional Brownian motion, processes with stationary increments, subfractional Brownian motion, bifractional Brownian motion and processes having a local variance. The rate of convergence in the central limit theorem obtained was the same for fractional, subfractional and bifractional Brownian motion and was argued to be optimal in some cases.

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